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Leibnizian, Robinsonian, and Boolean Valued Monads*[1]

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Abstract—This is an overview of the present-day versions of monadology with some applications to vector lattices and linear inequalities. Two approaches to combining nonstandard set-theoretic models are sketched and illustrated by order convergence, principal projection, and polyhedrality.

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The notion of monad is central to external set theory. Justifying the simultaneous use of infinitesimals and the technique of descending and ascending in vector lattice theory requires adaptation of monadology for the implementation of filters in Boolean valued universes. This is still a rather uncharted area of research. The two approaches are available now. One is to apply monadology to the descents of objects. The other consists in applying the standard monadology inside the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ over a complete Boolean algebra $\mathbb{B}$, while ascending and descending by the Escher rules (cp. [1] and [2]).

These approaches are sketched and illustrated by tests for order convergence and rules for fragmenting and projecting positive operators in vector lattices. Also, Lagrange's principle is shortly addressed in polyhedral environment with inexact data.

1. BASICS OF MONADOLOGY

The concept of monad stems from Ancient Greece. Monadology as a philosophical doctrine is a creation of Leibniz (cp. [3] and [4]). The general theory of the monads of filters was proposed by Luxemburg (cp. [5]) within Robinson's nonstandard analysis (cp. [6]).

Let $\mathcal{F}$ be a standard filter; ${}^\circ\mathcal{F}$, the standard core of $\mathcal{F}$; and ${}^a\mathcal{F} := \mathcal{F} \setminus {}^\circ\mathcal{F}$, the external set of *remote* elements of $\mathcal{F}$. Note that

$$\mu(\mathcal{F}) := \bigcap {}^\circ\mathcal{F} = \bigcup {}^a\mathcal{F}$$

is the *monad* of $\mathcal{F}$. Also, $\mathcal{F} = {}^*\operatorname{fil}(\{\mu(\mathcal{F})\})$; i.e., $\mathcal{F}$ is the standardization of the collection $\operatorname{fil}(\mu(\mathcal{F}))$ of all supersets of $\mu(\mathcal{F})$.

Let $\mathcal{A}$ be a filter on $X \times Y$, and let $\mathcal{B}$ be a filter on $Y \times Z$. Put

$$\mathcal{B} \circ \mathcal{A} := \operatorname{fil}\{B \circ A \mid A \in \mathcal{A},\ B \in \mathcal{B}\},$$

where we may assume all $B \circ A$ nonempty. Then

$$\mu(\mathcal{B} \circ \mathcal{A}) = \mu(\mathcal{B}) \circ \mu(\mathcal{A}).$$

Granted Horizon Principle. *Let X and Y be standard sets. Assume further that $\mathcal{F}$ and $\mathcal{G}$ are standard filters on X and Y respectively satisfying $\mu(\mathcal{F}) \cap {}^\circ X \neq \varnothing$. Distinguish a remote set F in ${}^a\mathcal{F}$. Given a standard correspondence $f \subset X \times Y$ meeting $\mathcal{F}$, the following are equivalent:*

(1) $f(\mu(\mathcal{F}) - F) \subset \mu(\mathcal{G})$;

(2) $(\forall F' \in {}^a\mathcal{F})\, f(F' - F) \subset \mu(\mathcal{G})$;

(3) $f(\mu(\mathcal{F})) \subset \mu(\mathcal{G})$.

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2. FILTERS WITHIN $\mathbb{V}^{(\mathbb{B})}$

Let $\mathbb{B}$ be a complete Boolean algebra. Given an ordinal α, put

$$V_\alpha^{(\mathbb{B})} := \{x \mid (\exists \beta \in \alpha)\, x : \operatorname{dom}(x) \to \mathbb{B}, \operatorname{dom}(x) \subset V_\beta^{(\mathbb{B})}\}.$$

The *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ is

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(\mathbb{B})},$$

with On the class of all ordinals.

The truth value $[\![\varphi]\!] \in \mathbb{B}$ is assigned to each formula φ of ZFC relativized to $\mathbb{V}^{(\mathbb{B})}$.

Let Q be the Stone space of a complete Boolean algebra $\mathbb{B}$. Denote the (separated) Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ by $\mathfrak{U}$. Given $q \in Q$, put

$$u \sim_q v \;\leftrightarrow\; q \in [\![u = v]\!].$$

Consider the bundle

$$V^Q := \big\{\big(q, \sim_q(u)\big) \mid q \in Q,\, u \in \mathfrak{U}\big\}$$

and denote $\big(q, \sim_q(u)\big)$ by $\widehat{u}(q)$. Hence, $\widehat{u} : q \mapsto \widehat{u}(q)$ is a section of V^Q for every $u \in \mathfrak{U}$. Note that to each $x \in V^Q$ there are $u \in \mathfrak{U}$ and $q \in Q$ satisfying $\widehat{u}(q) = x$. Moreover, we have $\widehat{u}(q) = \widehat{v}(q)$ if and only if $q \in [\![u = v]\!]$.

Make each fiber V^q of V^Q into an algebraic system of signature $\{\in\}$ by letting

$$V^q \models x \in y \;\leftrightarrow\; q \in [\![u \in v]\!],$$

where $u, v \in \mathfrak{U}$ are such that $\widehat{u}(q) = x$ and $\widehat{v}(q) = y$.

The class $\{\widehat{u}(A) \mid u \in \mathfrak{U}\}$, with A a clopen subset of Q, is a base for some topology on V^Q. Thus V^Q as a continuous bundle is called a *continuous polyverse*. By a *continuous section* of V^Q we mean a section that is a continuous function. Denote the class of all continuous sections of V^Q by $\mathfrak{C}$.

The mapping $u \mapsto \widehat{u}$ is a bijection between $\mathfrak{U}$ and $\mathfrak{C}$, yielding a convenient functional realization of the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$. This universal construction belongs to Gutman and Losenkov (cp. [7]).

The functional realization visualizes descending and ascending, the Escher rules, and the Gordon Theorem (cp. [8]).

