

The article focuses on subdifferential calculus. A discussion of sublinear operators is followed by convex operators and finally by general nonlinear operators and applications to extremal problems.

INTRODUCTION

Convex analysis as an independent mathematical discipline which crystallized only some two decades ago, in connection with the rapid penetration of mathematical methods into social sciences and primarily into economics. Since the early 1970s, we can distinguish between two relatively independent lines of research in convex analysis. The first constitutes the so-called local convex analysis (or convex analysis in the proper sense), which studies individual convex objects and their conjugates. The key concepts in this field are the subdifferential, the conjugate function, the polar, the dual model, etc. The second line of research constitutes the so-called global convex analysis which focuses on classes of convex objects and on the duality properties of these classes. The key concepts in this field are cones and spaces of convex functions and sets, Choquet order, Minkowski duality, geometric inequalities, etc.

The subject of this survey is local convex analysis. This choice was determined, first, by the rapid penetration of subdifferentials and Young–Fenchel transforms into various branches of mathematics and mathematical applications and, second, by the recent publication of a wealth of fruitful ideas in the sphere of local convex analysis.

The underlying theme of this article is subdifferential calculus, since it is in this area that, in our opinion, most of the interesting and important developments take place. The discussion is structured according to the following scheme: sublinear operators \rightarrow convex operators \rightarrow general nonlinear operators \rightarrow some applications to the theory of extremal problems.

CHAPTER I

SUBLINEAR OPERATORS

1.1. General Position of Cones

1.1.1. The cones K_1 and K_2 in a topological vector space (t.v.s.) X are in (topological) general position if there is a completable subspace $X_0 \subset X$ such that the set

$$(V \cap K_1 - V \cap K_2) \cap (V \cap K_2 - V \cap K_1)$$

is a neighborhood of 0 in X_0 for any neighborhood V of 0 in X . If $X_0 = X$, then we also say that K_1 and K_2 form an unflattened pair of cones. If these cones coincide, then the cone $K = K_1 = K_2$ is called unflattened, and the ordered t.v.s. (X, K) is called locally decomposable.

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Let X_1 and X_2 be subspaces of X , and M a subset in X . Denote by $\text{int}_{X_1} M$ the interior of M relative to X_1 , and by $X_1 \oplus X_2$ the subspace $X_1 + X_2$ with a topology in which the neighborhood filter at 0 is generated by sets of the form $V_1 + V_2$, with V_i going over the bases of the neighborhood filter at 0 in X_i .

1.1.2. Each of the following conditions is sufficient for the cones K_1 and K_2 to be in general position:

- (1) $\text{int}_{X_0} K_1 \cap K_2 \neq \emptyset$, where $X_0 = K_1 - K_2$ is completable;
- (2) $\text{int}_{X_1} K_1 \cap \text{int}_{X_2} K_2 \neq \emptyset$, where $X_i = K_i - K_i$, $i = 1, 2$, and $X_1 \oplus X_2$ is a completable subspace;
- (3) $K_1 - K_2$ is a finite-dimensional subspace and $\text{ri} K_1 \cap \text{ri} K_2 \neq \emptyset$. Here the symbol ri has the usual meaning of relative "algebraic" interior.

As we see, general position is realizable because of the existence of an interior point. However, this is not the only and far from the most typical reason for general position of cones. Let us now list some conditions of a different type. This will require the following concepts.

1.1.3. A locally convex space (l.c.s.) X is called hypercomplete if the space $\mathcal{F}(X)$ of all the closed convex subsets X containing 0 is complete relative to the Hausdorff uniformity (see [110, 173]). The base of the neighborhood filter in the Hausdorff uniformity consists of sets of the form

$$\{(A, B) \in \mathcal{F}(X) \times \mathcal{F}(X) : A \subset B + U, B \subset A + U\},$$

where U is an arbitrary neighborhood of 0 in X .

The cone K in a metrizable t.v.s. is called monotone complete if any Cauchy sequence in K increasing the sense of the cone K converges to some element in K .

1.1.4. THEOREM. Let K_1 and K_2 be cones in a t.v.s., $X_0 = K_1 - K_2$ a completable subspace in X . Each of the following conditions is sufficient for K_1 and K_2 to be in general position:

- (1) K_1 and K_2 are monotone complete, and X_0 is metrizable and of second category in itself;
- (2) K_1 and K_2 are analytic sets (i.e., continuous images of Polish spaces), and X_0 is of second category in itself;
- (3) K_1 and K_2 are closed, and X_0 is a hypercomplete barrelled l.c.s.;
- (4) X_0 is a bornological l.c.s. and for any bounded set $B_0 \subset X_0$ there is a bounded set $B \subset X$ such that $B_0 \subset B \cap K_1 - B \cap K_2$;
- (5) X_0 is a bornological l.c.s. and for any sequence (x_n) , $x_n \rightarrow 0$ there are sequences (k_n^1) in K_1 and (k_n^2) in K_2 such that $k_n^1 \rightarrow 0$, $k_n^2 \rightarrow 0$ and $x_n = k_n^1 - k_n^2$ for all $n = 0, 1, \dots$;
- (6) X_0 is a Mackey space (in particular X_0 is a barrelled space) and if the linear functional x^* is such that $x^* \leq x_1'$ in the sense of the cone K_1 and $x_2' \leq x^*$ in the sense of the cone K_2 for some linear continuous functionals x_1' and x_2' , then x^* is also continuous.

The notion of general position of cones can be extended by induction to n cones with $n > 2$.

1.1.5. The cones K_1, \dots, K_n are in general position if there is a permutation $\{i_1, \dots, i_n\}$ of the index set such that K_{i_s} and $\bigcap_{r=s+1}^n K_{i_r}$ are in general position for all $s = 1, \dots, n-1$.

1.1.6. If the cones K_1 and K_2 are in general position, then the cones $K_1 \times K_2$ and $\Delta_2(X)$ are also in general position (here $\Delta_n(X) = \{(x, \dots, x) \in X^n : x \in X\}$).

1.1.7. The notion of general position of cones was introduced in [23] as a direct generalization of the notion of unflattenedness, going back to M. G. Krein. In 1940, Shmul'yan [Smulian] generalizing one result due to Krein established unflattenedness of the closed generating cone in a Banach space [37]. In 1955, unflattenedness appeared in Bonsall's paper [116] in the form of the bounded decomposition property of a normed space. Subsequent developments are reflected in [11, 34, 170, 258].

Unflattenedness is one of the basic concepts of the theory of cones in t.v.s. It plays an important role in the study of the internal topics of this theory, and also in connection with the continuation of linear or sublinear operators and the study of nonlinear operators (see [11, 34, 170, 258]).

1.1.8. Proposition (1) of Theorem 1.1.4 generalizes the classical Krein-Shmul'yan theorem on unflattenedness of the closed generating cone in a Banach space. The central moment here is Klee's decomposition technique [174], analogous to Banach's method of the rolling sphere (see [11, 258]). Similar propositions were proved in [45, 208]. Proposition (2) was proved in [208]. Such theories are obtained by Baire's method of categories. The key point is Pettis's lemma (see [125], Chap. 5). Proposition (3) is obtained by Ptak's method. Propositions (4) and (6) for the case of coincident cones are well known [258].

1.2. Moreau-Rockafellar Formula. Sandwich

Consider an ordered vector space (o.v.s.) E , denoting by the symbol E^+ the set $EU(+\infty)$ with the ordinary operations of addition and multiplication (see [1]).

