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MODULES ADMITTING CONVEX ANALYSIS

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S. S. KUTATELADZE

The purpose of the present paper is to give a complete description of ordered unitary modules over lattice-ordered rings which admit convex analysis. The problem of constructing a convex analysis in modules has arisen because of the demands of the classical scalar theory of extremal problems. The fact is that Young transforms of convex operators have the property of modular convexity over the ideal center of the range. The operator convexity of subdifferentials reflects this fact. It is clear that any question of convex analysis in modules is in the end a problem concerning majorized extension of a modular homomorphism (cf. [1]). There are several theorems of this sort (cf. [2]). The general deficiency of these theorems is indicated below. Namely, it is established that there exists no specific modular convex analysis. More precisely, taking account of elementary stipulations, convex analysis exists if and only if we speak of K-spaces* considered as modules over rings of orthomorphisms; in this connection, additive support operators of a modularly sublinear operator turn out to be automatically modular homomorphisms.

1°. Let Y be a K-space and I_Y the identity operator in Y. We denote by $\mathrm{Orth}(Y)$ the component generated by I_Y in the K-space $L^r(Y)$ of regular operators. The elements of $\mathrm{Orth}(Y)$ are called *orthomorphisms* [3]. In $\mathrm{Orth}(Y)$ the smallest normal subspace $Z(Y)$ containing I_Y is distinguished. This subspace is called the *ideal center* of Y. We note that with their natural structures, $\mathrm{Orth}(Y)$ and $Z(Y)$ are function algebras; in this connection $Z(Y)$ is a foundation in $\mathrm{Orth}(Y)$ (i.e. an order dense ideal), and $\mathrm{Orth}(Y)$ is the centralizer of $Z(Y)$ in the ring $L^r(Y)$.

PROPOSITION 1. *For a positive operator $T \in L^r(Y)$ the following assertions are equivalent*:

(1) *T is an orthomorphism.*

(2) *$T + I_Y$ is a lattice homomorphism.*

(3) *$T + I_Y$ admits the Maharam property.*

2°. Let A be an arbitrary lattice-ordered ring (with positive identity). Moreover, let X be an A-module and Y an ordered A-module. All modules in this paper are assumed to be unitary. Let us adjoin a largest element $+\infty$ to Y, set $Y^\cdot = Y \cup \{+\infty\}$, and equip $Y^\cdot$ with the natural A^+-module structure. An operator $p: X \longrightarrow Y^\cdot$ is said to be *A-sublinear* if

$$p(\pi_1 x_1 + \pi_2 x_2) \leqslant \pi_1 p(x_1) + \pi_2 p(x_2), \quad x_1, x_2 \in X; \quad \pi_1, \pi_2 \in A^+.$$

The operator p is said to be *A^+-homogeneous* if $p(\pi x) = \pi p(x)$ for all $x \in X$ and $\pi \in A^+$. The symbol $\mathrm{Hom}_A(X, Y^\cdot)$ denotes the collection of those A-sublinear operators $p: X \longrightarrow Y^\cdot$

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Editor's note. In the Russian literature a K-space is a conditionally complete vector lattice.

for which $\mathrm{dom}(p) = \{x \in X: p(x) < +\infty\}$ is a submodule of X and the trace of p on $\mathrm{dom}(p)$ is an A-homomorphism. The set $\mathrm{Hom}_A(X, Y^{\cdot})$ is also equipped with a natural A^+-multimodule structure. For any sublinear operator $p: X \longrightarrow Y^{\cdot}$ we define a subdifferential and a subdifferential at a point:

$$\partial^A(p) = \{T \in \mathrm{Hom}_A(X, Y^{\cdot}): \; Tx \leqslant p(x), \; x \in X\};$$

$$\partial_x^A(p) = \{T \in \partial^A(p): \; Tx = p(x)\}.$$

Since X and Y are $\mathbf{Z}$-modules, the subdifferential $\partial^{\mathbf{Z}}(p)$ is defined; we will denote it simply by $\partial(p)$.

