
SOME APPLICATIONS OF BOOLEAN VALUED ANALYSIS

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ABSTRACT. This is an overview of the basic techniques and applications of Boolean valued analysis. Exposition focuses on the Boolean valued transfer principle for vector lattices and positive operators, Banach spaces and injective Banach lattices, AW^* -modules and AW^* -algebras, etc.

Keywords: Boolean valued universe, ascent, descent, transfer principle

1 Boolean Valued Requisites

In the beginning of the 1960s Cohen propounded his *method of forcing* and proved that the negation of the continuum hypothesis is consistent with the axioms of Zermelo–Fraenkel set theory (cp. [16]). The contemplation over the Cohen method gave rise to the *Boolean valued models of set theory*, which were first introduced by Scott and Solovay (see [115] and [129]). A systematic account of the theory of Boolean valued models and its applications to independence proofs can be found in [11], [40], [119], and [128].

Scott foresaw the role of Boolean valued models in mathematics and wrote as far back as in 1969 (see [116, p. 91]): “*We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good arguments.*” Some impressive arguments are available today (see, for example, [67], [68], [69], and [122]).

The term “Boolean valued analysis” appeared within the realm of mathematical logic. It was Takeuti, a renowned expert in proof theory, who introduced the term. Takeuti defined Boolean valued analysis in [122, p. 1] as “an application of Scott–Solovay’s Boolean valued models of set theory to analysis.” More precisely, Boolean valued analysis signifies the technique of studying the properties of an arbitrary mathematical object by comparison between its representations in two different set-theoretic models whose construction utilizes principally distinct Boolean algebras. As these models, the classical Cantorian paradise in the shape of the *von Neumann universe* \mathbb{V} and a specially-trimmed *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ are usually taken. Comparison analysis is carried out by some interplay between the universes \mathbb{V} and $\mathbb{V}^{(\mathbb{B})}$.

The needed information on the theory of Boolean valued analysis is briefly presented in [56, Chapter 9] and [69, Chapter 1]; details may be found in [67] and [68]. A short survey of the Boolean machinery is also in [78]. See more on the Boolean valued models and the independence proofs in [11], [40], and [128].

Throughout the sequel \mathbb{B} is a complete Boolean algebra with unity $\mathbb{1}$ and zero $\mathbb{0}$. A *partition of unity* in \mathbb{B} is a family $(b_\xi)_{\xi \in \Xi} \subset \mathbb{B}$ such that $\bigvee_{\xi \in \Xi} b_\xi = \mathbb{1}$ and $b_\xi \wedge b_\eta = \mathbb{0}$ whenever $\xi \neq \eta$. We let $:=$ denote the assignment by definition, while \mathbb{R} and \mathbb{C} symbolize the reals and the complexes. Recall also that ZFC is an abbreviation for *Zermelo–Fraenkel axiomatic set theory with the axiom of choice*.

1.1. Boolean valued universe and Boolean valued truth [69, § 1.2]. Given a complete Boolean algebra \mathbb{B} , we can define the *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$, the class of *\mathbb{B} -valued sets*. For making statements about $\mathbb{V}^{(\mathbb{B})}$ take an arbitrary formula $\varphi = \varphi(u_1, \dots, u_n)$ of the language of set theory and replace the variables u_1, \dots, u_n by elements $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$. Then we obtain some statement about the objects x_1, \dots, x_n . There is a natural way of assigning to each formula some element $\llbracket \varphi(x_1, \dots, x_n) \rrbracket \in \mathbb{B}$ that serves as the “*Boolean truth-value*” of $\varphi(u_1, \dots, u_n)$ in $\mathbb{V}^{(\mathbb{B})}$ and is defined by induction on the complexity of φ , using the naturally defined truth-values $\llbracket x \in y \rrbracket \in \mathbb{B}$ and $\llbracket x = y \rrbracket \in \mathbb{B}$, where $x, y \in \mathbb{V}^{(\mathbb{B})}$. We say that $\varphi(x_1, \dots, x_n)$ is *valid within* $\mathbb{V}^{(\mathbb{B})}$ provided that $\llbracket \varphi(x_1, \dots, x_n) \rrbracket = \mathbb{1}$. In this event, we will also write $\mathbb{V}^{(\mathbb{B})} \models \varphi(x_1, \dots, x_n)$.

1.2. Ascending–descending machinery [69, § 1.5, § 1.6, and § 2.2]. No comparison is feasible without some dialog between \mathbb{V} and $\mathbb{V}^{(\mathbb{B})}$. The relevant *technique of ascending and descending* bases on the operations of the canonical embedding, descent, and ascent.

(1) **THE CANONICAL EMBEDDING.** There is a *canonical embedding* of the von Neumann universe \mathbb{V} into the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ which sends $x \in \mathbb{V}$ to its *standard name* $x^\wedge \in \mathbb{V}^{(\mathbb{B})}$. The standard name sends \mathbb{V} onto $\mathbb{V}^{(2)}$, where

$2 := \{0, 1\} \subset \mathbb{B}$.

(2) DESCENT. Given a member x of a Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$, define the *descent* $x \downarrow$ of x by $x \downarrow := \{y \in \mathbb{V}^{(\mathbb{B})} : \llbracket y \in x \rrbracket = 1\}$. The class $x \downarrow$ is a set; i.e., $x \downarrow \in \mathbb{V}$ for every $x \in \mathbb{V}^{(\mathbb{B})}$.

(3) ASCENT. Assume that $x \in \mathbb{V}$ and $x \subset \mathbb{V}^{(\mathbb{B})}$. Then there exists a unique $x \uparrow \in \mathbb{V}^{(\mathbb{B})}$ such that $\llbracket u \in x \uparrow \rrbracket = \bigvee \{\llbracket u = y \rrbracket : y \in x\}$ for all $u \in \mathbb{V}^{(\mathbb{B})}$. The member $x \uparrow$ is the *ascent* of x .

The operations of descent, ascent, and canonical embedding can be naturally extended to mappings and relations, so that they are applicable to algebraic structures. The various functors of Boolean valued analysis thus arise whose interplay is of import in applications; see [67, Chapter 3] and [68, Chapter 5].

1.3. Principles of Boolean valued set theory [69, § 1.4]. The main properties of a Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ are collected in the four propositions:

(1) TRANSFER PRINCIPLE. If $\varphi(x_1, \dots, x_n)$ is a theorem of ZFC then so is the following formula: $(\forall x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}) \mathbb{V}^{(\mathbb{B})} \models \varphi(x_1, \dots, x_n)$.

(2) MAXIMUM PRINCIPLE. To each formula φ of ZFC there is a member x_0 of $\mathbb{V}^{(\mathbb{B})}$ satisfying $\llbracket (\exists x) \varphi(x) \rrbracket = \llbracket \varphi(x_0) \rrbracket$. In particular, if $\mathbb{V}^{(\mathbb{B})} \models (\exists x) \varphi(x)$, then there exists $x_0 \in \mathbb{V}^{(\mathbb{B})}$ such that $\mathbb{V}^{(\mathbb{B})} \models \varphi(x_0)$.

(3) MIXING PRINCIPLE. For every family $(x_\xi)_{\xi \in \Xi}$ in $\mathbb{V}^{(\mathbb{B})}$ and every partition of unity $(b_\xi)_{\xi \in \Xi}$ in \mathbb{B} there exists a unique $x \in \mathbb{V}^{(\mathbb{B})}$ satisfying $b_\xi \leq \llbracket x = x_\xi \rrbracket$ for all $\xi \in \Xi$. This unique x is the *mixing* of (x_ξ) by (b_ξ) and is denoted as follows: $x = \text{mix}_{\xi \in \Xi}(b_\xi x_\xi) = \text{mix}\{b_\xi x_\xi : \xi \in \Xi\}$.

A formula is *bounded* or *restricted* provided that each of its quantifiers occurs in the form $(\forall x \in y)$ or $(\exists x \in y)$ or if it can be proved to be equivalent in ZFC to such a formula.

(4) RESTRICTED TRANSFER PRINCIPLE. Given a restricted formula φ of ZFC and $x_1, \dots, x_n \in \mathbb{V}$, we have in ZFC that

$$\varphi(x_1, \dots, x_n) \iff \mathbb{V}^{(\mathbb{B})} \models \varphi(x_1^\wedge, \dots, x_n^\wedge).$$

The transfer principle tells us that all theorems of ZFC are true in $\mathbb{V}^{(\mathbb{B})}$; the maximum principle guarantees the existence of various “Boolean valued objects”; the mixing principle shows how these objects may be constructed. The transfer principle does not mean that if a theorem is true for an algebraic structure \mathfrak{A} within $\mathbb{V}^{(\mathbb{B})}$, then the theorem is true also for its descent $\mathfrak{A} \downarrow$ in \mathbb{V} . The question of when this happens was first studied by Gordon [25] and Jech [38].

1.4. Boolean valued technology. To prove the relative consistency of some set-theoretic propositions we use a Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ as follows: Let

\mathcal{T} and \mathcal{S} be some enrichments of Zermelo–Fraenkel theory ZF (without choice). Assume that the consistency of ZF implies the consistency of \mathcal{S} . Assume further that we can define \mathbb{B} so that $\mathcal{S} \models$ “ B is a complete Boolean algebra” and $\mathcal{S} \models \llbracket \varphi \rrbracket = \mathbb{1}$ for every axiom φ of \mathcal{T} . Then the consistency of ZF implies the consistency of \mathcal{T} . That is how we use $\mathbb{V}^{(\mathbb{B})}$ in foundational studies.

Other possibilities for applying $\mathbb{V}^{(\mathbb{B})}$ base on the fact that irrespective of the choice of a Boolean algebra \mathbb{B} , the universe is an arena for testing an arbitrary mathematical event. By the principles of transfer and maximum, every $\mathbb{V}^{(\mathbb{B})}$ has the objects that play the roles of numbers, groups, Banach spaces, manifolds, and whatever constructs of mathematics that are already introduced into practice or still remain undiscovered. These objects may be viewed as some nonstandard realizations of the relevant originals.

