Boolean Methods in Positivity[*][1][2]

A. G. Kusraev[1][**] and S. S. Kutateladze[2][***]

[1]*Institute of Applied Mathematics and Informatics, ul. Markusa 22, Vladikavkaz, 362027 Russia*
[2]*Sobolev Institute of Mathematics, pr. Akad. Koptyuga 4, Novosibirsk, 630090 Russia*
Received November 1, 2007

Abstract—This is an overview of the new possibilities that are open by Boolean valued analysis in positivity.

INTRODUCTION

Boolean valued analysis is a general mathematical method that rests on a special model-theoretic technique. This technique consists generally in comparison between the representations of arbitrary mathematical objects and theorems in two different set-theoretic models whose constructions start with principally distinct Boolean algebras. We usually take as these models the cosiest Cantorian paradise, the von Neumann universe of Zermelo–Fraenkel set theory, and a special universe of Boolean valued "variable" sets trimmed and chosen so that the traditional concepts and facts of mathematics acquire completely unexpected and bizarre interpretations. The use of two models, one of which is *formally* nonstandard, is a family feature of *nonstandard analysis*. For this reason, Boolean valued analysis means an instance of nonstandard analysis in common parlance. By the way, the term *Boolean valued analysis* was minted by G. Takeuti.

Proliferation of Boolean valued models is due to P. Cohen's final breakthrough in Hilbert's Problem Number One. His method of forcing was rather intricate and the inevitable attempts at simplification gave rise to the Boolean valued models by D. Scott, R. Solovay, and P. Vopěnka.

A grievous event is of relevance to this article: Saunders Mac Lane, a cofather of category theory, passed away in San Francisco on April 14, 2005. The power of mathematics rests heavily on the trick of socializing the objects and problems under consideration. The understanding of the social medium of set-theoretic models belongs to category theory.

A category is called an *elementary topos* provided that it is cartesian closed and has a subobject classifier. A. Grothendieck and F. W. Lawvere, the followers of Saunders Mac Lane, created topos theory in the course of "point elimination" and in the dream of invariance of the objects we operate in mathematics. It is on this road that we met the conception of variable sets, underlying the notion of topos and bringing about the understanding of the social medium of set-theoretic models. The Boolean valued models belong happily to the family of Boolean toposes enjoying the classical Aristotle logic.

Category theory, alongside set theory, serves as a universal language of modern mathematics. Categories, functors, and natural transformations are widely used in all areas of mathematics, allowing us to look uniformly and consistently on various constructions and formulate the general properties of diverse structures. The impact of category theory is irreducible to the narrow frameworks of its outstanding expressive conveniences. This theory has drastically changed our general outlook on the foundations of mathematics and widened the room of free thinking in mathematics.

---

[*]The text was submitted by the authors in English.
[1]This article is an extended version of a talk at the International Conference "Positivity IV: Theory and Applications" held by Technische Universität Dresden in 2005 (cp. [36]).
[2]In memory of Saunders Mac Lane (1909–2005)
[**]E-mail: kusraev@alanianet.ru
[***]E-mail: sskut@member.ams.org

Set theory, a great and ingenious creation of Georg Cantor, occupies in the common opinion of the twentieth century the place of the sole solid base of modern mathematics. Mathematics becomes sinking into a section of the Cantorian set theory. Most active mathematicians, teachers, and philosophers view as obvious and undisputable the thesis that mathematics cannot be grounded on anything but set theory. The set-theoretic stance transforms paradoxically into an ironclad dogma, a clear-cut forbiddance of thinking (as L. Feuerbach once put it wittily). Such an indoctrinated view of the foundations of mathematics is false and contradicts conspicuously to the leitmotif, nature, and pathos of the essence of all creative contribution of G. Cantor who wrote as far back as in 1883 that "denn das *Wesen* der *Mathematik* liegt gerade in ihrer *Freiheit*."

Topos theory provides a profusion of categories of which classical set theory is an ordinary member. Mathematics has thus acquired infinitely many new degrees of freedom. All these achievements rest on category theory.

"There remains to us, then, the pursuit of truth, by way of proof, the concatenation of those ideas which fit, and the beauty which results when they do fit." So wrote Saunders Mac Lane, a great genius, creator, master, and servant of mathematics. We reverently dedicate this article to the memory of this eternal and tragicomical mathematical Knight of the Sorrowful Figure and Category.

1. BOOLEAN REQUISITES

We start with recalling some auxiliary facts about the construction and treatment of Boolean valued models.

1.1. Let $\mathbb{B}$ be a complete Boolean algebra. Given an ordinal α, put

$$\mathbb{V}_\alpha^{(\mathbb{B})} := \{x : x \text{ is a function } \wedge \ (\exists\beta)(\beta < \alpha \ \wedge \operatorname{dom}(x) \subset \mathbb{V}_\beta^{(\mathbb{B})} \ \wedge \ \operatorname{im}(x) \subset \mathbb{B})\}.$$

After this recursive definition the *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ or, in other words, the *class of* $\mathbb{B}$-*sets* is introduced by

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha \in \operatorname{On}} \mathbb{V}_\alpha^{(\mathbb{B})},$$

with On standing for the class of all ordinals.

In case of the two element Boolean algebra $\mathbf{2} := \{\mathbb{O}, \mathbb{1}\}$ this procedure yields a version of the classical *von Neumann universe* $\mathbb{V}$ (see 2.1 (2)).

Let φ be an arbitrary formula of ZFC, Zermelo−Fraenkel set theory with choice. The *Boolean truth value* $[\![\varphi]\!] \in \mathbb{B}$ is introduced by induction on the length of a formula φ by naturally interpreting the propositional connectives and quantifiers in the Boolean algebra $\mathbb{B}$ and taking into consideration the way in which this formula is built up from atomic formulas. The Boolean truth values of the *atomic formulas* $x \in y$ and $x = y$, with $x, y \in \mathbb{V}^{(\mathbb{B})}$, are defined by means of the following recursion schema:

$$[\![x \in y]\!] = \bigvee_{t \in \operatorname{dom}(y)} y(t) \wedge [\![t = x]\!],$$

$$[\![x = y]\!] = \bigvee_{t \in \operatorname{dom}(x)} x(t) \Rightarrow [\![t \in y]\!] \wedge \bigvee_{t \in \operatorname{dom}(y)} y(t) \Rightarrow [\![t \in x]\!].$$

The sign $\Rightarrow$ symbolizes the implication in $\mathbb{B}$; i.e., $a \Rightarrow b := a^* \vee b$ where a^* is as usual the *complement* of a.

The universe $\mathbb{V}^{(\mathbb{B})}$ with the Boolean truth value of a formula is a model of set theory in the sense that the following statement is fulfilled.

1.2. Transfer Principle. *For every theorem φ of* ZFC*, we have* $[\![\varphi]\!] = \mathbb{1}$*; i.e., φ is true inside* $\mathbb{V}^{(\mathbb{B})}$*.*

Enter into the next agreement: If x is an element of $\mathbb{V}^{(\mathbb{B})}$ and $\varphi(\cdot)$ is a formula of ZFC then the phrase "x satisfies φ inside $\mathbb{V}^{(\mathbb{B})}$" or, briefly, "$\varphi(x)$ is true inside $\mathbb{V}^{(\mathbb{B})}$" means that $[\![\varphi(x)]\!] = \mathbb{1}$. This is sometimes written as $\mathbb{V}^{(\mathbb{B})} \models \varphi(x)$.

Given $x \in \mathbb{V}^{(\mathbb{B})}$ and $b \in \mathbb{B}$, define the function $bx : z \mapsto bx(z)$ $(z \in \mathrm{dom}(x))$. Here we presume that $b\varnothing := \varnothing$ for all $b \in \mathbb{B}$.

There is a natural equivalence relation $x \sim y \leftrightarrow [\![x = y]\!] = \mathbb{1}$ in the class $\mathbb{V}^{(\mathbb{B})}$. Choosing a representative of the smallest rank in each equivalence class or, more exactly, using the so-called "Frege–Russell–Scott trick," we obtain a *separated Boolean valued universe* $\overline{\mathbb{V}}^{(\mathbb{B})}$ in which

$$x = y \; \leftrightarrow \; [\![x = y]\!] = \mathbb{1}.$$

It is easily to see that the Boolean truth value of a formula remains unaltered if we replace in it each element of $\mathbb{V}^{(\mathbb{B})}$ by one of its equivalents. In this connection from now on we take $\mathbb{V}^{(\mathbb{B})} := \overline{\mathbb{V}}^{(\mathbb{B})}$ without further specification.

Observe that in $\overline{\mathbb{V}}^{(\mathbb{B})}$ the element bx is defined correctly for $x \in \overline{\mathbb{V}}^{(\mathbb{B})}$ and $b \in \mathbb{B}$ since

$$[\![x_1 = x_2]\!] = \mathbb{1} \; \to \; [\![bx_1 = bx_2]\!] = b \; \Rightarrow \; [\![x_1 = x_2]\!] = \mathbb{1}.$$

For a similar reason, we often write $\mathbb{O} := \varnothing$, and in particular $\mathbb{O}\varnothing = \varnothing = \mathbb{O}x$ for $x \in \mathbb{V}^{(\mathbb{B})}$.

1.3. Mixing Principle. *Let $(b_\xi)_{\xi \in \Xi}$ be a partition of unity in $\mathbb{B}$, i.e.*

$$\sup_{\xi \in \Xi} b_\xi = \sup \mathbb{B} = \mathbb{1}, \qquad \xi \neq \eta \; \to \; b_\xi \wedge b_\eta = \mathbb{O}.$$

To each family $(x_\xi)_{\xi \in \Xi}$ in $\mathbb{V}^{(\mathbb{B})}$ there exists a unique element x in the separated universe such that $[\![x = x_\xi]\!] \geq b_\xi$ $(\xi \in \Xi)$.

This element is called the *mixing* of $(x_\xi)_{\xi \in \Xi}$ by $(b_\xi)_{\xi \in \Xi}$ and is denoted by $\sum_{\xi \in \Xi} b_\xi x_\xi$.

1.4. Maximum Principle. *If φ is a formula of ZFC then there is a $\mathbb{B}$-valued set x_0 satisfying $[\![(\exists x)\varphi(x)]\!] = [\![\varphi(x_0)]\!]$.*

2. THE ESCHER RULES

Boolean valued analysis consists primarily in comparison of the instances of a mathematical object or idea in two Boolean valued models. This is impossible to achieve without some dialog between the universes $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$. In other words, we need a smooth mathematical toolkit for revealing interplay between the interpretations of one and the same fact in the two models $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$. The relevant *ascending-and-descending technique* rests on the functors of canonical embedding, descent, and ascent.