Let $\mathcal{G}$ be a filterbase on X, with $X \in \mathcal{P}(\mathbb{V}^{(\mathbb{B})})$. Put

$$\mathcal{G}' := \{F \in \mathcal{P}(X{\uparrow}){\downarrow} \mid (\exists G \in \mathcal{G})\, [\![\, F \supset G{\uparrow}\,]\!] = \mathbb{1}\},$$
$$\mathcal{G}'' := \{G{\uparrow} \mid G \in \mathcal{G}\}.$$

Then $\mathcal{G}'{\uparrow}$ and $\mathcal{G}''{\uparrow}$ are bases of the same filter $\mathcal{G}^{\uparrow}$ on $X{\uparrow}$ inside $\mathbb{V}^{(\mathbb{B})}$—the *ascent* of $\mathcal{G}$. If $\operatorname{fil}(\mathcal{G})$ is the set of all mixings of nonempty families of elements of $\mathcal{G}$ and $\mathcal{G}$ consists of cyclic sets; then $\operatorname{fil}(\mathcal{G})$ is a filterbase on X and $\mathcal{G}^{\uparrow} = \operatorname{fil}(\mathcal{G})^{\uparrow}$.

If $\mathcal{F}$ is a filter on X inside $\mathbb{V}^{(\mathbb{B})}$ then put $\mathcal{F}^{\downarrow} := \operatorname{fil}(\{F{\downarrow} \mid F \in \mathcal{F}{\downarrow}\})$. The filter $\mathcal{F}^{\downarrow}$ is the *descent* of $\mathcal{F}$. A filterbase $\mathcal{G}$ on $X{\downarrow}$ is *extensional* provided that $\operatorname{fil}(\mathcal{G}) = \mathcal{F}$ for some filter $\mathcal{F}$ on X.

The descent of an ultrafilter on X is a *proultrafilter* on $X{\downarrow}$. A filter with a base of cyclic sets is *cyclic*. Proultrafilters are maximal cyclic filters.

Fix a standard complete Boolean algebra $\mathbb{B}$ and think of $\mathbb{V}^{(\mathbb{B})}$ to be composed of internal sets. If A is external then the *cyclic hull* $\operatorname{fil}(A)$ of A consists of $x \in \mathbb{V}^{(\mathbb{B})}$ admitting an internal family $(a_\xi)_{\xi\in\Xi}$ of elements of A and an internal partition $(b_\xi)_{\xi\in\Xi}$ of unity in B such that x is the mixing of $(a_\xi)_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$; i.e., $b_\xi x = b_\xi a_\xi$ for $\xi \in \Xi$ or, equivalently, $x = \operatorname{fil}_{\xi\in\Xi}(b_\xi a_\xi)$.

Given a filter $\mathcal{F}$ on $X{\downarrow}$, let

$$\mathcal{F}{\uparrow}{\downarrow} := \operatorname{fil}(\{F{\uparrow}{\downarrow} \mid F \in \mathcal{F}\}).$$

Then $\operatorname{fil}(\mu(\mathcal{F})) = \mu(\mathcal{F}{\uparrow}{\downarrow})$ and $\mathcal{F}{\uparrow}{\downarrow}$ is the greatest cyclic filter coarser than $\mathcal{F}$.

The monad of $\mathcal{F}$ is called *cyclic* if $\mu(\mathcal{F}) = \mathrm{fil}(\mu(\mathcal{F}))$. Unfortunately, the cyclicity of a monad is not completely responsible for extensionality of a filter.

The *cyclic monad hull* $\mu_c(U)$ of an external set U is defined as follows:

$$x \in \mu_c(U) \leftrightarrow (\forall^{\mathrm{st}} V = V{\uparrow}{\downarrow}) V \supset U \rightarrow x \in \mu(V).$$

If $\mathbb{B} = \mathbb{2}$ then $\mu_c(U)$ is the monad of the standardization of the external filter of supersets of U, i.e., the (*discrete*) *monad hull* $\mu_d(U)$.

The cyclic monad hull of a set is the cyclic hull of its monad hull

$$\mu_c(U) = \mathrm{fil}(\mu_d(U)).$$

A special role is played by the *essential points* of $X{\downarrow}$ constituting the external set ${}^eX$. By definition, an essential point of ${}^eX$ belongs to the monad of some proultrafilter on $X{\downarrow}$. The collection ${}^eX$ of all essential points of X is usually external.

Test for Essentiality. *A point $x \in {}^eX$ if and only if x can be separated by a standard set from every standard cyclic set not containing x.*

If there is an essential point in the monad of an ultrafilter $\mathcal{F}$ then $\mu(\mathcal{F}) \subset {}^eX$; moreover, $\mathcal{F}{\uparrow}{\downarrow}$ is a proultrafilter.

A filter $\mathcal{F}$ is extensional if and only if $\mu(\mathcal{F}) = \mu_c({}^e\mu F)$. A standard set A is cyclic if and only if A is the cyclic monad hull of ${}^eA$.

Test for the Mixing of Filters. *Let $(\mathcal{F}_\xi)_{\xi\in\Xi}$ be a standard family of extensional filters, and let $(b_\xi)_{\xi\in\Xi}$ be a standard partition of unity. The filter $\mathcal{F}$ is the mixing of $(\mathcal{F}_\xi)_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$ if and only if*

$$(\forall^{\mathrm{St}} \xi \in \Xi)\, b_\xi \mu(\mathcal{F}) = b_\xi \mu(\mathcal{F}_\xi).$$

Properties of Essential Points. (1) *The image of an essential point under an extensional mapping is an essential point of the image;*

(2) *Let E be a standard set, and let X be a standard element of $\mathbb{V}^{(\mathbb{B})}$. Consider the product $X^{E^\wedge}$ inside $\mathbb{V}^{(\mathbb{B})}$, where $E^\wedge$ is the standard name of E in $\mathbb{V}^{(\mathbb{B})}$. If x is an essential point of $X^{E^\wedge}{\downarrow}$ then for every standard $e \in E$ the point $x{\downarrow}(e)$ is essential in $X{\downarrow}$;*