All ZFC theorems acquire interpretations for the members of $\mathbb{V}^{(\mathbb{B})}$, attaining the top truth-value. We thus obtain a new technology of comparison between the interpretations of mathematical facts in the universes over various complete Boolean algebras. Developing the relevant tools is the crux of Boolean valued analysis.

A general scheme of the method is as follows (see [68] and [69]). Assume that $\mathbf{X} \subset \mathbb{V}$ and $\mathcal{X} \subset \mathbb{V}^{(\mathbb{B})}$ are two classes of mathematical objects and we are able to prove the possibility of

Boolean Valued Representation: Each $X \in \mathbf{X}$ embeds into a Boolean valued model, becoming an object $\mathcal{X} \in \mathcal{X}$ within $\mathbb{V}^{(\mathbb{B})}$.

The *Boolean Valued Transfer Principle* tells us that every theorem about \mathcal{X} within Zermelo–Fraenkel set theory has its counterpart for the original object X interpreted as a Boolean valued object \mathcal{X} .

The *Boolean Valued Machinery* enables us to perform some translation of theorems from $\mathcal{X} \in \mathbb{V}^{(\mathbb{B})}$ to $X \in \mathbb{V}$ by using the appropriate general operations and the principles of Boolean valued analysis.

2 Vector Lattices

The reader can find the relevant information on the theory of vector lattices and order bounded operators in Aliprantis and Burkinshaw [4], Kusraev [56], Luxemburg and Zaanen [86], Meyer–Nieberg [89], Schaefer [114], Vulikh [130], and Zaanen [132].

Definition 1. A *vector lattice* or a *Riesz space* is a real vector space X equipped with a partial order \leq for which the *join* $x \vee y$ and the *meet* $x \wedge y$ exist for all $x, y \in X$, and such that the *positive cone* $X_+ := \{x \in X : 0 \leq x\}$ is closed under addition and multiplication by positive reals and for any $x, y \in X$ the relations $x \leq y$ and $0 \leq y - x$ are equivalent. A *Banach lattice* is a vector lattice that is

also a Banach space whose order is connected with the norm by the condition that $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in X$.

In the sequel, we assume that all vector lattices X are *Archimedean*; i.e., for every pair $x, y \in X$ it follows from $(\forall n \in \mathbb{N}) nx \leq y$ that $x \leq 0$. Most of the vector spaces that appear naturally in analysis (L^p , l^p , $C(K)$, c , c_0 , etc.) are Archimedean vector lattices with respect to the pointwise or coordinatewise order.

Definition 2. Two elements $x, y \in X$ are *disjoint* and write $x \perp y$ if $|x| \wedge |y| = 0$ where the *modulus* $|x|$ of x is defined as $|x| := x \vee (-x)$. A vector $0 < \mathbb{1} \in X$ said to be a *weak order unit* whenever $\mathbb{1}^\perp = \{0\}$. A *band* in a vector lattice X is a subset of the form $B := A^\perp := \{x \in X : (\forall a \in A) |x| \wedge |a| = 0\}$ for a nonempty $A \subset X$. The inclusion ordered set of all bands in X is a complete Boolean algebra denoted by $\mathbb{B}(X)$.

Definition 3. A band B in X such that $X = B \oplus B^\perp$ is referred to as a *projection band*, while the associated projection (onto B parallel to B^\perp) is a *band projection*. The set of all band projections $\mathbb{P}(X)$ in X also forms a Boolean algebra in which $\pi \leq \rho$ means $\pi(X) \subset \rho(X)$. If each band in X admits a band projection then $\mathbb{B}(X) \simeq \mathbb{P}(X)$.

Definition 4. A subset $U \subset X$ is *order bounded* if U lies in an *order interval* $[a, b] := \{x \in X : a \leq x \leq b\}$ for some $a, b \in X$. A vector lattice X is *Dedekind complete* (respectively, *laterally complete*) if every nonempty order bounded set (respectively, each nonempty set of pairwise disjoint positive vectors) U in X has a least upper bound $\sup(U) \in X$. The vector lattice that is laterally complete and Dedekind complete simultaneously is referred to as *universally complete*.

Definition 5. Say that a net (x_α) in a vector lattice X *o-converges* to $x \in X$ and write $x = o\text{-}\lim x_\alpha$ if there exists a decreasing net $(e_\beta)_{\beta \in \mathbb{B}}$ in X such that $\inf\{e_\beta : \beta \in \mathbb{B}\} = 0$ and for each $\beta \in \mathbb{B}$ there is $\alpha(\beta) \in \mathbb{A}$ with $|x_\alpha - x| \leq e_\beta$ for all $\alpha \geq \alpha(\beta)$.

Example 6. Assume that a measure space (Ω, Σ, μ) is semifinite; i.e., if $A \in \Sigma$ and $\mu(A) = \infty$ then there exists $B \in \Sigma$ with $B \subset A$ and $0 < \mu(A) < \infty$. The vector lattice $L^0(\mu) := L^0(\Omega, \Sigma, \mu)$ (of cosets) of μ -measurable functions on Ω is universally complete if and only if (Ω, Σ, μ) is *localizable*. In this event $L^p(\Omega, \Sigma, \mu)$ is Dedekind complete; see [21, 241G]. Observe that $\mathbb{P}(L^0(\Omega, \Sigma, \mu)) \simeq \Sigma/\mu^{-1}(0)$.

Example 7. Given a complete Boolean algebra \mathbb{B} of orthogonal projections in a Hilbert space H , denote by $\langle \mathbb{B} \rangle$ the space of all selfadjoint operators on H whose spectral resolutions are in \mathbb{B} ; i.e., $A \in \langle \mathbb{B} \rangle$ if and only if $A = \int_{\mathbb{R}} \lambda dE_\lambda$ and $E_\lambda \in \mathbb{B}$

for all $\lambda \in \mathbb{R}$. Define the partial order in $\langle \mathbb{B} \rangle$ by putting $A \geq B$ whenever $\langle Ax, x \rangle \geq \langle Bx, x \rangle$ holds for all $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$, where $\mathcal{D}(A) \subset H$ stands for the domain of A . Then $\langle \mathbb{B} \rangle$ is a universally complete vector lattice and $\mathbb{P}(\langle \mathbb{B} \rangle) \simeq \mathbb{B}$.

Applying the transfer principle and the maximum principle to the theorem of ZFC stating the existence of the field of real numbers, we find $\mathcal{R} \in \mathbb{V}^{(\mathbb{B})}$, the reals within $\mathbb{V}^{(\mathbb{B})}$ for which $\llbracket \mathcal{R} \text{ is the reals} \rrbracket = 1$. The fundamental result of Boolean valued analysis is the Gordon Theorem describing an interplay between \mathbb{R} , \mathbb{R}^\wedge , \mathcal{R} , and $\mathbf{R} = \mathcal{R} \downarrow$; see [69, §2.2–§2.4].

Theorem 8. (Gordon Theorem). *Let \mathbb{B} be a complete Boolean algebra, and let \mathcal{R} be the reals within $\mathbb{V}^{(\mathbb{B})}$. Endow \mathbf{R} with the descended operations and order. Then*

- (1) *The algebraic structure \mathbf{R} is a universally complete vector lattice.*
- (2) *The field $\mathcal{R} \in \mathbb{V}^{(\mathbb{B})}$ can be chosen so that $\llbracket \mathbb{R}^\wedge \text{ is a dense subfield of } \mathcal{R} \rrbracket = 1$.*
- (3) *There is a Boolean isomorphism χ from \mathbb{B} onto $\mathbb{P}(\mathbf{R})$ such that*

$$\begin{aligned} \chi(b)x &= \chi(b)y \iff b \leq \llbracket x = y \rrbracket, \\ \chi(b)x \leq \chi(b)y &\iff b \leq \llbracket x \leq y \rrbracket \\ &(x, y \in \mathbf{R}; b \in \mathbb{B}). \end{aligned}$$

As regards the further development of the theory of vector lattices on using Theorem 8; see Kusraev and Kutateladze [69, §2.2–§2.11]. Note that the versions of the Gordon Theorem which involve the multiplicative structure and complexification are true as well.

Definition 9. An *f*-algebra is a vector lattice X equipped with a distributive multiplication such that if $x, y \in X_+$ then $xy \in X_+$, and if $x \wedge y = 0$ then $(ax) \wedge y = (xa) \wedge y = 0$ for all $a \in X_+$. An *f*-algebra is *semiprime* provided that $xy = 0$ implies $x \perp y$ for all x and y . A *complex vector lattice* $X_{\mathbb{C}}$ is the complexification $X_{\mathbb{C}} := X \oplus iX$ (with i standing for the imaginary unity) of a real vector lattice X .

In the complex version of Example 7, $\langle \mathbb{B} \rangle$ consists of all normal operators $A + iB$ with $A, B \in \langle \mathbb{B} \rangle$ and the product AB is defined as the unique selfadjoint extension of the operator $x \mapsto A(Bx) = B(Ax)$ ($x \in \mathcal{D}(A) \cap \mathcal{D}(B)$).

Theorem 10. (1) *The universally complete vector lattice $\mathcal{R} \downarrow$ with the descended multiplication is a semiprime *f*-algebra with the ring unity $\mathbb{1} := 1^\wedge$. Moreover, for every $b \in \mathbb{B}$ the band projection $\chi(b) \in \mathbb{P}(\mathbf{R})$ acts as multiplication by $\chi(b)\mathbb{1}$.*

(2) *Let \mathcal{C} be the field of complex numbers within $\mathbb{V}^{(\mathbb{B})}$. Then the algebraic system $\mathcal{C} \downarrow$ is a universally complete complex *f*-algebra. Moreover, $\mathcal{C} \downarrow$ is the complexification of the universally complete real *f*-algebra $\mathcal{R} \downarrow$; i.e., $\mathcal{C} \downarrow = \mathcal{R} \downarrow \oplus i\mathcal{R} \downarrow$.*

Remark 11. If μ is a Maharam measure and \mathbb{B} in the Gordon Theorem is the algebra of all μ -measurable sets modulo μ -negligible sets, then $\mathcal{R}\downarrow$ is lattice isomorphic to $L^0(\mu)$; see Example 6. If \mathbb{B} is a complete Boolean algebra of projections in a Hilbert space H then $\mathcal{R}\downarrow$ is isomorphic to $\langle \mathbb{B} \rangle$; see Example 7. The two indicated particular cases of Gordon's Theorem were intensively and fruitfully exploited by Takeuti [122]–[125]. The object $\mathcal{R}\downarrow$ for general Boolean algebras was also studied by Jech [37], [38], and [39] who in fact rediscovered Gordon's Theorem. The difference is that in [37] a (complex) universally complete vector lattice with unit is defined by another system of axioms and is referred to as a complete *Stone algebra*. Selecting special \mathbb{B} 's, it is possible to obtain some properties of \mathcal{R} . For instance, Solovay proved the existence of \mathbb{B} such that all subsets of the reals are Lebesgue measurable in $\mathbb{V}^{(\mathbb{B})}$; see [118].

