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Lyapunov's Convexity Theorem, Zonoids, and Bang-Bang

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Abstract—This is a short overview of the connections of the Lyapunov Convexity Theorem with the modern sections of analysis, geometry, and optimal control.

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On the centenary of the birth of A. A. Lyapunov

The theory and practice of extremal problems, the choice of optimal control in a deterministic and stochastic environment, many techniques of mathematical economics rest on the fundamental ideas of functional analysis which are connected with convexity and measure.

The Lyapunov Convexity Theorem, proven in 1940 (see [1]), occupies a prominent place in modern mathematics, since it lies at the juncture of the theory of convex sets and measure theory. The Lyapunov Convexity Theorem became the starting point of numerous studies in the framework of mathematical analysis as well as in the realm of geometric research into the convex sets that are ranges of nonatomic vector measures.

The unexpectedness of the discovery by A. A. Lyapunov is due to the paradoxical and fragile balance of interactions between various finite- and infinite-dimensional ideas. The effects of the Lyapunov Convexity Theorem vanish or disintegrate in we admit into consideration nondiffuse (countably-additive) measures, or finitely-additive measures, or measures with values in infinite-dimensional spaces (e.g., see Lyapunov's second paper [2] and [16]). At the same time we should emphasize that, geometrically speaking, the Lyapunov Convexity Theorem addresses the image under some linear operator of the extreme points of a particular infinite-dimensional compact convex set. This very circumstance was vital in the exquisite proof which was found by J. Lindenstrauss in 1966 r. and which made the Lyapunov Convexity Theorem very popular (see [14]).

It is worth noting that today there are available many proofs of the Lyapunov Convexity Theorem that base on the basic facts of mathematical analysis (e.g., see [6], [8]). For instance, such is a rather elegant proof by D. Ross which was found in 2005 and bases only on the Intermediate Value Theorem (see [20]).

From the scratch the Lyapunov Convexity Theorem had raised the problem of describing the compact convex sets in finite-dimensional real spaces which serve as the ranges of diffuse measures. These compacta are known in the modern geometrical literature as *zonoids*. Among zonoids we distinguish the Minkowski sums of finitely many straight line segments. These sets, called *zonotopes*, fill a convex cone in the space of compact convex sets, and the cone of zonotopes is dense in the closed cone of all zonoids. The first description of the ranges of diffuse vector measures in the Lyapunov Convexity Theorem was firstly found by K. I. Chuĭkina practically in the modern terms (see[3]). Soon after that her result was somewhat supplemented and simplified by E. V. Glivenko in [4]. The zonotopes of the present epoch were called *parallelohedra* those days.

The significant further progress in studying the ranges of diffuse vector measures belong to Yu. G. Reshetnyak and V. A. Zalgaller who described zonoids as the results of mixing the linear elements of a rectifiable curve in a finite-dimensional space in 1954 (see [5]). In this paper they suggested a new proof of the Lyapunov Convexity Theorem and demonstrated that zonotopes are precisely those convex

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polyhedra whose two-dimensional faces have centers of symmetry. Unfortunately, these results remained practically unnoticed in the West. Analogous results were obtained by E. Bolker only fifteen years later in 1969 (see [7]).

We must mention the exceptional role of the Lyapunov Convexity Theorem in justification of the "bang-bang" principle in the theory of optimal control. The principle asserts that the optimal controls are implemented by the extreme point of the set of admissible controls.

The meaning of the bang-bang principle is as follows: For optimal transition in minimal time from one state of a system to the other in the conditions of limited resources we can use an extreme "bang-bang" control. In other words, if the system under control has an optimal control then it has an optimal "bang-bang" control (see [11, p. 47]). For extra information see, for instance, [10], [12], [13], [15], and [17].

In closing we mention that the history of the Lyapunov Convexity Theorem within functional analysis is displayed in some detail in [19]. About the place of the theorem and search into its generalizations within measure theory see [18]. As regards zonoids, see, e.g., [9].

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