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Domination, Discretization, and Scalarization

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Abstract—This is an overview of a few possibilities that are open by model theory in applied mathematics. The most attention is paid to the present state and frontiers of the Cauchy method of majorants, approximation of operator equations with finite-dimensional analogs, and the Lagrange multiplier principle in multiobjective decision making.

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The union of functional analysis and applied mathematics celebrates its sixtieth anniversary this year.[1] The present article focuses on the trends of interaction between model theory and the methods of domination, discretization, and scalarization.

1. PURE AND APPLIED MATHEMATICS

Provable counting is the art of calculus which is mathematics in modern parlance. Mathematics exists as a science more than two and a half millennia, and we can never mixed it with history or chemistry. In this respect our views of what is mathematics are independent of time.

The objects of mathematics are the quantitative forms of human reasoning. Mathematics functions as the science of convincing calculations. Once-demonstrated, the facts of mathematics will never vanish. Of course, mathematics renews itself constantly, while the stock increases of mathematical notions and constructions and the understanding changes of the rigor and technologies of proof and demonstration. The frontier we draw between pure and applied mathematics is also time-dependent.

Francis Bacon wrote in his celebrated book "The Advancement of Learning":[2]

> The Mathematics are either pure or mixed. To the Pure Mathematics are those sciences belonging which handle quantity determinate, merely severed from any axioms of natural philosophy; and these are two, Geometry and Arithmetic; the one handling quantity continued, and the other dissevered. Mixed hath for subject some axioms or parts of natural philosophy, and considereth quantity determined, as it is auxiliary and incident unto them. For many parts of nature can neither be invented with sufficient subtlety, nor demonstrated with sufficient perspicuity, nor accommodated unto use with sufficient dexterity, without the aid and intervening of the mathematics; of which sort are perspective, music, astronomy, cosmography, architecture, enginery, and divers others.
>
> In the Mathematics I can report no deficience, except it be that men do not sufficiently understand the excellent use of the Pure Mathematics, in that they do remedy and cure many defects in the wit and faculties intellectual. For if the wit be too dull, they sharpen it; if too wandering, they fix it; if too inherent in the sense, they abstract it. So that as tennis is a game of no use in itself, but of great use in respect it maketh a quick eye and a body ready to put itself into all postures; so in the Mathematics, that use which is collateral and intervenient is no less worthy than that which is principal and intended. . . . And as for the Mixed Mathematics, I may only make this prediction, that there cannot fail to be more kinds of them, as nature grows further disclosed.

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[1] Cp. [1, 2].

[2] The complete title was as follows: "The tvvoo bookes of Francis Bacon, of the proficience and aduancement of learning, diuine and humane. To the King. At London: Printed for Henrie Tomes, 1605."

After the lapse of 150 years Leonhard Euler used the words "pure mathematics" in the title of one of his papers *Specimen de usu observationum in mathesi pura* in 1761. It was practically at the same time that the term "pure mathematics" had appeared in the eldest *Encyclopaedia Britannica*. In the nineteenth century "mixed" mathematics became to be referred to as "applied."

The famous *Journal de Mathématiques Pures et Appliquées* was founded by Joseph Liouville in 1836 and *The Quarterly Journal of Pure and Applied Mathematics* started publication in 1857.

The intellectual challenge, beauty, and intrinsic logic of the topics under study are the impetus of many comprehensive and deep studies in mathematics which are customarily qualified as pure. Knowledge of the available mathematical methods and the understanding of their power underlie the applications of mathematics in other sciences. Any application of mathematics is impossible without creating some metaphors, models of the phenomena and processes under examination. Modeling is a special independent sphere of intellectual activities which is out of mathematics.

Application of mathematics resides beyond mathematics in much the same way as maladies exist in nature rather than inside medicine. Applied mathematics acts as an apothecary mixing drugs for battling illnesses.

The art and craft of mathematical techniques for the problems of other sciences are the content of applied mathematics.

2. LINEAR INEQUALITIES AND KANTOROVICH SPACES

Classical mechanics in the broadest sense of the words was the traditional sphere of applications of mathematics in the nineteenth century. This historical tradition is reflected in the numerous mechanics and mathematics departments of the best universities of Russia.