(3) *Let $\mathcal{F}$ be a cyclic filter in $X{\downarrow}$, and let ${}^e\mu(\mathcal{F}) := \mu(\mathcal{F}) \cap {}^eX$ be the set of essential points of its monad. Then ${}^e\mu(\mathcal{F}) = {}^e\mu(\mathcal{F}^{\uparrow\downarrow})$.*

Let $(X,\mathcal{U})$ be a uniform space inside $\mathbb{V}^{(\mathbb{B})}$. The descent $(X{\downarrow},\mathcal{U}^{\downarrow})$ is *procompact* or *cyclically compact* if $(X,\mathcal{U})$ is compact inside $\mathbb{V}^{(\mathbb{B})}$. A similar sense resides in the notion of *pro-total-boundedness* and so on.

Every essential point of $X{\downarrow}$ is nearstandard, i.e., infinitesimally close to a standard point, if and only if $X{\downarrow}$ is procompact.

Existence of many procompact but not compact spaces provides a lot of examples of inessential points.

Test for Proprecompactness. *A standard space is the descent of a totally bounded uniform space if and only if its every essential point is prenearstandard, i.e., belongs to the monad of a Cauchy filter.*

Let Y to be a universally complete vector lattice. By Gordon's Theorem, Y is the descent $\mathcal{R}{\downarrow}$ of the reals $\mathcal{R}$ inside $\mathbb{V}^{(\mathbb{B})}$ over the base $\mathbb{B} := \mathbb{B}(Y)$ of Y.

Denote by $\mathcal{E}$ the filter of order units in Y, i.e.,

$$\mathcal{E} := \{\varepsilon \in Y_+ \mid [\![\varepsilon = 0]\!] = \mathbb{0}\}.$$

Put $x \approx y \leftrightarrow (\forall^{\mathrm{st}} \varepsilon \in \mathcal{E})\,(|x - y| < \varepsilon)$. Given $a, b \in Y$, write $a < b$ if $[\![a < b]\!] = \mathbb{1}$; in other words, $a > b \leftrightarrow a - b \in \mathcal{E}$. Thus, there is some deviation from the understanding of the theory of ordered vector spaces. Clearly, this is done in order to adhere to the principles of introducing notation while descending and ascending.

Let ${}^{\approx}Y$ be the *nearstandard part* of Y. Given $y \in {}^{\approx}Y$, denote by ${}^{\circ}y$ (or by $\mathrm{st}(y)$) the *standard part* of y, i.e., the unique standard element infinitely close to y.

Tests for Order Convergence. *For a standard filter $\mathcal{F}$ in Y and a standard $z \in Y$, the following are true:*

(1) $\inf_{F\in\mathcal{F}} \sup F \le z \leftrightarrow (\forall y \in {}^{\cdot}\mu(\mathcal{F}\uparrow\downarrow))\, {}^{\circ}y \le z \leftrightarrow (\forall y \in {}^{e}\mu(\mathcal{F}\uparrow\downarrow))\, {}^{\circ}y \le z;$

(2) $\sup_{F\in\mathcal{F}} \inf F \ge z \leftrightarrow (\forall y \in {}^{\cdot}\mu(\mathcal{F}\uparrow\downarrow))\, {}^{\circ}y \ge z \leftrightarrow (\forall y \in {}^{e}\mu(\mathcal{F}\uparrow\downarrow))\, {}^{\circ}y \ge z;$

(3) $\inf_{F\in\mathcal{F}} \sup F \ge z \leftrightarrow (\exists y \in {}^{\cdot}\mu(\mathcal{F}\uparrow\downarrow))\, {}^{\circ}y \ge z \leftrightarrow (\exists y \in {}^{e}\mu(\mathcal{F}\uparrow\downarrow))\, {}^{\circ}y \ge z;$

(4) $\sup_{F\in\mathcal{F}} \inf F \le z \leftrightarrow (\exists y \in {}^{\cdot}\mu(\mathcal{F}\uparrow\downarrow))\, {}^{\circ}y \le z \leftrightarrow (\exists y \in {}^{e}\mu(\mathcal{F}\uparrow\downarrow))^{\circ}y \le z;$

(5) $S\,\mathcal{F} \overset{(o)}{z} \leftrightarrow (\forall y \in {}^{e}\mu(\mathcal{F}\uparrow\downarrow))y \approx z \leftrightarrow (\forall y \in \mu(\mathcal{F}^{\uparrow\downarrow}))y \approx z.$

Here

$$ {}^{\cdot}\mu(\mathcal{F}\uparrow\downarrow) := \mu(\mathcal{F}\uparrow\downarrow) \cap {}^{\approx}Y, $$

and, as usual, ${}^{e}\mu(\mathcal{F}\uparrow\downarrow)$ is the set of essential points of the monad $\mu(\mathcal{F}\uparrow\downarrow)$, i.e.,

$$ {}^{e}\mu(\mathcal{F}\uparrow\downarrow) = \mu(\mathcal{F}\uparrow\downarrow) \cap {}^{e}\mathcal{R}. $$

3. BOOLEAN VALUED MONADS

Let us follow the classical approach of Robinson inside $\mathbb{V}^{(\mathbb{B})}$. In other words, the classical and internal universes and the corresponding $*$-map (Robinson's standardization) are understood to be members of $\mathbb{V}^{(\mathbb{B})}$. Moreover, the nonstandard world is supposed to be properly saturated.

The descent of the $*$-map is referred to as *descent standardization*. Alongside the term "descent standardization" the expressions like "B-standardization," "prostandardization," etc. are in common parlance. Furthermore, denote the Robinson standardization of a B-set A by ${}^{*}A$.