2.1. We start with the canonical embedding of the von Neumann universe $\mathbb{V}$.

Given $x \in \mathbb{V}$, let $x^{\wedge}$ denote the *standard name* of x in $\mathbb{V}^{(\mathbb{B})}$; i.e., the element defined by the following recursion schema:

$$\varnothing^{\wedge} := \varnothing, \qquad \mathrm{dom}(x^{\wedge}) := \{y^{\wedge} : y \in x\}, \qquad \mathrm{im}(x^{\wedge}) := \{\mathbb{1}\}.$$

Observe some properties of the mapping $x \mapsto x^{\wedge}$ we need in the sequel:

(1) For an arbitrary $x \in \mathbb{V}$ and a formula φ of ZFC, we have

$$[\![(\exists y \in x^{\wedge})\,\varphi(y)]\!] = \bigvee_{z \in x} [\![\varphi(z^{\wedge})]\!], \qquad [\![(\forall y \in x^{\wedge})\,\varphi(y)]\!] = \bigwedge_{z \in x} [\![\varphi(z^{\wedge})]\!].$$

(2) If x and y are elements of $\mathbb{V}$ then, by transfinite induction, we establish

$$x \in y \; \leftrightarrow \; \mathbb{V}^{(\mathbb{B})} \models x^{\wedge} \in y^{\wedge}, \qquad x = y \; \leftrightarrow \; \mathbb{V}^{(\mathbb{B})} \models x^{\wedge} = y^{\wedge}.$$

In other words, the standard name can be considered as an embedding of $\mathbb{V}$ into $\mathbb{V}^{(\mathbb{B})}$. Moreover, it is beyond a doubt that the standard name sends $\mathbb{V}$ onto $\mathbb{V}^{(\mathbf{2})}$, which fact is demonstrated by the next proposition:

(3) The following holds: $(\forall u \in \mathbb{V}^{(\mathbf{2})})\,(\exists! x \in \mathbb{V})\, \mathbb{V}^{(\mathbb{B})} \models u = x^{\wedge}$.

A formula is called *bounded* or *restricted* if each bound variable in it is restricted by a bounded quantifier; i.e., a quantifier ranging over a particular set. The latter means that each bound variable x is restricted by a quantifier of the form $(\forall x \in y)$ or $(\exists x \in y)$ for some y.

2.2. Restricted Transfer Principle. *For each bounded formula φ of* ZFC *and every collection* $x_1, \ldots, x_n \in \mathbb{V}$ *the following holds:* $\varphi(x_1, \ldots, x_n) \leftrightarrow \mathbb{V}^{(\mathbb{B})} \models \varphi(x_1^\wedge, \ldots, x_n^\wedge)$.

Henceforth, working in the separated universe $\overline{\mathbb{V}}^{(\mathbb{B})}$, we agree to preserve the symbol $x^\wedge$ for the distinguished element of the class corresponding to x.

For example, observe that the restricted transfer principle yields:

$$\text{``}\Phi \text{ is a correspondence from } x \text{ to } y\text{''}$$

$$\leftrightarrow \ \mathbb{V}^{(\mathbb{B})} \models \text{``}\Phi^\wedge \text{ is a correspondence from } x^\wedge \text{ to } y^\wedge\text{''};$$

$$\text{``}f : x \to y\text{''} \ \leftrightarrow \ \mathbb{V}^{(\mathbb{B})} \models \text{``}f^\wedge : x^\wedge \to y^\wedge\text{''}$$

(moreover, $f(a)^\wedge = f^\wedge(a^\wedge)$ for all $a \in x$). Thus, the standard name can be considered as a covariant functor of the category of sets (or correspondences) inside $\mathbb{V}$ to an appropriate subcategory of $\mathbb{V}^{(2)}$ in the separated universe $\mathbb{V}^{(\mathbb{B})}$.

2.3. A set X is *finite* if X coincides with the image of a function on a finite ordinal. In symbols, this is expressed as $\mathrm{fin}(X)$; hence,

$$\mathrm{fin}(X) := (\exists n)(\exists f)(n \in \omega \wedge f \text{ is a function} \wedge \mathrm{dom}(f) = n \wedge \mathrm{im}(f) = X)$$

(as usual $\omega := \{0, 1, 2, \dots\}$). Obviously, the above formula is not bounded. Nevertheless there is a simple transformation rule for the class of finite sets under the canonical embedding. Let $\mathcal{P}_{\mathrm{fin}}(X)$ denote the class of all finite subsets of X; i.e., $\mathcal{P}_{\mathrm{fin}}(X) := \{Y \in \mathcal{P}(X) : \mathrm{fin}(Y)\}$. For an arbitrary set X the following holds: $\mathbb{V}^{(\mathbb{B})} \models \mathcal{P}_{\mathrm{fin}}(X)^\wedge = \mathcal{P}_{\mathrm{fin}}(X^\wedge)$.

2.4. Given an arbitrary element x of the (separated) Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$, we define the *descent $x{\downarrow}$* of x as $x{\downarrow} := \{y \in \mathbb{V}^{(\mathbb{B})} : [\![y \in x]\!] = \mathbb{1}\}$. We list the simplest properties of descending:

(1) The class $x{\downarrow}$ is a set, i.e., $x{\downarrow} \in \mathbb{V}$ for all $x \in \mathbb{V}^{(\mathbb{B})}$. If $[\![x \neq \varnothing]\!] = \mathbb{1}$ then $x{\downarrow}$ is a nonempty set.

(2) Let $z \in \mathbb{V}^{(\mathbb{B})}$ and $[\![z \neq \varnothing]\!] = \mathbb{1}$. Then, for every formula φ of ZFC, we have

$$[\![(\forall x \in z)\, \varphi(x)]\!] = \bigwedge_{x \in z{\downarrow}} [\![\varphi(x)]\!], \qquad [\![(\exists x \in z)\, \varphi(x)]\!] = \bigvee_{x \in z{\downarrow}} [\![\varphi(x)]\!].$$

Moreover, there exists $x_0 \in z{\downarrow}$ such that $[\![\varphi(x_0)]\!] = [\![(\exists x \in z)\, \varphi(x)]\!]$.

(3) Let Φ be a correspondence from X to Y in $\mathbb{V}^{(\mathbb{B})}$. Thus, Φ, X, and Y are elements of $\mathbb{V}^{(\mathbb{B})}$ and, moreover, $[\![\Phi \subset X \times Y]\!] = \mathbb{1}$. There is a unique correspondence $\Phi{\downarrow}$ from $X{\downarrow}$ to $Y{\downarrow}$ such that $\Phi{\downarrow}(A{\downarrow}) = \Phi(A){\downarrow}$ for every nonempty subset A of X inside $\mathbb{V}^{(\mathbb{B})}$. The correspondence $\Phi{\downarrow}$ from $X{\downarrow}$ to $Y{\downarrow}$ of the above proposition is called the *descent* of the correspondence Φ from X to Y inside $\mathbb{V}^{(\mathbb{B})}$.

(4) The descent of the composite of correspondences inside $\mathbb{V}^{(\mathbb{B})}$ is the composite of their descents:

$$(\Psi \circ \Phi){\downarrow} = \Psi{\downarrow} \circ \Phi{\downarrow}.$$

(5) If Φ is a correspondence inside $\mathbb{V}^{(\mathbb{B})}$ then $(\Phi^{-1}){\downarrow} = (\Phi{\downarrow})^{-1}$.

(6) Let Id_X be the identity mapping inside $\mathbb{V}^{(\mathbb{B})}$ of a set $X \in \mathbb{V}^{(\mathbb{B})}$. Then $(\mathrm{Id}_X){\downarrow} = \mathrm{Id}_{X{\downarrow}}$.

(7) Suppose that $X, Y, f \in \mathbb{V}^{(\mathbb{B})}$ are such that $[\![f : X \to Y]\!] = \mathbb{1}$, i.e., f is a mapping from X to Y inside $\mathbb{V}^{(\mathbb{B})}$. Then $f{\downarrow}$ is a unique mapping from $X{\downarrow}$ to $Y{\downarrow}$ satisfying $[\![f{\downarrow}(x) = f(x)]\!] = \mathbb{1}$ for all $x \in X{\downarrow}$.

By virtue of (1)–(7), we can consider the descent operation as a functor from the category of $\mathbb{B}$-valued sets and mappings (correspondences) to the category of the usual sets and mappings (correspondences) (i.e., in the sense of $\mathbb{V}$).

(8) Given $x_1, \ldots, x_n \in \mathbb{V}^{(\mathbb{B})}$, denote the corresponding ordered n-tuple inside $\mathbb{V}^{(\mathbb{B})}$ by $(x_1, \ldots, x_n)^{\mathbb{B}}$. Assume that P is an n-ary relation on X inside $\mathbb{V}^{(\mathbb{B})}$; i.e., $X, P \in \mathbb{V}^{(\mathbb{B})}$ and $[\![P \subset X^{n^\wedge}]\!] = \mathbb{1}$, where $n \in \omega$. Then there exists an n-ary relation P' on $X{\downarrow}$ such that

$$(x_1, \ldots, x_n) \in P' \; \leftrightarrow \; [\![(x_1, \ldots, x_n)^{\mathbb{B}} \in P]\!] = \mathbb{1}.$$

Slightly abusing notation, we denote the relation P' by the same symbol $P{\downarrow}$ and call it the *descent* of P.

2.5. Let $x \in \mathbb{V}$ and $x \subset \mathbb{V}^{(\mathbb{B})}$; i.e., let x be some set composed of $\mathbb{B}$-valued sets or, in other words, $x \in \mathcal{P}(\mathbb{V}^{(\mathbb{B})})$. Put $\varnothing{\uparrow} := \varnothing$ and $\mathrm{dom}(x{\uparrow}) := x$, $\mathrm{im}(x{\uparrow}) := \{\mathbb{1}\}$ if $x \neq \varnothing$. The element $x{\uparrow}$ (of the separated universe $\mathbb{V}^{(\mathbb{B})}$, i.e., the distinguished representative of the class $\{y \in \mathbb{V}^{(\mathbb{B})} : [\![y = x{\uparrow}]\!] = \mathbb{1}\}$) is called the *ascent* of x.

(1) For all $x \in \mathcal{P}(\mathbb{V}^{(\mathbb{B})})$ and every formula φ we have the following:

$$[\![(\forall z \in x{\uparrow})\, \varphi(z)]\!] = \bigwedge_{y \in x} [\![\varphi(y)]\!], \qquad [\![(\exists z \in x{\uparrow})\, \varphi(z)]\!] = \bigvee_{y \in x} [\![\varphi(y)]\!].$$

Introducing the ascent of a correspondence $\Phi \subset X \times Y$, we have to bear in mind a possible distinction between the domain of departure X and the domain $\mathrm{dom}(\Phi) := \{x \in X : \Phi(x) \neq \varnothing\}$. This circumstance is immaterial for the sequel; therefore, speaking of ascents, we always imply total correspondences; i.e., $\mathrm{dom}(\Phi) = X$.

(2) Let $X, Y, \Phi \in \mathbb{V}^{(\mathbb{B})}$, and let Φ be a correspondence from X to Y. There exists a unique correspondence $\Phi{\uparrow}$ from $X{\uparrow}$ to $Y{\uparrow}$ inside $\mathbb{V}^{(\mathbb{B})}$ such that $\Phi{\uparrow}(A{\uparrow}) = \Phi(A){\uparrow}$ is valid for every subset A of $\mathrm{dom}(\Phi)$ if and only if Φ is *extensional*; i.e., satisfies the condition

$$y_1 \in \Phi(x_1) \rightarrow [\![x_1 = x_2]\!] \leq \bigvee_{y_2 \in \Phi(x_2)} [\![y_1 = y_2]\!]$$

for $x_1, x_2 \in \mathrm{dom}(\Phi)$. In this event, $\Phi{\uparrow} = \Phi'{\uparrow}$, where $\Phi' := \{(x, y)^{\mathbb{B}} : (x, y) \in \Phi\}$. The element $\Phi{\uparrow}$ is called the *ascent* of the initial correspondence Φ.

