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CONVEX ϵ-PROGRAMMING

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Let X be a vector space, and let $Y \cup \{+\infty\}$ be an ordered vector space Y with adjoined greatest element $+\infty$. Consider a convex operator $F: X \longrightarrow Y \cup \{+\infty\}$, a point x belonging to the effective domain $\operatorname{dom}(F) = \{x \in X |\ Fx < +\infty\}$ and a positive element $\epsilon \in Y^+$. The set

$$\partial_{x,\epsilon}(F) = \{ A \in L(X, Y): Ax' - Ax \leqslant Fx' - Fx + \epsilon, x' \in X \},$$

where $L(X, Y)$ is the space of linear operators from X into Y, is called the *ϵ-subdifferential* of F at x. The point x is called *ϵ-optimal* for F if $0 \in \partial_{x,\epsilon}(F)$. In this paper, we announce some formulae to calculate ϵ-subdifferentials and corresponding ϵ-optimality criteria which show that suitable versions of the Lagrange principle are valid for ϵ-programming. The classical convex programming theory arises naturally if we set $\epsilon = 0$.

Composition of convex operators. Let $G: X \longrightarrow Z \cup \{+\infty\}$ be an increasing convex operator, and let Z be a K-space.* If $F[\operatorname{dom}(F)]$ contains an interior point of $\operatorname{dom}(G)$, then for any $\epsilon \in Z^+$

$$\partial_{x,\epsilon}(G \circ F) = \bigcup_{\substack{\epsilon_1 \geqslant 0,\, \epsilon_2 \geqslant 0 \\ \epsilon_1 + \epsilon_2 = \epsilon}} \ \bigcup_{B \in \partial_{Fx',\, \epsilon_1}(G)} \partial_{x,\epsilon_2}(B \circ F).$$

Sum of convex operators. Let $F_1, \ldots, F_n: X \longrightarrow X \cup \{+\infty\}$ be convex operators, and let Y be a K-space. If the domains of the Hörmander transforms of $F_1, \ldots, F_n$ are in general position [1], then

$$\partial_{x,\epsilon}(F_1 + \ldots + F_n) = \bigcup_{\substack{\epsilon_1 \geqslant 0, \ldots, \epsilon_n \geqslant 0 \\ \epsilon_1 + \ldots + \epsilon_n = \epsilon}} (\partial_{x,\epsilon_1}(F_1) + \ldots + \partial_{x,\epsilon_n}(F_n)).$$

For scalar functions on a finite dimensional space, this formula was announced in [2].

Maximum of convex operators. Let Y be a vector lattice, and let $F_1, \ldots, F_n$: $X \longrightarrow Y \cup \{+\infty\}$ be convex operators whose Hörmander transforms have domains in general position. If Z is a K-space and $A \in L^+(Y, Z)$ is a positive linear operator, then for any $\epsilon \in Z^+$

$$\partial_{x,\epsilon}(A \circ (F_1 \vee \ldots \vee F_n)) =$$

$$= \Big\{ \sum_{k=1}^{n} \partial_{x,\epsilon_k}(A_k \circ F_k): A_k \in L^+(Y, Z), \ \sum_{k=1}^{n} A_k = A;$$

$$\epsilon_k \geqslant 0, \ \sum_{k=1}^{n} \epsilon_k = \epsilon; \ \sum_{k=1}^{n} A_k \circ F_k x \geqslant A \circ F_1 x \vee \ldots \vee F_n x - \epsilon_{n+1} \Big\}.$$

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**Translator's note.* In Western linteature, K-spaces are usually called conditionally (or boundedly) complete vector lattices.

Composition with an affine operator. Let X, X_1 be vector spaces, let Y be a K-space, and let $F\colon X \longrightarrow Y \cup \{+\infty\}$ be a convex operator whose domain contains an interior point belonging to the range of an affine mapping $A_x\colon x_1 \longrightarrow Ax_1 + x$, where $A \longrightarrow L(X_1, X)$, $x \in X$. Then

$$\partial_{x_1,\epsilon}(F \circ A_x) = \partial_{A_x x_1,\epsilon}(F) \circ A,$$

whenever $x_1 \in X_1$ is such that $A_x x_1 \in \mathrm{dom}(F)$.