The *descent standardization* of a set A with B-structure, i.e., a subset of $\mathbb{V}^{(\mathbb{B})}$, is defined as $({}^{*}(A\uparrow))\downarrow$ and is denoted by ${}_{*}A$ (it is meant here that $A\uparrow$ is an element of the standard universe located inside $\mathbb{V}^{(\mathbb{B})}$).

Thus, ${}^{*}a \in {}_{*}A \leftrightarrow a \in A\uparrow\downarrow$. The *descent standardization* ${}_{*}\Phi$ of an *extensional correspondence* Φ is also defined in a natural way.

Considering the descent standardizations of the standard names of elements of the von Neumann universe $\mathbb{V}$, use the abbreviations ${}^{*}x := {}^{*}(x^{\wedge})$ and ${}_{*}x := ({}^{*}x)\downarrow$ for $x \in \mathbb{V}$. The rules of placing and omitting asterisks (by default) in descent standardization are also assumed as liberal as those for the Robinson $*$-map.

Transfer. *Let $\varphi = \varphi(x, y)$ be a formula of* ZFC *without any free variables other than x and y. Then*

$$ (\exists x \in {}_{*}F)\, [\![\, \varphi(x, {}^{*}z)\,]\!] = \mathbb{1} \leftrightarrow (\exists x \in F\downarrow)\, [\![\, \varphi(x, z)\,]\!] = \mathbb{1}; $$

$$ (\forall x \in {}_{*}F)\, [\![\, \varphi(x, {}^{*}z)\,]\!] = \mathbb{1} \leftrightarrow (\forall x \in F\downarrow)\, [\![\, \varphi(x, z)\,]\!] = \mathbb{1} $$

for a nonempty element F in $\mathbb{V}^{(\mathbb{B})}$ and for every z.

Idealization. *Let $X\uparrow$ and Y be classical elements of $\mathbb{V}^{(\mathbb{B})}$, and let $\varphi = \varphi(x, y, z)$ be a formula of* ZFC. *Then*

$$ (\forall^{\mathrm{fin}} A \subset X)\, (\exists y \in {}_{*}Y)\, (\forall x \in A)\, [\![\, \varphi({}^{*}x, y, z)\,]\!] = \mathbb{1} \leftrightarrow (\exists y \in {}_{*}Y)\, (\forall x \in X)\, [\![\, \varphi({}^{*}x, y, z)\,]\!] = \mathbb{1} $$

for an internal element z in $\mathbb{V}^{(\mathbb{B})}$.

Given a filter $\mathcal{F}$ of sets with B-structure, define the *descent monad* $m(\mathcal{F})$ of $\mathcal{F}$ as

$$ m(\mathcal{F}) := \bigcap_{F\in\mathcal{F}} {}_{*}F. $$

Meets of Descent Monads. *Let $\mathcal{E}$ be a set of filters, and let $\mathcal{E}^{\uparrow} := \{\mathcal{F}^{\uparrow} \mid \mathcal{F} \in \mathcal{E}\}$ be its ascent to $\mathbb{V}^{(\mathbb{B})}$. The following are equivalent:*

(1) *the set of cyclic hulls $\mathcal{E}$, i.e., $\mathcal{E}{\uparrow}{\downarrow} := \{\mathcal{F}{\uparrow}{\downarrow} \mid \mathcal{F} \in \mathcal{E}\}$, is bounded above;*

(2) *$\mathcal{E}^{\uparrow}$ is bounded above inside $\mathbb{V}^{(\mathbb{B})}$;*

(3) $\bigcap\{m(\mathcal{F}) \mid \mathcal{F} \in \mathcal{E}\} \neq \varnothing$.

Moreover, in this event

$$m(\sup \mathcal{E}{\uparrow}{\downarrow}) = \bigcap\{m(\mathcal{F}) \mid \mathcal{F} \in \mathcal{E}\}; \qquad \sup \mathcal{E}^{\uparrow} = (\sup \mathcal{E})^{\uparrow}.$$

It is worth noting that, for an infinite set of descent monads, its union and even the cyclic hull of this union is not a descent monad in general. The situation here is the same as for ordinary monads.

Nonstandard Tests for a Proultrafilter. *The following are equivalent:*

(1) *$\mathcal{U}$ is a proultrafilter;*

(2) *$\mathcal{U}$ is an extensional filter with inclusion-minimal descent monad;*

(3) *the representation $\mathcal{U} = (x)^{\downarrow} := \mathrm{fil}\,(\{U{\uparrow}{\downarrow} \mid x \in {}_*A\})$ holds for each point x of the descent monad $m(\mathcal{U})$;*

(4) *$\mathcal{U}$ is an extensional filter whose descent monad is easily caught by a cyclic set; i.e., either $m(\mathcal{U}) \subset {}_*U$ or $m(\mathcal{U}) \subset {}_*(X \setminus U)$ for every $U = U{\uparrow}{\downarrow}$;*

(5) *$\mathcal{U}$ is a cyclic filter satisfying the condition: for every cyclic U, if ${}_*U \cap m(\mathcal{A}) \neq \varnothing$ then $U \in \mathcal{U}$.*

Nonstandard Test for the Mixing of Filters. *Let $(\mathcal{F}_\xi)_{\xi\in\Xi}$ be a family of filters, let $(b_\xi)_{\xi\in\Xi}$ be a partition of unity, and let $\mathcal{F} = \mathrm{fil}_{\xi\in\Xi}(b_\xi \mathcal{F}_\xi^{\uparrow})$ be the mixing of $\mathcal{F}_\xi^{\uparrow}$ by b_ξ. Then*

$$m(\mathcal{F}^{\downarrow}) = \mathrm{fil}_{\xi\in\Xi}(b_\xi m(\mathcal{F}_\xi)).$$

A point y of ${}_*X$ is called *descent-nearstandard* or simply *nearstandard* if there is no danger of misunderstanding whenever ${}^*x \approx y$ for some $x \in X{\downarrow}$; i.e., $(x, y) \in m(\mathcal{U}^{\downarrow})$, with $\mathcal{U}$ the uniformity on X.

