Докл. Акад. Наук СССР
Том 251 (1980), № 2

Soviet Math. Dokl.
Vol. 21 (1980), No. 2

ROCKAFELLAR CONVOLUTION AND A CHARACTERIZATION OF OPTIMAL TRAJECTORIES

UDC 513.88 + 519.95

A. G. KUSRAEV AND S. S. KUTATELADZE

In the present paper we announce formulas for the calculation of the Young transformation and the ϵ-subdifferential of the Rockafellar convolution (composition of bifunctions [1]) and the characteristics of optimal trajectories connected with it in dynamical problems [2], [3]. It is shown how the natural qualification of general position [4] of convex sets in pseudotopological vector spaces guarantees the applicability of the algebraic techniques of local analysis of convex operators developed in [5] and [6] in the continuous case.

Preliminary conventions. Everywhere in what follows the term *vector space* means pseudotopological vector space. Analogously, by a *K-space* we mean a pseudotopological K-space* with normal cone of positive elements. The term *linear operator* means continuous linear operator. The symbols $\mathcal{L}(X, Y)$ and $\mathcal{L}^+(X, Y)$ mean the set of linear operators and the set of positive linear operators acting from X to Y.

The space $\mathcal{L}(X_1, Y) \times \mathcal{L}(X_2, Y)$ is identified with $\mathcal{L}(X_1 \times X_2, Y)$ by the formula $(A_1, A_2)(x_1, x_2) = A_1x_1 - A_2x_2$.

For a K-space Y the symbol $Y^{\cdot}$ denotes Y with the adjunction of a largest element; moreover, for a mapping $F: X \longrightarrow Y^{\cdot}$, as usual, $\operatorname{dom}(F) = \{x \in X: Fx \in Y\}$ is the *effective set* of F.

General position of convex sets. Let H_1 and H_2 be cones in a vector space X. The *cones* H_1 *and* H_2 *are in general position* if, first, they are in general position algebraically, i.e. $H_1 - H_2 = H_2 - H_1$, second, the space $X_0 = H_1 - H_2$ is complemented in X, and third, every filter in X_0 converging to zero contains a filter with base comprised of the sets

$$(G \cap H_1 - G \cap H_2) \cap (G \cap H_2 - G \cap H_1),$$

where G runs through a base of a filter in X_0 converging to zero.

The *convex sets* G_1 and G_2 in the space X are in *general position* if their Hörmander transforms are in general position, i.e. the conic hulls of the sets $G_1 \times \{1\}$ and $G_2 \times \{1\}$ are in general position in $X \times R$. The latter sets are effective sets of the Hörmander transforms of the indicator operators of G_1 and G_2.

As in [5], the general position of a finite family of sets $G_1, \ldots, G_n$ can be defined in a natural way. We note that, in particular, a sufficient condition for $G_1, \ldots, G_n$ to be in general position is the presence of an interior point of each of them in their intersection, with the possible exception of one G_i. It is also well known that in a Fréchet space X the algebraic general position of the closed cones H_1 and H_2 such that the subspace $H_1 - H_2$ is complemented in X guarantees the general position of the cones H_1 and H_2 in the sense indicated above.

Moreau's formula. *The general position of the graphs of the convex operators* $F_1, \ldots, F_n$: $X \longrightarrow Y'$ *guarantees the sharpness of Moreau's formula in the class of continuous linear*

1980 *Mathematics Subject Classification.* Primary 52A05; Secondary 90A16.