(3) The composite of extensional correspondences is extensional. Moreover, the ascent of a composite is equal to the composite of the ascents inside $\mathbb{V}^{(\mathbb{B})}$: On assuming that $\mathrm{dom}(\Psi) \supset \mathrm{im}(\Phi)$, we have

$$\mathbb{V}^{(\mathbb{B})} \vDash (\Psi \circ \Phi){\uparrow} = \Psi{\uparrow} \circ \Phi{\uparrow}.$$

Note that if Φ and Φ^{-1} are extensional then $(\Phi{\uparrow})^{-1} = (\Phi^{-1}){\uparrow}$. However, in general, the extensionality of Φ in no way guarantees the extensionality of Φ^{-1}.

(4) It is worth mentioning that if an extensional correspondence f is a function from X to Y then the ascent $f{\uparrow}$ of f is a function from $X{\uparrow}$ to $Y{\uparrow}$. Moreover, the extensionality property can be stated as follows: $[\![x_1 = x_2]\!] \leq [\![f(x_1) = f(x_2)]\!]$ for all $x_1, x_2 \in X$.

2.6. Given a set $X \subset \mathbb{V}^{(\mathbb{B})}$, we denote the set of all mixings of the form $\mathrm{mix}(b_\xi x_\xi)$ by the symbol $\mathrm{mix}(X)$, where $(x_\xi) \subset X$ and (b_ξ) is an arbitrary partition of unity. The following propositions are referred to as the *arrow cancellation rules* or *ascending-and-descending rules*. There are many good reasons to call them simply the *Escher rules* [17].

(1) Let X and X' be subsets of $\mathbb{V}^{(\mathbb{B})}$ and let $f : X \to X'$ be an extensional mapping. Suppose that $Y, Y', g \in \mathbb{V}^{(\mathbb{B})}$ are such that $[\![Y \neq \varnothing]\!] = [\![g : Y \to Y']\!] = \mathbb{1}$. Then $X{\uparrow}{\downarrow} = \mathrm{mix}(X)$, $Y{\downarrow}{\uparrow} = Y$, $f{\uparrow}{\downarrow} = f$, and $g{\downarrow}{\uparrow} = g$.

(2) From 2.3(8) we easily infer the useful relation: $\mathcal{P}_{\mathrm{fin}}(X{\uparrow}) = \{\theta{\uparrow} : \theta \in \mathcal{P}_{\mathrm{fin}}(X)\}{\uparrow}$.

Suppose that $X \in \mathbb{V}$, $X \neq \varnothing$; i.e., X is a nonempty set. Let the letter ι denote the standard name embedding $x \mapsto x^\wedge$ $(x \in X)$. Then $\iota(X){\uparrow} = X^\wedge$ and $X = \iota^{-1}(X^\wedge{\downarrow})$. Using the above relations, we may extend the descent and ascent operations to the case in which Φ is a correspondence from X to $Y{\downarrow}$ and $[\![\Psi$ is a correspondence from $X^\wedge$ to $Y]\!] = \mathbb{1}$, where $Y \in \mathbb{V}^{(\mathbb{B})}$. Namely, we put $\Phi{\Uparrow} := (\Phi \circ \iota){\uparrow}$ and $\Psi{\Downarrow} := \Psi{\downarrow} \circ \iota$. In this case, $\Phi{\Uparrow}$ is called the *modified ascent* of Φ, and $\Psi{\Downarrow}$ is called the *modified descent* of Ψ. (If the context excludes ambiguity then we briefly speak of ascents and descents using

simple arrows.) It is easy to see that $\Psi\uparrow$ is a unique correspondence inside $\mathbb{V}^{(\mathbb{B})}$ satisfying the relation $[\![\Phi\uparrow(x^\wedge) = \Phi(x)\uparrow]\!] = \mathbb{1}$ $(x \in X)$. Similarly, $\Psi\downarrow$ is a unique correspondence from X to $Y\downarrow$ satisfying the equality $\Psi\downarrow(x) = \Psi(x^\wedge)\downarrow$ $(x \in X)$, If $\Phi := f$ and $\Psi := g$ are functions then these relations take the form $[\![f\uparrow(x^\wedge) = f(x)]\!] = \mathbb{1}$ and $g\downarrow(x) = g(x^\wedge)$ for all $x \in X$.

2.7. Various function spaces reside in functional analysis, and so the problem is natural of replacing an abstract Boolean valued system by some function-space analog, a model whose elements are functions and in which the basic logical operations are calculated "pointwise." An example of such a model is given by the class $\mathbb{V}^Q$ of all functions defined on a fixed nonempty set Q and acting into $\mathbb{V}$. The truth values on $\mathbb{V}^Q$ are various subsets of Q: The truth value $[\![\varphi(u_1, \ldots, u_n)]\!]$ of $\varphi(t_1, \ldots, t_n)$ at functions $u_1, \ldots, u_n \in \mathbb{V}^Q$ is calculated as follows:

$$[\![\varphi(u_1, \ldots, u_n)]\!] = \{q \in Q : \varphi(u_1(q), \ldots, u_n(q))\}.$$

A. G. Gutman and G. A. Losenkov solved the above problem by the concept of continuous polyverse which is a continuous bundle of models of set theory. It is shown that the class of continuous sections of a continuous polyverse is a Boolean valued system satisfying all basic principles of Boolean valued analysis and, conversely, each Boolean valued algebraic system can be represented as the class of sections of a suitable continuous polyverse. More details are collected in [35, Chapter 6].

2.8. Every Boolean valued universe has the collection of mathematical objects in full supply: available in plenty are all sets with extra structure: groups, rings, algebras, normed spaces, etc. Applying the descent functor to such *internal* algebraic systems of a Boolean valued model, we distinguish some bizarre entities or recognize old acquaintances, which leads to revealing the new facts of their life and structure.

This technique of research, known as *direct Boolean valued interpretation*, allows us to produce new theorems or, to be more exact, to extend the semantical content of the available theorems by means of slavish translation. The information we so acquire might fail to be vital, valuable, or intriguing, in which case the direct Boolean valued interpretation turns out into a leisurely game.

It thus stands to reason to raise the following questions: What structures significant for mathematical practice are obtainable by the Boolean valued interpretation of the most typical algebraic systems? What transfer principles hold true in this process? Clearly, the answers should imply specific objects whose particular features enable us to deal with their Boolean valued representation which, if understood duly, is impossible to implement for arbitrary algebraic systems.

An *abstract Boolean set* or *set with $\mathbb{B}$-structure* is a pair (X, d), where $X \in \mathbb{V}$, $X \neq \varnothing$, and d is a mapping from $X \times X$ to $\mathbb{B}$ such that

$$d(x, y) = \mathbb{O} \leftrightarrow x = y, \qquad d(x, y) = d(y, x), \qquad d(x, y) \leq d(x, z) \vee d(z, y)$$

for all $x, y, z \in X$.

To obtain an easy example of an abstract $\mathbb{B}$-set, given $\varnothing \neq X \subset \mathbb{V}^{(\mathbb{B})}$ put

$$d(x, y) := [\![x \neq y]\!] = \neg[\![x = y]\!]$$

for $x, y \in X$.

Another easy example is a nonempty X with the *discrete $\mathbb{B}$-metric* d; i.e., $d(x, y) = \mathbb{1}$ if $x \neq y$ and $d(x, y) = \mathbb{O}$ if $x = y$.

Let (X, d) be some abstract $\mathbb{B}$-set. There exist an element $\mathcal{X} \in \mathbb{V}^{(\mathbb{B})}$ and an injection $\iota : X \to X' := \mathcal{X}\downarrow$ such that $d(x, y) = [\![\iota x \neq \iota y]\!]$ for all $x, y \in X$ and every element $x' \in X'$ admits the representation $x' = \text{mix}_{\xi \in \Xi}(b_\xi \iota x_\xi)$, where $(x_\xi)_{\xi \in \Xi} \subset X$ and $(b_\xi)_{\xi \in \Xi}$ is a partition of unity in $\mathbb{B}$. The element $\mathcal{X} \in \mathbb{V}^{(\mathbb{B})}$ is referred to as the *Boolean valued realization* of X.

If X is a discrete abstract $\mathbb{B}$-set then $\mathcal{X} = X^\wedge$ and $\iota x = x^\wedge$ for all $x \in X$. If $X \subset \mathbb{V}^{(\mathbb{B})}$ then $\iota\uparrow$ is an injection from $X\uparrow$ to $\mathcal{X}$ (inside $\mathbb{V}^{(\mathbb{B})}$). A mapping f from a $\mathbb{B}$-set (X, d) to a $\mathbb{B}$-set (X', d') is said to be *contractive* if $d(x, y) \geq d'(f(x), f(y))$ for all $x, y \in X$.

We see that an abstract $\mathbb{B}$-set X embeds in the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ so that the Boolean distance between the members of X becomes the Boolean truth value of the negation of their equality. The corresponding element of $\mathbb{V}^{(\mathbb{B})}$ is, by definition, the *Boolean valued representation* of X.

In case a $\mathbb{B}$-set X has some a priori structure we may try to furnish the Boolean valued representation of X with an analogous structure, so as to apply the technique of ascending and descending to the study of the original structure of X. Consequently, the above questions may be treated as instances of the unique problem of searching a well-qualified Boolean valued representation of a $\mathbb{B}$-set with some additional structure.

We call these objects *algebraic $\mathbb{B}$-systems*. Located at the epicenter of exposition, the notion of an algebraic $\mathbb{B}$-system refers to a nonempty $\mathbb{B}$-set endowed with a few contractive operations and $\mathbb{B}$-predicates, the latter meaning $\mathbb{B}$-valued contractive mappings.

The Boolean valued representation of an algebraic $\mathbb{B}$-system appears to be a conventional two valued algebraic system of the same type. This means that an appropriate completion of each algebraic $\mathbb{B}$-system coincides with the descent of some two valued algebraic system inside $\mathbb{V}^{(\mathbb{B})}$.

On the other hand, each two valued algebraic system may be transformed into an algebraic $\mathbb{B}$-system on distinguishing a complete Boolean algebra of congruences of the original system. In this event, the task is in order of finding the formulas holding true in direct or reverse transition from a $\mathbb{B}$-system to a two valued system. In other words, we have to seek here for some versions of the transfer or identity preservation principle of long standing in some branches of mathematics.

3. BOOLEAN VALUED NUMBERS

Boolean valued analysis stems from the fact that each internal field of reals of a Boolean valued model descends into a universally complete Kantorovich space. Thus, a remarkable opportunity opens up to expand and enrich the treasure-trove of mathematical knowledge by translating information about the reals to the language of other noble families of functional analysis. We will elaborate upon the matter in this section.

3.1. Recall a few definitions. Two elements x and y of a vector lattice E are called *disjoint* (in symbols $x \perp y$) if $|x| \wedge |y| = 0$. A *band* of E is defined as the *disjoint complement*

$$M^{\perp} := \{x \in E : (\forall y \in M)\, x \perp y\}$$

of a nonempty set $M \subset E$.

The inclusion-ordered set $\mathfrak{B}(E)$ of all bands in E is a complete Boolean algebra with the Boolean operations:

$$L \wedge K = L \cap K, \qquad L \vee K = (L \cup K)^{\perp\perp}, \qquad L^* = L^{\perp} \qquad (L, K \in \mathfrak{B}(E)).$$

The Boolean algebra $\mathfrak{B}(E)$ is often referred as to the *base* of E.