Composition with a regular convex operator. Let X be a vector space, let Y be a K-space, and let $\mathfrak{A}$ be a weakly order bounded set in $L(X, Y)$. As usual, the symbol $\langle \mathfrak{A} \rangle$ denotes the linear operator which carries X into the K-space $(Y^{\mathfrak{A}})_\infty$ (the space of order bounded Y-valued functions on $\mathfrak{A}$) according to the rule $\langle \mathfrak{A} \rangle x\colon A \longmapsto Ax$, $A \in \mathfrak{A}$. The symbol $\Delta_{\mathfrak{A}}$ stands for the natural identification of Y with the diagonal of $(Y^{\mathfrak{A}})_\infty$ and $\epsilon_{\mathfrak{A}}$ stands for the canonical sublinear operator

$$\epsilon_{\mathfrak{A}}\colon (Y^{\mathfrak{A}})_\infty \to Y;\ \ \epsilon_{\mathfrak{A}} f = \sup\{f(A)\colon A \in \mathfrak{A}\}.$$

Now assume that $F\colon X \longrightarrow Y \cup \{+\infty\}$ is a regular convex operator, i.e. one which can be represented as $F = \epsilon_{\mathfrak{A}} \circ \langle \mathfrak{A} \rangle_y$ for suitably chosen $\mathfrak{A} \subset L(X, Y)$, $y \in (Y^{\mathfrak{A}})_\infty$. Further assume that Z is a K-space and $G\colon Y \longrightarrow Z \cup \{+\infty\}$ is an increasing convex mapping. If the range $F[X]$ contains an interior point of the effective domain $\mathrm{dom}(G)$ and $Fx \in \mathrm{dom}(G)$ for certain $x \in X$, then for any $\epsilon \in Z^+$

$$\partial_{x,\epsilon}(G \circ F) = \{B \circ \langle \mathfrak{A} \rangle\colon B \circ \Delta_{\mathfrak{A}} \in \partial_{Fx,\epsilon-\epsilon'}(G);\ B \in L^+((Y^{\mathfrak{A}})_\infty, Z);$$
$$0 \leqslant B \circ \Delta_{\mathfrak{A}} Fx - B \circ \langle \mathfrak{A} \rangle_y x \leqslant \epsilon' \leqslant \epsilon\}.$$

ϵ-optimality for regular programs. Consider a regular convex program

$$Gx \leqslant 0,\ \ Fx \to \inf.$$

In other words, $G, F\colon X \longrightarrow Y \cup \{+\infty\}$ are convex operators and Y is assumed to be a K-space. For simplicity we also assume that $\mathrm{dom}(F) = \mathrm{dom}(G) = X$ and for any $x \in X$ either $Gx \leqslant 0$ or $Gx \geqslant 0$. Finally, we assume that there is $x_0 \in X$ such that $-Gx_0$ is a unit in Y.

A feasible point x is ϵ-optimal for a regular problem if and only if the following system of conditions is consistent:

$$\alpha, \beta \in L^+(Y, Y);\ \ \alpha + \beta = I_Y;\ \ \mathrm{Ker}(\alpha) = \{0\};$$
$$\epsilon_1 \geqslant 0,\ \ \epsilon_2 \geqslant 0;\ \ \epsilon_1 + \epsilon_2 \leqslant \alpha\epsilon + \beta \circ Gx;$$
$$0 \in \partial_{x,\epsilon_1}(\alpha \circ F) + \partial_{x,\epsilon_2}(\beta \circ G).$$

Here I_y is the identity operator in Y.

ϵ-optimality for Slater-regular problems. Consider the convex problem

$$Ax = Ax_0,\ \ Gx \leqslant 0,\ \ Fx \to \inf,$$

where X_1, X are vector spaces, $A \in L(X, Y)$ is a linear operator, $G\colon X \longrightarrow Z \cup \{+\infty\}$ and $F\colon X \longrightarrow Y \cup \{+\infty\}$ are convex operators. For simplicity we assume that $\mathrm{dom}(G) = \mathrm{dom}(F) = X$. Now assume that the problem under consideration is Slater regular; that is, Z is an Archimedean ordered vector space, T is a K-space of bounded elements, and there is a feasible point x_0 such that $-Gx_0$ belongs to the interior of the cone Z^+.