We say that an A-module Y has the *A-extension property* if for any A-sublinear operator $p: X \longrightarrow Y^{\cdot}$ such that $\mathrm{dom}(p) = X$ and any A-submodule X_0 of X we have the following nonsymmetric Hahn-Young formula:

$$\partial^A(p + \delta_Y(X_0)) = \partial^A(p) + \partial^A(\delta_Y(X_0)),$$

where, as usual, $\delta_Y(X_0)$ is the indicator operator of X_0. If, in addition, for every $x \in X$ the subdifferential $\partial_x^A(p)$ is not empty, then we say that *the A-module Y admits convex analysis.*

3°. Groups obtained from K-spaces by ignoring multiplications by real numbers are called *erased K-spaces.*

We agree that the symbol Y_b will denote the group $Y^+ - Y^+$, where Y^+ is the semigroup of positive elements in Y.

PROPOSITION 2. *If an ordered A-module Y has the A-extension property, then Y_b is an erased K-space.*

PROPOSITION 3. *If an A-module Y has the A-extension property and for all $y \in Y^+$*

$$\pi_1 \pi_2^+ y = (\pi_1 \pi_2)^+ y, \qquad \pi_2^+ \pi_1 y = (\pi_2 \pi_1)^+ y, \qquad \pi_1 \in A^+, \qquad \pi_2 \in A,$$

then the natural linear representation of A in Y_b is a lattice homomorphism. Moreover, for every $\pi \in A^+$ the Maharam property holds: $\pi[0, y] = [0, \pi y], y \in Y^+$.

REMARK. An analogue of Proposition 2 was established by A. D. Ioffe in the theory of fans. Concerning the case $A = \mathbf{Z}$, see [4], as well.

4°. From the naive approach to the Hahn-Banach formula [5] we obtain the following assertion, needed in what follows:

PROPOSITION 4. *Let X be an Abelian group ($\mathbf{Z}$-module) and Y an ordered $\mathbf{Z}$-module. Let Y_b be a K-space. If $p: X \longrightarrow Y^{\cdot}$ is a $\mathbf{Z}$-sublinear operator and* $\mathrm{dom}(p) = X$, *then the operator* $p_h(x) = \sup\{Tx: T \in \partial(p)\}$ *is $\mathbf{Z}$-sublinear and $\mathbf{Z}^+$-homogeneous. Then*

$$\mathrm{dom}(p_h) = \mathrm{dom}(p), \qquad \partial(p) = \partial(p_h), \qquad \partial(p)(x) = [-p_h(-x), p_h(x)].$$

Moreover, for every $x \in X$ the set $\partial_x(p_h)$ is not empty, and

$$\partial_x(p_h) = \partial((p_h)'(x)), \qquad (p_h)'(x)(x') = \inf_{m \in \mathbf{N}} (p_h(mx + x') - p_h(mx)).$$

COROLLARY. *The set $\partial(p)$ is the smallest subdifferential containing all boundary points of $\partial(p)$ (cf.* [6]).

5°. In the present section Y is an ordered A-module, Y_b is a K-space, and A is a subring and sublattice of the ring of orthomorphisms of Y_b naturally acting in Y_b. As usual, the symbol $(Y^{\mathfrak{A}})_\infty$, where $\mathfrak{A}$ is an arbitrary set, denotes the set of bounded Y-valued functions on $\mathfrak{A}$. This set is equipped with a natural A-module structure (it is an A-submodule of the product $Y^{\mathfrak{A}}$).