A *band projection* in E is a linear idempotent operator in $\pi : E \to E$ satisfying the inequalities $0 \leq \pi x \leq x$ for all $0 \leq x \in E$. The set $\mathfrak{P}(E)$ of all band projections ordered by $\pi \leq \rho \iff \pi \circ \rho = \pi$ is a Boolean algebra with the Boolean operations:

$$\pi \wedge \rho = \pi \circ \rho, \qquad \pi \vee \rho = \pi + \rho - \pi \circ \rho, \qquad \pi^* = I_E - \pi \qquad (\pi, \rho \in (E)).$$

Let $u \in E_+$ and $e \wedge (u - e) = 0$ for some $0 \leq e \in E$. Then e is a *fragment* or *component* of u. The set $\mathfrak{E}(u)$ of all fragments of u with the order induced by E is a Boolean algebra where the lattice operations are taken from E and the Boolean complement has the form $e^* := u - e$.

3.2. A Dedekind complete vector lattice is also called a *Kantorovich space* or *K-space*, for short. A K-space E is *universally complete* if every family of pairwise disjoint elements of E is order bounded.

(1) **Theorem.** *Let E be an arbitrary K-space. Then the correspondence $\pi \mapsto \pi(E)$ determines an isomorphism of the Boolean algebras $\mathfrak{P}(E)$ and $\mathfrak{B}(E)$. If there is an order unity $\mathbb{1}$ in E then the mappings $\pi \mapsto \pi\mathbb{1}$ from $\mathfrak{P}(E)$ into $\mathfrak{E}(E)$ and $e \mapsto \{e\}^{\perp\perp}$ from $\mathfrak{E}(E)$ into $\mathfrak{B}(E)$ are isomorphisms of Boolean algebras too.*

(2) **Theorem.** *Each universally complete K-space E with order unity $\mathbb{1}$ can be uniquely endowed by multiplication so as to make E into a faithful f-algebra and $\mathbb{1}$ into a ring unity. In this f-algebra each band projection $\pi \in \mathfrak{P}(E)$ is the operator of multiplication by $\pi(\mathbb{1})$.*

(3) *there is an o-continuous lattice isomorphism* $\iota : E \to E'$ *such that* $\iota(E)$ *is a coinitial sublattice of* $\mathcal{R}{\downarrow}$;

(4) *for every* $b \in \mathbb{B}$ *the band projection in* $\mathcal{R}{\downarrow}$ *onto* $\{\iota(\jmath(b))\}^{\perp\perp}$ *coincides with* $\chi(b)$.

Note also that $\mathcal{E}$ and $\mathcal{R}$ coincide if and only if E is Dedekind complete. Thus, each theorem about the reals within Zermelo–Fraenkel set theory has an analog in an arbitrary Kantorovich space. Translation of theorems is carried out by appropriate general functors of Boolean valued analysis. In particular, the most important structural properties of vector lattices such as the functional representation, spectral theorem, etc. are the ghosts of some properties of the reals in an appropriate Boolean valued model. More details and references are collected in [35].

3.8. The theory of vector lattices with a vast field of applications is thoroughly covered in many monographs (see [5, 7, 24, 25, 43, 54, 55, 65, 67]). The credit for finding the most important instance among ordered vector spaces, an order complete vector lattice or K-space, is due to L. V. Kantorovich. This notion appeared in Kantorovich's first article on this topic [23] where he wrote: "In this note, I define a new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in such a space) as linear functionals."

Thus the *heuristic transfer principle* was stated for K-spaces which becomes the Ariadna thread of many subsequent studies. The depth and universality of Kantorovich's principle are explained within Boolean valued analysis.

3.9. Applications of Boolean valued models to functional analysis stem from the works by E. I. Gordon [11, 12] and G. Takeuti [59]. If $\mathbb{B}$ in 3.6 is the algebra of μ-measurable sets modulo μ-negligible sets then $\mathcal{R}{\downarrow}$ is isomorphic to the universally complete K-space $L^0(\mu)$ of measurable functions. This fact (for the Lebesgue measure on an interval) was already known to D. Scott and R. Solovay (see [35]). If $\mathbb{B}$ is a complete Boolean algebra of projections in a Hilbert space then $\mathcal{R}{\downarrow}$ is isomorphic to the space of selfadjoint operators $\mathfrak{A}(\mathbb{B})$. These two particular cases of Gordon's Theorem were intensively and fruitfully exploited by G. Takeuti (see [59] and the bibliography in [35]). The object $\mathcal{R}{\downarrow}$ for general Boolean algebras was also studied by T. Jech [18]–[20] who in fact rediscovered Gordon's Theorem. The difference is that, in [21], a (complex) universally complete K-space with unity is defined by another system of axioms and is referred to as a *complete Stone algebra*. Theorem 3.7 was obtained by A. G. Kusraev [30]. A close result (in other terms) is presented in T. Jech's article [20], where some Boolean valued interpretation is revealed of the theory of linearly ordered sets. More details can be found in [35].

4. BAND PRESERVING OPERATORS

This section deals with the class of band preserving operators. Simplicity of these operators notwithstanding, the question about their order boundedness is far from trivial.

4.1. Recall that a complex K-space is the complexification $G_{\mathbb{C}} := G \oplus iG$ of a real K-space G (see [54]). A linear operator $T : G_{\mathbb{C}} \to G_{\mathbb{C}}$ is *band preserving* or *contractive* or a *stabilizer* if, for all $f, g \in G_{\mathbb{C}}$, from $f \perp g$ it follows that $Tf \perp g$. Disjointness in $G_{\mathbb{C}}$ is defined just as in G (see 3.1), whereas

$$|z| := \sup\{\operatorname{Re}(e^{i\theta} z) : 0 \le \theta \le \pi\}, \qquad z \in G_{\mathbb{C}}.$$

(1) Let $\operatorname{End}_N(G_{\mathbb{C}})$ stand for the set of all band preserving linear operators in $G_{\mathbb{C}}$, with $G := \mathcal{R}{\downarrow}$. Clearly, $\operatorname{End}_N(G_{\mathbb{C}})$ is a complex vector space. Moreover, $\operatorname{End}_N(G_{\mathbb{C}})$ becomes a faithful unitary module over the ring $G_{\mathbb{C}}$ if we define gT as

$$gT : x \mapsto g \cdot Tx \quad \text{for all } x \in G.$$

This follows from the fact that multiplication by a member of $G_{\mathbb{C}}$ is a band preserving operator and the composite of band preserving operators is band preserving too.

(2) Let $\operatorname{End}_{\mathbb{C}^\wedge}(\mathcal{C})$ denote the element of $\mathbb{V}^{(\mathbb{B})}$ representing the space of all $\mathbb{C}^\wedge$-linear mappings from $\mathcal{C}$ to $\mathcal{C}$. Then $\operatorname{End}_{\mathbb{C}^\wedge}(\mathcal{C})$ is a vector space over $\mathbb{C}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$, and $\operatorname{End}_{\mathbb{C}^\wedge}(\mathcal{C}){\downarrow}$ is a faithful unitary module over $G_{\mathbb{C}}$.

4.2. Following [31] it is easy to prove that a linear operator T in the K-space $G_{\mathbb{C}}$ is band preserving if and only if T is extensional. Since each extensional mapping has an ascent, $T \in \mathrm{End}_N(G_{\mathbb{C}})$ has the ascent $\tau := T\uparrow$ which is a unique internal functional from $\mathcal{C}$ to $\mathcal{C}$ such that $[\![\tau(x) = Tx]\!] = \mathbb{1}$ $(x \in G_{\mathbb{C}})$. We thus arrive at the following assertion:

The modules $\mathrm{End}_N(G_{\mathbb{C}})$ *and* $\mathrm{End}_{\mathbb{C}^\wedge}(\mathcal{C})\!\downarrow$ *are isomorphic by sending each band preserving operator to its ascent.*

By Gordon's Theorem this assertion means that the problem of finding a band preserving operator in G amounts to solving (for $\tau : \mathcal{C} \to \mathcal{C}$) inside $\mathbb{V}^{(\mathbb{B})}$ the *Cauchy functional equation*: $\tau(x + y) = \tau(x) + \tau(y)$ $(x, y \in \mathcal{C})$ under the subsidiary condition $\tau(\lambda x) = \lambda\tau(x)$ $(x \in \mathcal{C}, \lambda \in \mathbb{C}^\wedge)$.

As another subsidiary condition we may consider the *Leibniz rule* $\tau(xy) = \tau(x)y + x\tau(y)$(in which case τ is called a $\mathbb{C}^\wedge$-*derivation*) or multiplicativity $\tau(xy) = \tau(x)\tau(y)$. These situations are addressed in 4.5.

4.3. An element $g \in G^+$ is *locally constant* with respect to $f \in G^+$ if $g = \bigvee_{\xi \in \Xi} \lambda_\xi \pi_\xi f$ for some numeric family $(\lambda_\xi)_{\xi \in \Xi}$ and a family $(\pi_\xi)_{\xi \in \Xi}$ of pairwise disjoint band projections. A universally complete K-space $G_{\mathbb{C}}$ is called *locally one-dimensional* if all elements of G^+ are locally constant with respect to some order unity of G (and hence each of them). Clearly, a K-space $G_{\mathbb{C}}$ is locally one-dimensional if each $g \in G_{\mathbb{C}}$ may be presented as $g = o\text{-}\sum_{\xi \in \Xi} \lambda_\xi \pi_\xi \mathbb{1}f$ with some family $(\lambda_\xi)_{\xi \in \Xi} \subset \mathbb{C}$ and partition of unity $(\pi_\xi)_{\xi \in \Xi} \subset \mathfrak{P}(G)$.

4.4. A σ-complete Boolean algebra $\mathbb{B}$ is called σ-*distributive* if

$$\bigvee_{n \in \mathbb{N}} \bigwedge_{m \in \mathbb{N}} b_{n,m} = \bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \bigvee_{n \in \mathbb{N}} b_{n,\varphi(n)}.$$

for every double sequence $(b_{n,m})_{n,m \in \mathbb{N}}$ in $\mathbb{B}$. Other equivalent definitions are collected in [57]. As an example of a σ-distributive Boolean algebra we may take a complete atomic Boolean algebra, i.e., the boolean of a nonempty set. It is worth observing that there are nonatomic σ-distributive complete Boolean algebras (see [32, 5.1.8]).

We now address the problem which is often referred to in the literature as *Wickstead's problem*: Characterize the universally complete vector lattices spaces in which every band preserving linear operator is order bounded.

According to 4.2, Boolean valued analysis reduces Wickstead's problem to that of order boundedness of the endomorphisms of the field $\mathcal{C}$ viewed as a vector space and algebra over the field $\mathbb{C}^\wedge$.

4.5. Theorem. *Let* $\mathbb{P}$ *be a dense subfield of the field of complexes* $\mathbb{C}$*. The following are equivalent*:

(1) $\mathbb{P} = \mathbb{C}$;

(2) *every* $\mathbb{P}$-*linear function on* $\mathbb{C}$ *is order bounded*;

(3) *there are no nontrivial* $\mathbb{P}$-*derivations on* $\mathbb{C}$;

(4) *each* $\mathbb{P}$-*linear endomorphism on* $\mathbb{C}$ *is the zero or identity function*;

(5) *there is no* $\mathbb{P}$-*linear automorphism on* $\mathbb{C}$ *other than the identity.*

The equivalence $(1) \leftrightarrow (2)$ is checked by using a Hamel basis of the vector space $\mathbb{C}$ over $\mathbb{P}$. The remaining equivalences rest on replacing a Hamel basis with a transcendence basis (for details see [33]).