1.2.1. Let X be a vector space, E an o.v.s. An operator $P: X \rightarrow E^+$ is called sublinear if

$$\begin{aligned} P(\lambda x) &\leq \lambda P x \quad \text{for all } x \in X, \lambda \geq 0; \\ P(x_1 + x_2) &\leq P x_1 + P x_2 \quad \text{for all } x_1, x_2 \in X. \end{aligned}$$

The sets $\text{dom } P = \{x \in X : P(x) < +\infty\}$ and $\text{epi } P = \{(x, y) \in X \times E : y \geq P x\}$ are respectively called the effective domain and the epigraph of the operator P .

1.2.2. The linear operator $T: X \rightarrow E$ is called the support operator of P if $P(x) \geq T x$ for all $x \in X$. The symbol ∂P denotes the set of all the support operators of P . The set ∂P is called the support set or the subdifferential of the operator P . If X is a t.v.s. and E an o.t.v.s., then $\partial^c P = \partial P \cap \mathcal{L}(X, E)$, where $\mathcal{L}(X, E)$ is the space of linear continuous operators from X into E .

1.2.3. A cone K in a t.v.s. X is called normal if sets of the form $(U+K) \cap (U-K)$, where U is the neighborhood of 0 , form a fundamental system of neighborhoods of 0 in X .

If E is an o.t.v.s. with a normal cone of positive elements and $\text{dom } P = X$, then $\partial^c P = \partial P$, i.e., the algebraic and the continuous cases coincide.

One of the main results of the theory of sublinear operators is formulated by the following theorem.

1.2.4. THEOREM. Let X be a vector space, E some K -space, and $P_1, P_2: X \rightarrow E$ sublinear operators. If $\text{dom } P_1 = \text{dom } P_2 = \text{dom } P_2 - \text{dom } P_1$, i.e., the effective domains of the operators P_1 and P_2 are in algebraic general position, then we have the representation

$$\partial(P_1 + P_2) = \partial P_1 + \partial P_2,$$

which is known as the Moreau-Rockafellar formula.

Theorem 1.2.4 is a form of the Hahn-Banach-Kantorovich theorem [1, 10, 32] and it therefore only guarantees the existence of some linear operators, without ensuring, in general, their continuity [1]. In order to derive a continuous version of the Moreau-Rockafellar formula, we require additional tools based on the notion of general position of cones.

Let σ_n be a coordinate permutation mapping which realizes a natural isomorphism of the spaces $(X \times Y)^n$ and $X^n \times Y^n$. An order-complete ordered t.v.s. with a normal cone of positive elements will be called a topological K -space.

1.2.5. THEOREM. Let X be a t.v.s., E a topological K -space, P_1 and $P_2: X \rightarrow E$ sublinear operators. If the cones $\sigma_2(\text{epi } P_1 \times \text{epi } P_2)$ and $\Delta_2(X) \times E^2$ are in general position, we have the Moreau-Rockafellar formula

$$\partial^c(P_1 + P_2) = \partial^c P_1 + \partial^c P_2.$$

Proof. The inclusion \supset is obvious; let us prove \subset . Let $K = \sigma_2(\text{epi } P_1 \times \text{epi } P_2)$ and $X_0 = \text{dom } P_1 - \text{dom } P_2$. We can show that $K - \Delta_2(X) \times E^2 = [X_0^2 + \Delta_2(X)] \times E^2$ and $Z_0 = X_0^2 + \Delta_2(X)$ is completable in X^2 . If $(x, y) \in Z_0$, then $(x, y) = (h_1, h_2) - (h, h) = (k, k) - (k_1, k_2)$ for some $h_1, k_1 \in \text{dom } P_1$, $h_2, k_2 \in \text{dom } P_2$, $h, k \in X$. But then for $T \in \partial^c(P_1 + P_2)$ we have

$$-P_1(k - x) - P_2(k - y) + Tk \leq P_1(h + x) + P_2(h + y) - Th.$$

Therefore the following operator is well-defined:

$$P_0(x, y) = \inf \{ P_1(x + h) + P_2(y + h) - Th : x + h \in \text{dom } P_1, y + h \in \text{dom } P_2 \}.$$

It is easily seen that the operator P_0 is sublinear and if $B: X^2 \rightarrow Z_0$ is a continuous projector on Z_0 and $(T_1, T_2) \in \partial(P_0 \circ B)$, then $T_1 \in \partial P_1$, $T_2 \in \partial P_2$ and $T_1 + T_2 = T$. Thus, it suffices to show that the operator P_0 is continuous at 0, since in this case the Hahn-Banach-Kantorovich theorem and the normality of the cone E^+ imply that $\partial(P_0 \circ B) = \partial^c(P_0 \circ B) \neq \emptyset$.

Let V_1 be an arbitrary neighborhood of 0 in E and $(V' + E^+) \cap (V' - E^+) \subset V_1$, where $V' \supset V + V + T[U]$ for some symmetric neighborhoods of 0 V in E and U in X . Denote $W = K \cap U^2 \times V^2 - \Delta_2(X) \times E^2 \cap U^2 \times V$. Since K and $\Delta_2(X) \times E^2$ are in general position, there are neighborhoods of 0 $U_0 \subset X$ and $V_0 \subset E$ such that $U_0^2 \times V_0^2 \subset W \cap (-W)$. If $(x_1, x_2) \in U_0^2$ and $(y_1, y_2) \in V_0^2$, then for some $h \in U$ and $k_1, k_2 \in V$ we have

$$(x_i + h, k_i + y_i) \in \text{epi } P_i \cap U \times V \quad (i=1, 2),$$

and so

$$P_0(x_1, x_2) \in P_1(x_1 + h) + P_2(x_2 + h) - Th - E^+ \subset y_1 + y_2 + k_1 + k_2 - Th - E^+ \subset V' - E^+.$$

Similarly, $(h' - x_i, k_i' - y_i) \in \text{epi } P_i \cap U \times V$ for some $h' \in U, k_1', k_2' \in V$. Therefore,

$$P_0(x_1, x_2) \in -P_1(h' - x_1) - P_2(h' - x_2) + Th' + E^+ \subset -y_1 - y_2 + k_1' + k_2' + Th' + E^+ \subset V' + E^+.$$

Thus, $P_0[U_0 \times V_0] \subset V_1$, which proves continuity of P_0 at 0. Q.E.D.

Theorems 1.2.4 and 1.2.5 can be combined into one proposition with the aid of the theory of pseudotopological vector spaces [44]. We will not follow this course here, but in future we will agree not to distinguish between the symbols ∂P and $\partial^{\circ} P$. It is implied that the following results are equally valid in both the algebraic and the topological version, assuming the corresponding interpretation of general position.

1.2.6. COROLLARY. If the conditions of Theorem 1.2.5 hold and also $P_1x + P_2x \geq 0$ for all $x \in X$, then there is an operator $T \in \mathcal{L}(X, E)$ such that

$$P_2x \leq Tx \leq P_1x$$

for all $x \in X$.

Propositions of the form 1.2.6 are known in the literature as sandwich theorems. For convex operators such propositions were formulated in [1, 96, 205, 262]. Various sandwich theorems for semigroups and distributive lattices can be found in [149, 172, 174, 179, 232]. We give here one of the results (see [179]).