The beginning of the twentieth century was marked with a broad enlargement of the sphere of applications of mathematics. Quantum mechanics appeared, requesting for new mathematical tools. The theory of operators in Hilbert spaces and distribution theory were oriented primarily to adapting the heuristic methods of the new physics. At the same time the social phenomena became the object of the nonverbal research requiring the invention of especial mathematical methods. The demand for the statistical treatment of various data grew rapidly. Founding new industries as well as introducing of promising technologies and new materials, brought about the necessity of elaboration of the technique of calculations. The rapid progress of applied mathematics was facilitated by the automation and mechanization of accounting and standard calculations.

In the 1930s applied mathematics rapidly approached functional analysis. Of profound importance in this trend was the research of John von Neumann in the mathematical foundations of quantum mechanics and game theory as a tool for economic studies. Leonid Kantorovich was a pioneer and generator of new synthetic ideas in Russia.

Kantorovich considered as his principal mathematical achievement in functional analysis the introduction of the special class of Dedekind complete vector lattices which are referred to as *K-spaces* or *Kantorovich spaces* in the Russian literature.[3]

It was already in his first paper of 1935 in this new area of mathematics that Kantorovich wrote:[4]

> In this note, I define a new type of space that I call a semiordered linear space. The introduction of this kind of spaces allows us to study linear operations of one abstract class (those with values in these spaces) in the same way as linear functionals.

So was firstly formulated the major methodological rule that is now referred to as *Kantorovich's heuristic principle*. It is worth observing that Kantorovich included Axiom I_6 of relative order completeness into his definition of semiordered linear space. Kantorovich demonstrated the role of K-spaces by the example of the Hahn–Banach theorem. He proved that this central principle of functional analysis admits the replacement of reals with elements of an arbitrary K-space while substituting linear and sublinear operators with range in this space for linear and sublinear functionals. These observations

[3] He wrote about "my spaces" in his working notebooks.

[4] Cp. [3] and [9, pp. 49–50].

laid grounds for the universal heuristics based on his intuitive belief that the members of an abstract Kantorovich space are a sort of generalized numbers.

Kantorovich spaces have provided the natural framework for developing the theory of linear inequalities which was a practically uncharted area of research those days. The concept of inequality is obviously relevant to approximate calculations where we are always interested in various estimates of the accuracy of results. Another challenging source of interest in linear inequalities was the stock of problems of economics. The language of partial comparison is rather natural in dealing with what is reasonable and optimal in human behavior when means and opportunities are scarce. Finally, the concept of linear inequality is inseparable with the key idea of a convex set. Functional analysis implies the existence of nontrivial continuous linear functional over the space under consideration, while the presence of a functional of this type amounts to the existence of nonempty proper open convex subset of the ambient space. Moreover, each convex set is generically the set of solutions of an appropriate system of simultaneous linear inequalities.

Linear programming is a technique of maximizing a linear functional over the positive solutions of a system of linear inequalities. It is no wonder that the discovery of linear programming was immediate after the foundation of the theory of Kantorovich spaces.

At the end of the 1940s Kantorovich formulated and explicated the thesis of interdependence between functional analysis and applied mathematics:[5]

> There is now a tradition of viewing functional analysis as a purely theoretical discipline far removed from direct applications, a discipline which cannot deal with practical questions. This article[6] is an attempt to break with this tradition, at least to a certain extent, and to reveal the relationship between functional analysis and the questions of applied mathematics. . . .

He distinguished the three techniques: the Cauchy method of majorants also called *domination*, the method of finite-dimensional approximations, and the Lagrange method for the new optimization problems motivated by economics.

Kantorovich based his study of the Banach space versions of the Newton method on domination in general ordered vector spaces.

Approximation of infinite-dimensional spaces and operators by their finite-dimensional analogs, which is *discretization*, must be considered alongside the marvelous universal understanding of computational mathematics as the science of finite approximations to general (not necessarily metrizable) compacta.[7]

The novelty of the extremal problems arising in social sciences is connected with the presence of multidimensional contradictory utility functions. This raises the major problem of agreeing conflicting aims. The corresponding techniques may be viewed as an instance of *scalarization* of vector-valued targets.