Remark 12. Interpretation of an arbitrary field in a Boolean valued model leads to the class of rationally complete semiprime commutative rings (see Lambek [82] for the definitions). Gordon proved in [26] that if K is a rationally complete semiprime commutative ring and \mathbb{B} stands for the Boolean algebra of all annihilator ideals of K , then there is an internal field $\mathcal{K} \in \mathbb{V}^{(\mathbb{B})}$, the *Boolean valued representation* of K , such that the ring K is isomorphic to $\mathcal{K}\downarrow$. It follows that the Horn theories of fields and rationally complete semiprime commutative rings coincide. Details may be found in [67, Theorems 4.5.6 and 4.5.7] and [68, Theorems 8.3.1 and 8.3.2]. Note also that Smith in [120] established an equivalence between the category of commutative regular rings and the category of Boolean valued fields. Boolean valued rings, integral domains, and fields were examine also by Nishimura [97] and [103]. Here we also point out the article by Nishimura [90] on the Boolean-valued analysis of continuous geometries and the article by Chupin [15] with a solution to Problem 18 in the book by Goodearl [22, p. 346].

Remark 13. In another article [27], Gordon found the following description of the class of modules arising as descents of vector spaces from Boolean valued models: Assume that K and \mathcal{K} are the same as in Remark 12. For every strongly unital injective K -module M there exists $\mathcal{M} \in \mathbb{V}^{(\mathbb{B})}$, the *Boolean valued representation* of the module M , such that M is isomorphic to $\mathcal{M}\downarrow$; also see [67, 4.5.10 (5)]. Now, if \mathcal{M} and \mathcal{M}' are Boolean valued representations of M and M' , respectively, then by the transfer principle, \mathcal{M} and \mathcal{M}' are isomorphic if and only if they have Hamel bases of the same cardinality. Using the descent functor and the description of Boolean valued cardinals enables us to obtain a classification of strongly unitary injective modules. The result was obtained recently by Chilin and Karimov [14] with the superfluous assumption $K = L^0(\mu)$ (but without any instance of Boolean valued analysis).

3 Positive Operators

The aim of this section is to establish some variants of the Boolean valued transfer principle from functionals to operators between vector lattices.

Let X and Y be vector lattices. By $L(X, Y)$ we denote the space of all linear operators from X to Y . Take $T \in L(X, Y)$. Call T *positive* and write $T \geq 0$ provided that $T(X_+) \subset Y_+$. Call T *order bounded* or *o-bounded* whenever T sends each order bounded subset of X to an order bounded subset of Y .

The set of all order bounded operators from X to Y is denoted by $L^\sim(X, Y)$. The order relation in $L^\sim(X, Y)$ is defined as follows: $S \geq T \iff S - T \geq 0$.

The celebrated Riesz–Kantorovich Theorem tells us that if X and Y are vector lattices with Y Dedekind complete, then $L^\sim(X, Y)$ is a Dedekind complete vector lattice. Moreover, in this event every order bounded operator T is *regular*; i.e., T can be presented as a difference of two positive operators.

The fact that X is a vector lattice over the ordered field \mathbb{R} may be rewritten as a restricted formula, say, $\varphi(X, \mathbb{R})$. Hence, recalling the restricted transfer principle, we come to the identity $\llbracket \varphi(X^\wedge, \mathbb{R}^\wedge) \rrbracket = \mathbb{1}$ which amounts to saying that X^\wedge is a vector lattice over the ordered field \mathbb{R}^\wedge within $\mathbb{V}^{(B)}$. Similarly, the positive cone X_+ is defined by a restricted formula; hence $\mathbb{V}^{(B)} \models (X^\wedge)_+ = (X_+)^\wedge$. By the same reason $|x^\wedge| = |x|^\wedge$, $(x \vee y)^\wedge = x^\wedge \vee y^\wedge$, $(x \wedge y)^\wedge = x^\wedge \wedge y^\wedge$ for all $x, y \in X$, since the lattice operations \vee , \wedge , and $|\cdot|$ in X are defined by restricted formulas.

Let $X^{\wedge\sim} := L_{\mathbb{R}^\wedge}^\sim(X^\wedge, \mathcal{R})$ be the space of regular \mathbb{R}^\wedge -linear functionals from X^\wedge to \mathcal{R} . More precisely, \mathcal{R} is considered as a vector space over the field \mathbb{R}^\wedge and by the maximum principle there exists $X^{\wedge\sim} \in \mathbb{V}^{(B)}$ such that $\llbracket X^{\wedge\sim} \rrbracket$, the set of \mathbb{R}^\wedge -linear order bounded functionals from X^\wedge to \mathcal{R} , is a vector space over \mathcal{R} ordered by the cone of positive functionals $\llbracket \tau \geq 0 \rrbracket = \mathbb{1}$. A functional $\tau \in X^{\wedge\sim}$ is positive whenever $\llbracket \tau \geq 0 \rrbracket = \mathbb{1}$.

Definition 14. Let $X \in \mathbb{V}$ and $Y \in \mathbb{V}^{(B)}$ be such that $X \neq \emptyset$ and $\llbracket Y \neq \emptyset \rrbracket = \mathbb{1}$. Given an operator $T : X \rightarrow Y \downarrow$, there exists a unique $T \uparrow \in \mathbb{V}^{(B)}$ (called the *modified ascent* of T) such that $\llbracket T \uparrow : X^\wedge \rightarrow Y \rrbracket = \mathbb{1}$ and $\llbracket T \uparrow(x^\wedge) = T(x) \rrbracket = \mathbb{1}$ for all $x \in X$. Given a member $\tau \in \mathbb{V}^{(B)}$ with $\llbracket \tau : X^\wedge \rightarrow Y \rrbracket = \mathbb{1}$, there exists a unique $\tau \downarrow : X \rightarrow Y \downarrow$ (called the *modified descent* of τ) with $\llbracket \tau(x^\wedge) = \tau \downarrow(x) \rrbracket = \mathbb{1}$ for all $x \in X$.

Definition 15. A linear operator T from X to Y is a *lattice homomorphism* whenever $T(x_1 \vee x_2) = Tx_1 \vee Tx_2$ for all $x_1, x_2 \in X$. Say that T is *disjointness preserving* if $|x| \wedge |y| = 0$ implies $|T(x)| \wedge |T(y)| = 0$ for all $x, y \in X$. Two vector lattices X and Y are said to be *lattice isomorphic* if there is a lattice isomorphism from X onto Y . Let $\text{Hom}(X, Y)$ and $L_{dp}^\sim(X, Y)$ stand for the sets of all lattice homomorphisms and all disjointness preserving operators from X to Y , respectively.

Theorem 16. *Let X and Y be vector lattices with Y universally complete and represented as $Y = \mathcal{R}\downarrow$. Given $T \in L^\sim(X, Y)$, the modified ascent $T\uparrow$ is an order bounded \mathbb{R}^\wedge -linear functional on X^\wedge within $\mathbb{V}^{(\mathbb{B})}$; i.e., $\llbracket T\uparrow \in X^{\wedge\sim} \rrbracket = \mathbb{1}$. The mapping $T \mapsto T\uparrow$ is a lattice isomorphism between the Dedekind complete vector lattices $L^\sim(X, Y)$ and $X^{\wedge\sim}\downarrow$.*

As an example of the application of Theorem 16, we will describe some property of an order bounded operator $T \in L^\sim(X, Y)$ in terms of the kernels $\ker(bT) = \{x \in X : b \circ Tx = 0\}$ of its *stratum* bT with $b \in \mathbb{P}(Y)$. To this end, assume $Y = \mathcal{R}\downarrow$, put $\tau := T\uparrow$, and observe that $T \in \text{Hom}(X, Y)$ if and only if $\llbracket \tau \in \text{Hom}(X^\wedge, \mathcal{R}) \rrbracket = \mathbb{1}$ and $T \in L_{dp}^\sim(X, Y)$ if and only if $\llbracket \tau \in (X^{\wedge\sim})_{dp} \rrbracket = \mathbb{1}$. Moreover, X_0 is an order ideal (or sublattice, or Grothendieck subspace) in X if and only if $\llbracket \text{so is } X_0^\wedge \text{ in } X^\wedge \rrbracket = \mathbb{1}$. Recall that a subspace $X_0 \subset X$ is a *Grothendieck subspace* provided that $x \vee y \vee 0 + x \wedge y \wedge 0 \in X_0$ for all $x, y \in X_0$. Combining the above, we can reduce the problem about the operator T to studying the functional τ . The following result is due to Kutateladze [76] and [77]; also see [69, §3.4–§3.6].

Theorem 17. *Let X and Y be vector lattices with Y Dedekind complete, $\mathbb{B} := \mathbb{P}(Y)$, and let $T : X \rightarrow Y$ be an order bounded operator. The following assertions hold:*

- (1) *T is disjointness preserving if and only if the kernel of each stratum bT of T with $b \in \mathbb{P}(Y)$ is an order ideal in X .*
- (2) *An operator T is the difference of two lattice homomorphisms if and only if the kernel of each stratum bT of T with $b \in \mathbb{B}$ is a vector sublattice of X .*
- (3) *The modulus $|T|$ of T is the sum of some pair of lattice homomorphisms if and only if the kernel of each stratum bT of T with $b \in \mathbb{B}$ is a Grothendieck subspace of X .*

The modified ascent mapping $T \mapsto T\uparrow$ has the disadvantage that it does not preserve order continuity. Now consider an embedding into $\mathbb{V}^{(\mathbb{B})}$ preserving o -continuity.