1.2.7. THEOREM. Let X be a semigroup, E some K -space, and let the operators $P_1, P_2: X \rightarrow E$ satisfy the following conditions:

- (1) P_1 is subadditive, i.e., $P_1(x+y) \leq P_1x + P_2y$ for all $x, y \in X$;
- (2) P_2 is superadditive, i.e., $P_2(x+y) \geq P_2x + P_2y$ for all $x, y \in X$;
- (3) $P_2x \leq P_1x$ for all $x \in X$.

Then there is an additive operator $T: X \rightarrow E$ such that

$$P_2x \leq Tx \leq P_1x \quad (x \in X).$$

1.2.8. Normality as the dual concept of unflattenedness was introduced by Krein in 1940 [36, 37]. It generalizes the classical "two policemen" principle and plays an important role in the theory of cones [11, 98, 258, 1, 44, 45].

1.2.9. Theorem 1.2.4 for the case $\text{dom } P_1 = \text{dom } P_2 = X$ was established in [62] and for the case $\text{dom } P_1 = \text{dom } P_2$ in [96]. Theorem 1.2.5 for operators continuous in their domains was proved in [42]. If $\text{epi } P_1$ and $\text{epi } P_2$ are in general position, then the theorem is also valid (see [44, 45]).

1.2.10. Theorem 1.2.5 remains valid if the space E is topologically complete, and the condition of general position of the cones K and $\Delta_2(X) \times E^2$ is somewhat relaxed, premising completability of the subspace $\bar{Z}_0 = \overline{\Delta_2(X) + X_0^2}$. Indeed, the operator P_0 constructed in the proof of the theorem is continuously continuable to \bar{Z}_0 and all the arguments remain in force. If $E = \mathbb{R}$ and X is a l.c.s., the completability condition can be dropped altogether.

1.2.11. From Theorem 1.2.5 we can derive all the basic formulas of the calculus of support sets (maximum, superposition, etc.) of sublinear operators in t.v.s. (see [45]). These formulas can be extracted from the results of Chap. II.

1.2.12. The Hanh-Banach-Kantorovich theorem, the Moreau-Rockafellar theorem, and the sandwich theorem are mutually equivalent and are necessarily connected with order completeness of the space E (see Sec. 1.5). The equivalence of order completeness and of the property

of majorized continuation of operators was established in [115, 248]. On other equivalent properties, see [144, 189, 206].

1.3. Canonical Operator Method

Among all the sublinear operators, we distinguish relatively simple canonical operators, so that with each cardinality and each K-space we associate in a standard fashion a unique canonical operator. Any other everywhere defined sublinear operator is formed as a superposition of a canonical and a linear operator. Thus, the main topics in the theory of sublinear operators (in particular, the calculus of support operators for everywhere defined operators) may be reduced to the analysis of the support set of a canonical operator. The detailed structure of the canonical operator naturally depends on the specific properties of the K-space. Some new concepts will be needed for rigorous formulations.

Consider a K-space E , an arbitrary nonempty set \mathfrak{A} , and let the space $E^{\mathfrak{A}}$ be endowed with canonical (coordinatewise) structure of a K-space. Let $\Delta_{\mathfrak{A}}: E \rightarrow E^{\mathfrak{A}}$ be a diagonal mapping, i.e., $\Delta_{\mathfrak{A}}: y \mapsto (y)_{A \in \mathfrak{A}}$. Denote by $(E^{\mathfrak{A}})_{\infty}$ the ideal in $E^{\mathfrak{A}}$, generated by the set $\Delta_{\mathfrak{A}}[E]$, i.e.,

$$(E^{\mathfrak{A}})_{\infty} = (\Delta_{\mathfrak{A}}[E] + E_+^{\mathfrak{A}}) \cap (\Delta_{\mathfrak{A}}[E] - E_+^{\mathfrak{A}}).$$

Clearly $(E^{\mathfrak{A}})_{\infty}$ is a K-space with respect to the order induced from the product $E^{\mathfrak{A}}$.

1.3.1. The sublinear operator $\varepsilon_{\mathfrak{A}}: (E^{\mathfrak{A}})_{\infty} \rightarrow E$ acting according to the rule

$$\varepsilon_{\mathfrak{A}} = \varepsilon_{\mathfrak{A}, E}: (y_A)_{A \in \mathfrak{A}} \mapsto \sup \{y_A: A \in \mathfrak{A}\}$$

is called a canonical operator.

1.3.2. The support set of the operator $\varepsilon_{\mathfrak{A}}$ has the form

$$\partial \varepsilon_{\mathfrak{A}} = \{\alpha \in L^+((E^{\mathfrak{A}})_{\infty}, E): \alpha \circ \Delta_{\mathfrak{A}} = I_E\},$$

where I_E is the identity mapping of E onto itself.

Consider the set $\mathfrak{A} \subset L(X, E)$ and assume that the set $\{Ax: A \in \mathfrak{A}\}$ is bounded for any $x \in X$. Then we can define a sublinear operator $P_{\mathfrak{A}}: X \rightarrow E$ by the formula

$$P_{\mathfrak{A}}: x \mapsto \sup \{Ax: A \in \mathfrak{A}\}.$$

1.3.3. The support set of the sublinear operator $P_{\mathfrak{A}}$ is called a support hull of the set \mathfrak{A} and is denoted by $\text{cop } \mathfrak{A}$.

Note that $\text{cop } \mathfrak{A}$ is the least support set which includes \mathfrak{A} .

1.3.4. Let X be a vector space, E some K-space, $P: X \rightarrow E$ a sublinear operator, and $\partial P = \text{cop } \mathfrak{A}$ for some set $\mathfrak{A} \subset L(X, E)$. Then we have the representation

$$P = \varepsilon_{\mathfrak{A}} \circ \langle \mathfrak{A} \rangle, \quad \partial P = \partial \varepsilon_{\mathfrak{A}} \circ \langle \mathfrak{A} \rangle,$$

where $\langle \mathfrak{A} \rangle: X \rightarrow (E^{\mathfrak{A}})_{\infty}$ is a linear operator defined by the relationship $\langle \mathfrak{A} \rangle: x \mapsto (Ax)_{A \in \mathfrak{A}}$.

The canonical operator method is useful in virtually all the parts of the theory of sublinear operators. We give here one of the most general formulas for calculating the support set of a superposition of sublinear operators, based on the canonical operator method.

1.3.5. THEOREM. Let $P_1: X \rightarrow Y$ be a sublinear operator, and $P_2: Y \rightarrow E$ an increasing sublinear operator. Then

$$\partial(P_2 \circ P_1) = \{A \circ \langle \partial P_1 \rangle: A \in L^+((Y^{\partial P_1})_{\infty}, E), A \circ \Delta_{\partial P_1} \in \partial P_2\}.$$

If $\partial P_1 = \text{cop } \mathfrak{A}_1$ and $\partial P_2 = \text{cop } \mathfrak{A}_2$, then

$$\partial(P_2 \circ P_1) = \{A \circ \langle \mathfrak{A}_1 \rangle: A \in L^+((Y^{\mathfrak{A}_1})_{\infty}, E), \exists \alpha \in \partial \varepsilon_{\mathfrak{A}_2} A \circ \Delta_{\mathfrak{A}_1} = \alpha \circ \langle \mathfrak{A}_2 \rangle\}.$$

Convolution of the "integral" formulas 1.3.5 leads to various corollaries for the calculation of support sets. We have, in particular, the following proposition which is proved in [1].