3. DOMINATION

Let X and Y be real vector spaces lattice-normed with Kantorovich spaces E and F. In other words, given are some lattice-norms $|\cdot|_X$ and $|\cdot|_Y$. Assume further that T is a linear operator from X to Y and S is a positive operator from X into Y satisfying

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ |\cdot|_X \downarrow & & \downarrow |\cdot|_Y \\ E & \xrightarrow[S]{} & F \end{array}$$

[5] Cp. [2] and [7]. The excerpt is taken from [10, p. 171].

[6] Implied is the article [2] which appeared in the citation of the Stalin Prize of Second Degree with prize money of 100,000 rubles which was awarded to Kantorovich in 1948.

[7] This revolutionary definition was given in the joint talk [5] submitted by S. L. Sobolev, L. A. Lyusternik, and L. V. Kantorovich at the Third All-Union Mathematical Congress in 1956. Also see [6, pp. 443–444].

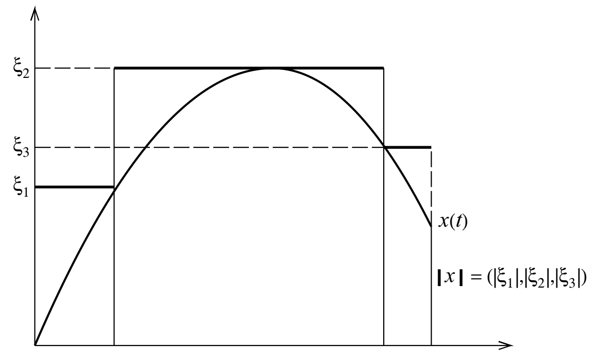

Fig. 1.

Moreover, in case

$$|Tx|_Y \leq S|x|_X \quad (x \in X),$$

we call S the *dominant* or *majorant* of T. If the set of all dominants of T has the least element, then the latter is called the *abstract norm* or *least dominant* of T and denoted by $|T|$. Hence, the least dominant $|T|$ is the least positive operator from E to F such that

$$|Tx| \leq |T|\big(|x|\big) \quad (x \in X).$$

Kantorovich wrote about this matter as follows:[8]

> The abstract norm enables us to estimate an element much sharper that a single number, a real norm. We can thus acquire more precise (and more broad) boundaries of the range of application of successive approximations. For instance, as a norm of a continuous function we can take the set of the suprema of its modulus in a few partial intervals.... This allows us to estimate the convergence domain of successive approximations for integral equations. In the case of an infinite system of equations we know that each solution is as a sequence and we can take as the norm of a sequence not only a sole number but also finitely many numbers; for instance, the absolute values of the first entries and the estimation of the remainder:
>
> $$|(\xi_1, \xi_2, \dots)| = (|\xi_1|, |\xi_2|, \dots, |\xi_{N-1}|, \sup_{k \geq N} |\xi_k|) \in \mathbb{R}^N.$$
>
> This enables us to specify the conditions of applicability of successive approximations for infinite simultaneous equations. Also, this approach allows us to obtain approximate (surplus or deficient) solutions of the problems under consideration with simultaneous error estimation. I believe that the use of members of semiordered linear spaces instead of reals in various estimations can lead to essential improvement of the latter.

It is worth recalling that Kantorovich carried out his classical studies of the Newton method by using the most general domination technique.[9]

These days the development of domination proceeds within the frameworks of Boolean valued analysis.[10] The modern technique of mathematical modeling opened an opportunity to demonstrate that the principal properties of lattice normed spaces represent the Boolean valued interpretations of the relevant properties of classical normed spaces. The most important interrelations here are as follows: Each Banach space inside a Boolean valued model becomes a universally complete Banach–Kantorovich space in result of the external deciphering of constituents. Moreover, each lattice normed space may be realized as a dense subspace of some Banach space in an appropriate Boolean valued model. Finally, a Banach space X results from some Banach space inside a Boolean valued model by a special machinery of bounded descent if and only if X admits a complete Boolean algebra of norm-one projections which enjoys the cyclicity property. The latter amounts to the fact that X is a Banach–Kantorovich space and X is furnished with a mixed norm.[11]

[8] Cp. [1].
[9] Cp. [4].
[10] Cp. [12].