Nonstandard Test for Procompactness. *A set $A{\uparrow}{\downarrow}$ is procompact if and only if every point of ${}_*A$ is descent-nearstandard.*

Truth Value on a Proultrafilter. *Let $\varphi = \varphi(x)$ be a formula of* ZFC. *The truth value of φ is constant on the descent monad of every proultrafilter $\mathcal{A}$; i.e.,*

$$(\forall x, y \in m(\mathcal{A}))\, [\![\, \varphi(x) \,]\!] = [\![\, \varphi(y) \,]\!].$$

Let $\varphi = \varphi(x, y, z)$ be a formula of ZFC, and let $\mathcal{F}$ and $\mathcal{G}$ be filters of sets with B-structure.

Rules of Descent Standardization. *The following quantification rules are valid (for internal y and z in $\mathbb{V}^{(\mathbb{B})}$):*

(1) $(\exists x \in m(\mathcal{F}))\, [\![\, \varphi(x, y, z) \,]\!] = \mathbb{1} \leftrightarrow (\forall F \in \mathcal{F})\, (\exists x \in {}^*F)\, [\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$;

(2) $(\forall x \in m(\mathcal{F}))\, [\![\, \varphi(x, y, z) \,]\!] = \mathbb{1} \leftrightarrow (\exists F \in \mathcal{F}^{\uparrow\downarrow})(\forall x \in {}_*F)\, [\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$;

(3) $(\forall x \in m(\mathcal{F}))\, (\exists y \in m(\mathcal{G}))[\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$
$\leftrightarrow (\forall G \in \mathcal{G})\, (\exists F \in \mathcal{F}^{\uparrow\downarrow})\, (\forall x \in {}^*F)\, (\exists y \in {}^*G)\, [\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$;

(4) $(\exists x \in m(\mathcal{F}))\, (\forall y \in m(\mathcal{G}))\, [\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$
$\leftrightarrow (\exists G \in \mathcal{G}^{\uparrow\downarrow})\, (\forall F \in \mathcal{F})\, (\exists x \in {}^*F)\, (\forall y \in {}^*G)\, [\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$.

(5) $(\exists x \in m(\mathcal{F}))\, [\![\, \varphi(x, {}^*y, {}^*z) \,]\!] = \mathbb{1} \leftrightarrow (\forall F \in \mathcal{F})\, (\exists x \in F{\uparrow}{\downarrow})\, [\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$;

(6) $(\forall x \in m(\mathcal{F}))\, [\![\, \varphi(x, {}^*y, {}^*z) \,]\!] = \mathbb{1} \leftrightarrow (\exists F \in \mathcal{F}^{\uparrow\downarrow})\, (\forall x \in F)\, [\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$;

(7) $(\forall x \in m(\mathcal{F}))\, (\exists y \in m(\mathcal{G}))\, [\![\, \varphi(x, y, {}^*z) \,]\!] = \mathbb{1}$
$\leftrightarrow (\forall G \in \mathcal{G})\, (\exists F \in \mathcal{F}^{\uparrow\downarrow})\, (\forall x \in F)\, (\exists y \in G{\uparrow}{\downarrow})\, [\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$;

(8) $(\exists x \in m(\mathcal{F}))\, (\forall y \in m(\mathcal{G}))\, [\![\, \varphi(x, y, {}^*z) \,]\!] = \mathbb{1}$
$\leftrightarrow (\exists G \in \mathcal{G}^{\uparrow\downarrow})\, (\forall F \in \mathcal{F})\, (\exists x \in F{\uparrow}{\downarrow})\, (\forall y \in G)\, [\![\, \varphi(x, y, z) \,]\!] = \mathbb{1}$.

4. THE ESCHER RULES IN VECTOR LATTICES

The fact that E is a vector lattice is a restricted formula, say, $\varphi(E,\mathbb{R})$. Hence, recalling the restricted transfer principle, we come to the equality $[\![\varphi(E^\wedge,\mathbb{R}^\wedge)]\!]=\mathbb{1}$; i.e., $E^\wedge$ is a vector lattice over the ordered field $\mathbb{R}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$.

Let $E^{\wedge\sim}$ be the space of regular $\mathbb{R}^\wedge$-linear functionals from $E^\wedge$ to $\mathcal{R}$. It is easy that $E^{\wedge\sim}:=L^\sim(E^\wedge,\mathcal{R})$ is a K-space, i.e., a Dedekind complete vector lattice, inside $\mathbb{V}^{(\mathbb{B})}$. Since $E^{\wedge\sim}$ is a K-space, the descent $E^{\wedge\sim}{\downarrow}$ of $E^{\wedge\sim}$ is a K-space too.

Turn to the universally complete vector lattice $F:=\mathcal{R}{\downarrow}$. For every operator $T\in L^\sim(E,F)$ the ascent $T{\uparrow}$ is defined by the equality $[\![Tx=T{\uparrow}(x^\wedge)]\!]=\mathbb{1}$ for all $x\in E$. If $\tau\in E^{\wedge\sim}$ then $[\![\tau:E^\wedge\to\mathcal{R}]\!]=\mathbb{1}$; hence, the operator $\tau{\downarrow}:E\to F$ is available. Moreover, $\tau{\downarrow}{\uparrow}=\tau$. On the other hand, $T{\uparrow}{\downarrow}=T$.

For every $T\in L^\sim(E,F)$, the ascent $T{\uparrow}$ is a regular $\mathbb{R}^\wedge$-functional on $E^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$; i.e.,

$$[\![T{\uparrow}\in E^{\wedge\sim}]\!]=\mathbb{1}.$$

The mapping $T\mapsto T{\uparrow}$ is a linear and lattice isomorphism between $L^\sim(E,F)$ and $E^{\wedge\sim}{\downarrow}$.