Definition 18. An operator $T : X \rightarrow Y$ between vector lattices is *order continuous* provided that $o\text{-lim } Tx_\alpha = 0$ in Y for every net (x_α) with $o\text{-lim } x_\alpha = 0$ in X . A positive operator $T : X \rightarrow Y$ enjoys the *Maharam property* (or is *order interval preserving*) whenever $T[0, x] = [0, Tx]$ for every $0 \leq x \in X$; i.e., if for all $0 \leq x \in X$ and $0 \leq y \leq Tx$ there is some $0 \leq u \in X$ such that $Tu = y$ and $0 \leq u \leq x$. A *Maharam operator* is an order continuous linear operator whose modulus has the Maharam property.

Definition 19. A positive operator $T : X \rightarrow Y$ has the *Levi property* if $Y = T(X)^{\perp\perp}$ and $\sup x_\alpha$ exists in X for every increasing net $(x_\alpha) \subset X_+$, provided that the net (Tx_α) is order bounded in Y . Given an order bounded order continuous operator

T from X to Y , denote by $\mathcal{D}_m(T)$ the largest ideal of the universal completion X^u onto which we may extend T by order continuity. For a positive order continuous operator T we have $X = \mathcal{D}_m(T)$ if and only if T has the Levi property.

The following result states that each Maharam operator is representable as an order continuous linear functional in an appropriate Boolean valued model. This Boolean valued status of the concept of Maharam operator was found by Kusraev [50] and [51].

Theorem 20. *Let X be a Dedekind complete vector lattice, $Y := \mathcal{R}\downarrow$, and let $T : X \rightarrow Y$ be a positive Maharam operator with $Y = T(X)^{\perp\perp}$. Then there are \mathcal{X} and $\tau \in \mathbb{V}^{(\mathbb{B})}$ such that*

- (1) $\llbracket \mathcal{X} \text{ is a Dedekind complete vector lattice and } \tau : \mathcal{X} \rightarrow \mathcal{R} \text{ is an order continuous strictly positive functional with the Levi property} \rrbracket = \mathbb{1}$.
- (2) $\mathcal{X}\downarrow$ is a Dedekind complete vector lattice and a unital f -module over the f -algebra $\mathcal{R}\downarrow$.
- (3) $\tau\downarrow : \mathcal{X}\downarrow \rightarrow \mathcal{R}\downarrow$ is a strictly positive Maharam operator with the Levi property and an $\mathcal{R}\downarrow$ -module homomorphism.
- (4) There exists an order continuous lattice homomorphism $\varphi : X \rightarrow \mathcal{X}\downarrow$ such that $\varphi(X)$ is order dense ideal of $\mathcal{X}\downarrow$ and $T = \tau\downarrow \circ \varphi$.

Remark 21. The Maharam operators stem from the theory of Maharam’s “full-valued” integrals which was developed in 1949–1953 (see the survey [87]). Luxemburg in the joint articles with de Pagter [84] and Schep [85] extended some portion of Maharam’s theory to the case of positive operators in Dedekind complete vector lattices; in particular, some operator versions of the Hahn Decomposition Theorem and the Radon–Nikodým Theorem were obtained in [85]. The Maharam ideas were transferred to the convex operators by Kusraev [48] and [49]. More results, applications, and references on Maharam operators can be found in [56], [66], and [69].

Remark 22. Suppose that X is a vector lattice over a dense subfield $\mathbb{F} \subset \mathbb{R}$ and $\varphi : X \rightarrow \mathbb{R}$ is a strictly positive \mathbb{F} -linear functional. Then the completion X^φ of the normed lattice $(X, \|\cdot\|_\varphi)$ with $\|x\|_\varphi := \varphi(|x|)$ is an AL -space that includes X . This simple constriction interpreted within a Boolean valued model yields an extension of an arbitrary positive operator to a Maharam operator, i.e. the *Maharam extension*. This was done by Akilov, Kolesnikov, and Kusraev in [5] and [6]. Later, Luxemburg and de Pagter [84] constructed the Maharam extension for a given ideal of operators in $L^\sim(X, Y)$ without using Boolean valued analysis.

Remark 23. In 1935 Kantorovich in his first definitive article on vector lattices (see [41]) wrote: “In this note, I define the new type of space that I call a semiordered

linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in such a space) as linear functionals.” Here Kantorovich stated an important *heuristic transfer principle*; Theorems 16 and 20 present two instances of the mathematical implementation of this principle.

4 Boolean Valued Banach Spaces

In this section we discuss Banach spaces within a Boolean valued universe. We start with the concept of Banach–Kantorovich space (not to be confused with that of *Kantorovich–Banach space* or, shortly, *KB-space* which is by definition a Banach lattice with an order continuous Levi norm; see [2, p. 89] and [89, Definition 2.4.11].)

Definition 24. Consider a vector space X and a real vector lattice Λ . A Λ -valued norm is a mapping $|\cdot| : X \rightarrow \Lambda_+$ such that $|x| = 0$ implies $x = 0$, $|\lambda x| = |\lambda| |x|$, and $|x + y| \leq |x| + |y|$ for all $x, y \in X$ and $\lambda \in \mathbb{R}$. A Λ -valued norm is *decomposable* if, for each decomposition $|x| = \lambda_1 + \lambda_2$ with $\lambda_1, \lambda_2 \in \Lambda_+$ and $x \in X$, there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $|x_k| = \lambda_k$ ($k := 1, 2$).

Definition 25. A *Banach–Kantorovich space* over a Dedekind complete vector lattice Λ is a vector space X with a decomposable norm $|\cdot| : X \rightarrow \Lambda$ which is norm complete in the sense that, given a net $(x_\alpha)_{\alpha \in A}$ in X with $(|x_\alpha - x_\beta|)_{(\alpha, \beta) \in A \times A}$ o -convergent to the zero of Λ , there exists $x \in X$ such that $(|x_\alpha - x|)_{\alpha \in A}$ is o -convergent to the zero of Λ .

Definition 26. A Banach–Kantorovich space over Λ is *universally complete* in case Λ is universally complete. By a *universal completion* of a Λ -normed space $(X, |\cdot|)$ we mean a universally complete Banach–Kantorovich space Y over Λ^u together with a linear isometry $\iota : X \rightarrow Y$ (i.e., $|\iota(x)| = |x|$ for all $x \in X$) such that each universally complete subspace of Y containing $\iota(X)$ coincides with Y .

Definition 27. A linear operator $T : X \rightarrow Y$ between Banach–Kantorovich spaces over Λ is Λ -bounded if $|Tx| \leq \lambda |x|$ ($x \in X$) for some $\lambda \in \Lambda_+$; the least such λ is denoted by $|T|$. Define $\mathcal{L}_\Lambda(X, Y)$ as the space of Λ -bounded operators from X to Y .

The following two theorems stating that the category of Banach–Kantorovich spaces over $\Lambda = \mathcal{R}\downarrow$ and Λ -bounded linear operators is equivalent to the category of Banach spaces and bounded linear operators within $\mathbb{V}^{(\mathbb{B})}$ were established by Kusraev [51] (see [52], [56], and [67] for full details).

Theorem 28. *Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space within the model $\mathbb{V}^{(\mathbb{B})}$. If $X := \mathcal{X}\downarrow$ and $|\cdot| := \|\cdot\|\downarrow$, then $(X, |\cdot|)$ is a universally complete Banach–Kantorovich space*

over $\mathcal{R}\downarrow$; moreover, the relations $b \leq \llbracket x = 0 \rrbracket$ and $\chi(b)x = 0$ are equivalent for all $b \in \mathbb{B}$ and $x \in X$. Conversely, for every lattice-normed space $(X, |\cdot|)$ with $\mathbb{B} \simeq \mathbb{P}(|X|^{\perp\perp})$, there exists a unique (up to a linear isometry) Banach space \mathcal{X} within $\mathbb{V}^{(\mathbb{B})}$, for which the descent $\mathcal{X}\downarrow$ is a universal completion of X .

Theorem 29. Let \mathcal{X} and \mathcal{Y} be Boolean valued representations of Banach-Kantorovich spaces X and Y over some universally complete vector lattice Λ . Let $\mathcal{L}^{\mathbb{B}}(\mathcal{X}, \mathcal{Y})$ be the space of bounded linear operators from \mathcal{X} into \mathcal{Y} within $\mathbb{V}^{(\mathbb{B})}$, where $\mathbb{B} := \mathbb{P}(E)$. The descent and ascent operations implement linear isometries between the Banach-Kantorovich spaces $\mathcal{L}_{\Lambda}(X, Y)$ and $\mathcal{L}^{\mathbb{B}}(\mathcal{X}, \mathcal{Y})\downarrow$.

Let $\Lambda := \mathcal{R}\downarrow$ be the bounded part of the vector lattice $\mathcal{R}\downarrow$; i.e., Λ consists of all $x \in \mathcal{R}\downarrow$ with $|x| \leq C\mathbb{1}$ for some $C \in \mathbb{R}$, where $\mathbb{1} := 1^{\wedge} \in \mathcal{R}\downarrow$. Take a Banach space \mathcal{X} within $\mathbb{V}^{(\mathbb{B})}$ and put $\mathcal{X}\downarrow := \{x \in \mathcal{X}\downarrow : |x| \in \Lambda\}$. Endow $\mathcal{X}\downarrow$ with a mixed norm

$$\|x\| := \|\lvert x \rvert\|_{\infty} := \inf\{0 < C \in \mathbb{R} : |x| \leq C\mathbb{1}\}.$$

We will write $\Lambda = \Lambda(\mathbb{B})$ if $\mathcal{R} \in \mathbb{V}^{(\mathbb{B})}$ and $\bar{\Lambda} := \mathcal{C}\downarrow = \Lambda \oplus i\Lambda$; i.e., $\bar{\Lambda}$ is the complexification of Λ .

Definition 30. The normed space $(\mathcal{X}\downarrow, \|\cdot\|)$ is the bounded descent of \mathcal{X} . If $\tau : \mathcal{X} \rightarrow \mathcal{Y}$ is a bounded linear operator then $\tau\downarrow$ denotes the restriction of $\tau\downarrow$ to $\mathcal{X}\downarrow$.