4. DISCRETIZATION

Summarizing his research into the general theory of approximation methods, Kantorovich wrote:[12]

> There are many distinct methods for various classes of problems and equations, and constructing and studying them in each particular case presents considerable difficulties. Therefore, the idea arose of evolving a general theory that would make it possible to construct and study them with a single source. This theory was based on the idea of the connection between the given space, in which the equation to be studied is specified, and a more simple one into which the initial space is mapped. On the basis of studying the "approximate equation" in the simpler space the possibility of constructing and studying approximate methods in the initial space was discovered. . . .
>
> It seems to me that the main idea of this theory is of a general character and reflects the general gnoseological principle for studying complex systems. It was, of course, used earlier, and it is also used in systems analysis, but it does not have a rigorous mathematical apparatus. The principle consists simply in the fact that to a given large complex system in some space a simpler, smaller dimensional model in this or a simpler space is associated by means of one-to-one or one-to-many correspondence. The study of this simplified model turns out, naturally, to be simpler and more practicable. This method, of course, presents definite requirements on the quality of the approximating system.

The classical scheme of discretization as suggested by Kantorovich for the analysis of the equation $Tx = y$, with $T : X \to Y$ a bounded linear operator between some Banach spaces X and Y, consists in choosing finite-dimensional approximating subspaces X_N and Y_N and the corresponding embeddings $\imath_N$ and $\jmath_N$:

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \imath_N \downarrow & & \downarrow \jmath_N \\ X_N & \xrightarrow[T_N]{} & Y_N \end{array}$$

In this event, the equation

$$T_N x_N = y_N$$

is viewed as a finite-dimensional approximation to the original problem.

Boolean valued analysis enables us to expand the range of applicability of Banach–Kantorovich spaces and more general modules for studying extensional equations. Many promising possibilities are open by the new method of *hyperapproximation* which rests on the ideas of infinitesimal analysis. The classical discretization approximates an infinite-dimensional space with the aid of finite-dimensional subspaces. Arguing within nonstandard set theory we may approximate an infinite-dimensional vector space with external finite-dimensional spaces. Undoubtedly, the dimensions of these hyperapproximations are given as actually infinite numbers.

The tentative scheme of hyperapproximation is reflected by the following diagram:

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ \varphi_E \downarrow & & \downarrow \varphi_F \\ E^{\#} & \xrightarrow[T^{\#}]{} & F^{\#} \end{array}$$

Here E and F are normed vector space over the same scalars; T is a bounded linear operator from E to F; and $^{\#}$ symbolizes the taking of the relevant nonstandard hull.

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[11] The modern theory of dominated operators is thoroughly set forth in the book [11] by A. G.Kusraev.

[12] Cp. [8] and [9, pp. 49–50].

Let E be an internal vector space over ${}^*\mathbb{F}$, where $\mathbb{F}$ is the *basic field* of scalars; i.e., the *reals* $\mathbb{R}$ or *complexes* $\mathbb{C}$, while * is the symbol of the Robinsonian standardization. Hence, we are given the two internal operations

$$+ : E \times E \to E, \qquad \cdot : {}^*\mathbb{F} \times E \to E$$

satisfying the usual axioms of a vector space. Since $\mathbb{F} \subset {}^*\mathbb{F}$, the internal vector space E is a vector space over $\mathbb{F}$ as well. In other words, E is an external vector space which is not a normed nor Hilbert space externally even if E is endowed with either structure as an internal space. However, with each normed or pre-Hilbert space we can associate some external Banach or Hilbert space.