A feasible point x is ϵ-optimal in a Slater regular program if and only if the following

*Translator's note. In other words $Y = C(Q)$ for a certain extremally disconnected compact Q.

system of conditions is consistent:

$$\gamma \in L^+(Z, Y); \ \mu \in L(X_1, Y);$$

$$\epsilon_1 \geqslant 0, \ \epsilon_2 \geqslant 0; \ \epsilon_1 + \epsilon_2 \leqslant \gamma \circ Gx + \epsilon;$$

$$0 \in \partial_{x,\epsilon_1}(F) + \partial_{x,\epsilon_2}(\gamma \circ G) + \mu \circ A.$$

Pareto ϵ-optimality. Consider a Slater regular program and a positive number ϵ. A feasible point x is called *Pareto ϵ-optimal* (with respect to a strong unit **1** in Y) if for any feasible point x' such that $Fx' - Fx \leqslant -\epsilon\mathbf{1}$ we have $Fx' = Fx - \epsilon\mathbf{1}$.

If a feasible point x is Pareto ϵ-optimal for a Slater regular program and $0 \leqslant \epsilon < 1$, then there are linear functionals α, β, γ on Y, Z, and X_1 respectively for which the following system of conditions is consistent

$$\alpha > 0, \ \beta \geqslant 0; \ \epsilon_1 \geqslant 0, \ \epsilon_2 \geqslant 0;$$

$$\epsilon_1 + \epsilon_2 \leqslant \epsilon + \beta \circ Gx;$$

$$0 \in \partial_{x,\epsilon_1}(\alpha \circ F) + \partial_{x,\epsilon_2}(\beta \circ G) + \gamma \circ A.$$

Conversely, if these conditions are fulfilled for a feasible point x and $\alpha(\mathbf{1}) = 1$, then x is Pareto ϵ-optimal.

Generalized ϵ-solutions. Let Y be a K-space, and let $F_0: X \longrightarrow Y \cup \{+\infty\}$ be a convex operator. Let a convex set U_0 be contained in $\mathrm{dom}(F_0)$. A subset $U \subset U_0$ is called a *generalized ϵ-solution* to the program $x \in U_0$, $F_0x \longrightarrow \inf$ if $\inf F_0[U_0] \geqslant \inf F_0[U] - \epsilon$.

Consider the space X^U and the operator

$$F: X^U \to Y^U \cup \{+\infty\}; \ F\mathfrak{X}: x \to F_0\mathfrak{X}(x).$$

Let $\mathfrak{X}: x \longrightarrow x$ and assume that for any $\mathfrak{X}_0$ belonging to $(\mathrm{dom}(F_0))^U$ the relation $F\mathfrak{X}_0 \in (Y^U)_\infty$ holds. Assume also that $\mathfrak{X}$ is an interior point of $(\mathrm{dom}(F_0))^U$.

A set U is a generalized ϵ-solution to the program $x \in U_0$, $F_0x \longrightarrow \inf$ if and only if the following system of conditions is consistent

$$\alpha \in L^+((Y^U)_\infty, Y); \ \ \alpha \circ \Delta_U = I_Y;$$

$$\alpha \circ F\mathfrak{X} = \inf_{x \in U} F_0x; \ \ \epsilon_1 \geqslant 0, \ \epsilon_2 \geqslant 0, \ \ \epsilon_1 + \epsilon_2 = \epsilon;$$

$$0 \in \partial_{\mathfrak{X},\epsilon_1}(\alpha \circ F) + \partial_{\mathfrak{X},\epsilon_2}(\delta_Y((U_0)^U)).$$

As usual, $\delta_Y(V)$ is the indicator operator of the set V.

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BIBLIOGRAPHY

1. G. P. Akilov and S. S. Kutateladze, *Ordered vector spaces*, Novosibirsk, 1978. (Russian)

2. V. F. Dem'janov and V. K. Šomesova, Dokl. Akad. Nauk SSSR **242** (1978), 753; English transl. in Soviet Math. Dokl. **19** (1978).

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