The bounded descent of an internal Banach space is a Banach space. Thus, the natural question arises: *Which Banach spaces are linearly isometric to the bounded descents of internal Banach spaces?* The answer is given in terms of \mathbb{B} -cyclic Banach spaces. Let \mathcal{B} be a complete Boolean algebra of norm one projections in a Banach space X with the Boolean operations: $\pi \wedge \rho := \pi \circ \rho = \rho \circ \pi$, $\pi \vee \rho = \pi + \rho - \pi \circ \rho$, $\pi^* = I_X - \pi$ ($\pi, \rho \in \mathcal{B}$), and the zero and identity operators in X serve as the zero and unity of the Boolean algebra \mathcal{B} .

Definition 31. If $(b_{\xi})_{\xi \in \Xi}$ is a partition of unity in \mathcal{B} and $(x_{\xi})_{\xi \in \Xi}$ is a family in X , then the element $x \in X$ with $b_{\xi}x_{\xi} = b_{\xi}x$ for all $\xi \in \Xi$ is a *mixing* of (x_{ξ}) with respect to (b_{ξ}) . A Banach space X is \mathbb{B} -cyclic if \mathbb{B} is a complete Boolean algebra isomorphic to \mathcal{B} and the mixing of every family in the unit ball of X with respect to every partition of unity in \mathbb{B} (with the same index set) exists in the unit ball and is unique; see [56, Definitions 7.3.1 and 7.3.3]. In the sequel we will identify \mathcal{B} and \mathbb{B} .

Let X and Y be Banach spaces with $\mathbb{B} \subset \mathcal{L}(X)$ and $\mathbb{B} \subset \mathcal{L}(Y)$. An operator $T : X \rightarrow Y$ is \mathbb{B} -linear, whenever T is linear and commutes with all projections in

\mathbb{B} , i.e. in the case that $b \circ T = T \circ b$. The set of all bounded \mathbb{B} -linear operators from X into Y denote by $\mathcal{L}_{\mathbb{B}}(X, Y)$. The terms \mathbb{B} -isomorphism and \mathbb{B} -isometry are self-evident. The space $X^{\#} := \mathcal{L}_{\mathbb{B}}(X, \Lambda)$, where $\Lambda = \Lambda(\mathbb{B})$, is \mathbb{B} -dual to X .

The following result can be easily deduced from Theorem 28 and the fact that a Banach lattice $(X, \|\cdot\|)$ is \mathbb{B} -cyclic with respect to a complete Boolean algebra \mathbb{B} of projections if and only if X is a Banach–Kantorovich space with a $\Lambda(\mathbb{B})$ -valued norm $|\cdot|$ such that $\|x\| = \|\lvert x \rvert\|_{\infty}$ for all $x \in X$; see Kusraev [53].

Theorem 32. *The bounded descent of a Banach space from the model $\mathbb{V}^{(\mathbb{B})}$ is a \mathbb{B} -cyclic Banach space. Conversely, if X is a \mathbb{B} -cyclic Banach space, then in the model $\mathbb{V}^{(\mathbb{B})}$ there is a Banach space \mathcal{X} unique up to an isometric isomorphism whose bounded descent $\mathcal{X} \downarrow$ is \mathbb{B} -isometric to X .*

The element $\mathcal{X} \in \mathbb{V}^{(\mathbb{B})}$ from Theorem 32 is the *Boolean valued representation* of X . Let \mathcal{X} and \mathcal{Y} be the Boolean valued representations of \mathbb{B} -cyclic Banach spaces X and Y , respectively. Denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ an element in $\mathbb{V}^{(\mathbb{B})}$ representing the space of bounded linear operators from \mathcal{X} into \mathcal{Y} . As in Theorem 29, the bounded descent of the Banach space $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ and the \mathbb{B} -cyclic Banach space $\mathcal{L}_{\mathbb{B}}(X, Y)$ are isometrically \mathbb{B} -isomorphic. Moreover, the functor of bounded descent establishes an equivalence of the category of Banach spaces and bounded linear operators within $\mathbb{V}^{(\mathbb{B})}$ with the category of \mathbb{B} -cyclic Banach spaces and norm bounded \mathbb{B} -linear operators.

Definition 33. Let $\bar{\Lambda} = \bar{\Lambda}(\mathbb{B})$ with unity $\mathbb{1}$ and consider a unital $\bar{\Lambda}$ -module X . The mapping $\langle \cdot | \cdot \rangle : X \times X \rightarrow \bar{\Lambda}$ is a $\bar{\Lambda}$ -valued inner product if, for all $x, y, z \in X$ and $\lambda \in \bar{\Lambda}$, the following are satisfied:

- (1) $\langle x | x \rangle \geq 0$; $\langle x | x \rangle = 0 \iff x = 0$;
- (2) $\langle x | y \rangle = \langle y | x \rangle^*$;
- (3) $\langle \lambda x | y \rangle = \lambda \langle x | y \rangle$;
- (4) $\langle x + y | z \rangle = \langle x | z \rangle + \langle y | z \rangle$.

Using a $\bar{\Lambda}$ -valued inner product, we introduce the norm by $\|x\| := \sqrt{\|\langle x | x \rangle\|}$ ($x \in X$) and the decomposable Λ -valued norm by $\lvert x \rvert := \sqrt{\langle x | x \rangle}$ ($x \in X$). Obviously, $\|x\| = \|\lvert x \rvert\|_{\infty}$ for all $x \in X$, and so X is a space with mixed norm.

Definition 34. Let X be a $\bar{\Lambda}$ -module with an inner product $\langle \cdot | \cdot \rangle : X \times X \rightarrow \bar{\Lambda}$. If X is complete with respect to the mixed norm $\|\cdot\|$ then X is a C^* -module over $\bar{\Lambda}$. A unitary C^* -module X over $\bar{\Lambda}(\mathbb{B})$ is a *Kaplansky–Hilbert module* or *AW*-module* if X enjoys one (hence, both) of the equivalent conditions: (1) $(X, \|\cdot\|)$ is a \mathbb{B} -cyclic Banach space and (2) $(X, |\cdot|)$ is a Banach–Kantorovich space over $\Lambda(\mathbb{B})$.

The equivalence (1) \iff (2) in Definition 34 follows from Theorem 32 and it is clear that some counterparts of Theorems 28 and 29 are true for Kaplansky–Hilbert modules. This result was obtained by Ozawa in [104] and [106].

Theorem 35. *The bounded descent functor establishes an equivalence of the category of Hilbert spaces and bounded linear operators within $\mathbb{V}^{(\mathbb{B})}$ with the category of Kaplansky–Hilbert modules over $\bar{\Lambda}(\mathbb{B})$ and bounded \mathbb{B} -linear operators.*

Remark 36. The concept of vector space normed by the elements of a vector lattice was introduced by Kantorovich in 1936 [42]. The first applications of vector norms and metrics were related to the method of successive approximations in numerical analysis. The modern theory of lattice-normed spaces and dominated operators on them is presented in Kusraev [56].

Remark 37. The bounded descent of 30 appeared in the research by Takeuti into von Neumann algebras and C^* -algebras within Boolean valued models; see [126] and [127]. Then the technique was developed in the research by Ozawa into the Boolean valued interpretation of the theory of Hilbert spaces; see [104] and [106]. Theorem 32 is due to Kusraev in [51], [53]; also see [52] and [56]. Similar results were obtained by Ozawa [111, Theorem 5.2]; the difference is in the fact that Ozawa [111] deals with Banach spaces possessing an extra module structure over $\Lambda(\mathbb{B})$ which may be recovered in each \mathbb{B} -cyclic Banach space. Nishimura [100] established the Boolean valued transfer principle from L^* -algebras to AL^* -algebras in the spirit of the Takeuti–Ozawa theory of AW^* -modules; also see [95]. (An L^* -algebra is a complex Lie algebra whose vector space is a Hilbert space endowed with an involution and some axiom connecting the Lie bracket, inner product, and involution.)

Remark 38. In [106] Ozawa found a complete system of isomorphism invariants for Kaplansky–Hilbert modules: There is one-to-one correspondence between the isomorphism classes of Kaplansky–Hilbert modules over $\bar{\Lambda}(\mathbb{B})$ and the cardinals in $\mathbb{V}^{(\mathbb{B})}$. At the same time each Kaplansky–Hilbert module admits a direct sum decomposition into homogeneous components. Using these results, Kusraev obtained the following functional representation: To each Kaplansky–Hilbert module X there exist a set of cardinals Γ and a family of nonempty extremally disconnected compact spaces $(Q_\gamma)_{\gamma \in \Gamma}$ such that there is a unitary equivalence $X \simeq \sum_{\gamma \in \Gamma}^{\oplus} C_{\#}(Q_\gamma, l_2(\gamma))$. (Here $C_{\#}(Q, X)$ is the space of cosets of X -valued bounded continuous functions defined on comeager subsets of Q ; see [67, 6.4.1] and [69, 5.13.3].) The representation is not unique and, as discovered Ozawa in [106], the reason for this is the *cardinal shift* phenomena in $\mathbb{V}^{(\mathbb{B})}$: Given two infinite cardinals $\varkappa < \lambda$, there is a complete Boolean algebra \mathbb{B} such that $\mathbb{V}^{(\mathbb{B})} \models |\varkappa^\wedge| = |\lambda^\wedge|$, and so the injective Banach lattices $C_{\#}(K, l_2(\varkappa))$ and $C_{\#}(K, l_2(\lambda))$ are lattice \mathbb{B} -isometric with K the Stone representation space for \mathbb{B} ; see [67] and [68].

5 Injective Banach Lattices

In this section we present the instance of the Boolean valued transfer principle from AL -spaces to injective Banach lattices which states that each injective Banach lattice is embedded into an appropriate Boolean valued model, becoming an AL -space; see Kusraev [59], [60], [61], and [62]. First we consider Boolean valued Banach lattices.

Definition 39. A Banach lattice X is an AL -space (resp., AM -space) if $\|x + y\| = \|x\| + \|y\|$ (resp., $\|x \vee y\| = \max\{\|x\|, \|y\|\}$) whenever $x \wedge y = 0$. An AM -space has a (strong order) *unit* $u \geq 0$ if the order interval $[-u, u]$ is the unit ball of X .

