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EXTREMAL OPERATORS

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The aim of this work is to discern and investigate properties of a special class of operators whose supports play a role which is analogous to the role of extremal subsets of a convex set. The role which such sets play in the theory of integral representations of convex compacta—Choquet theory—is well known, but the approach which we will use and the results which we obtain are new even in classical cases. The exposition will be confined to the theory of Kantorovic spaces* (see [2] and [3]), and we will not discuss the interesting possibilities of further developments and generalizations. In particular, the theory of our class of objects overlaps with the concepts of the Pareto boundary, superextremal subsets of convex compacta, minimal bounds of bounded functionals in Banach spaces, maximal operators, and Choquet boundaries in K-spaces. Hereafter without further explanations we will use the terminology and results of Choquet theory in K-spaces (see [4]). In some instances, for the sake of simplicity we will not present the results in their full generality.

1. Let X be a K-lineal (vector lattice), and let H be a (convex) cone in X which is an upper lattice. The symbol $\succ$ will denote the Choquet ordering generated by H. For a regular operator $T \in \mathfrak{L}(X, Y)$, where Y is a K-space, we let $N(T)$ denote the null lattice of T; that is,

$$N(T)=\{x\in X:\ |T|\,|x|=0\}.$$

Let $\mathfrak{L}$ denote a component (band) in the K-space $\mathfrak{L}(X, Y)$ and, as usual, let $\mathfrak{L}_+$ denote its positive cone.

An operator $T \in \mathfrak{L}_+$ is said to be *extremal* (in $\mathfrak{L}$ with respect to the cone H) if for each $S \in \mathfrak{L}_+$ such that $S \succ T$, we have $N(S) \supset N(T)$. The set of all such operators will be denoted by $\mathfrak{E}(H, \mathfrak{L})$, or simply $\mathfrak{E}$ if it is clear which sets H and $\mathfrak{L}$ are being considered.

Theorem 1. *An upper lattice $\mathfrak{E}$ is a cone in $\mathfrak{L}(X, Y)$. If $0 \in \mathfrak{E}$, then $\mathfrak{E}$ is proper upwards; that is, for an arbitrary subset $U \subset \mathfrak{E}$ which is bounded from above in $\mathfrak{L}(X, Y)$ we have* $\sup U \in \mathfrak{E}$.

A cone $\mathfrak{E}$ need not be proper downwards; that is, it need not contain the infima (in $\mathfrak{L}(X, Y)$) of its own subsets. As a rule $\mathfrak{E}$ is not even a lower lattice. The simplest universal (with respect to H) sufficient condition for being a lattice is that X be normally imbedded in $\mathfrak{L}(\mathfrak{L}(X, Y), Y)$. This condition is equivalent to the condition that X

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**Editor's note.* This is the Russian terminology for a conditionally complete vector lattice.

is a K-space, all operators in $\mathfrak{L}(X, Y)$ are completely linear (order continuous), and operators $T_1, T_2 \in \mathfrak{L}_+(X, Y)$ are disjoint if and only if their components $N(T_1)^d$ and $N(T_2)^d$ of essential positivity are disjoint. It is clear that for $Y \neq R$ such a condition cannot in general be satisfied. We note here that in nontrivial cases $\mathfrak{E}$ is not a normal set.

2. The investigation of extremal operators by the usual methods proceeds by means of a lifting to some "test" K-space Z. Namely, one fixes an operator $T_0 \in \mathfrak{L}_+(X, Z)$, and then an operator $T \in \mathfrak{L}_+(Z, Y)$ is said to be T_0-*extremal* (in $\mathfrak{L}$ with respect to H) if $TT_0 \in \mathfrak{E}(H, \mathfrak{L})$. We let $\mathfrak{E}(T_0)$ or $\mathfrak{E}(T_0, H, \mathfrak{L})$ denote the set of all T_0-extremal operators.

Theorem 2. *In the case $\mathfrak{L} = \mathfrak{L}(X, Y)$ we have the following assertions:*

(1) *If $\overline{H + X_+} = X$, then $\mathfrak{E}(T_0)$ is a proper cone in $\mathfrak{L}(Z, Y)$.*

(2) *If T is completely linear and $T \in \mathfrak{E}(T_0)$, then the projection P_T onto the component $N(T)^d$ is in $\mathfrak{E}(T_0)$.*

(3) *If H is coinitial with X, then $T \in \mathfrak{E}(T_0)$ if and only if $q_{H, TT_0} = q_{H(TT_0), TT_0}$, where for an operator $S \in \mathfrak{L}_+(X, Y)$ we set*

$$q_{M, S}(x) = \sup\{Sm:\ m \in M,\ m \leq x\}$$

for a coinitial cone M in X and $M(S) = M + N(S)$.