Let $(E, \|\cdot\|)$ be an internal normed space over ${}^*\mathbb{F}$. As usual, $x \in E$ is a *limited* element provided that $\|x\|$ is a limited real (whose modulus has a standard upper bound by definition). If $\|x\|$ is an infinitesimal (=infinitely small real) then x is also referred to as an *infinitesimal*. Denote the external sets of limited elements and infinitesimals of E by $\operatorname{ltd}(E)$ and $\mu(E)$. The set $\mu(E)$ is the *monad* of the origin in E. Clearly, $\operatorname{ltd}(E)$ is an external vector space over $\mathbb{F}$, and $\mu(E)$ is a subspace of $\operatorname{ltd}(E)$. Denote the factor-space $\operatorname{ltd}(E)/\mu(E)$ by $E^\#$. The space $E^\#$ is endowed with the natural norm by the formula

$$\|\varphi x\| := \|x^\#\| := \operatorname{st}(\|x\|) \in \mathbb{F} \quad (x \in \operatorname{ltd}(E)).$$

Here

$$\varphi := \varphi_E := (\cdot)^\# : \operatorname{ltd}(E) \to E^\#$$

is the canonical homomorphism, and st stands for the taking of the standard part of a limited real. In this event $(E^\#, \|\cdot\|)$ becomes an external normed space that is called the *nonstandard hull* of E. If $(E, \|\cdot\|)$ is a standard space then the nonstandard hull of E is by definition the space $({}^*E)^\#$ corresponding to the Robinsonian standardization ${}^*E$.

If $x \in E$ then $\phi({}^*x) = ({}^*x)^\#$ belongs to $({}^*E)^\#$. Moreover, $\|x\| = \|({}^*x)^\#\|$. Therefore, the mapping $x \mapsto ({}^*x)^\#$ is an isometric embedding of E in $({}^*E)^\#$. It is customary to presume that $E \subset ({}^*E)^\#$.

Suppose now that E and F are internal normed spaces and $T : E \to F$ is an internal bounded linear operator. The set of reals

$$c(T) := \{C \in {}^*\mathbb{R} : (\forall x \in E)\, \|Tx\| \le C\|x\|\}$$

is internal and bounded. Recall that $\|T\| := \inf c(T)$.

If the norm $\|T\|$ of T is limited then the classical normative inequality $\|Tx\| \le \|T\|\,\|x\|$ valid for all $x \in E$, implies

$$T(\operatorname{ltd}(E)) \subset \operatorname{ltd}(F), \qquad T(\mu(E)) \subset \mu(F).$$

Consequently, we may soundly define the descent of T to the factor space $E^\#$ as the external operator $T^\# : E^\# \to F^\#$, acting by the rule

$$T^\# \varphi_E x := \varphi_F T x \quad (x \in E).$$

The operator $T^\#$ is linear (with respect to the members of $\mathbb{F}$) and bounded; moreover, $\|T^\#\| = \operatorname{st}(\|T\|)$. The operator $T^\#$ is called the *nonstandard hull* of T. It is worth noting that $E^\#$ is automatically a Banach space for each internal (possible, incomplete) normed space E. If the internal dimension of an internal normed space E is finite then E is referred to as a *hyperfinite-dimensional* space. To each normed vector space E there is a hyperfinite-dimensional subspace $F \subset {}^*E$ containing all standard members of the internal space ${}^*E$.

Infinitesimal methods also provide new schemes for hyperapproximation of general compact spaces. As an approximation to a compact space we may take an arbitrary internal subset containing all standard elements of the space under approximation.[13]

[13] Cp. [17].

5. SCALARIZATION

Scalarization in the most general sense means reduction to numbers. Since each number is a measure of quantity, the idea of scalarization is clearly of a universal importance to mathematics. The deep roots of scalarization are revealed by the Boolean valued validation of the Kantorovich heuristic principle. We will dwell upon the aspects of scalarization most important in applications and connected with the problems of multicriteria optimization.[14]

Kantorovich observed as far back as in 1948 as follows:[15]

> Many mathematical and practical problems lead to the necessity of finding "special" extrema. On the one hand, those are boundary extrema when some extremal value is attained at the boundary of the domain of definition of an argument. On the other hand, this is the case when the functional to be optimized is not differential. Many problems of these sorts are encountered in mathematics and its applications, whereas the general methods turn out ineffective in regard to the problems.