An operator $S\in L^\sim(E,F)$ is a *fragment* or *component* of $0\le T\in L^\sim(E,F)$ if $S\wedge(T-S)=0$. Say that T is *F-discrete* whenever $[0,T]=[0,I_F]\circ T$; i.e., for every $0\le S\le T$ there is an operator $0\le\alpha\le I_F$ satisfying $S=\alpha\circ T$. Let $L_a^\sim(E,F)$ be the band of $L^\sim(E,F)$ generated by F-discrete operators, and write $L_d^\sim(E,F):=L_a^\sim(E,F)^\perp$. The bands $(E^{\wedge\sim})_a$ and $(E^{\wedge\sim})_d$ are introduced similarly. The elements of $L_d^\sim(E,F)$ are usually referred to as *F-diffuse* operators. The $\mathbb{R}$-discrete or $\mathbb{R}$-diffuse operators are called for the sake of brevity *discrete* or *diffuse* functionals.

Rules of Descending. *Consider $S,T\in L^\sim(E,F)$ and put $\tau:=T{\uparrow}$; $\sigma:=S{\uparrow}$. The following are true:*

(1) $T\ge 0\leftrightarrow[\![\tau\ge 0]\!]=\mathbb{1}$;

(2) (S *is a fragment of* T) $\leftrightarrow$ $[\![\sigma$ *is a fragment of* $\tau]\!]=\mathbb{1}$;

(3) (T *is F-discrete*) $\leftrightarrow$ $[\![\tau$ *is discrete*$]\!]=\mathbb{1}$;

(4) $T\in L_a^\sim(E,F)\leftrightarrow[\![\tau\in(E^{\wedge\sim})_a]\!]=\mathbb{1}$;

(5) $T\in L_d^\sim(E,F)\leftrightarrow[\![\tau\in(E^{\wedge\sim})_d]\!]=\mathbb{1}$.

(6) (T *is a lattice homomorphism*) $\leftrightarrow$ $[\![\tau$ *is a lattice homomorphism*$]\!]=\mathbb{1}$.

Let E stand for a vector lattice and F, for a K-space. A set $\mathcal{P}$ of band projections in $L^\sim(E,F)$ *generates the fragments of* T, $0\le T\in L^\sim(E,F)$, provided that $Tx^+=\sup\{pTx\mid p\in\mathcal{P}\}$ for all $x\in E$. If this happens for all $0\le T\in L^\sim(E,F)$ then $\mathcal{P}$ is a *generating* set.

Put $F:=\mathcal{R}{\downarrow}$ and let p be a band projection in $L^\sim(E,F)$. Then there is a unique element $p{\uparrow}\in\mathbb{V}^{(\mathbb{B})}$ such that $[\![\,p{\uparrow}$ is a band projection in $E^{\wedge\sim}]\!]=\mathbb{1}$ and $(pT){\uparrow}=p{\uparrow}\,T{\uparrow}$ for all $T\in L^\sim(E,F)$.

Rules of Fragmenting. *Consider some set $\mathcal{P}$ of band projections in $L^\sim(E,F)$ and a positive operator $T\in L^\sim(E,F)$. Put $\tau:=T{\uparrow}$ and $\mathcal{P}{\uparrow}:=\{p{\uparrow}\mid p\in\mathcal{P}\}{\uparrow}$. Then*

$$[\![\mathcal{P}{\uparrow}\ \text{ is a set of band projections in }\ E^{\wedge\sim}]\!]=\mathbb{1}$$

and the following are true:

(1) ($\mathcal{P}$ *generates the fragments of* T) $\leftrightarrow$ $[\![\mathcal{P}{\uparrow}$ *generates the fragments of* $\tau]\!]=\mathbb{1}$;

(2) ($\mathcal{P}$ *is a generating set*) $\leftrightarrow$ $[\![\mathcal{P}{\uparrow}$ *is a generating set*$]\!]=\mathbb{1}$.

Given a set A in a K-space, denote by $A^\vee$ the result of adjoining to A suprema of every nonempty finite subset of A. Let $A^\uparrow$ stand for the result of adjoining to A suprema of nonempty increasing nets of elements of A. The symbols $A^{\uparrow\downarrow}$ and $A^{\uparrow\downarrow\uparrow}$ are understood naturally (cp. [9]–[11]).

Put $\mathcal{P}(f):=\{pf\mid p\in\mathcal{P}\}$ and note that E will for a time being stand for a vector lattice over a dense subfield of $\mathbb{R}$ while $\mathcal{P}$ is a set of band projections in $E^\sim$. Let $\mathfrak{E}(f)$ stand for the set of all fragments of f.

Up-Down Theorem. *The following are equivalent:*

(1) $\mathcal{P}(f)^{\vee(\uparrow\downarrow\uparrow)}=\mathfrak{E}(f)$;

(2) $\mathcal{P}$ *generates the fragments of* f;

(3) $(\forall x \in {}^{\circ}E)(\exists p \in \mathcal{P})pf(x) \approx f(x^{+})$;

(4) *a functional g in $[0, f]$ is a fragment of f if and only if*

$$\inf_{p\in\mathcal{P}}(p^{\perp}g(x) + p(f-g)(x)) = 0$$

for every $0 \le x \in E$;

(5) $(\forall g \in {}^{\circ}\mathfrak{E}(f))(\forall x \in {}^{\circ}E_{+})(\exists p \in \mathcal{P})|pf - g|(x) \approx 0$;

(6) $\inf\{|pf-g|(x) \mid p \in \mathcal{P}\} = 0$ *for all fragments $g \in \mathfrak{E}(f)$ and $x \ge 0$;*

(7) *for $x \in E_{+}$ and $g \in \mathcal{E}(f)$, there is an element $p \in \mathcal{P}(f)^{\vee(\uparrow\downarrow\uparrow)}$ satisfying*

$$|pf - g|(x) = 0.$$

Proof. The implications (1) → (2) → (3) are obvious.