(4) *If H is coinitial with X and the operator T is completely linear, then*

$$T \in \mathfrak{E}(T_0) \Leftrightarrow P_T \in \mathfrak{E}(T_0).$$

In applications an important role is played by operators in the class $\mathfrak{E}(T_0, H, \mathfrak{L} \cap \mathfrak{B}(Y))$, where $\mathfrak{B}(Y)$ is a component of the boundary operators (in the sense of Choquet). This class will be denoted by $\mathfrak{E}_s(T_0)$ or in extended form by $\mathfrak{E}_s(T_0, H, \mathfrak{L})$. Operators in $\mathfrak{E}_s(T_0)$ are said to be *superextremal* (in $\mathfrak{L}$ with respect to H). This terminology is connected with the obvious inclusion $\mathfrak{E}_s(T_0) \supset \mathfrak{E}(T_0)$. We note that this inclusion is usually proper.

Theorem 3. *Let T be a completely linear operator and let $\mathfrak{L} = \mathfrak{L}(X, Y)$.*

(1) *If H is coinitial with X and $T \in \mathfrak{E}_s(T_0)$, then $P_T \in \mathfrak{E}_s(T_0)$.*

(2) *If H is a simplicial cone and $P_T \in \mathfrak{E}(T_0)$, then $T \in \mathfrak{E}_s(T_0)$.*

3. The results in the preceding section illustrate the role of extremal projections. In a certain sense fully linear operators are T_0-extremal if and only if their supports (components of essential positivity) are T_0-extremal. In this regard we will discuss some properties of these projections related to typical applications.

Let H be an arbitrary cone in a K-space Z. Let $\mathfrak{L}$ be a component in $\mathfrak{L}(Z, Z)$ which consists of completely linear operators, where $\mathrm{B}(Z) \subset \mathfrak{L}$. Here $\mathrm{B}(Z)$ is a base of the K-space Z; that is, a Boolean algebra of projections in Z. The symbol $\mathrm{Ext}(H, \mathfrak{L})$ will denote the set of extremal projections $\mathrm{B}(Z) \cap \mathfrak{E}(P(H), \mathfrak{L})$, where $P(H)$ is the smallest upper lattice generated by H. We note that $P \in \mathrm{Ext}(H, \mathfrak{L})$ if and only if for each $A \in \mathfrak{L}_+$ such that $Ah \geq Ph$ for all $h \in H$ we have $AP = A$.

Theorem 4. *The set $\mathrm{Ext}(H, \mathfrak{L})$ is a proper downwards sublattice of the base $\mathrm{B}(Z)$. If $0 \in \mathrm{Ext}(H, \mathfrak{L})$, then $\mathrm{Ext}(H, \mathfrak{L})$ is a proper sublattice of $\mathrm{B}(Z)$.*

This theorem can serve as a foundation for various topological considerations in connection with extremal projections. We will describe such a possibility in a very simple case. Let Z, a discrete K-space, be a fundament (order dense ideal) in a product of lines R^Q. Suppose that $0 \in \mathrm{Ext}(H, \mathfrak{L})$,aand let the closed sets in Q be precisely those sets which correspond to extremal projectors. The corresponding topology in Q is called the *extremal*, or *strongest, boundary topology*. It is clear that this topology coincides with the right interval topology for some uniquely determined pre-order relation on Q. In Choquet theory and convex analysis we are often interested in the boundary structure of Q. For the corresponding H, $\mathfrak{L}$ and Z this is the quotient set of Q modulo an equivalence relation connected with the indicated pre-order.

4. In conclusion we present some facts concerning the case of extremal Radon measures. These facts can be regarded as an illustration of the preceding results, but unfortunately their proof requires special techniques.

Let H be an adapted cone in the space $C(Q)$ of continuous functions on a metrizable compactum Q. We will be interested in *extremal* and *superextremal measures*; that is, elements of the sets $\mathfrak{E} = \mathfrak{E}(H, C'(Q))$ and $\mathfrak{E}_s = \mathfrak{E}_s(H, C'(Q))$. When we speak of extremal or superextremal sets in Q we will have in mind extremality or superextremality of the corresponding projections in R^Q. The symbol $\mathrm{Spr}(\epsilon_x, H)$ will denote, as usual, the positive sprout of Dirac measures ϵ_x on the cone H.

Theorem 5. *If the Choquet boundary* $\mathrm{Ch}(H)$ *is closed and the mapping* $x \mapsto \mathrm{Spr}(\epsilon_x, H) \cap \mathfrak{B}(R)$ *is lower semicontinuous in the vague Hausdorff topology, then a measure is in* $\mathfrak{E}_s$ *if and only if its support is superextremal.*

Theorem 6. *If the mapping* $x \mapsto \mathrm{Spr}(\epsilon_x, H)$ *is lower semicontinuous in the vague Hausdorff topology, then a measure is in* $\mathfrak{E}$ *if and only if its support is extremal.*

In the case of sufficiency the assertions in Theorems 5 and 6 are true without any additional assumptions about continuity of the corresponding H-convex hulls. In the case of necessity in general it is not so simple for negligible sets. We can assert that the support of a superextremal measure contains a superextremal subset of full measure. This can be shown even in the case where the whole support is not a superextremal set.

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