1.3.6. If $P_1: X \rightarrow Y$ is a sublinear operator, $P_2: Y \rightarrow E$ an increasing sublinear operator, then

$$\partial(P_2 \circ P_1) = \bigcup \{ \partial(A \circ P_1) : A \in \partial P_2 \}.$$

1.3.7. We should stress the fundamental qualitative difference between the formulas in 1.3.5 and 1.3.6. The formula in 1.3.6 does not contain the support set of the canonical operator, which dropped out during the convolution of 1.3.5. As a result, 1.3.6 is meaningless for a linear P_2 , whereas 1.3.5 describes the construction of a support set in this case also. Unfortunately, universal integral formulas of the form 1.3.5 are not available for operators which are not everywhere defined.

1.3.8. The canonical operator method and formulas of the form 1.3.5 were proposed in [47] and are considered in detail, e.g., in [1, 51, 88].

1.4. Structure of Support Sets

Let X be a vector space, E a K -space, and $P: X \rightarrow E$ a sublinear operator. The symbol $\text{Ch}(P)$ denotes the set of extreme points of the subdifferential ∂P of the operator P . By $\Lambda(E) = [0, I_E] = \partial(y \mapsto y^+)$ we denote the set of all the multipliers in E [9].

1.4.1. THEOREM. The operator T is included in $\text{Ch}(P)$ if and only if for any operators $T_1, \dots, T_n \in \partial P$ and multipliers $\alpha_1, \dots, \alpha_n$ such that $\sum_{k=1}^n \alpha_k = I_E$ and $T = \sum_{k=1}^n \alpha_k \circ T_k$ we have $\alpha_k \circ T = \alpha_k \circ T_k$ for each $k = 1, \dots, n$.

The extreme points of the support set are thus operator extreme points. We now describe the set of extreme points of the canonical operator.

1.4.2. THEOREM. The set of extreme points of the subdifferential of the canonical operator $e_{\mathfrak{A}}$ coincides with the set of lattice homomorphisms of the space $(E^{\mathfrak{A}})_{\infty}$ in E which are included in $\partial e_{\mathfrak{A}}$.

Finally, let us present the operator versions of the Krein-Mil'man theorem and its converse.

1.4.3. THEOREM. (1) (Krein-Mil'man) Each subdifferential is a support hull of the set of its extreme points.

(2) (Mil'man) If the subdifferential ∂P is the support hull of the set \mathfrak{A} , then $\text{Ch}(P) \subset \text{Ch}(e_{\mathfrak{A}}) \circ \langle \mathfrak{A} \rangle$.

Note that the Krein-Mil'man theorem remains valid under weaker assumptions. On the one hand, it suffices to stipulate that the space E has the property of chain completeness, and on the other hand, the subdifferential is reconstructable not only from the entire set of its extreme points, but also from the subset of universally defined o-extreme points [53].

From Theorem 1.4.3 (1) it follows that if $P: X \rightarrow E$ is a sublinear operator, then for any $x \in X$ we have

$$Px = \sup \{ Ax : A \in \text{Ch}(P) \}.$$

However, if we drop the condition of (chain) completeness of the space E , then (in general) it becomes impossible to reconstruct the operator P even if the entire $\partial(P)$ is

given [68]. Moreover, the support set may prove to be empty. Various classes of sublinear operators (defined on the entire space or on a cone) described in [69, 70] can be reconstructed from their support sets. The main method of analysis in these cases is the type selection theorems [105, 109].

We reproduce here one typical result (for details see [72, 88]).

1.4.4. THEOREM. Let X be a separable Banach space, E the KB-linear of bounded elements, and $P: X \rightarrow E$ a continuous sublinear operator. Then

$$Px = \sup \{Ax: A \in \partial P\}$$

for all $x \in X$.

Another cycle of problems is associated with the internal characterization of support sets, i.e., finding the conditions on the set Ω in the operator space such that this set coincides with the support set of some sublinear operator. The problem of factorizing the operation cop in terms of the closure operation and the operator convex hull is also related to this category. The most advanced results in this case were obtained assuming that E is a K -space with sufficiently many order-continuous functionals. On this topic, see [26, 47, 59, 60, 61, 67, 68, 71, 93].

1.4.5. THEOREM. Let X be a l.c.s., E a locally convex K -space in which the order intervals are weakly compact. Then the weakly order bounded set $\Omega \subset \mathcal{L}(X, E)$ is a support set if and only if it is operator convex, equicontinuous, and closed in the precompact convergence topology of the space $\mathcal{L}(X, E)$.

1.4.6. The classical Krein-Mil'man theorem was established in 1940 in [180]. Since then, an enormous body of research has been published on this theorem and its various generalizations and modifications. Detailed surveys of the literature were published in [78, 97]. The description of the extreme points of convex sets in operator spaces apparently began in the 1960s (see, e.g., [97]). Morris and Phelps [198] proved the Krein-Mil'man theorem for a ball in the space of operators with values in $C(X)$; for the case of support sets of operators with values in K -space with sufficiently many order-continuous functionals, the theorem was proved by Rubinov [87] who used Fan's method [146]. A new phenomenon was discovered in this work: reconstructability from weakly extreme points. All these authors appeal to compactness. However, as we see from [53], the Krein-Mil'man theorem is associated with order completeness. Apart from Krein and Mil'man, this fact apparently was first discovered by Nachbin [202, 203], who proved the Krein-Mil'man theorem for a closed convex set in a l.c.s. with the positive binary intersection property (see also [31, 32]). Solving one of Nachbin's problems, Oates [212] proved the Krein-Mil'man theorem for support sets of special type.

1.4.7. Theorem 1.4.3 constitutes "pure versions" of the Krein-Mil'man theorem and its Mil'man converse, in the sense that these are the most general propositions associated with the property of order completeness. Further refinement of these results in algebraic-topological terms involves factorization of the operator cop [1, 58].

1.4.8. Choquet's operator theory in K -spaces is related to the subject of this section (see [1, 46]).

1.4.9. Extreme points and the Krein-Mil'man theorem for a special support set — the set of all positive extensions of a positive operator — have been intensively studied by a number of authors (see [185, 186]).

1.5. Convex Analysis in Modules

Scalar duality theorems play an important role in local convex analysis. These are bipolar theorems, theorems of Fenchel-Moreau type on involution of the transition to the conjugate function, duality theorems for the values of extremal problems, etc. The basis for all these theorems is provided by "duality in the wide sense" which stipulates equivalence of points and hyperplanes (vectors and functional covectors). When we pass to vector-valued problems, this equivalence is lost, in particular because the space of operators acting into the K-space of "objectives" allows multiplication by the multipliers, whereas the original space of "states" does not allow this multiplication. Qualitatively speaking, the dual problem in the vector version is characterized by a modular structure (e.g., the subdifferential is no longer a simple convex set, but necessarily an operator convex set).

In view of the above, we have to elucidate to what extent and on what modules the main propositions of local convex analysis are preserved.

1.5.1. Let A be an arbitrary lattice-ordered ring with positive unit. Let, further, X be an A -module and E an ordered A -module. All the modules, as always, are assumed unitary. In this setting, we naturally define the notion of A -sublinear and A^+ -homogeneous operators. It is clear that in general not every A -sublinear operator is A^+ -homogeneous (in distinction from the previously considered case $A = \mathbb{R}$). As always, we use the symbol $\text{Hom}_A(X, E)$ to denote the set of all A -homomorphisms from X into E or, putting it differently, the space of A -linear operators. The subdifferential (= support set) has the following obvious meaning:

$$\partial^A P = \{T \in \text{Hom}_A(X, E) : Tx \leq Px \quad \forall x \in X\}.$$

Since each A -module, in particular, is a \mathbb{Z} -module, i.e., an abelian group, the subdifferential $\partial^{\mathbb{Z}} P$ is defined, and we simply denote it by ∂P .