(3) → (4): We will work within the *standard entourage*; i.e., we presume that all free variables are standard. Note first that validity of the sought equality for all functionals g and f satisfying $0 \le g \le f$ amounts to existence of $p \in \mathcal{P}$, given a standard $x \ge 0$, such that $p^{\perp}g(x) \approx 0$ and $p(f-g)(x) \approx 0$. (As usual, $p^{\perp}$ is the *complementary band projection* to p.) Thus,

$${}^{\circ}p(g \wedge (f-g))(x) \le {}^{\circ}p(f-g)(x) = 0, \qquad {}^{\circ}p^{\perp}((f-g) \wedge g)(x) \le {}^{\circ}p^{\perp}g(x) = 0,$$

i.e., $g \wedge (f-g) = 0$.

Prove now that, on assuming (3), the sought equality ensues from the conventional criterion for disjointness:

$$\inf\{g(x_1) + (f-g)(x_2) \mid x_1 \ge 0,\ x_2 \ge 0,\ x_1 + x_2 = x\} = 0.$$

Given a standard x, find internal positive x_1 and x_2 such that $x = x_1 + x_2$ and, moreover, $g(x_1) \approx 0$ and $f(x_2) \approx g(x_2)$. By (3), it follows from the Kreĭn–Milman Theorem that the fragment g belongs to the weak closure of $\mathcal{P}(f)$. In particular, there is an element $p \in \mathcal{P}$ satisfying $g(x_1) \approx pf(x_1)$ and $g(x_2) \approx pf(x_2)$. Thus, $p^{\perp}g(x_2) \approx 0$, because $p^{\perp}g \le p^{\perp}f$. Finally, $p^{\perp}g(x) \approx 0$. Hence,

$$p(f-g)(x) = pf(x_2) + pf(x_1) - pg(x) \approx g(x_2) + g(x_1) - pg(x) \approx p^{\perp}g(x) \approx 0.$$

This yields the claim.

(4) → (5): Using the equality $|pf - g|(x) = p^{\perp}g(x) + p(f-g)(x)$, we may find $p \in \mathcal{P}$ so that $p^{\perp}g(x) \approx 0$ and $p(f-g)(x) \approx 0$. This justifies the claim.

The equivalence (5) ↔ (6) is clear. The implications (5) → (7) → (1) are standard. The proof is complete. □

We now turn to principal bands. For positive functionals f and g and for a generating set of band projections $\mathcal{P}$, the following are equivalent:

(1) $g \in \{f\}^{\perp\perp}$;

(2) If x is a *limited* element of E, i.e. $x \in {}^{\mathrm{fin}}E := \{x \in E \mid (\exists \overline{x} \in {}^{\circ}E)|x| \le \overline{x}\}$, then $pg(x) \approx 0$ whenever $pf(x) \approx 0$ for $p \in \mathcal{P}$;

(3) $(\forall x \in E_{+})\,(\forall \varepsilon > 0)(\exists \delta > 0)\,(\forall p \in \mathcal{P})\,pf(x) \le \delta \ \rightarrow\ pg(x) \le \varepsilon$.

With the principal bands available, we may proceed to the principal projections.

Let f and g be positive functionals on E, and let x be a positive element of E. Denote the band projection to $\{f\}^{\perp\perp}$ by b_f.

Principal Projection on a Functional. *The following representations hold:*

(1) $b_f g(x) \risingdotseq \inf{}^{*}\{{}^{\circ}pg(x) \mid p^{\perp}f(x) \approx 0,\ p \in \mathcal{P}\}$,
where $\risingdotseq$ means that the formula is exact, i.e., equality is attained;

(2) $b_f g(x) = \sup_{\varepsilon>0} \inf\{pg(x) \mid p^{\perp}f(x) \le \varepsilon,\ p \in \mathcal{P}\}$;

(3) $b_f g(x) \risingdotseq \inf{}^{*}\{{}^{\circ}g(y) \mid f(x-y) \approx 0,\ 0 \le y \le x\}$;

(4) $(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall p \in \mathcal{P})\,pf(x) < \delta \rightarrow b_f g(x) \le p^{\perp}g(x) + \varepsilon$;

$(\forall\varepsilon > 0)\,(\forall\delta > 0)\,(\exists p \in \mathcal{P})\,pf(x) < \delta \wedge p^{\perp}g(x) \leq b_f g(x) + \varepsilon$;

(5) $(\forall\varepsilon > 0)\,(\exists\delta > 0)\,(\forall 0 \leq y \leq x)\,f(x-y) \leq \delta \rightarrow b_f g(x) \leq g(y) + \varepsilon$;

$(\forall\varepsilon > 0)\,(\forall\delta > 0)\,(\exists 0 \leq y \leq x)\,f(x-y) \leq \delta \wedge g(y) \leq b_f g(x) + \varepsilon$.

Ascending to and descending from the appropriate Boolean valued universe, we implement principal bands in the operator case:

For a set of band projections $\mathcal{P}$ in $L^{\sim}(E,F)$ and $0 \leq S \in L^{\sim}(E,F)$, the following are equivalent:

(1) $\mathcal{P}(S)^{\vee(\uparrow\downarrow\uparrow)} = \mathfrak{E}(S)$;

(2) *$\mathcal{P}$ generates the fragments of S;*

(3) *$T \in [0,S]$ is a fragment of S if and only if*

$$\inf_{p\in\mathcal{P}} (p^{\perp}Tx + p(S-T)x) = 0$$

for all $0 \leq x \in E$;

(4) $(\forall x \in {}^{\circ}E)\,(\exists p \in \mathcal{P}{\uparrow\downarrow})\,pSx \approx Sx^{+}$.