Definition 40. A band projection π in a Banach lattice X is an M -projection if $\|x\| = \max\{\|\pi x\|, \|\pi^\perp x\|\}$ for all $x \in X$, where $\pi^\perp := I_X - \pi$. The collection of all M -projections forms a subalgebra $\mathfrak{M}(X)$ of $\mathfrak{P}(X)$ in X . A Banach lattice X is \mathbb{B} -cyclic whenever X is a \mathbb{B} -cyclic Banach space for a complete subalgebra $\mathbb{B} \subset \mathfrak{M}(X)$. A \mathbb{B} -isometric lattice homomorphism is referred to as *lattice \mathbb{B} -isometry*.

Theorem 41. *The bounded descent of a Banach lattice from the model $\mathbb{V}^{(\mathbb{B})}$ is a \mathbb{B} -cyclic Banach lattice. Conversely, if X is a \mathbb{B} -cyclic Banach lattice, then in the model $\mathbb{V}^{(\mathbb{B})}$ there is a Banach lattice \mathcal{X} unique up to an isometric isomorphism whose bounded descent is lattice \mathbb{B} -isometric to X . Moreover, $\pi \mapsto \pi \downarrow$ is an isomorphism of Boolean algebras $\mathfrak{M}(\mathcal{X}) \downarrow$ and $\mathfrak{M}(X)$; in symbols, $\mathfrak{M}(\mathcal{X}) \downarrow \simeq \mathfrak{M}(X)$.*

Definition 42. A real Banach lattice X is *injective* whenever, for every Banach lattice Y , every closed vector sublattice $Y_0 \subset Y$, and every positive linear operator $T_0 : Y_0 \rightarrow X$ there exists a positive linear extension $T : Y \rightarrow X$ with $\|T_0\| = \|T\|$.

Thus, the injective Banach lattices are the injective objects in the category of Banach lattices with the positive contractions as morphisms. Arendt [7, Theorem 2.2] proved that the injective objects are the same if the regular operators with contractive modulus are taken as morphisms.

The first example of an injective Banach lattice was indicated by Abramovich in [1] without introducing the term: *A Dedekind complete AM -space with unit is an injective Banach lattice*. Later this fact was rediscovered by Lotz in [83], where the concept of injective Banach lattice was introduced. Lotz also proved that *each AL -space is an injective Banach lattice*; see [83, Proposition 3.2]. This shows that there is an essential difference between the injective Banach lattices and injective Banach spaces, since $C(K)$ with an extremally disconnected compact set K is the only injective object (up to isomorphism) in the category of Banach spaces and linear contractions (see the Nachbin–Goodner–Kelley–Hasumi Theorem [81, Theorem 6]) An important contribution to the study of injective Banach lattices was made by

Cartwright [13] who found the *order intersection property* and proved that a Banach lattice X is injective if and only if X has the order intersection property and there exists a positive contractive projection in X'' onto X (the property (P)); see [69, Definition 5.10.9 (3), Theorems 5.10.10, and 5.10.11]. Another significant advance is due to Haydon [30]. He discovered that an injective Banach space has a mixed AM - AL -structure and proved three representation theorems [30, Theorems 5C, 6H, and 7B].

Theorem 43. *The bounded descent $\mathcal{X} \downarrow$ of an AL -space \mathcal{X} from $\mathbb{V}^{(\mathbb{B})}$ is an injective Banach lattice with $\mathbb{B} \simeq \mathbb{M}(\mathcal{X} \downarrow)$. Conversely, if X is an injective Banach lattice and $\mathbb{B} \simeq \mathbb{M}(X)$, then there exists an AL -space \mathcal{X} within $\mathbb{V}^{(\mathbb{B})}$ whose bounded descent is lattice \mathbb{B} -isometric to X ; in symbols, $X \simeq_{\mathbb{B}} \mathcal{X} \downarrow$.*

According to Theorem 43, each theorem about AL -spaces within Zermelo–Fraenkel set theory has its counterpart for injective Banach lattices. Translation of theorems from AL -spaces to injective Banach lattices is carried out by the functors of Boolean valued analysis. Combining Theorems 20 and 43 yields the following result.

Theorem 44. *If Φ is some strictly positive Maharam operator with the Levi property that takes values in a Dedekind complete AM -space Λ with unit and $\|x\| = \|\Phi(|x|)\|_{\infty}$ ($x \in L^1(\Phi)$), then $(L^1(\Phi), \|\cdot\|)$ is an injective Banach lattice and there is a Boolean isomorphism φ from $\mathbb{B} := \mathbb{P}(\Lambda)$ onto $\mathbb{M}(L^1(\Phi))$ such that $\pi \circ \Phi = \Phi \circ \varphi(\pi)$ for all $\pi \in \mathbb{B}$. Conversely, every injective Banach lattice X is lattice \mathbb{B} -isometric to $(L^1(\Phi), \|\cdot\|)$ for some strictly positive Maharam operator Φ with the Levi property that takes values in a Dedekind complete AM -space Λ with unit, where $\mathbb{B} = \mathbb{P}(\Lambda) \simeq \mathbb{M}(X)$.*

Consider the question of the functional representation of injective Banach lattices. For every cardinal γ , there exists a canonical measure on the unit cube $[0, 1]^{\gamma}$, i.e. the γ th power of Lebesgue’s measure on $[0, 1]$. The associated Banach lattice of integrable functions will be denoted by $L_1([0, 1]^{\gamma})$. The celebrated Kakutani–Maharam representation result tells us that for each AL -space \mathcal{X} there exists a unique family of cardinals $(\delta_{\gamma})_{\gamma \in \Gamma \cup \{0\}}$ with Γ a set of infinite cardinals such that δ_{γ} is either equal to 1 or is uncountable for all $\gamma \in \Gamma$ and

$$\mathcal{X} \simeq l_1(\gamma_0) \oplus \sum_{\gamma \in \Gamma}^{\oplus} \delta_{\gamma} L_1([0, 1]^{\gamma}), \tag{1}$$

where \simeq stands for lattice isometry, while \oplus and \sum^{\oplus} denote l_1 -joins, and δY denotes the l_1 -join of δ copies of Y ; see [81] and [117]. Thus, the Banach lattices $l^1(\gamma_0)$ and

$L_1([0, 1]^\gamma)$ are the “building blocks” for AL -spaces. By transfer the result is true for a Boolean valued representation \mathcal{X} of an injective Banach lattice X . Having worked with the descent and ascent functors, we can find that the building blocks for X are injective Banach lattices $C_\#(K, l^1(\alpha))$ and $C_\#(K, L_1([0, 1]^\gamma))$. Every injective Banach lattice is lattice \mathbb{B} -isometric to a *injective direct sum* of these building blocks. For an injective Banach lattice X there exist families $(K_{\beta\gamma})_{\beta \in B(\gamma)}$ ($\gamma \in \Gamma$) and $(K_\alpha)_{\alpha \in A}$, where Γ is a set of infinite cardinals, A and $B(\gamma)$ are the sets of cardinals, and each element of $B(\gamma)$ is either equal to 1 or is uncountable for all $\gamma \in \Gamma$, such that $K_{\beta\gamma}$ and $K_{\beta\gamma}$ make up the partition of unity in the Boolean algebra of clopen subsets of the Stone representation space of $\mathbb{M}(X)$ and the representation holds:

$$X \simeq_{\mathbb{B}} \left(\sum_{\alpha \in A} C_\#(K_\alpha, l^1(\alpha)) \right)_\infty \boxplus \sum_{\gamma \in \Gamma}^{\boxplus} \left(\sum_{\beta \in B(\gamma)} \beta \diamond C_\#(K_{\beta\gamma}, L^1([0, 1]^\gamma)) \right)_\infty, \quad (2)$$

where $\beta \diamond Y$ stands for the injective direct sum of β copies of Y and \sum denotes the l_∞ -join. The formula (2) is the descent of the internal representation (1), while the injective direct sum \sum^{\boxplus} of injective Banach lattices can be defined as the descent of the internal l^1 -join within $\mathbb{V}^{(\mathbb{B})}$. For more details see [60], [61]. The representation (2) of an injective Banach lattice is not unique in general for the same reason as in Remark 38: If $\varkappa < \lambda$ and $\mathbb{V}^{(\mathbb{B})} \models |\varkappa^\wedge| = |\lambda^\wedge|$, then $C_\#(K, L^1([0, 1]^\varkappa))$ and $C_\#(K, L^1([0, 1]^\lambda))$ are lattice \mathbb{B} -isometric. The above enables us to give a complete isometric classification of injective Banach lattices; see [60] and [61].

Remark 45. We indicate a few more results obtained by using the Boolean valued transfer principle for injective Banach lattices. The Daugavet equation in injective Banach lattices, injective Banach lattices of operators, the Boolean valued interpretation of the theory of cone absolutely summing operators, and the operators factoring through injective Banach lattices are examined in Kusraev [63]; Kusraev and Wickstead [72] (also see [69]). The following Boolean value version of Ando’s Theorem was obtained by Kusraev and Kutateladze [70, Theorem 6.4]: Each closed \mathbb{B} -complete sublattice in a \mathbb{B} -cyclic Banach lattice X admits a positive contractive projection commuting with projections from $\mathbb{B} = \mathbb{M}(X)$ if and only if there exists a partition of unity $(\pi_\gamma)_{\gamma \in \Gamma \cup \{0\}}$ in \mathbb{B} with Γ being a nonempty set of cardinals such that $\pi_0 X \simeq_{\pi_0 \mathbb{B}} L^p(\Phi)$ for some $1 \leq p \in \Lambda^u$ and injective Banach lattice $L := L^1(\Phi)$, for which $\mathbb{M}(L) \simeq \pi_0 \mathbb{B}$, and $\pi_\gamma X \simeq_{\pi_\gamma \mathbb{B}} C_\#(Q_\gamma, c_0(\gamma))$ for all $\gamma \in \Gamma$, where Q_γ is a clopen subset of the Stone representation space Q of \mathbb{B} corresponding to the projection π_γ .