1.5.2. The A -module E is A -continuable if for every A -sublinear operator $P: X \rightarrow E$ and an arbitrary A -homomorphism $T \in \text{Hom}_A(Y, X)$ the Hahn-Banach formula applies (in asymmetric form):

$$\partial^A (P \circ T) = \partial^A P \circ T.$$

If, moreover, the operator P is A^+ -homogeneous, we say that the A -module E admits convex analysis.

In order to describe these modules, we will require some concepts from the theory of K -spaces.

1.5.3. Let E be a K -space and I_E the identity operator in E . The component generated by I_E in the K -space of regular endomorphisms of E is denoted by $\text{Orth}(E)$. The elements of $\text{Orth}(E)$ are orthomorphisms. The least normal subspace $Z(E)$ in $\text{Orth}(E)$ containing I_E is called the ideal center of E .

The properties of orthomorphisms were studied in detail in [168]. We only note here that with regard to the natural ring and order structure $\text{Orth}(E)$ and $Z(E)$ are functional algebras (f -algebras). $Z(E)$ is the fundament of $\text{Orth}(E)$, and $\text{Orth}(E)$ is the centralizer of $Z(E)$ in the ring $\text{Orth}(E)$.

1.5.4. The groups obtained from K-spaces when the real multiplication structure is ignored are called erased K-spaces.

1.5.5. A subring A of the orthomorphism ring is called almost rational if for every $n \in \mathbb{N}$ there is a decreasing net of multipliers (α_i) from A such that for any $y \in E^+$ we have $(1/n)y = o\text{-}\lim \alpha_i y = \inf \alpha_i y$.

1.5.6. THEOREM. Let A be a d-ring, i.e., for any $\alpha_1 \in A$ and $\alpha_2 \in A^+$, we have $(\alpha_1 \alpha_2)^+ = \alpha_1^+ \alpha_2$, $(\alpha_2 \alpha_1)^+ = \alpha_2 \alpha_1^+$.

An ordered A-module E is A-continuable if and only if the group $E_b = E^+ - E^+$ is an erased K-space and the natural linear representation of A in E_b is a ring and lattice homomorphism on a subring and a sublattice of the ring $\text{Orth}(E_b)$. For any A-sublinear operator $P: X \rightarrow E$ in this case we have $\partial P = \partial^A P$.

1.5.7. The a priori conditions imposed on the ring A may be modified, but we cannot completely eliminate assumptions of this kind if we want to preserve A^+ -homogeneity of a Z^+ -homogeneous A-sublinear operator. Also note that Theorem 1.5.6 shows that the continuability property necessarily holds in the strong form, i.e., the group homomorphism defined on a subgroup and majorized by the modular sublinear operator is continuable to modular homomorphism.

1.5.8. THEOREM. An ordered A-module E admits convex analysis if and only if E_b is an erased K-space and the natural linear representation of A in E_b is a ring and lattice homomorphism on the almost rational orthomorphism ring.

1.5.9. The above theorems from [55] outline the applicability region of the universal methods of local convex analysis. In particular, they show that, in fact, specific modular convex analysis simply does not exist (see [113, 117, 252]). More precisely, allowing for some elementary qualifications, convex analysis applies if and only if we are dealing with K-spaces regarded as modules over the algebras of their orthomorphisms. Moreover, additive supports of a modular-sublinear operator are automatically modular homomorphisms.

CHAPTER II

CONVEX OPERATORS

2.1. Fenchel-Young Transforms of a Sum and a Convolution

In addition to the largest element $+\infty$, we also adjoin the smallest element $-\infty$ to the K-space E. The resulting set will be denoted by \bar{E} . The operations of addition and multiplication by a scalar are extended from E to \bar{E} according to the following rules:

$$\begin{aligned} (-1) \cdot (+\infty) &= -\infty, & (-1) \cdot (-\infty) &= +\infty, \\ x + y &= \inf \{x' + y' : x', y' \in E, x' \geq x, y' \geq y\}. \end{aligned}$$

Thus, all the undefined expressions are assigned the value $+\infty$.

2.1.1. The Fenchel-Young transform of the mapping $f: X \rightarrow \bar{E}$ is the operator $f^*: \mathcal{L}(X, E) \rightarrow \bar{E}$ acting according to the rule

$$f^*(T) = \sup \{Tx - f(x) : x \in X\} \quad (T \in \mathcal{L}(X, E)).$$

The second Fenchel-Young transform f^{**} is similarly defined, specifically,

$$f^{**}(x) = \sup \{Tx - f^*(T) : T \in \mathcal{L}(X, E)\} \quad (x \in X).$$

Finding the rules of evaluation of the Fenchel–Young transform is one of the central problems of local convex analysis. In this and the following section we present a scheme for the derivation of such rules and some of the basic formulas.

2.1.2. The convex sets F_1 and F_2 in a t.p.s. X are in general position if their Hörmander transforms, i.e., the canonical hulls of the sets $F_1 \times \{1\}$ and $F_2 \times \{1\}$ in the space $X \times \mathbb{R}$, are in general position.

As in Sec. 1.1, we can naturally define general position for a finite family of convex sets. Note, in particular, that a sufficient condition for general position of convex sets is the inclusion in their intersection of an interior point of each of the sets, with the possible exception of one.

2.1.3. THEOREM. If $f_1, \dots, f_n: X \rightarrow E$ are convex operators and the sets $\text{epi} f_1, \dots, \text{epi} f_n$ are in general position, then

$$(f_1 + \dots + f_n)^*(T) = f_1^* \oplus \dots \oplus f_n^*(T) = \inf \{f_1^*(T_1) + \dots + f_n^*(T_n)\},$$

where the infimum is over all the representations $T = T_1 + \dots + T_n$, $T_i \in \mathcal{L}(X, E)$, $i = 1, \dots, n$. This formula is exact, i.e., for any $T \in \text{dom}(f_1 + \dots + f_n)^*$ there are operators $T_i \in \mathcal{L}(X, E)$, $i = 1, \dots, n$, such that

$$\begin{aligned} T &= T_1 + \dots + T_n, \\ (f_1 + \dots + f_n)^*(T) &= f_1^*(T_1) + \dots + f_n^*(T_n). \end{aligned}$$

The proof is by reduction to the case of sublinear operators with subsequent application of Theorem 1.2.5 [126].