The main particularity of the extremal problems of economics consists in the presence of numerous conflicting ends and interests which are to be harmonized. In fact, we encounter the instances of multicriteria optimization whose characteristic feature is a vector-valued target. Seeking for an optimal solution in these circumstances, we must take into account various contradictory preferences which combine into a sole compound aim. Further more, it is impossible as a rule to distinguish some particular scalar target and ignore the rest of the targets without distorting the original statement of the problem under study. This circumstance involves the specific difficulties that are untypical in the scalar case: we must specify what we should call a solution of a vector program and we must agree upon the method of conforming versatile ends provided that some agreement is possible in principle. Therefore, it is actual to seek for the reasonable concepts of optimality in multiobjective decision making. Among these we distinguish the concepts of ideal and generalized optimum alongside Pareto-optimum as well as approximate and infinitesimal optimum.

Assume that X is a vector space, E is an ordered vector space,

$$f : X \to E^{\bullet} := E \cup +\infty$$

is a convex operator, and $C \subset X$ is a convex set. A *vector program* we call a pair (C, f) which is written in symbols as follows:

$$x \in C, \ \ f(x) \to \inf.$$

A vector program is often referred to as a *multiobjective* or *multicriteria problem*. The operator f is the *target* of (C, f), and C is the *constraint* of (C, f). The members $x \in C$ are *feasible elements* or, rarely, *plans* of (C, f).

The above record of a vector program reflects the fact that under consideration is the following *extremal problem*: Find the least upper bound of the values of f at the members of C. In case $C = X$, we speak about an unconditional or unconstrained problem.

The constraint of an extremal problem may be of a compound nature possibly including equalities and inequalities. Let $g : X \to F^{\bullet}$ be a convex operator and Λ be a linear operator from X to Y, and $y \in Y$, with Y a vector space and F a preordered vector space. If the constraints C_1 and C_2 have the form

$$C_1 := \{x \in C : \ g(x) \le 0\}, \qquad C_2 := \{x \in X : \ g(x) \le 0, \ \Lambda x = y\},$$

then we will rephrase (C_1, f) and (C_2, f) as (C, g, f) and (Λ, g, f) or even more impressively as

$$x \in C, \qquad g(x) \le 0, \qquad f(x) \to \inf;$$
$$\Lambda x = y, \qquad g(x) \le 0, \qquad f(x) \to \inf.$$

The element $e := \inf_{x \in C} f(x)$ (if existent) is called the *value* of (C, f). A feasible element x_0 is an *ideal oprimum* or *solution* provided that $e = f(x_0)$. Hence, x_0 is an ideal optimum if and only if $f(x_0)$ is a least element of the image $f(C)$; i.e., $x_0 \in C$ and $f(C) \subset f(x_0) + E^{+}$.

[14] More details are in [16].
[15] Cp. [1].

It is immediate by definition that x_0 is a solution of the unconditional problem $f(x) \to \inf$ if and only if the zero operator belongs to the subdifferential $\partial f(x_0)$. In other words,

$$f(x_0) = \inf_{x \in X} f(x) \quad \leftrightarrow \quad 0 \in \partial f(x_0).$$

Variational analysis distinguishes between *local* and *global* optima. This subtlety is usually irrelevant for the problems of minimization of convex operators which will be addressed further.

Let $x_0 \in C$ be a local ideal optimum of (C, f) in the following rather weak sense: There is an absorbing set $U \subset X$ such that

$$f(x_0) = \inf\{f(x) : x \in C \cap (x_0 + U)\}.$$

Then $f(x_0) = \inf\{f(x) : x \in C\}$.

It is an easy matter to see from rather elementary examples that this is a rare event to observe an ideal optimum in a vector program. This drives us to try and introduce some concepts of optimality that are reasonable for particular classes of problems. Among these is listed *approximate optiomality* useful already in the scalar situation; i.e., in the problems with a numeric target.