* *Editor's note.* The underlying algebraic structure is that of a conditionally complete vector lattice.

operators. In other words, if

$$F^*A = \sup\{Ax - Fx:\ x \in \operatorname{dom}(F)\},\quad A \in \mathcal{L}(X, Y),$$

is the Young transform of the operator F, then for any $A \in \operatorname{dom}((F_1 + \cdots + F_n)^*)$ *there exist operators* $A_1, \ldots, A_n \in \mathcal{L}(X, Y)$ *such that*

$$A = A_1 + \ldots + A_n,\qquad (F_1 + \ldots + F_n)^*A = F_1^*A_1 + \ldots + F_n^*A_n.$$

Rockafellar convolution. Let X, X_1 and X_2 be vector spaces, let Y be a K-space, and let $F_1: X_1 \times X \longrightarrow Y^{\cdot}$ and $F_2: X \times X_2 \longrightarrow Y^{\cdot}$ be convex operators. We assume that the formula

$$F_1 \vartriangle F_2(x_1, x_2) = \inf_{x \in X} (F_1(x_1, x) + F_2(x, x_2))$$

defines an operator from $X_1 \times X_2$ into Y with effective domain

$$\operatorname{dom}(F_1 \vartriangle F_2) = \operatorname{dom}(F_2) \circ \operatorname{dom}(F_1).$$

It is clear that $F_1 \vartriangle F_2$ is convex. This operator is called the *Rockafellar convolution* of F_1 and F_2.

If in the space $X_1 \times X \times X_2 \times Y$ *the sets*

$$\{(x_1, x, x_2, y):\ y \geqslant F_1(x_1, x)\},\qquad \{(x_1, x, x_2, y):\ y \geqslant F_2(x, x_2)\}$$

are in general position, then

$$(F_1 \vartriangle F_2)^* = F_1^* \vartriangle F_2^*.$$

Moreover, the latter formula is best possible, i.e. for any $A_1 \in \mathcal{L}(X_1, Y)$ and $A_2 \in \mathcal{L}(X_2, Y)$ there exists an $A \in \mathcal{L}(X, Y)$ such that

$$(F_1 \vartriangle F_2)^*(A_1, A_2) = F_1^*(A_1, A) + F_2^*(A, A_2).$$

If, moreover, the Rockafellar convolution $F_1 \vartriangle F_2$ *is best possible at some point* $(x_1, x_2) \in \operatorname{dom}(F_1 \vartriangle F_2)$, *i.e. for some* $x \in X$ *such that* $(x_1, x) \in \operatorname{dom}(F_1)$ *and* $(x_1, x_2) \in \operatorname{dom}(F_2)$ *we have*

$$F_1 \vartriangle F_2(x_1, x_2) = F_1(x_1, x) + F_2(x, x_2),$$

then for every positive element ϵ *in Y*

$$\partial_\epsilon(F_1 \vartriangle F_2)(x_1, x_2) = \bigcup_{\substack{\epsilon_1 \geqslant 0,\ \epsilon_2 \geqslant 0 \\ \epsilon_1 + \epsilon_2 = \epsilon}} \partial_{\epsilon_2}(F_2)(x, x_2) \circ \partial_{\epsilon_1}(F_1)(x_1, x).$$

Here, as usual, for the operator $F: X \longrightarrow Y^{\cdot}$ the set

$$\partial_\epsilon(F)(x) = \{A \in \mathcal{L}(X, Y):\ Ax_0 - Ax \leqslant Fx_0 - Fx + \epsilon,\quad x_0 \in X\}$$

is defined and called the ϵ*-subdifferential* of F at the point $x \in \operatorname{dom}(F)$.

The validity of the formulas of local convex analysis. If we identify a convex process with its indicator operator, then to the superposition of convex processes there corresponds the Rockafellar convolution. Besides, the graph of the superposition of a convex operator with an increasing convex operator coincides with the superposition of the graphs of the operators under consideration. Consequently, the above facts concerning the Rockafellar convolution contain a rule for calculating the Young transform of a superposition and the corresponding chain rule for determining the subdifferentials. This enables us to draw the following conclusion:

All formulas of local convex analysis—the rules of the change of variables in the Young transformation, the formulas for the calculation of ϵ-subdifferentials, criteria for optimality of convex programs, and so on—hold in pseudotopological spaces. The formulation of these formulas mentioned in [5] and [6] can be preserved verbatim in the general case taking account of the preliminary conventions made above.