Recall that a linear operator $D : G_{\mathbb{C}} \to G_{\mathbb{C}}$ is a $\mathbb{C}$-*derivation* if $D(fg) = D(f)g + fD(g)$ for all $f, g \in G_{\mathbb{C}}$. It can be easily checked that every $\mathbb{C}$-derivation is band preserving.

Interpreting Theorem 4.5 in $\mathbb{V}^{(\mathbb{B})}$, we arrive at

4.6. Theorem. *If* $\mathbb{B}$ *is a complete Boolean algebra then the following are equivalent:*

(1) $\mathcal{C} = \mathbb{C}^\wedge$ *inside* $\mathbb{V}^{(\mathbb{B})}$;

(2) *every band preserving linear operator is order bounded in the complex vector lattice* $\mathcal{C}\!\downarrow$;

(3) *there is no nontrivial* $\mathbb{C}$-*derivation in the complex f-algebra* $\mathcal{C}\!\downarrow$;

(4) *each band preserving endomorphism is a band projection in* $\mathcal{C}\!\downarrow$;

(5) *there is no band preserving automorphism other than the identity in* $\mathcal{C}\!\downarrow$.

(6) *the K-space $\mathcal{R}\downarrow$ is locally one-dimensional;*

(7) $\mathbb{B}$ *is σ-distributive.*

4.7. The question was raised by A. W. Wickstead in [66] whether every band preserving linear operator in a universally complete vector lattice is automatically order bounded. The first example of an unbounded band preserving linear operator was suggested by Yu. A. Abramovich, A. I. Veksler, and A. V. Koldunov in [1, 2]. The equivalence $(1) \leftrightarrow (6)$ is trivial, whereas $(2) \leftrightarrow (6)$ combines a result of Yu. A. Abramovich, A. I. Veksler, and A. V. Koldunov [1, Theorem 2.1] and that of P. T. N. McPolin and A. W. Wickstead [45, Theorem 3.2]. The equivalence $(6) \leftrightarrow (7)$ was obtained by A. E. Gutman [15]; he also found an example of a purely nonatomic locally one-dimensional Dedekind complete vector lattice (see [14]). The equivalences $(1) \leftrightarrow (3) \leftrightarrow (4) \leftrightarrow (5)$ belong to A. G. Kusraev [33].

5. BOOLEAN VALUED POSITIVE FUNCTIONALS

A linear functional on a vector space is determined up to a scalar from its zero hyperplane. In contrast, a linear operator is recovered from its kernel up to a simple multiplier on a rather special occasion. Fortunately, Boolean valued analysis prompts us that some operator analog of the functional case is valid for each operator with target a Kantorovich space, a Dedekind complete vector lattice. We now proceed along the lines of this rather promising approach.

5.1. Let E be a vector lattice, and let F be a K-space with base a complete Boolean algebra $\mathbb{B}$. By 3.2, we may assume that F is a nonzero space embedded as an order dense ideal in the universally complete Kantorovich space $\mathcal{R}\downarrow$ which is the descent of the reals $\mathcal{R}$ inside the separated Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ over $\mathbb{B}$.

An operator T is *F-discrete* if $[0, T] = [0, I_F] \circ T$; i.e., for all $0 \le S \le T$ there is some $0 \le \alpha \le I_F$ satisfying $S = \alpha \circ T$. Let $L_a^\sim(E, F)$ be the band in $L^\sim(E, F)$ spanned by F-discrete operators and $L_d^\sim(E, F) := L_a^\sim(E, F)^\perp$. By analogy we define $(E^{\wedge\sim})_a$ and $(E^{\wedge\sim})_d$. The members of $L_d^\sim(E, F)$ are usually called *F-diffuse*.

5.2. As usual, we let $E^\wedge$ stand for the standard name of E in $\mathbb{V}^{(\mathbb{B})}$. Clearly, $E^\wedge$ is a vector lattice over $\mathbb{R}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$. Let $\tau := T\uparrow$ be the ascent of T to $\mathbb{V}^{(\mathbb{B})}$. Clearly, τ acts from $E^\wedge$ to the ascent $F\uparrow = \mathcal{R}$ of F inside the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$. Therefore, $\tau(x^\wedge) = Tx$ inside $\mathbb{V}^{(\mathbb{B})}$ for all $x \in E$, which means in terms of truth values that

$$[\![\tau : E^\wedge \to \mathcal{R}]\!] = \mathbb{1}, \qquad (\forall x \in E) \, [\![\tau(x^\wedge) = Tx]\!] = \mathbb{1}.$$

Let $E^{\wedge\sim}$ stand for the space of all order bounded $\mathbb{R}^\wedge$-linear functionals from $E^\wedge$ to $\mathcal{R}$. Clearly, $E^{\wedge\sim} := L^\sim(E^\wedge, \mathcal{R})$ is a K-space inside $\mathbb{V}^{(\mathbb{B})}$. The descent $E^{\wedge\sim}\downarrow$ of $E^{\wedge\sim}$ is a K-space. Given $S, T \in L^\sim(E, F)$, put $\tau := T\uparrow$ and $\sigma := S\uparrow$.

5.3. Theorem. *For each $T \in L^\sim(E, F)$, the ascent $T\uparrow$ of T is an order bounded $\mathbb{R}^\wedge$-linear functional on $E^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$; i.e., $[\![T\uparrow \in E^{\wedge\sim}]\!] = \mathbb{1}$. The mapping $T \mapsto T\uparrow$ is a lattice isomorphism of $L^\sim(E, F)$ and $E^{\wedge\sim}\downarrow$. Moreover, the following hold:*

(1) $T \ge 0 \ \leftrightarrow\ [\![\tau \ge 0]\!] = \mathbb{1}$;

(2) *S is a fragment of T* $\leftrightarrow\ [\![\sigma$ *is a fragment of* $\tau]\!] = \mathbb{1}$;

(3) *T is a lattice homomorphism if and only if so is τ inside $\mathbb{V}^{(\mathbb{B})}$;*

(4) *T is F-diffuse* $\leftrightarrow\ [\![\tau$ *is diffuse*$]\!] = \mathbb{1}$;

(5) $T \in L_a^\sim(E, F) \ \leftrightarrow\ [\![\tau \in (E^{\wedge\sim})_a]\!] = \mathbb{1}$;

(6) $T \in L_d^\sim(E, F) \ \leftrightarrow\ [\![\tau \in (E^{\wedge\sim})_d]\!] = \mathbb{1}$.

Since τ, the ascent of an order bounded operator T, is defined up to a scalar from $\ker(\tau)$, we infer the following analog of the Sard Theorem:

5.4. Theorem. *Let S and T be linear operators from E to F. Then $\ker(bS) \supset \ker(bT)$ for all $b \in \mathbb{B}$ if and only if there is an orthomorphism α of F such that $S = \alpha T$.*

We see that a linear operator T is, in a sense, determined up to an orthomorphism from the family of the kernels of the *strata bT* of T. This remark opens a possibility of studying some properties of T in terms of the kernels of the strata of T.

5.5. Theorem. *An order bounded operator T from E to F may be presented as the difference of some lattice homomorphisms if and only if the kernel of each stratum bT of T is a vector sublattice of E for all $b \in \mathbb{B}$.*

Straightforward calculations of truth values show that $T_+\!\uparrow = \tau_+$ and $T_-\!\uparrow = \tau_-$ inside $\mathbb{V}^{(\mathbb{B})}$. Moreover, $[\![\ker(\tau)$ is a vector sublattice of $E^\wedge]\!] = \mathbb{1}$ whenever so are $\ker(bT)$ for all $b \in \mathbb{B}$. Since the ascent of a sum is the sum of the ascents of the summands, we reduce the proof of Theorem 5.5 to the case of the functionals on using 5.3(3).

5.6. Recall that a subspace H of a vector lattice E is a *G-space* or *Grothendieck subspace* (cp. [13, 41]) provided that H enjoys the following property:

$$(\forall x, y \in H)\,(x \vee y \vee 0 + x \wedge y \wedge 0 \in H).$$

By simple calculations of truth values we infer that $[\![\ker(\tau)$ is a Grothendieck subspace of $E^\wedge]\!] = \mathbb{1}$ if and only if the kernel of each stratum bT is a Grothendieck subspace of E. We may now assert that the following appears as a result of "descending" its scalar analog.

5.7. Theorem. *The modulus of an order bounded operator $T : E \to F$ is the sum of some pair of lattice homomorphisms if and only if the kernel of each stratum bT of T with $b \in \mathbb{B}$ is a Grothendieck subspace of the ambient vector lattice E.*

To prove the relevant scalar versions of Theorems 5.5 and 5.7, we use one of the formulas of subdifferential calculus (cp. [34]):

5.8. Decomposition Theorem. *Assume that $H_1, \ldots, H_N$ are cones in a vector lattice E. Assume further that f and g are positive functionals on E. The inequality*

$$f(h_1 \vee \cdots \vee h_N) \geq g(h_1 \vee \cdots \vee h_N)$$

holds for all $h_k \in H_k$ $(k := 1, \ldots, N)$ if and only if to each decomposition of g into a sum of N positive terms $g = g_1 + \cdots + g_N$ there is a decomposition of f into a sum of N positive terms $f = f_1 + \cdots + f_N$ such that $f_k(h_k) \geq g_k(h_k)$ $(h_k \in H_k, k := 1, \ldots, N)$.

5.9. Theorems 5.5 and 5.7 were obtained by S. S. Kutateladze in [39, 40]. Theorem 5.8 appeared in this form in [37]. Note that the sums of lattice homomorphisms were first described by S. J. Bernau, C. B. Huijsmans, and B. de Pagter in terms of n-disjoint operators in [9]. A survey of some conceptually close results on n-disjoint operators is given in [32].

6. BOOLEAN VALUED BANACH SPACES

In this section we discuss the transfer principle of Boolean valued analysis in regard to lattice-normed spaces. It turns out that the interpretation of a Banach space inside an arbitrary Boolean valued model is a Banach–Kantorovich space. Conversely, the universal completion of each lattice-normed space becomes a Banach space on ascending in a suitable Boolean valued model. This open up an opportunity to transfer the available theorems on Banach spaces to analogous results on lattice-normed spaces by the technique of Boolean valued analysis.

6.1. Consider a vector space X and a real vector lattice E. Note that all vector lattices under consideration are assumed Archimedean. An *E-valued norm* is a mapping $|\!|\cdot|\!| : X \to E_+$ such that

(1) $|\!|x|\!| = 0 \Longleftrightarrow x = 0$ $(x \in X)$;

(2) $|\!|\lambda x|\!| = |\lambda| |\!|x|\!|$ $(\lambda \in \mathbb{R}, x \in X)$;

(3) $|\!|x + y|\!| \leq |\!|x|\!| + |\!|y|\!|$ $(x, y \in X)$.

A vector norm is *decomposable* if

(4) for all $e_1, e_2 \in E_+$ and $x \in X$, from $|\!|x|\!| = e_1 + e_2$ it follows that there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $|\!|x_k|\!| = e_k$ $(k := 1, 2)$.

If (4) is valid only for disjoint $e_1, e_2 \in E_+$ then the norm is *d-decomposable*. A triple $(X, |\!|\cdot|\!|, E)$ as well as briefer versions is a *lattice-normed space* over E whenever $|\!|\cdot|\!|$ is an E-valued norm on X.