Using the simplest Escher rules and Nelson's algorithm yields the description of the principal band generated by an operator:

For positive operators S and T and a generating set $\mathcal{P}$ of band projections in $L^{\sim}(E,F)$, the following are equivalent:

(1) $T \in \{S\}^{\perp\perp}$;

(2) $(\forall x \in {}^{\mathrm{fin}}E)\,(\forall p \in \mathcal{P})\,(\forall b \in \mathbb{B})\,bpSx \approx 0 \rightarrow bpTx \approx 0$;

(3) $(\forall x \in {}^{\mathrm{fin}}E)\,(\forall b \in \mathbb{B})\,bSx \approx 0 \rightarrow bTx \approx 0$;

(4) $(\forall x \geq 0)\,(\forall\varepsilon \in \mathcal{E})\,(\exists\delta \in \mathcal{E})\,(\forall p \in \mathcal{P})\,(\forall b \in \mathbb{B})\,bpSx \leq \delta \rightarrow bpTx \leq \varepsilon$;

(5) $(\forall x \geq 0)\,(\forall\varepsilon \in \mathcal{E})\,(\exists\delta \in \mathcal{E})\,(\forall b \in \mathbb{B}) bSx \leq \delta \rightarrow bTx \leq \varepsilon$.

Let E be a vector lattice, and let F be a K-space having the filter of order units $\mathcal{E}$ and the base $\mathbb{B}$. Suppose that S and T are positive operators in $L^{\sim}(E,F)$ and R is the band projection of T to the band $\{S\}^{\perp\perp}$.

Theorem of Principal Projection. *For a positive $x \in E$, the following are valid:*

(1) $Rx = \sup_{\varepsilon\in\mathcal{E}} \inf\{bTy + b^{\perp}Sx \mid 0 \leq y \leq x, b \in \mathbb{B},\ bS(x-y) \leq \varepsilon\}$;

(2) $Rx = \sup_{\varepsilon\in\mathcal{E}} \inf\{(bp)^{\perp}Tx \mid bpSx \leq \varepsilon,\ p \in \mathcal{P}, b \in \mathbb{B}\}$,

where $\mathcal{P}$ is a generating set of band projections in F.

In closing, turn to the revisited Farkas Lemma (cp. [8], [12], and [13]). Let X be a Y-seminormed real vector space, with Y a K-space. Given are some dominated polyhedral sublinear operators $P_1,\dots,P_N$ from X to Y and a dominated sublinear operator $P : X \rightarrow Y$.

Polyhedral Lagrange Principle. *The finite value of the constrained problem*

$$P_1(x) \leq u_1, \dots, P_N(x) \leq u_N, \qquad P(x) \rightarrow \inf$$

is the value of the unconstrained problem for an appropriate Lagrangian without any constraint qualification but polyhedrality.

Polyhedrality is omnipresent and so finds applications in inexact data processing (cp. [14]). Let X be a Y-seminormed real space, with Y a K-space. Assume given a dominated polyhedral sublinear operator $P : X \rightarrow Y$, a dominated sublinear operator $Q : X \rightarrow Y$, and $u, v \in Y$. Assume further that $\{P \leq u\} \neq \varnothing$.

Interval Farkas Lemma. *The following are equivalent:*

(1) *for all $b \in \mathbb{B}$, with $\mathbb{B}$ the base of Y, the sublinear operator inequality $bQ\circ\sim(x) \geq -bv$ is a consequence of the polyhedral sublinear operator inequality $bP(x) \leq bu$, i.e.,*

$$\{bP \leq bu\} \subset \{bQ\circ \sim \geq -bv\},$$

with $\sim(x) := -x$ *for all* $x \in X$;

(2) *there are* $A \in \partial(P)$, $B \in \partial(Q)$, *and a positive orthomorphism* $\alpha \in \mathrm{Orth}(m(Y))$ *on the universal completion* $m(Y)$ *of* Y *satisfying*

$$B = \alpha A, \qquad \alpha u \leq v.$$

REFERENCES

1. A. G. Kusraev and S. S. Kutateladze, *Introduction to Boolean Valued Analysis* (Nauka, Moscow, 2005) [in Russian].
2. A. G. Kusraev and S. S. Kutateladze, "Boolean Methods in Positivity," J. Appl. Indust. Math. **2** (1), 81–99 (2008).
3. G. W. Leibniz, "Monadology," In: *Collected Works*, Vol. 1 (Mysl', Moscow, 1982), pp. 143–429 [in Russian].
4. S. S. Kutateladze, "The Mathematical Background of Lomonosov's Contribution," J. Appl. Indust. Math. **5** (2), 155–162 (2011).
5. W. A. J. Luxemburg, "A General Theory of Monads," In: *Applications of Model Theory to Algebra, Analysis and Probability* (Holt, Rinehart and Minston, New York, 1966), pp. 18–86.
6. J. W. Dauben, *The Creation of Nonstandard Analysis. A Personal and Mathematical Odyssey* (Princeton University Press, Princeton, 1995).
7. A. E. Gutman and G. A. Losenkov, "Functional Representation of a Boolean Valued Universe," In: *Nonstandard Analysis and Vector Lattices*, Ed. by S. S. Kutateladze (Kluwer Academic Publ., Dordrecht, 2000), pp. 81–104.
8. S. S. Kutateladze, "Boolean Trends in Linear Inequalities," J. Appl. Indust. Math. **4** (3), 340–348 (2010).
9. B. de Pagter, "The Components of a Positive Operator," Indag. Math. **45** (2), 229–241 (1983).
10. A. G. Kusraev and S. S. Kutateladze, "Nonstandard Methods and Kantorovich Spaces," In: *Nonstandard Analysis and Vector Lattices*, Ed. by S.S. Kutateladze (Kluwer Acad. Publ., Dordrecht, 2000), pp. 1–79.
11. A. G. Kusraev and S. S. Kutateladze, "On the Calculus of Order Bounded Operators," Positivity **9** (3), 327–339 (2005).
12. S. S. Kutateladze, "The Farkas Lemma Revisited," Sibirsk. Mat. Zh. **51** (1), 98–109 (2010) [Siberian Math. J. **51** (1), 78–87 (2010)].
13. S. S. Kutateladze, "The Polyhedral Lagrange Principle," Sibirsk. Mat. Zh. **52** (3), 615–618 (2011) [Siberian Math. J. **52** (3), 484–486 (2011)].
14. M. Fiedler et al., *Linear Optimization Problems with Inexact Data* (Springer, New York, 2006).