2.1.4. Let X, X_1, X_2 be vector spaces. A Rockafellar convolution of the operators $f_1: X_1 \times X \rightarrow \bar{E}$ and $f_2: X \times X_2 \rightarrow \bar{E}$ is the mapping $f_1 \Delta f_2: X_1 \times X_2 \rightarrow \bar{E}$ defined by the relationship

$$(f_1 \Delta f_2)(x_1, x_2) = \inf_{x \in X} (f_1(x_1, x) + f_2(x, x_2)) \quad ((x_1, x_2) \in X_1 \times X_2).$$

If f_1 and f_2 are convex operators, then $f_1 \Delta f_2$ is also a convex operator, and $\text{dom}(f_1 \Delta f_2) = \text{dom}(f_1) \circ \text{dom}(f_2)$. The convolution operation is anticommutative and associative, i.e.,

$$f_1 \Delta f_2(x_1, x_2) = f_2 \Delta f_1(x_2, x_1) \quad ((x_1, x_2) \in X_1 \times X_2),$$

and if $f_3: X_2 \times X_3 \rightarrow \bar{E}$ is another mapping, then

$$f_1 \Delta (f_2 \Delta f_3) = (f_1 \Delta f_2) \Delta f_3.$$

2.1.5. THEOREM. Let X, X_1 , and X_2 be t.v.s., E a topological K -space, $f_1: X_1 \times X \rightarrow E$ and $f_2: X \times X_2 \rightarrow E$ convex operators. If in the space $X_1 \times X \times X_2 \times E$ the sets

$$(\text{epi } f_1)_{X_2} = \{(x_1, x, x_2, y): y \geq f_1(x_1, x)\},$$

$$(\text{epi } f_2)_{X_1} = \{(x_1, x, x_2, y): y \geq f_2(x, x_2)\}$$

are in general position, then

$$(f_1 \Delta f_2)^* = f_1^* \Delta f_2^*.$$

The last formula is exact, i.e., for any $T_i \in \mathcal{L}(X_i, E)$, $i = 1, 2$ there is $T \in \mathcal{L}(X, E)$ such that

$$(f_1 \Delta f_2)^*(T_1, T_2) = f_1^*(T_1, T) + f_2^*(T, T_2).$$

2.1.6. The Fenchel–Young transform has a long history, which is reflected in [28, 29, 35, 50, 85, 92, 101, 102, 119, 147, 148, 195–197]. The conjugate of a convex function was introduced and studied by Fenchel [147, 148]. The Fenchel–Young transform for (vector-valued) operators apparently was first introduced in [219], and then in [63]. General formulas for

the substitution of a variable in a Fenchel–Young transform and methods of their derivation were proposed in [49].

2.1.7. The convolution operation defined in 2.1.4 is the product of bifunctions introduced by Rockafellar [85, 222]. This operation apparently provides the most convenient tool for the derivation of subdifferentiation formulas from Theorem 1.2.2. Moreover, the properties of the convolution are essential in the duality theory of convex correspondences and, in particular, of dynamic economic models [33, 45, 64, 77, 78, 82, 222].

2.2. Some Rules for Substitution of Variables in the Fenchel–Young Transform

When a convex set is identified with its indicator operator, a superposition of convex correspondences passes into a Rockafellar convolution. On the other hand, the epigraph of the superposition of a convex operator with an increasing convex operator coincides with the superposition of the epigraphs of these operators. Thus, the results of Sec. 2.1 include a rule for the evaluation of the Fenchel–Young transformation of a superposition and a corresponding chain rule of subdifferentiation. The same conclusion applies to other formulas of local convex analysis.

2.2.1. The indicator operator $\delta_E(F)$ of the set $F \subset X$ is the mapping from X to E which is equal to zero on F and to $+\infty$ on the complement of F . The Fenchel–Young transform of the operator $\delta_E(F)$ is called the (E) -support operator of the set F and is denoted by F^* , where $F^*(T) = \delta_E(F)^*(T) = \sup\{Tx : x \in F\}$ for any $T \in \mathcal{L}(X, E)$.

The effective domain and the epigraph of an arbitrary mapping are defined as in Sec. 1.2 for a sublinear operator.

2.2.2. Let X be a vector space, Y an o.v.s., E a K -space. Let $f: X \rightarrow Y$ be a convex operator which is not identically equal to $+\infty$ and $F = \text{epi } f$.

(1) The pair (A, B) , where $A \in \mathcal{L}(X, E)$ and $B \in \mathcal{L}(Y, E)$, is included in $\text{dom } F^*$ if and only if the operator B is positive and $A \in \text{dom } (B \circ f)^*$.

(2) If $C \in \mathcal{L}^+(Y, E)$, then

$$F^*(A, C) = (C \circ f)^*(A).$$

The evaluation of the Fenchel–Young transform of convex operators is thus reduced to the evaluation of support operators of their epigraphs. This fact, combined with the results of Sec. 2.1, makes it possible to derive all the basic rules for the evaluation of Fenchel–Young transforms. We now list some of these rules. In the following theorems, X_1, X are t.v.s., Y is an o.t.v.s., E is a topological K -space.

2.2.3. THEOREM. Let $f: X \rightarrow Y$ be a convex operator, $g: Y \rightarrow E$ an increasing convex operator, and the convex sets $\text{epi } f \times \bar{E}$ and $X \times \text{epi } g$ are in the general position. For any $A \in \mathcal{L}(X, E)$ we have the exact formula

$$(g \circ f)^*(A) = \inf \{ (B \circ f)^*(A) + g^*(B) : B \in \mathcal{L}(Y, E) \}.$$

Here A_x denotes the operator $y \rightarrow Ay + x$.

2.2.4. THEOREM. Let $f: X \rightarrow E$ be a convex operator, A a continuous linear operator from X_1 to X , and $x \in X$. If the sets $\text{graf } A_x \times E$ and $Y \times \text{epi } f$ are in general position, then for any $B \in \mathcal{L}(X_1, E)$ we have the exact formula

$$(f \circ A_x)^*(B) = \inf \{ f^*(C) - Cx : B = C \circ A \}.$$

2.2.5. COROLLARY. Let Y be a lattice and $A \in \mathcal{L}^+(Y, E)$. If the epigraphs of the convex operators $f_1, \dots, f_n: X \rightarrow Y^+$ are in general position, then

$$(A \circ f_1 \vee \dots \vee f_n)^* = \inf \left\{ \bigoplus_{i=1}^n (A_i \circ f_i)^*: A_i \in \mathcal{L}^+(Y, E), \sum_{i=1}^n A_i = A \right\}.$$

Suppose that the cone $\mathcal{L}^+(Y, E)$ isolates a cone of positive elements Y^+ , i.e., $y \in Y^+$ if and only if $Ay \geq 0$ for any $A \in \mathcal{L}^+(Y, E)$. For a convex operator $f: X \rightarrow Y$ define $\{f \leq y\} = \{x \in X: f(x) \leq y\}$ which is termed the Lebesgue set of the operator f .

2.2.6. THEOREM. If the convex sets $\text{epi } f$ and $-X \times Y^+$ are in general position, then

$$(\{f \leq y\})^*(A) = \inf \{(B \circ f)^*(A) + By: B \in \mathcal{L}^+(Y, E)\}.$$

A vector minimax theorem is obtained as another corollary from 2.2.3.

2.2.7. THEOREM. Let $f: X \rightarrow Y$ be a convex operator and $P: Y \rightarrow E$ a continuous sublinear operator. Then the following minimax relationship holds:

$$\inf_{x \in \text{dom } f} \sup_{B \in \delta P} B \circ f(x) = \sup_{B \in \delta P} \inf_{x \in \text{dom } f} B \circ f(x).$$

2.2.8. The minimax theory goes back to von Neumann [207]. A survey of the literature and the results can be found in [15, 17, 18, 85, 91]. On minimax for (quasi)convex-concave operators, see also [187].

2.3. ϵ -Subdifferentials of Convex Operators

In this section we show that the rules for the evaluation of the Fenchel-Young transform lead to the basic formulas of ϵ -subdifferentiation.