6.2. By a *Boolean algebra of projections* in a vector space X we mean a set $\mathcal{B}$ of commuting idempotent linear operators in X. Moreover, the Boolean operations have the following form:

$$\pi \wedge \rho := \pi \circ \rho = \rho \circ \pi, \qquad \pi \vee \rho = \pi + \rho - \pi \circ \rho,$$
$$\pi^* = I_x - \pi \qquad (\pi, \rho \in \mathcal{B}),$$

and the zero and identity operators in X serve as the zero and unity of the Boolean algebra $\mathcal{B}$.

Suppose that E is a vector lattice with the projection property and

$$E = |X|^{\perp\perp} := \{|x| : x \in X\}^{\perp\perp}.$$

If (X, E) is a d-decomposable lattice-normed space then there exists a complete Boolean algebra $\mathcal{B}$ of band projections in X and an isomorphism h from $\mathfrak{P}(E)$ onto $\mathcal{B}$ such that

$$b|x| = |h(b)x| \quad \big(b \in \mathfrak{P}(E), \ x \in X\big).$$

We identify the Boolean algebras $\mathfrak{P}(E)$ and $\mathcal{B}$ and write $\pi|x| = |\pi x|$ for all $x \in X$ and $\pi \in \mathfrak{P}(E)$.

6.3. A net $(x_\alpha)_{\alpha \in A}$ in X is *bo-convergent* to $x \in X$ (in symbols: $x = bo\text{-}\lim x_\alpha$) if $(|x - x_\alpha|)_{\alpha \in A}$ is o-convergent to zero. A lattice-normed space X is *bo-complete* if each net $(x_\alpha)_{\alpha \in A}$ is *bo*-convergent to some element of X provided that $(|x_\alpha - x_\beta|)_{(\alpha, \beta) \in A \times A}$ is o-convergent to zero. A decomposable *bo*-complete lattice-normed space $(X, |\cdot|, E)$ is called a *Banach−Kantorovich space*. If E is a universally complete Kantorovich space then X is also referred to as universally complete. By a *universal completion* of a lattice-normed space (X, E) we mean a universally complete Banach−Kantorovich space $(Y, m(E))$ together with a linear isometry $\imath : X \to Y$ such that each universally complete *bo*-complete subspace of $(Y, m(E))$ containing $\imath(X)$ coincides with Y. Here $m(E)$ is a universal completion of E.

6.4. Theorem. *Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space inside $\mathbb{V}^{(\mathbb{B})}$. Put $X := \mathcal{X}{\downarrow}$ and $|\cdot| := \|\cdot\|{\downarrow}(\cdot)$. Then $(X, |\cdot|, \mathcal{R}{\downarrow})$ is a universally complete Banach−Kantorovich space. Moreover, X can be endowed with the structure of a faithful unitary module over the ring $\Lambda := \mathcal{C}{\downarrow}$ so that $|ax| = |a| \, |x|$ and $b \leq [\![x = 0]\!] \leftrightarrow \chi(b)x = 0$ for all $a \in \mathcal{C}{\downarrow}$, $x \in X$, and $b \in \mathbb{B}$, where χ is an isomorphism of $\mathbb{B}$ onto $\mathfrak{P}(X)$.*

6.5. Theorem. *To each lattice-normed space $(X, |\cdot|)$, there exists a unique Banach space (up to a linear isometry) $\mathcal{X}$ inside $\mathbb{V}^{(\mathbb{B})}$, with $\mathbb{B} \simeq \mathfrak{B}\big(|X|^{\perp\perp}\big)$, such that the descent $\mathcal{X}{\downarrow}$ of $\mathcal{X}$ is a universal completion of X.*

As in 3.1, we call $x \in X$ and $y \in Y$ *disjoint* and write $x \perp y$ whenever $|x| \wedge |y| = 0$. Let X and Y be Banach−Kantorovich spaces over some K-space G.

An operator T is *band preserving* if $x \perp y$ implies $Tx \perp y$ for all $x \in X$ and $y \in Y$. Let $\mathcal{L}_G(X, Y)$ denote the space of all band preserving operators $T : X \to Y$ that send all norm-o-bounded sets into norm-o-bounded sets.

6.6. Theorem. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Boolean valued representations for Banach−Kantorovich spaces X and Y normed by some universally complete K-space $G := \mathcal{R}{\downarrow}$. Let $\mathcal{L}^{\mathbb{B}}(\mathcal{X}, \mathcal{Y})$ be the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$ inside $\mathbb{V}^{(\mathbb{B})}$, where $\mathbb{B} := \mathfrak{B}(G)$. The descent and ascent mappings (for operators) implement linear isometries between the lattice-normed spaces $\mathcal{L}_G(X, Y)$ and $\mathcal{L}^{\mathbb{B}}(\mathcal{X}, \mathcal{Y}){\downarrow}$.*

6.7. The concept of lattice-normed space was suggested by L. V. Kantorovich in 1936 [23]. It is worth stressing that [23] is the first article with the unusual decomposability axiom for an abstract norm. Paradoxically, this axiom was often omitted as inessential in the further papers by other authors. The profound importance of 6.1 (4) was revealed by Boolean valued analysis. The connection between the decomposability and existence of a Boolean algebra of projections in a lattice-normed space was discovered in [28, 29]. The theory of lattice-normed spaces and dominated operators is set forth in [32]. As regards the Boolean valued approach, see [35].

7. BOOLEAN VALUED ORDER CONTINUOUS FUNCTIONALS

We now address the class of o-continuous order bounded operators that turn into o-continuous functionals on ascending to a suitable Boolean valued model.

7.1. Assume that a lattice-normed space X is simultaneously a vector lattice. The norm $|\cdot| : X \to E_+$ of X is *monotone* if from $|x| \le |y|$ it follows that $|x| \le |y|$ $(x, y \in X)$. In this event, X is a *lattice-normed vector lattice*. Moreover, if X is a Banach–Kantorovich space then X is called a *Banach–Kantorovich lattice*.

We say that the norm $|\cdot|$ in X is *additive* if $|x + y| = |x| + |y|$ for all $x, y \in X_+$; it is *order semicontinuous* or *o-semicontinuous* for short if $\sup|x_\alpha| = |\sup x_\alpha|$ for each increasing net $(x_\alpha) \subset X$ with the least upper bound $x \in X$; and it is *order continuous* or *o-continuous* if $\inf|x_\alpha| = 0$ for every decreasing net $(x_\alpha) \subset X$ with $\inf_\alpha x_\alpha = 0$.

The Boolean valued interpretation of Banach–Kantorovich lattices proceeds along the lines of the previous section.

7.2. Theorem. *Let $(X, |\cdot|)$ be a Banach–Kantorovich space and let $(\mathcal{X}, \| \cdot \|) \in \mathbb{V}^{(\mathbb{B})}$ stand for its Boolean valued realization. Then*

(1) *X is a Banach–Kantorovich lattice if and only if $\mathcal{X}$ is a Banach lattice inside $\mathbb{V}^{(\mathbb{B})}$;*

(2) *X is an order complete Banach–Kantorovich lattice if and only if $\mathcal{X}$ is an order complete Banach lattice inside $\mathbb{V}^{(\mathbb{B})}$;*

(3) *the norm $|\cdot|$ is o-continuous (order semicontinuous, monotone complete, or additive) if and only if the norm $\| \cdot \|$ is o-continuous (order semicontinuous, monotone complete, or additive) inside $\mathbb{V}^{(\mathbb{B})}$.*

7.3. Let E be a vector lattice, let F be some K-space, and let T be a positive operator from E to F.

Say that T possesses the *Maharam property* if, for all $x \in E_+$ and $0 \le f \le Tx \in F_+$, there is some $0 \le e \le x$ satisfying $f = Te$. An o-continuous positive operator with the Maharam property is a *Maharam operator*.

Observe that $T \in L(E, F)_+$ possesses the Maharam property if only if the equality $T([0, x]) = [0, Tx]$ holds for all $x \in E_+$. Thus, a Maharam operator is exactly an o-continuous order-interval preserving positive operator.

Let T be an essentially positive operator from E to F enjoying the Maharam property. Put $|e| := T(|x|)$ $(e \in E)$. Then $(E, |\cdot|)$ is a disjointly decomposable lattice-normed space over F.

Put $F_T := \{T(|x|) : x \in E\}^{\perp\perp}$, and let $\mathcal{D}_m(T)$ stand for the greatest order dense ideal of the universal completion $m(E)$ of E among those to which T can be extended by o-continuity. In other words, $z \in \mathcal{D}_m(T)$ if and only if $z \in m(E)$ and the set $\{T(x) : x \in E, 0 \le x \le |z|\}$ is bounded in F. In this event there exists a minimal extension of T to $\mathcal{D}_m(T)$ presenting an o-continuous positive operator.

Let E and F be some K-spaces, and let $T : E \to F$ be a Maharam operator. Put $X := \mathcal{D}_m(T)$ and $|x| := \Phi(|x|)$ $(x \in X)$, where Φ is an o-continuous extension of T to X. Then $(X, |\cdot|)$ is a Banach–Kantorovich lattice whose norm is o-continuous and additive.

7.4. Theorem. *Let X be an arbitrary K-space and let E be a universally complete K-space $\mathcal{R}\downarrow$. Assume that $\Phi : X \to E$ is a Maharam operator such that $X = X_\Phi = \mathcal{D}_m(\Phi)$ and $E = E_\Phi$. Then there are elements $\mathcal{X}$ and φ in $\mathbb{V}^{(\mathbb{B})}$ satisfying*

(1) *$[\![\mathcal{X}$ is a K-space, $\varphi : \mathcal{X} \to \mathcal{R}$ is a positive o-continuous functional, and $\mathcal{X} = \mathcal{X}_\varphi = \mathcal{D}_m(\varphi)]\!] = \mathbb{I}$;*

(2) *if $X' := \mathcal{X}\downarrow$ and $\Phi' = \varphi\downarrow$ then X' is a K-space and $\Phi' : X' \to E$ is a Maharam operator;*

(3) *there is a linear and lattice isomorphism h from X onto X' such that $\Phi = \Phi' \circ h$;*

(4) *for a linear operator Ψ, the containment $\Psi \in \{\Phi\}^{\perp\perp}$ is true if and only if there is $\psi \in \mathbb{V}^{(\mathbb{B})}$ such that $\psi \in \{\varphi\}^{\perp\perp}$ inside $\mathbb{V}^{(\mathbb{B})}$ and $\Psi = (\psi\downarrow) \circ h$.*

Theorem 7.4 enables us to claim that each fact about o-continuous positive linear functionals in K-spaces has a parallel version for Maharam operators which can be revealed by using 7.4. For instance, we state the abstract

7.5. Radon–Nikodým Theorem. *Let E and F be K-spaces. Assume further that S and T are o-continuous positive operators from E to F, with T enjoying the Maharam property. Then the following are equivalent:*

(1) $S \in \{T\}^{\perp\perp}$;

(2) $Sx \in \{Tx\}^{\perp\perp}$ *for all* $x \in E_+$;

(3) *there is an extended orthomorphism $0 \leq \rho \in \mathrm{Orth}^\infty(E)$ satisfying $Sx = T(\rho x)$ for all $x \in E$ such that $\rho x \in E$;*

(4) *there is a sequence of orthomorphisms $(\rho_n) \subset \mathrm{Orth}(E)$ such that $Sx = \sup_n T(\rho_n x)$ for all $x \in E$.*

7.6. A brief description for Maharam's approach to studying positive operators in the spaces of measurable functions and the main results in this area are collected in [44]. W. A. J. Luxemburg and A. R. Schep [42] extended a portion of Maharam's theory on the Radon–Nikodým Theorem to the case of positive operators in vector lattices.