Fix a positive element $\varepsilon \in E$. A feasible point x_0 is an *ε-solution* or *ε-optimum* if (C, f) provided that $f(x_0) \leq e + \varepsilon$, with e the value of (C, f). Therefore, x_0 is an ε-solution of (C, f) if and only if $x_0 \in C$ and $f(x_0) - \varepsilon$ is a lower bound for the image $f(C)$ or, which is the same,

$$f(C) + \varepsilon \subset f(x_0) + E^+.$$

Obviously, x_0 is an ε-solution of the unconditional program $f(x) \to \inf$ if and only if zero belongs to $\partial_\varepsilon f(x_0)$; i.e.,

$$f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon \leftrightarrow 0 \in \partial_\varepsilon f(x_0).$$

Here we see the *ε-subdifferential* $\partial_\varepsilon f(x_0)$. Recall that a member l of the latter is a linear operator from X to E satisfying

$$(\forall x \in X) l(x - x_0) \leq f(x) - f(x_0) + \varepsilon.$$

A *generalized ε-solution* of (C, f) is a set $\mathfrak{A} \subset C$ provided that $\inf_{x \in \mathfrak{A}} f(x) \leq e + \varepsilon$, with e the value of (C, f). If $\varepsilon = 0$ then we speak about a *generalized solution*. Some generalized ε-solution is always available (for instance, $\mathfrak{A} = C$), but we are interested in a more reasonable instances. An inclusion-minimal generalized ε-solution is an ideal ε-optimum for $\mathfrak{A} = \{x_0\}$. It is curious that each generalized ε-solution is a ε-solution of a relevant vector problem of convex programming.

The above concepts of optimality are connected with the greatest lower bound of the target over the set of feasible elements; i.e., with the *value* of the program under study. The concept of *minimal element* leads to a principally different concept of optimality.

It will be useful for us to assume now that E is a preordered vector space; i.e., the positive cone of E is not necessarily *salient*. In other words, the subspace $E_0 := E^+ \cap (-E^+)$, may differ from the origin in general. Given $u \in E$, put

$$[u] := \{v \in E : u \leq v, \ v \leq u\}.$$

The record $u \sim v$ means that $[u] = [v]$.

A feasible point x_0 is called *ε-optimal in the sense of Pareto* or *ε-Pareto-optimal* for (C, f) provided that $f(x_0)$ is a minimal element of the set $U + \varepsilon$, with $U := f(C)$; i.e.,

$$(f(x_0) - E^+) \cap (f(C) + \varepsilon) = [f(x_0)].$$

In more detail, the ε-Pareto optimality of x_0 means that $x_0 \in C$ and for all $x \in C$ from $f(x_0) \geq f(x) + \varepsilon$ it follows that $f(x_0) \sim f(x) + \varepsilon$. If $\varepsilon = 0$ then we speak about Pareto optimality, omitting any indication of ε.

Study of Pareto optimality proceed often by *scalarization*; i.e., reduction of the original vector program to a scalar extremal problem with a sole numerical target. There are a few approaches to scalarization. We will discuss just one of the possibilities.

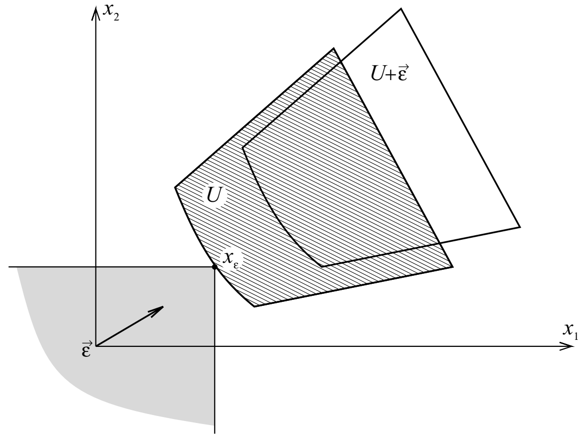

Fig. 2.