A characteristic of optimal trajectories. For the sake of simplicity, consider a finite-step terminal convex dynamical problem in the form as in [3]. Let $X_0, X_1, \ldots, X_n$ be vector spaces and $G_i \subset X_{i-1} \times X_i$ convex processes, $i = 1, \ldots, n$. Consider the resulting dynamical family of processes

$$G_{i,j} = G_j \circ \ldots \circ G_{i+1}, \quad j > i+1$$

$$G_{i,i+1} = G_{i+1}, \quad i = 0, 1, \ldots, n-1.$$

As usual, the trajectory of the family is, by definition, a collection of elements $(x_0, x_1, \ldots, x_n)$ such that $x_j \in G_{ij}(x_i)$ for $i < j \leqslant n$. Let $F: X \to Y^{\cdot}$ be a convex operator, ϵ a positive element of Y and $x_0 \in X_0$. Consider the program

$$x \in G_{0,n}(x_0), \quad Fx \to \inf.$$

The trajectories corresponding to the ϵ-solutions of this program are called the *ϵ-optimal trajectories* of the initial dynamical family of processes. In other words, a trajectory $(x_0, x_1, \ldots, x_n)$ is ϵ-optimal if and only if for any trajectory $(t_0, t_1, \ldots, t_n)$ we have the inequality $Fx_n \leqslant Ft_n + \epsilon$. An n-tuple $(A_1, \ldots, A_n)$ of continuous linear operators, where $A_i \in \mathcal{L}(X_i, Y)$, is called an *$\epsilon$-characteristic* of the trajectory $(x_0, x_1, \ldots, x_n)$ if there are positive elements $\epsilon_1, \ldots, \epsilon_n$ such that $\epsilon_1 + \cdots + \epsilon_n \leqslant \epsilon$ and

$$A_i x - A_j t \leqslant A_i x_i - A_j x_j + \epsilon_{i+1} + \ldots + \epsilon_{j+1}, \quad (x, t) \in G_{i,j}.$$

We assume that, first, the graph of the operator F and the set $G_{0,n}(x_0) \times Y$ are in general position, second, the sets $G_{0,n}$ and $X_0 \times \{x_n\}$ are in general position, and third, for every $i = 1, \ldots, n$ the sets $G_i \times X_n$ and $X_{i-1} \times G_{i,n}$ are in general position. Under the above regularity assumptions we have the following assertion.

A trajectory $(x_0, x_1, \ldots, x_n)$ is ϵ-optimal if and only if for some $0 \leqslant \delta \leqslant \epsilon$ it admits a δ-characteristic $(A_1, \ldots, A_n)$ such that $A_n \in \partial_{\epsilon-\delta}(F)(x_n)$.

Institute of Mathematics
Siberian Branch, Academy of Sciences of the USSR
Received 11/NOV/79

BIBLIOGRAPHY

1. R. T. Rockafellar, *Convex analysis,* Princeton Univ. Press, Princeton, N.J., 1970.

2. V. L. Makarov and A. M. Rubinov, *Mathematical theory of economic dynamics and equilibria,* "Nauka", Moscow, 1973; English transl., Springer-Verlag, 1977.

3. A. M. Rubinov, *Superlinear multivalued mappings and their applications to economic-mathematical problems,* Moscow, 1980. (Russian)

4. A. G. Kusraev, *On the subdifferentiation of nonsmooth operators,* Novosibirsk, 1979. (Russian)

5. S. S. Kutateladze, Uspehi Mat. Nauk **34** (1979), no. 1 (205), 167; English transl. in Russian Math. Surveys **34** (1979).

6. ———, Dokl. Akad. Nauk SSSR **245** (1979), 1048; English transl. in Soviet Math. Dokl. **20** (1979).