Theorem 7.2 and 7.4 were obtained by A. G. Kusraev [27] and Theorem 7.5, by W. A. J. Luxemburg and A. R. Schep [42]. About various applications of the above results on Maharam operators and some extension of this theory to sublinear and convex operators see [29, 32, 34, 35].

8. SPACES WITH MIXED NORM

The definitions of various objects of functional analysis rest often on some blending of the norm and order properties. Among these are listed the spaces with mixed norm and the classes of linear operators between them.

8.1. If (X, E) is a lattice-normed space whose *norm lattice* E is a Banach lattice. Since, by definition, $|x| \in E$ for $x \in X$, we may introduce the *mixed norm* on X by the formula

$$\|x\| := \| |x| \| \quad (x \in X).$$

In this situation, the normed space $(X, \|\cdot\|)$ is called a *space with mixed norm*. A *Banach space with mixed norm* is a pair (X, E) with E a Banach lattice and X a br-complete lattice-normed space with E-valued norm. The following proposition justifies this definition.

Let E be a Banach lattice. Then $(X, \|\cdot\|)$ is a Banach space if and only if the lattice-normed space (X, E) is relatively uniformly complete.

8.2. Let Λ be the *bounded part* of the universally complete K-space $\mathcal{R}\downarrow$, i.e. Λ is the order-dense ideal in $\mathcal{R}\downarrow$ generated by the order unity $\mathbb{1} := 1^\wedge \in \mathcal{R}\downarrow$. Take a Banach space $\mathscr{X}$ inside $\mathbb{V}^{(\mathbb{B})}$. Put

$$\mathscr{X}\Downarrow := \{x \in \mathscr{X}\downarrow : |x| \in \Lambda\},$$
$$\|x\| := \| |x| \|_\infty := \inf\{0 < \lambda \in \mathbb{R} : |x| \leq \lambda\mathbb{1}\}.$$

Then $\mathscr{X}\Downarrow$ is a Banach–Kantorovich space called the *bounded descent* of $\mathscr{X}$. Since Λ is an order complete AM-space with unity, $\mathscr{X}\Downarrow$ is a Banach space with mixed norm over Λ.

Thus, we came to the following natural question: Which Banach spaces are linearly isometric to the bounded descents of internal Banach spaces? The answer is given in terms of $\mathbb{B}$-cyclic Banach spaces.

8.3. Let X be a normed space. Suppose that $\mathcal{L}(X)$ has a complete Boolean algebra of norm one projections $\mathcal{B}$ which is isomorphic to $\mathbb{B}$. In this event we will identify the Boolean algebras $\mathcal{B}$ and $\mathbb{B}$,

writing $\mathbb{B} \subset \mathcal{L}(X)$. Say that X is a *normed $\mathbb{B}$-space* if $\mathbb{B} \subset \mathcal{L}(X)$ and for every partition of unity $(b_\xi)_{\xi \in \Xi}$ in $\mathbb{B}$ the two conditions hold:

(1) If $b_\xi x = 0$ $(\xi \in \Xi)$ for some $x \in X$ then $x = 0$;

(2) If $b_\xi x = b_\xi x_\xi$ $(\xi \in \Xi)$ for $x \in X$ and a family $(x_\xi)_{\xi \in \Xi}$ in X then

$$\|x\| \leq \sup\{\|b_\xi x_\xi\| : \xi \in \Xi\}.$$

Given a partition of unity (b_ξ), we refer to $x \in X$ satisfying the condition $(\forall\, \xi \in \Xi)\, b_\xi x = b_\xi x_\xi$ as a *mixing* of (x_ξ) by (b_ξ). If (1) holds then there is a unique mixing x of (x_ξ) by (b_ξ). In these circumstances we naturally call x the *mixing* of (x_ξ) by (b_ξ). Condition (2) maybe paraphrased as follows: The unit ball U_X of X is closed under mixing.

A normed $\mathbb{B}$-space X is *$\mathbb{B}$-cyclic* if we may find in X a mixing of each norm-bounded family by each partition of unity in $\mathbb{B}$. It is easy to verify that X is a $\mathbb{B}$-cyclic normed space if and only if, given a partition of unity $(b_\xi) \subset \mathbb{B}$ and a family $(x_\xi) \subset U_X$, we may find a unique element $x \in U_X$ such that $b_\xi x = b_\xi x_\xi$ for all ξ.

A linear operator (linear isometry) S between normed $\mathbb{B}$-spaces is *$\mathbb{B}$-linear* (*$\mathbb{B}$-isometry*) if S commutes with the projections in $\mathbb{B}$; i.e., $\pi \circ S = S \circ \pi$ for all $\pi \in \mathbb{B}$. Let $\mathcal{L}_\mathbb{B}(X, Y)$ denote the set of all bounded $\mathbb{B}$-linear operators from X to Y. We call $X^{\#} := \mathcal{L}_\mathbb{B}(X, \mathbb{B}(\mathbb{R}))$ the *$\mathbb{B}$-dual* of X. If $X^{\#}$ and Y are $\mathbb{B}$-isometric to each other then we say that Y is a *$\mathbb{B}$-dual space* and X is a *$\mathbb{B}$-predual* of Y.

8.4. Theorem. *A Banach space X is linearly isometric to the bounded descent of some Banach space $\mathcal{X}$ inside $\mathbb{V}^{(\mathbb{B})}$ (called a Boolean valued representation of X) if and only if X is $\mathbb{B}$-cyclic. If X and Y are $\mathbb{B}$-cyclic Banach spaces and $\mathcal{X}$ and $\mathcal{Y}$ stand for some Boolean valued representations of X and Y then the space $\mathcal{L}_\mathbb{B}(X, Y)$ is $\mathbb{B}$-isometric to the bounded descent of the internal space $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$.*

8.5. Let Λ be a Stone algebra with unity $\mathbb{1}$ ($=$ an order complete complex AM-space with strong order unity $\mathbb{1}$ and uniquely defined multiplicative structure) and consider a unitary Λ-module X. The mapping $\langle \cdot \,|\, \cdot \rangle : X \times X \to \Lambda$ is a *Λ-valued inner product*, if for all $x, y, z \in X$ and $a \in \Lambda$ the following are satisfied:

(1) $\langle x \,|\, x \rangle \geq \mathbb{0}$; $\langle x \,|\, x \rangle = \mathbb{0} \leftrightarrow x = \mathbb{0}$;

(2) $\langle x \,|\, y \rangle = \langle y \,|\, x \rangle^*$;

(3) $\langle ax \,|\, y \rangle = a\langle x \,|\, y \rangle$;

(4) $\langle x + y \,|\, z \rangle = \langle x \,|\, z \rangle + \langle y \,|\, z \rangle$.

Using a Λ-valued inner product, we may introduce the norm of $x \in X$ by

$$\|x\| := \sqrt{\|\langle x|x \rangle\|}$$

and the decomposable vector norm of $x \in X$ by

$$|x| := \sqrt{\langle x|x \rangle}.$$

Obviously, $\|x\| = \big\||x|\big\|$ for all $x \in X$, and so X is a space with mixed norm.

8.6. Let X be a Λ-module with an inner product $\langle \cdot \,|\, \cdot \rangle : X \times X \to \Lambda$. If X is complete with respect to the mixed norm $\|\cdot\|$ then X is called a *C^*-module* over Λ. It can be proved (see [32]) that for a C^*-module X the pair $(X, \|\cdot\|)$ is a $\mathbb{B}$-cyclic Banach space if and only if $(X, |\cdot|)$ is a Banach$-$Kantorovich space over Λ. If a unitary C^*-module satisfies one of these equivalent conditions then it is called a *Kaplansky$-$Hilbert module*.

8.7. Theorem. *The bounded descent of a Hilbert space in $\mathbb{V}^{(\mathbb{B})}$ is a Kaplansky$-$Hilbert module over the Stone algebra $\mathcal{C}{\Downarrow}$. Conversely, if X is a Kaplansky$-$Hilbert module over $\mathcal{C}{\Downarrow}$ then there is a Hilbert space $\mathcal{X}$ in $\mathbb{V}^{(\mathbb{B})}$ whose bounded descent is unitarily equivalent with X. This space is unique up to unitary equivalence inside $\mathbb{V}^{(\mathbb{B})}$.*

8.8. Theorem. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces inside $\mathbb{V}^{(\mathbb{B})}$. Suppose that X and Y are the bounded descents of $\mathcal{X}$ and $\mathcal{Y}$. Then the space $\mathcal{L}_{\mathbb{B}}(X, Y)$ of all $\mathbb{B}$-linear bounded operators is a $\mathbb{B}$-cyclic Banach space $\mathbb{B}$-isometric to the bounded descent of the internal Banach space $\mathcal{L}^{\mathbb{B}}(\mathcal{X}, \mathcal{Y})$ of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$.*

8.9. Boolean valued analysis approach gives rise to an interesting concept of cyclically compact operator in a Banach $\mathbb{B}$-space [32, 8.5.5]. Without plunging into details we formulate a result on the general form of cyclically compact operators in Kaplansky–Hilbert modules.

Theorem. *Let X and Y be Kaplansky–Hilbert modules over a Stone algebra Λ and let T be a cyclically compact operator from X to Y. There are orthonormal families $(e_k)_{k \in \mathbb{N}}$ in X, $(f_k)_{k \in \mathbb{N}}$ in Y, and a family $(\mu_k)_{k \in \mathbb{N}}$ in Λ such that the following hold:*

(1) $\mu_{k+1} \leq \mu_k$ $(k \in \mathbb{N})$ *and* $o\text{-}\lim_{k \to \infty} \mu_k = 0$;

(2) *there exists a projection π_∞ in Λ such that $\pi_\infty \mu_k$ is a weak order unity in $\pi_\infty \Lambda$ for all $k \in \mathbb{N}$;*

(3) *there exists a partition $(\pi_k)_{k=0}^\infty$ of the projection $\pi_\infty^\perp$ such that $\pi_0 \mu_1 = 0$, $\pi_k \leq \mu_k$, and $\pi_k \mu_{k+1} = 0$ for all $k \in \mathbb{N}$;*

(4) *the representation is valid*

$$T = \pi_\infty \, bo\text{-}\sum_{k=1}^\infty \mu_k e_k^\# \otimes f_k, \qquad +bo\text{-}\sum_{n=1}^\infty \pi_n \sum_{k=1}^n \mu_k e_k^\# \otimes f_k.$$

8.10. The bounded descent of 8.2 appeared in the research by G. Takeuti into von Neumann algebras and C^*-algebras within Boolean valued models [61, 62] and in the research by M. Ozawa into Boolean valued interpretation of the theory of Hilbert spaces [46]. Theorems 8.4 and 8.9 were obtained by A. G. Kusraev in [28, 29, 32]. Theorems 8.7 and 8.8 were proved by M. Ozawa [46].