2.3.1. Let $f: X \rightarrow E$ be a convex operator from the t.v.s. X to the o.t.v.s. E , $x \in \text{dom } f$ and $\epsilon \in E^+$. Then the ϵ -subdifferential of f at the point x is the set

$$\partial_\epsilon f(x) = \{T \in \mathcal{L}(X, E): T x' - T x \leq f(x') - f(x) + \epsilon, x' \in X\}.$$

For $\epsilon = 0$ the set $\partial_\epsilon f(x)$ is called the subdifferential of f at the point x and is denoted by $\partial f(x)$.

2.3.2. The operator $T \in \mathcal{L}(X, E)$ is included in $\partial_\epsilon f(x)$ if and only if $f(x) + f^*(T) \leq T x + \epsilon$.

This simple fact enables us to associate with each formula for the Fenchel-Young transform a corresponding formula for ϵ -subdifferentials. We now demonstrate this point for the case of convex operators.

2.3.3. THEOREM. Let $f_1, \dots, f_n: X \rightarrow E$ be convex operators whose epigraphs $\text{epi } f_1, \dots, \text{epi } f_n$ are in general position. Then we have the representation

$$\partial_\epsilon (f_1 + \dots + f_n)(x) = \bigcup_{\substack{\epsilon_1 > 0, \dots, \epsilon_n > 0 \\ \epsilon_1 + \dots + \epsilon_n = \epsilon}} \{\partial_{\epsilon_1} f_1(x) + \dots + \partial_{\epsilon_n} f_n(x)\}.$$

Proof. If $T \in \partial_\epsilon (f_1 + \dots + f_n)(x)$, then

$$\epsilon + T x \geq (f_1 + \dots + f_n)^*(T) + f_1(x) + \dots + f_n(x).$$

By Theorem 2.1.1, there are $T_1, \dots, T_n \in \mathcal{L}(X, E)$ such that $T = T_1 + \dots + T_n$; $e_1 + \dots + e_n \leq \epsilon$, where

$$f_i^*(T_i) + f_i(x) - T_i x = e_i, \quad (i = 1, \dots, n).$$

Since the elements e_1, \dots, e_n are positive, for some $\epsilon_1, \dots, \epsilon_n \in E^+$ we have the relationships

$$\epsilon_1 \geq e_1, \dots, \epsilon_n \geq e_n; \quad \epsilon_1 + \dots + \epsilon_n = \epsilon.$$

This implies that $T_i \in \partial_{\varepsilon_i} f_i(x)$ for $i = 1, \dots, n$, i.e., $T \in \partial_{\varepsilon_1} f_1(x) + \dots + \partial_{\varepsilon_n} f_n(x)$. The converse inclusion is obvious.

2.3.4. THEOREM. Let $f_1: X_1 \times X \rightarrow E$, $f_2: X \times X_2 \rightarrow E$ be convex operators and for some $(x_1, x) \in \text{dom} f_1$ and $(x, x_2) \in \text{dom} f_2$ let the following equality hold:

$$(f_1 \Delta f_2)(x_1, x_2) = f_1(x_1, x) + f_2(x, x_2).$$

If the sets $(\text{epi} f_1)_{x_2}$ and $(\text{epi} f_2)_{x_1}$ are in general position, then

$$\partial_{\varepsilon}(f_1 \Delta f_2)(x_1, x_2) = \bigcup_{\substack{\varepsilon_1 > 0, \varepsilon_2 > 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \partial_{\varepsilon_1} f_1(x_1, x) \circ \partial_{\varepsilon_2} f_2(x, x_2).$$

2.3.5. THEOREM. Let $f: X \rightarrow Y$ be a convex operator, $g: Y \rightarrow E$ an increasing convex operator, and the convex sets $\text{epi} f \times E$ and $X \times \text{epi} g$ are in general position. Then for any $x \in \text{dom} g \circ f$ we have

$$\partial_{\varepsilon}(g \circ f)(x) = \bigcup_{\substack{T \in \partial_{\varepsilon_1} g(f(x)) \\ \varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_1 + \varepsilon_2 = \varepsilon}} \partial_{\varepsilon}(T \circ f)(x).$$

2.3.6. Consider a convex operator $f: X \rightarrow E$ and let $x \in \text{int} \text{dom} f$. For every element $h \in X$, define a ε -derivative of the operator f at the point x in the direction h by the equality

$$f^{\varepsilon}(x)h = \inf_{t > 0} t^{-1} [f(x + th) - f(x)].$$

The operator $f^{\varepsilon}(x): X \rightarrow E$ is clearly sublinear, and $\partial_{\varepsilon} f(x) = \partial f^{\varepsilon}(x)$. The explicit evaluation of ε -derivatives is much more involved than that of ε -subdifferentials. This is so because, unlike the ordinary directional derivative, the ε -derivative with $\varepsilon > 0$ is actually a nonlocal concept. At the same time, the formulas for ε -derivatives are easily obtained by the technique of the Fenchel-Young transform (Secs. 2.1, 2.2) if the Minkowski duality property holds, as in the case of everywhere defined operators.

2.3.7. Note that if a convex operator $f: X \rightarrow E$ is continuous, then $f^{\varepsilon}(x)$ is a continuous sublinear operator for every $x \in X$. Thus, from 2.3.3 and 2.3.5, by Minkowski duality, we obtain, e.g., the following formulas (if all the relevant operators are everywhere defined and continuous):

$$(f_1 + \dots + f_n)^{\varepsilon}(x) = \sup_{\substack{\varepsilon_1 > 0, \dots, \varepsilon_n > 0 \\ \varepsilon_1 + \dots + \varepsilon_n = \varepsilon}} (f_1^{\varepsilon_1}(x) + \dots + f_n^{\varepsilon_n}(x));$$

$$(g \circ f)^{\varepsilon}(x) = \sup_{\substack{\varepsilon_1 > 0, \varepsilon_2 > 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \sup_{T \in \partial_{\varepsilon_1} g(f(x))} (T \circ f)^{\varepsilon_2}(x).$$

2.3.8. The ε -subdifferential was introduced by Rockafellar [85]. Nurminskii [80] established that the ε -subdifferential mapping has the continuity property for any $\varepsilon > 0$. Some formulas of ε -subdifferentiation for convex functions were derived in [21, 163]. The general ε -subdifferential calculus in the class of convex operators was constructed in [52, 54] (see also [45]). Note that a rule of ε -subdifferentiation for a sum of convex operators was derived independently in [240].

It should be stressed that the case $\varepsilon = 0$ is of course the best developed. On this topic, see additionally [1, 27, 48, 50, 62, 86, 87, 101, 120, 164, 195, 197, 239, 251, 259], and also Chap. III.

2.39. The Minkowski duality theory and its applications are discussed in the monograph [56], where a bibliography is also given.

2.4. Openness of Convex Correspondences

A correspondence from X into Y , as always, is defined as the triple (F, X, Y) (or simply F), where X and Y are t.v.s. and $F \subset X \times Y$. The correspondence F will be identified with the mapping which associates with the element $x \in X$ the set $F[x] = \{y \in Y : (x, y) \in F\}$, i.e., the multi-valued mapping $F: x \mapsto F[x]$.

2.4.1. Let F be a correspondence from X to Y . If F is a closed set, then the correspondence (F, X, Y) is called closed. The correspondence F is called open (almost open) at the point $(x, y) \in F$ (or at 0, if $x = 0, y = 0$) if for any neighborhood U of the point x the set $F[U] = \cup \{F[x] : x \in U\}$ (respectively the closure of the set $F[U]$) is a neighborhood of the point y .