6 C^* -Algebras and AW^* -Algebras

This section deals with a transfer principle for C^* -algebras and AW^* -algebras and a classification of type I AW^* -algebras. We start with C^* -algebras. See Berberian [12], Sakai [113], and Takesaki [121] for the needed information on the topic.

Definition 46. A \mathbb{B} -cyclic C^* -algebra or \mathbb{B} - C^* -algebra A is a C^* -algebra that is a \mathbb{B} -cyclic Banach space and for each projection $\pi \in \mathbb{B}$ we have $\pi(xy) = \pi(x)y = x\pi(y)$ and $\pi(x^*) = \pi(x)^*$ for all $x, y \in A$. An element $z \in A$ is *central* provided that z commutes with every member of A . The *center* of a T^* -algebra A is the set $\mathcal{Z}(A)$ of all central elements. Clearly, $\mathcal{Z}(A)$ is a commutative C^* -subalgebra of A and $\mathbb{C}\mathbb{1} \subset \mathcal{Z}(A)$.

The Boolean valued transfer principle for C^* -algebras, discovered by Takeuti [127], is stated below in terms of the complete Boolean algebra of projections. As regards other formulations that use a module structure, see Ozawa [109, Theorem 2], [111, Theorem 6.3] and Takeuti [127, Theorem 1.1]).

Theorem 47. *If \mathcal{A} is a C^* -algebra within $\mathbb{V}^{(\mathbb{B})}$ then $A := \mathcal{A} \Downarrow$ is a \mathbb{B} - C^* -algebra. Conversely, for each \mathbb{B} - C^* -algebra A there exists C^* -algebra \mathcal{A} within $\mathbb{V}^{(\mathbb{B})}$ such that A is $*$ - \mathbb{B} -isomorphic to $\mathcal{A} \Downarrow$.*

Definition 48. An AW^* -algebra is a C^* -algebra presenting a Baer $*$ -algebra. More explicitly, an AW^* -algebra is a C^* -algebra A whose every *right annihilator* $M^\perp := \{y \in A : (\forall x \in M) xy = 0\}$ has the form pA , with p a projection. A *projection* p is a hermitian ($p^* = p$) idempotent ($p^2 = p$) element. If $\mathcal{Z}(A) = \{\lambda\mathbb{1} : \lambda \in \mathbb{C}\}$ then the AW^* -algebra A is an AW^* -factor.

The symbol $\mathbb{P}(A)$ stands for the set of all projections of an involutive algebra A . Denote the set of all central projections by $\mathbb{P}_c(A)$. Observe that $\bar{\Lambda} := \mathcal{C} \Downarrow$ is a commutative AW^* -algebra and $\mathbb{P}(\bar{\Lambda}) = \mathbb{P}_c(\bar{\Lambda})$. If $\bar{\Lambda} = \mathcal{Z}(A)$ then $\bar{\Lambda} = \bar{\Lambda}(\mathbb{B})$ with $\mathbb{B} = \mathbb{P}_c(A)$. An AW^* -algebra A is a \mathbb{B} -cyclic C^* -algebra for every order closed subalgebra \mathbb{B} of the complete Boolean algebra $\mathbb{P}_c(A)$. This fact together with Theorem 32 yields the following result due to Ozawa [109].

Theorem 49. *If \mathcal{A} is an AW^* -algebra within $\mathbb{V}^{(\mathbb{B})}$ then $A := \mathcal{A} \Downarrow$ is also an AW^* -algebra and $\mathbb{P}_c(A)$ has an order closed subalgebra isomorphic to \mathbb{B} . Conversely, if A is an AW^* -algebra and \mathbb{B} is an order closed subalgebra of the Boolean algebra $\mathbb{P}_c(A)$ then there is an AW^* -algebra \mathcal{A} within $\mathbb{V}^{(\mathbb{B})}$ such that $\mathcal{A} \Downarrow$ is $*$ - \mathbb{B} -isomorphic with A . Moreover, \mathcal{A} is an AW^* -factor if and only if $\mathbb{B} := \mathbb{P}_c(A)$.*

The classification of an AW^* -algebra into types is determined from the structure of its lattice of projections; see [56] and [113]. It is important to emphasize the *absoluteness* of types; i.e., the Boolean valued representation preserves this classification; see Takeuti [126] and Ozawa [109]. Similar absoluteness theorems in a completely lattice-theoretical framework were established by Nishimura [93]. We recall only the definition of type I AW^* -algebra.

Definition 50. A projection $\pi \in A$ is *abelian* provided that the algebra $\pi A \pi$ is commutative. An algebra A has *type I*, if each nonzero projection in A contains a nonzero abelian projection. Say that a C^* -algebra A is \mathbb{B} -*embeddable* whenever there are a type I AW^* -algebra N with $\mathbb{B} = \mathbb{P}_c(N)$ and a $*$ -monomorphism $\pi : A \rightarrow N$ such that $\pi(A)$ coincides with the bicommutant $\pi(S)''$ of $\pi(A)$ in N . Furthermore, if $\mathbb{B} = \mathbb{P}_c(A)$ then A is *centrally embeddable*.

Definition 51. A \mathbb{B} -cyclic Banach space Y is \mathbb{B} -*dual* or \mathbb{B} -*bidual* provided that, respectively, $Y \simeq_{\mathbb{B}} X^\#$ or $Y \simeq_{\mathbb{B}} X^{\#\#}$ for some \mathbb{B} -cyclic Banach space X , where $\simeq_{\mathbb{B}}$ stands for isometric \mathbb{B} -isomorphy. (Recall that $X^\# := \mathcal{L}_{\mathbb{B}}(X, \mathbb{B}(\Lambda))$ and $\Lambda = \Lambda(\mathbb{B})$.) Say that Y is a \mathbb{B} -*predual* of X if $Y^\# \simeq_{\mathbb{B}} X$ and Y is \mathbb{B} -*selfdual* if $Y \simeq_{\mathbb{B}} Y^\#$.

Ozawa [111, Theorems A, B, and C] characterized those C^* -algebras that are \mathbb{B} -*dual*, \mathbb{B} -*bidual*, and \mathbb{B} -*selfdual* (in terms of the $\bar{\Lambda}(\mathbb{B})$ -module instead of the Boolean algebra of projections \mathbb{B}). He also proved that a \mathbb{B} -embeddable C^* -algebra has a predual unique up to \mathbb{B} -isometry which is a Kaplansky–Hilbert module over $\bar{\Lambda}(\mathbb{B})$; see [111, Theorem D]).

Let X be a Kaplansky–Hilbert module over $\bar{\Lambda}$ and denote by $B_{\bar{\Lambda}}(X)$ the space of all continuous $\bar{\Lambda}$ -linear operators in X . Since a $\bar{\Lambda}$ -linear operator is continuous if and only if it has an adjoint, $B_{\bar{\Lambda}}(X)$ is an AW^* -algebra of type I with center isomorphic to $\bar{\Lambda}$. As it was shown by Kaplansky [45], a type I AW^* -algebra A is isomorphic to $B_{\bar{\Lambda}}(X)$ for some Kaplansky–Hilbert module X over $\bar{\Lambda}(\mathbb{B})$ with $\mathbb{B} = \mathbb{P}_c(A)$. Taking into account Theorem 35, we arrive at the following transfer principle from von Neumann algebras to embeddable AW^* -algebras (see Ozawa [107, Theorem 2.3] and [109, Theorem 6]):

Theorem 52. *Let \mathcal{A} be a C^* -algebra within $\mathbb{V}^{(\mathbb{B})}$ and let A be the bounded descent of \mathcal{A} . Then A is a \mathbb{B} -embeddable AW^* -algebra if and only if \mathcal{A} is a von Neumann algebra within $\mathbb{V}^{(\mathbb{B})}$. The algebra A is centrally embeddable if and only if \mathcal{A} is a von Neumann factor within $\mathbb{V}^{(\mathbb{B})}$.*

We now present a complete system of $*$ -isomorphism invariants for type I AW^* -algebras due to Ozawa [106]. Every automorphism π of a complete Boolean algebra \mathbb{B} can be extended to a Boolean truth-value preserving automorphism π^* of $\mathbb{V}^{(\mathbb{B})}$; see [69, § 1.3].

Definition 53. Two internal cardinals $\alpha, \beta \in \mathbb{V}(\mathbb{B})$ are said to be *congruent* if there is an automorphism π of \mathbb{B} with $\beta = \pi^*(\alpha)$. The *congruence class* of α is defined as $[\alpha] := \{\pi^*(\alpha) : \pi \text{ is an automorphism of } \mathbb{B}\}$. Given a type I AW^* -algebra A with center isomorphic to $\bar{\Lambda}(\mathbb{B})$, define the *degree* $\text{Deg}(A)$ of A as $[\text{Dim}(X)]$, where X is a Kaplansky–Hilbert module over $\bar{\Lambda}(\mathbb{B})$ such that A is $*$ -isomorphic to $B_{\Lambda}(X)$ and $\text{Dim}(X) \in \mathbb{V}(\mathbb{B})$ is the dimension of the Boolean valued representation $\mathcal{X} \in \mathbb{V}(\mathbb{B})$ of X .

Theorem 54. *Two type I AW^* -algebras are $*$ -isomorphic if and only if their centers are $*$ -isomorphic and they have the same degree. For every nonzero cardinal α within $\mathbb{V}(\mathbb{B})$ there is a type I AW^* -algebra A with $\mathcal{L}(A)$ isomorphic to $\bar{\Lambda}(\mathbb{B})$ and $\text{Deg}(A) = [\alpha]$.*

Remark 55. The modern structural theory of AW^* -algebras originates with the articles [43]–[45] by Kaplansky. These objects appear naturally by way of algebraization of the theory of von Neumann operator algebras. The study of C^* -algebras and von Neumann algebras by Boolean valued models was started by Takeuti with [125] and [126]. See Korol’ and Chilin [46], Nishimura [91], [94], [98], [101], and Ozawa [104]–[111] for further related developments.