REFERENCES

1. Yu. A. Abramovich, A. I. Veksler, and A. V. Koldunov, "On Disjointness Preserving Operators," Dokl. Akad. Nauk SSSR, **289** (5), 1033–1036 (1979) [Sov. Math., Dokl. **20**, 1089-1093 (1979)].
2. Yu. A. Abramovich, A. I. Veksler, and A. V. Koldunov, "Disjointness Preserving Operators, Their Continuity, and Multiplicative Representation," in *Linear Operators and Their Applications* (Leningrad. Ped. Inst., Leningrad, 1981), pp. 3–34 [in Russian].
3. J. Aczél and J. Dhombres, *Functional Equations in Several Variables* (Cambridge Univ. Press, Cambridge, 1989).
4. Sh. A. Ayupov, "Jordan Operator Algebras," in *Modern Problems of Mathematics: Recent Advances*, Vol. 27 (Vsesoyuzn. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1985), pp. 67–97 [J. Sov. Math. **37**, 1422–1448 (1987)].
5. G. P. Akilov and S. S. Kutateladze, *Ordered Vector Spaces* (Nauka, Novosibirsk, 1978) [in Russian].
6. E. M. Alfsen, F. W. Shultz, and E. Størmer, "A Gelfand–Neumark Theorem for Jordan Algebras," Adv. in Math. **28** (1), 11–56 (1978).
7. C. D. Aliprantis and O. Burkinshaw, *Positive Operators* (Academic Press, New York, 1985).
8. J. L. Bell, *Boolean-Valued Models and Independence Proofs in Set Theory* (Clarendon Press, Oxford, 1985).
9. S. J. Bernau, C. B. Huijsmans, and B. de Pagter, "Sums of Lattice Homomorphisms," Proc. Amer. Math. Soc. **115** (1), 151–156 (1992).
10. K. R. Goodearl, *Von Neumann Regular Rings* (Pitman, London, 1979).
11. E. I. Gordon, "Real Numbers in Boolean Valued Models of Set Theory and K-Spaces," Dokl. Akad. Nauk SSSR **237** (4), 773–775 (1977) [Sov. Math., Dokl. **18**, 1481–1484 (1978)].
12. E. I. Gordon, "K-Spaces in Boolean Valued Models of Set Theory," Dokl. Akad. Nauk SSSR **258** (4), 777–780 (1981) [Sov. Math., Dokl. **23**, 579–582 (1981)].
13. A. Grothendieck, "Une caractérisation vectorielle-métrique des espaces L^1," Canad. J. Math. **4**, 552–561 (1955).
14. A. E. Gutman, "Locally One-Dimensional K-Spaces and σ-Distributive Boolean Algebras," Mat. Trudy **5** (2), 99–121 (1995) [Siberian Adv. Math. **5** (1), 42-48 (1995)].

15. A. E. Gutman, "Disjointness Preserving Operators," in *Vector Lattices and Integral Operators*, Ed. by S. S. Kutateladze (Kluwer Academic, Dordrecht, 1996), pp. 361−454.

16. H. Hanche-Olsen and E. Størmer, *Jordan Operator Algebras* (Pitman, Boston, 1984).

17. D. R. Hofstedter, *Gödel, Escher, Bach: an Eternal Golden Braid* (Basic Books, New York, 1999).

18. T. J. Jech, "Abstract Theory of Abelian Operator Algebras: an Application of Forcing," Trans. Amer. Math. Soc. **289** (1), 133−162 (1985).

19. T. J. Jech, "First Order Theory of Complete Stonean Algebras (Boolean Valued Real and Complex Numbers)," Canad. Math. Bull. **30** (4), 385−392 (1987).

20. T. J. Jech, "Boolean-Linear Spaces," Adv. in Math. **81** (2), 117−197 (1990).

21. T. J. Jech, *Lectures in Set Theory with Particular Emphasis on the Method of Forcing* (Springer, Berlin, 1971).

22. P. Jordan, J. von Neumann, and E. Wigner, "On an Algebraic Generalization of the Quantum Mechanical Formalism," Ann. of Math. (2) **35** (1), 29−64 (1934).

23. L. V. Kantorovich, "On Semiordered Linear Spaces and Their Applications to the Theory of Linear Operations," Dokl. Akad. Nauk SSSR **4** (1−2), 11−14 (1935).

24. L. V. Kantorovich and G. P. Akilov, *Functional Analysis* (Pergamon Press, Oxford, 1982; Nauka, Moscow, 1984).

25. L. V. Kantorovich, B. Z. Vulikh, and A. G. Pinsker, *Functional Analysis in Partially Ordered Spaces*, (Gostekhizdat, Moscow and Leningrad, 1950) [in Russian].

26. I. Kaplansky, "Modules over Operator Algebras," Amer. J. Math. **75** (4), 839−858 (1953).

27. A. G. Kusraev, "General Disintegration Formulas," Dokl. Akad. Nauk SSSR **265** (6), 1312−1316 (1982) [Sov. Math., Dokl. **26**, 255−259 (1982)]

28. A. G. Kusraev, "On Banach−Kantorovich Spaces," Sibirsk. Mat. Zh. **26** (2), 119−126 (1985) [Siberian Math. J. **26** (2), 254−259 (1985)].

29. A. G. Kusraev, *Vector Duality and Its Applications* (Nauka, Novosibirsk, 1985) [in Russian].

30. A. G. Kusraev, "Numeric Systems in Boolean Valued Models of Set Theory," in *Proceedings of the VIII All-Union Conference on Mathematical Logic, Moscow, 1986* (Moscow, 1986), p. 99.

31. A. G. Kusraev, "On Band Preserving Operators," Vladikavkaz. Mat. Zh. **6** (3), 48−58 (2004).

32. A. G. Kusraev, *Dominated Operators* (Kluwer Academic, Dordrecht, 2000).

33. A. G. Kusraev, "Automorphisms and Derivations on a Universally Complete Complex f-Algebra," Sibirsk. Mat. Zh. **47** (1), 97−107 (2006) [Siberian Math. J. **47** (1), 77−85 (2006)].

34. A. G. Kusraev and S. S. Kutateladze, *Subdifferentials: Theory and Applications*, (Nauka, Novosibirk, 1992; Kluwer Academic, Dordrecht, 1995).

35. A. G. Kusraev and S. S. Kutateladze, *Introduction to Boolean Valued Analysis* (Nauka, Moscow, 2005) [in Russian].

36. A. G. Kusraev and S. S. Kutateladze, "Boolean Valued Analysis and Positivity," in Proceedings of the Conference Positivity IV—Theory and Applications. (Technische Universität, Dresden, 2006), pp. 93−106.

37. S. S. Kutateladze , "Choquet Boundaries in K-Spaces," Uspekhi Mat. Nauk **30** (4), 107−146 (1975) [Russ. Math. Surveys **30** (4), 115−155 (1975)].

38. S. S. Kutateladze , "Nonstandard Tools for Convex Analysis," Math. Japon. **43** (2), 391−410 (1996).

39. S. S. Kutateladze , "On Differences of Riesz Homomorphisms," Sibirsk. Mat. Zh. **46** (2), 390−393 (2005) [Siberian Math. J. **46** (2), 305−307 (2005)].

40. S. S. Kutateladze , "On Grothendieck Subspaces," Sibirsk. Mat. Zh. **46** (3), 620−624 (2005) [Siberian Math. J. **46** (3), 489−493 (2005)].

41. J. Lindenstrauss and D. E. Wulbert, "On the Classification of the Banach Spaces Whose Duals are L_1-Spaces," J. Funct. Anal. **4** (3), 322−249 (1969).

42. W. A. J. Luxemburg and A. R. Schep, "A Radon−Nikodým Type Theorem for Positive Operators and a Dual," Indag. Math. **40**, 357−375 (1978).

43. W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces*, Vol. 1 (North-Holland, Amsterdam, 1971).

44. D. Maharam, "On Positive Operators," Contemp. Math. **26**, 263−277 (1984).

45. P. T. N. McPolin and A. W. Wickstead, "The Order Boundedness of Band Preserving Operators on Uniformly Complete Vector Lattices," Math. Proc. Cambridge Philos. Soc. **97** (3), 481−487 (1985).

46. M. Ozawa, "Boolean Valued Interpretation of Hilbert Space Theory," J. Math. Soc. Japan **35** (4), 609−627 (1983).

47. M. Ozawa, "Boolean Valued Analysis and Type I AW^*-Algebras," Proc. Japan Acad. Ser. A Math. Sci. **59** (8), 368−371 (1983).

48. M. Ozawa, "A Classification of Type I AW^*-Algebras and Boolean Valued Analysis," J. Math. Soc. Japan **36** (4), 589−608 (1984).

49. M. Ozawa, "A Transfer Principle from von Neumann Algebras to AW^*-Algebras," J. London Math. Soc., II. Ser. **32** (1), 141−148 (1985).

50. M. Ozawa, "Nonuniqueness of the Cardinality Attached to Homogeneous AW^*-Algebras," Proc. Amer. Math. Soc. **93**, 681−684 (1985).
51. M. Ozawa, "Boolean Valued Analysis Approach to the Trace Problem of AW^*-Algebras," J. London Math. Soc., II. Ser. **33** (2), 347−354 (1986).
52. M. Ozawa, "Boolean Valued Interpretation of Banach Space Theory and Module Structures of von Neumann Algebras," Nagoya Math. J. **117**, 1−36 (1990).
53. S. Sakai, *C^*-Algebras and W^*-Algebras* (Springer, Berlin, 1971).
54. H. H. Schaefer, *Banach Lattices and Positive Operators* (Springer, Berlin, 1974).
55. H.-U. Schwarz, *Banach Lattices and Operators* (Teubner, Leipzig, 1984).
56. F. W. Shultz, "On Normed Jordan Algebras Which Are Banach Dual Spaces," J. Funct. Anal. **31**, 360−376 (1979).
57. R. Sikorski, *Boolean Algebras* (Springer, Berlin, 1964).
58. E. Størmer, "Jordan Algebras of Type I," Acta Math. **115** (3−4), 165−184 (1966).
59. G. Takeuti, *Two Applications of Logic to Mathematics* (Princeton Univ. Press, Princeton, 1978).
60. G. Takeuti, "Boolean Valued Analysis," in *Lecture Notes in Math.*, Vol. 753: *Applications of Sheaves: Proceedings Res. Symposium on Application of Sheaf Theory to Logic, Algebra and Analysis, University Durham, Durham, 1977* (Springer, Berlin, 1979), pp. 714−731.
61. G. Takeuti, "Von Neumann Algebras and Boolean Valued Analysis," J. Math. Soc. Japan **35** (1), 1−21 (1983).
62. G. Takeuti, "C^*-Algebras and Boolean Valued Analysis," Japan. J. Math. (N.S.) **9** (2), 207−246 (1983).
63. G. Takeuti and W. M. Zaring, *Axiomatic Set Theory* (Springer, New York, 1973).
64. D. M. Topping, "Jordan Algebras of Self-Adjoint Operators," Mem. Amer. Math. Soc. **53**, 48 (1965).
65. B. Z. Vulikh, *Introduction to the Theory of Partially Ordered Spaces* (Fizmatgiz, Moscow, 1961; Wolters-Noordhoff Scientific, Groningen, 1967).
66. A. W. Wickstead, "Representation and Duality of Multiplication Operators on Archimedean Riesz Spaces," Compositio Math. **35** (3), 225−238 (1977).
67. A. C. Zaanen, *Riesz Spaces*, Vol. 2 (North-Holland, Amsterdam, 1983).