PROPOSITION 5. *Let the positive operator* $\alpha: (Y^{\mathfrak{A}})_\infty \longrightarrow Y$ *be such that* $\alpha\Delta_{\mathfrak{A}} = I_Y$, *where* $\Delta_{\mathfrak{A}}: Y \longrightarrow (Y^{\mathfrak{A}})_\infty$ *is the natural identification of* Y *with the diagonal of* $Y^{\mathfrak{A}}$. *Then* α *is an* $\mathrm{Orth}(Y_b)$*-homomorphism.*

PROPOSITION 6. *For any A-sublinear operator* $p: X \longrightarrow Y^{\cdot}$ *with* $\mathrm{dom}(p) = X$

$$\partial(p) = \partial^A(p) = \partial^A(p_h) = \partial(p_h).$$

COROLLARY. *The operator* p_h *is* A^+*-homogeneous.*

PROPOSITION 7. *Let the A-sublinear operators* $p_1, p_2: X \longrightarrow Y^{\cdot}$ *be such that their effective sets* $\mathrm{dom}(p_1)$ *and* $\mathrm{dom}(p_2)$ *are in general position in the strong sense, i.e., for any A-submodule* X_0 *in* X *containing* $\mathrm{dom}(p_1) \cap \mathrm{dom}(p_2)$, *the equality*

$$X_0 = \mathrm{dom}(p_1) \cap X_0 - \mathrm{dom}(p_2) \cap X_0$$

is satisfied. Then the Hahn-Banach symmetric formula holds:

$$\partial^A(p_1 + p_2) = \partial^A(p_1) + \partial^A(p_2).$$

PROPOSITION 8. *Let* $\pi \in A^+$ *and* $\pi \geqslant I_{Y_b}$. *For* $\gamma \in A^+$ *set*

$$[\pi^{-1}]\gamma = \inf\{\delta \in A^+: \ \delta\pi \geqslant \gamma\}.$$

Then $[\pi^{-1}]$ *is an increasing A-sublinear operator, and* $\gamma = [\pi^{-1}]\pi\gamma$ *for all* $\gamma \in A^+$.

A ring A is said to be *almost rational* if for every $m \in \mathbf{N}$ there is a decreasing net (π_ξ) of elements of A such that

$$0 \leqslant \pi_\xi \leqslant I_{Y_b}, \quad o\text{-}\lim_\xi \pi_\xi y = (1/m)y, \qquad y \in Y_b.$$

PROPOSITION 9. *Let the ring A be almost rational. Any A-sublinear operator is* $\mathbf{Z}^+$*-homogeneous.*

6°. By means of the above propositions, we obtain the following fundamental results.

THEOREM 1. *Let A be a d-ring, i.e.,* $\pi_1\pi_2^+ = (\pi_1\pi_2)^+$ *and* $(\pi_2\pi_1)^+ = \pi_2^+\pi_1$ *for* $\pi_1 \in A^+$ *and* $\pi_2 \in A$.

An ordered A-module Y has the A-extension property if and only if Y_b *is an erased K-space and the natural representation of A in* Y_b *is a ring and lattice homomorphism onto the subring and sublattice* $\mathrm{Orth}(Y)$ *of orthomorphisms. Then the Hahn-Banach symmetric formula holds. Moreover,* $\partial(p) = \partial^A(p)$ *for any A-sublinear operator* $p: X \longrightarrow Y^{\cdot}$ *such that* $\mathrm{dom}(p) = X$.

REMARK. The condition on the ring A can be altered but not weakened, if we wish to preserve the A^+-homogeneity of a $\mathbf{Z}^+$-homogeneous A-sublinear operator. We also note that the equality $\partial(p) = \partial^A(p)$, in particular, means that the extension property is satisfied in a strengthened form, i.e. the additive operator defined on a subgroup and majorized by p admits a majorized extension to a modular homomorphism.

THEOREM 2. *An ordered A-module Y admits convex analysis if and only if* Y_b *is an erased K-space and the natural representation of A in* Y_b *is a lattice homomorphism onto an almost rational ring of orthomorphisms.*

Institute of Mathematics
Siberian Branch, Academy of Sciences of the USSR

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