Remark 56. Combining the results about the Boolean valued representations of AW^* -algebras with the analytical representations for dominated operators, we come to some functional representations of AW^* -algebras (see Kusraev [56]): *To each type I AW^* -algebra A there exist a set of cardinals Γ and a family of nonempty extremally disconnected compact spaces $(Q_{\gamma})_{\gamma \in \Gamma}$ such that there is a $*$ - \mathbb{B} -isomorphism:*

$$A \simeq \sum_{\gamma \in \Gamma}^{\oplus} SC_{\#}(Q_{\gamma}, B(l_2(\gamma))).$$

Remark 57. Boolean valued analysis of AW^* -algebras yields a negative solution to the Kaplansky problem of unique decomposition of a type I AW^* -algebra into the direct sum of homogeneous components. Ozawa gave this solution in [106] and [108]. The lack of uniqueness is tied with the effect of the *cardinal shift*. The cardinal shift is impossible in the case when the Boolean algebra of central idempotents \mathbb{B} under study satisfies the countable chain condition, and so the decomposition in question is unique. Kaplansky established the uniqueness of the decomposition on assuming that \mathbb{B} satisfies the countable chain condition and conjectured that uniqueness fails in general; see [45].

Remark 58. The concept of Kaplansky–Hilbert module was introduced by Kaplansky in [45] under the name AW^* -module. In the introduction he wrote: “... *the new*

idea is to generalize Hilbert space by allowing the inner product to take values in a more general ring than the complex numbers. After the appropriate preliminary theory of these AW^ -modules has been developed, one can operate with a general AW^* -algebra of type I in almost the same manner as with the factor.”* In other words, the central elements of an AW^* -algebra can be taken as complex numbers and one can work with factors rather than general AW^* -algebras. Needless to say, this is a version of Kantorovich’s heuristic principle; see Remark 23.

7 Miscellany

7.1 The Wickstead problem

An operator in a vector lattice is *band preserving* if each band is its invariant subspace. The following question was raised by Wickstead in [131]: Which vector lattices have the property (sometimes called the *Wickstead property*) that every linear band preserving operator in them is automatically order bounded? One of the principal technical tools is the concept of *d-basis* which is presented in the memoir [3, Section 4]. Boolean valued analysis reduces the Wickstead problem to that of order boundedness of the endomorphisms of the field \mathcal{R} or \mathcal{C} viewed as a vector lattice and algebra over the field \mathbb{R}^\wedge or \mathbb{C}^\wedge , respectively; see [69, §4.2]. In particular, each *d-basis* is just a Boolean valued Hamel basis [69, §4.5]. Gutman [33] proved that a vector lattice X has the Wickstead property if and only if the Boolean algebra $\mathbb{P}(X)$ is σ -distributive if and only if \mathcal{R} and \mathbb{R}^\wedge coincide within $\mathbb{V}^{(\mathbb{B})}$. Kusraev [57] established that in a universally complete complex vector lattice X with a fixed f -algebra multiplication the Wickstead property is equivalent to each of the following assertions: (1) there is no nonzero derivation in X ; (2) every band preserving endomorphism in X is a band projection; (3) there is no nontrivial band preserving automorphism in X . The history and state of the art of the Wickstead problem are presented in [34] and [69, Chapter 4]. It worth mentioning here that the question of automatic continuity of homomorphisms from a Banach algebra of continuous functions into an arbitrary Banach algebra is independent of ZFC; see Dales and Woodin [18] as well as Dales and Oliveri [17].

7.2 A transfer principle in harmonic analysis

In [124] Takeuti introduced the Fourier transform for the mappings defined on a locally compact abelian group and having as values pairwise commutable normal operators in a Hilbert space. By applying the transfer principle, he developed a general technique for translating classical results to operator-valued functions. In particu-

lar he established a version of the Bochner Theorem describing the set of all inverse Fourier transforms of positive operator-valued Radon measures. Similar results were obtained by Gordon and Lyubetskii within their theory of the Boolean extension of a uniform space; see [28] and [29]. Nishimura [92] extended Takeuti's Boolean valued approach to abstract harmonic analysis on locally compact abelian groups to locally compact groups (not abelian in general). Kusraev and Malyugin in [71] improved Takeuti's results in the following directions: more general arrival spaces (including Banach spaces and Dedekind complete vector lattices) were considered, the class of dominated mappings was identified with the set of all inverse Fourier transforms of order bounded quasi-Radon vector measures, and the construction of a Boolean valued universe was eliminated from the definitions and statements of the results.

7.3 Boolean compactness

Combining the notions of Boolean mixing and compactness yields the concept of *mix-compactness* (or *cyclic compactness*) and the corresponding class of linear operators. Consider a Λ -metric space (X, ρ) with $\Lambda = \mathcal{R}\downarrow$. A subset $K \subset X$ is *mix-compact* if K is mix-complete and for every sequence $(x_n)_{n \in \mathbb{N}} \subset K$ there is $x \in K$ such that $\inf_{n \geq k} \rho(x_n, x) = 0$ for all $k \in \mathbb{N}$. Clearly, in case $\Lambda = \mathbb{R}$ mix-compactness is equivalent to compactness in the metric topology. The concept of cyclic compactness was first studied by Kusraev [47] and [52]. Section 8.5 in [56] deals with the cyclically compact linear operators on \mathbb{B} -cyclic Banach spaces. Gönüllü [31] and [32] found the Lidskii trace formula and the Rayleigh–Ritz minimax formula for cyclically compact operators in Kaplansky–Hilbert modules. The equivalent concept of mix-compact subset of a lattice-normed space was introduced in Gutman and Lisovskaya [35]. Basing on Boolean valued analysis, they proved some counterparts of the three classical theorems for arbitrary lattice-normed spaces over universally complete vector lattices, namely, the boundedness principle, the Banach–Steinhaus Theorem, and the uniform boundedness principle for a compact convex set; see [35, Theorems 2.4, 2.6, and 3.3]. In [63] and [72] Kusraev and Wickstead examine the question of when the space of compact operators is a vector lattice or an injective vector lattice. Moreover, a Dodds–Fremlin–Wickstead type domination result for cyclically compact operators was obtained in [72, Theorem 8.13].

7.4 *JB*-algebras

The *JB*-algebras are nonassociative real analogs of C^* -algebras and von Neumann operator algebras. The theory of these algebras exists as a branch of functional analysis since the mid 1960s; see [10] and [36]. The Boolean valued approach to

JB -algebras is outlined by Kusraev [54] and [55]. In [54] a \mathbb{B} - JB -algebra is defined as a JB -algebra that is a \mathbb{B} -cyclic Banach space with respect to a complete Boolean algebra of central idempotents \mathbb{B} and, naturally, it turns out that \mathbb{B} - JL -algebras are the bounded descents of JB -algebras from $\mathbb{V}^{(\mathbb{B})}$ [54, Theorem 3.1]. Then it is proved that a \mathbb{B} - JB -algebra A is a \mathbb{B} -dual space if and only if A is monotone complete and admits a separating set of $\Lambda(\mathbb{B})$ -valued normal states [54, Theorem 4.2]. An algebra A satisfying one of these equivalent conditions is a \mathbb{B} - JBW -algebra. Each \mathbb{B} - JBW -factor A admits a unique decomposition $A = eA \oplus e^*A$ with a central projection $e \in \mathbb{B}$, $e^* := 1 - e$, such that the algebra eA has a faithful representation in the algebra of selfadjoint operators on a Kaplansky–Hilbert module and e^*A is isomorphic to $C(Q, M_3^{\mathbb{S}})$, where Q is the Stone representation space of the Boolean algebra $e^*B := [0, e^*]$ and $M_3^{\mathbb{S}} := M_3(\mathbb{O})$ is the algebra of hermitian (3×3) -matrices over the Cayley numbers \mathbb{O} ; see [54, Theorem 4.6]. A full classification of type I_2 AJW -algebras was obtained in [55]. More details and references are collected in [54], [58], and [68].

7.5 Convex analysis

One of the most important concepts in convex analysis is that of *support set* or *subdifferential at zero*, i.e. the convex set of linear operators majorized by a sublinear operator; see [66]. The intrinsic characterization of subdifferentials was first formulated as a conjecture by Kutateladze in [73] and then it was proved by Kusraev and Kutateladze (see [64] and [65]): *A weakly order bounded set of operators is a subdifferential if and only if it is operator convex and closed with respect to pointwise order convergence.* The result is well known for functionals and the Boolean valued transfer principle enables one to translate the result to the operators taking values in the universally complete vector lattice that is the descent of the reals. Similarly, we can recover a subdifferential from its extreme points on using the classical Krein–Milman Theorem and its Milman’s inversion. Kutateladze in [74] and [75] weakened the boundedness assumption in the spirit of the classical theory of caps which was developed by Choquet and his followers; see [8] and [112]. The peculiarity of his approach consists in working with the new notion of operator cap. An operator cap is not a cap in the classical sense in general but becomes a usual cap in the scalar case. More precisely, when studying convex sets of operators it is appropriate to use operator caps rather than conventional caps, i.e. the descents of scalar caps from a suitable Boolean valued model; see [66] for details. Recently Kutateladze applied Boolean valued analysis to deriving the operator versions of the classical Farkas Lemma in the theory of simultaneous linear inequalities and proved the Lagrange principle for dominated polyhedral sublinear operators; see [79] and [80].

7.6 Mathematical finance

In order to provide an analytical basis to some problems of *mathematical finance* in a multiperiod setup with a dynamic flow of information, the two approaches were proposed: randomized convex analysis (Filipovic, Kupper, and Vogelpoth [20]) and conditional set theory (Drapeau, Jamnesahn, Karliczek, and Kupper [19]). It is proved in Avilés and Zapata [9, Theorems 2.2 and 3.1] that: (1) the category of mix-complete L^0 -convex modules and continuous L^0 -linear operators is equivalent to the category of locally convex spaces and continuous linear operators within $\mathbb{V}^{(\mathbb{B})}$; (2) the category of conditional sets and conditional mappings is equivalent to the category of sets and mappings within $\mathbb{V}^{(\mathbb{B})}$; also see [133]. Thus, *Boolean valued analysis* provides a natural framework for the study of *locally L^0 -convex analysis* and *conditional set theory* and to explore new applications to conditional risk measures, equilibrium theory, optimal stochastic control, financial preferences, etc. More details and references are collected in [9], [133], and [134].

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