Assume that the preorder $\leq$ on E is given by the formula:

$$u \leq v \leftrightarrow (\forall l \in \partial q)\, lu \leq lv,$$

with $q : E \to \mathbb{R}$ a sublinear functional. In other words, the cone E^+ has the form

$$E^+ := \{u \in E : (\forall l \in \partial q)\, lu \geq 0\}.$$

A feasible point x_0 is then an ε-Pareto-optimal for (C, f) if and only if for all $x \in C$ we have $f(x_0) \sim f(x) + \varepsilon$ or there is $l \in \partial q$ satisfying $lf(x_0) < l(f(x) + \varepsilon)$. In particular,

$$\inf_{x \in C} q(f(x) - f(x_0) + \varepsilon) \geq 0$$

for an ε-Pareto-optimal point $x_0 \in C$. The converse fails in general since the above inequality amounts to the weaker concept of optimality: $x_0 \in C$ is *weakly ε-Pareto-optimal* provided that to each $x \in C$ there is $l \in \partial q$ satisfying $l(f(x) - f(x_0) + \varepsilon) \geq 0$; i.e., the system of strict inequalities

$$lf(x_0) < l(f(x) + \varepsilon) \qquad (l \in \partial q)$$

is inconsistent for any $x \in C$. Clearly, the weak ε-Pareto-optimality of x_0 may be rephrased as

$$q(f(x) - f(x_0) + \varepsilon) \geq 0 \quad \text{for all} \quad x \in C,$$

and this concept is nontrivial only if $0 \notin \partial q$.

The role of ε-subdifferentials is revealed in particular by the fact that an ε-solution with a sufficiently small ε may be viewed as a candidate for a "practical optimum," a practically reasonable solution of the original problem. The calculus of ε-subdifferentials is a formal apparatus for calculating the error bounds for a solution of an extremal problem. The relevant technique is now rather perfect and we may even call it exquisite and subtle. At the same time, the corresponding exact formulas are rather bulky and do not agree fully with the practical optimization technique that rests on the heuristic rules for "neglecting infinitely small errors." Available is an adequate apparatus of infinitesimal subdifferentials,[16] free of these shortcomings, which bases on the modern opportunities open up by nonstandard set theory.

Assume that E has some downward-filtered set $\mathcal{E}$ of strictly positive elements. Assume further X, E, and $\mathcal{E}$ are standard objects. Take a standard convex operator $f : X \to E^\bullet$ and a standard convex set $C \subset X$. Recall that the record $e_1 \approx e_2$ means the validity of the inequality $-\varepsilon \leq e_1 - e_2 \leq \varepsilon$ for all standard $\varepsilon \in \mathcal{E}$.

Suppose that the value $e := \inf_{x \in C} f(x)$ of (C, f) is limited. A feasible point x_0 is an *infinitesimal solution* of (C, f) provided that $f(x_0) \approx e$; i.e.,

$$f(x_0) \leq f(x) + \varepsilon \quad \text{for all} \quad x \in C \quad \text{and all standard} \quad \varepsilon \in \mathcal{E}.$$

[16] Cp. [15].

A point $x_0 \in X$ is an infinitesimal solution of the unconditional program $f(x) \to \inf$ if and only if $0 \in Df(x_0)$, where $Df(x_0)$ is the external union of the corresponding ε-subdifferentials over all infinitesimal ε.

The ideas of scalarization and convex ε-programming, formulated already at the end of the 1970s,[17] turn out rather relevant.[18]

6. VISTAS OF THE FUTURE

Adaptation of the modern ideas of model theory to functional analysis projects among the most important directions of developing the synthetic methods of pure and applied mathematics. This approach yields new models of numbers, spaces, and types of equations. The content expands of all available theorems and algorithms. The whole methodology of mathematical research is enriched and renewed, opening up absolutely fantastic opportunities. We can now use actual infinities and infinitesimals, transform matrices into numbers, spaces into straight lines, and noncompact spaces into compact spaces, yet having still uncharted vast territories of new knowledge.

Quite a long time had passed until the classical functional analysis occupied its present position of the language of continuous mathematics. Now the time has come of the new powerful technologies of model theory in mathematical analysis. Not all theoretical and applied mathematicians have already gained the importance of modern tools and learned how to use them. However, there is no backward traffic in science, and the modern methods are doomed to reside in the realm of mathematics for ever and in a short time they will become as elementary and omnipresent in calculuses and calculations as Banach spaces and linear operators.

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[17] Cp. [13], [14].
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