Let X and Y be real l.c.s. With the correspondence $F \subset X \times Y$ we can associate a family of conjugate correspondences

$$F_t^* = \{(x^*, y^*) \in X^* \times Y^* : F^*(x^*, y^*) \leq t\} \quad (t \in \mathbb{R}),$$

where $X^* = L(X, \mathbb{R})$ and $Y^* = L(Y, \mathbb{R})$ are the algebraic conjugates of X and Y , and F^* is an \mathbb{R} -valued support function.

2.4.2. THEOREM. Let F be a convex correspondence from X to Y , $(0, 0) \in F$ and $F[X]$ is the neighborhood of 0 in Y . Then the following conditions are equivalent:

- (1) the conical hull of the convex set $\{1\} \times X \times F$ and the cone $\mathbb{R}_+ \times \Delta(X^2) \times \{0\}$ constitute an unflattened pair in $\mathbb{R} \times X^2 \times Y$;
- (2) for every real number t and every equicontinuous set $S \subset X'$, the set $F_t^*[S]$ is also equicontinuous;
- (3) the correspondence F is almost open at 0 and for any $x' \in X'$ the set $\{y^* \in Y^* : F^*(x', y^*) < +\infty\}$ is included in Y' ;
- (4) the correspondence F is almost open at 0 and for any $x' \in X'$ and $y_0 \in \text{int } F[X]$ we have

$$(F^{-1}[y_0])^*(x') = \inf \{F^*(x', y') - (y_0, y') : y' \in Y'\},$$

where the infimum in the right-hand side is attainable;

- (5) the correspondence F is open at 0.

Here $X' = \mathcal{L}(X, \mathbb{R})$ and $Y' = \mathcal{L}(Y, \mathbb{R})$ are the topologically conjugate spaces of X and Y , respectively.

We have the following form of the classical Banach principle.

2.4.3. THEOREM (Openness Principle). Let X and Y be locally convex spaces, and X is hypercomplete. Suppose that F is a closed convex correspondence from X to Y , which is almost open at some point (x_0, y_0) . Furthermore, let $y \in \text{int } F[X]$ and $x \in F^{-1}[y]$. Then for any neighborhood of 0 U in X there is a neighborhood of 0 V in Y such that for all $t \in [0, 1]$ we have the inclusion

$$F[x+tU] \supset y+tV.$$

Note that the Frechet space (and in particular a Banach space) is hypercomplete, and therefore, the above openness principle applies.

Theorem 2.4.3 leads to the following continuity result for convex operators.

2.4.4. THEOREM. Let X and Y be l.c.s., where X is a barrelled space and Y is hypercomplete and ordered by a normal cone. Suppose that $f: X \rightarrow Y$ is a convex operator with a

closed epigraph and $x_0 \in \text{int dom } f$. Then for any continuous seminorm q on Y there are a continuous seminorm p on X and a neighborhood of 0 U in X such that

$$q[F(x_1) - f(x_2)] \leq p(x_1 - x_2)$$

for all $x_1, x_2 \in x_0 + U$.

The Banach-Steinhaus theorem is also true for convex operators.

2.4.5. THEOREM. Let X be an ultrabarrelled t.v.s., Y an o.t.b.s. with a normal cone of positive elements. Consider a family $(f_\alpha)_{\alpha \in A}$ of convex operators $f_\alpha: X \rightarrow Y^*$. Let all f_α be continuous at the point $x_0 \in \text{int} \bigcap_{\alpha \in A} \text{dom } f_\alpha$ and let the set $\{f_\alpha(x): \alpha \in A\}$ be bounded for all x from some neighborhood of the point x_0 . Then for any neighborhood of 0 $V \subset Y$ there is a neighborhood U of the point x_0 such that

$$f_\alpha(x_1) - f_\alpha(x_2) \in V$$

for all $x_1, x_2 \in U$.

2.4.6. The equivalences (5) \Leftrightarrow (3) \Leftrightarrow (2) in 2.4.3 constitute well-known characterizations of openness if F coincides with the graph of some linear operator; proposition (4) in this case reduces to the scalar version of the lemma of three (= the Sard quotient theorem) [29, 92, 164]. The dual characterization of openness, joined with Ptak's ideas, leads to Theorem 2.4.3, which constitutes a generalization of the classical openness principle. For Banach spaces this was established in [221] and [250]. Also note that hypercompleteness is traceable to Ptak's β -completeness [218]; a somewhat different notion of hypercompleteness was introduced by Kelly in [173]. Some theorems on automatic continuity associated with hypercompleteness were derived in [110].

Other generalizations of the classical openness principle, differing from Ptak's approach, and a bibliography on the subject can be found for t.v.s. in [178] (on the same topic in topological groups, see, e.g., [122]). Theorems on open mapping and closed graph in complete topological spaces in the Cech sense without additional algebraic structure will be found in [123, 215, 257].

2.4.7. Theorem 2.4.5 for the case of sublinear operators was proved in [255], where some interesting applications of this result to measure theory are given. On this subject, see also [176, 177].

2.5. Structure of Subdifferential Mappings

In this section we present the results on the structure of the subdifferential mapping

$$\partial f: x \mapsto \partial f(x), \quad x \in X,$$

where $f: X \rightarrow E$ is a convex operator.

2.5.1. A correspondence ρ from X to $L(X, E)$ is called cyclically monotone if for any natural number n the following condition is satisfied: any set of pairs $(x_0, T_0), \dots, (x_n, T_n)$ such that $x_i \in X$ and $T_i \in \rho(x_i)$, $i = 1, \dots, n$ satisfies the inequality

$$T_0(x_1 - x_0) + T_1(x_2 - x_1) + \dots + T_n(x_0 - x_n) \leq 0.$$

The maximal elements of the set of all cyclically monotone correspondences ordered by inclusions are called the maximal cyclically monotone correspondences.

2.5.2. The convex operator $f: X \rightarrow E^*$ is called closed if there are families $\{T_\sigma\}_{\sigma \in \Sigma} \subset L(X, E)$ and $\{y_\sigma\}_{\sigma \in \Sigma} \subset E$ such that

$$f(x) = \sup_{\sigma \in \Sigma} \{T_\sigma x + y_\sigma\}, \quad x \in X.$$

2.5.3. THEOREM. The subdifferential mappings of closed convex operators from X to E and only they are maximal cyclically monotone correspondences from X to $L(X, E)$. A convex closed operator is determined by its subdifferential mapping up to an additive element from E , i.e., if $f, g: X \rightarrow E^*$ are convex closed operators and $\partial f = \partial g$, then there is an element $y \in E$ such that $f(x) = g(x) + y$ for all $x \in X$.

2.5.4. If in Definition 2.5.1 the relevant condition holds only for $n = 2$, then we say that the correspondence ρ is monotone. A maximal monotone correspondence is similarly defined. The structure of maximal monotone correspondences and also their relationship with subdifferential mappings in the general case are unclear, although they have been studied in detail for some types of spaces (e.g., finite-dimensional spaces) [192, 193, 194, 223, 226, 228, 230, 231].

2.5.5. THEOREM. Let X be a Banach space and $f: X \rightarrow \mathbb{R}^*$ a semicontinuous convex proper function. Then the mapping ∂f is maximal monotone.

Note, however, that not every maximal monotone correspondence is a subdifferential mapping.

A large body of research deals with the question of univalence of the subdifferential mapping $x \mapsto \partial f(x)$. Phelps showed that in the nonscalar case $E \neq \mathbb{R}$ this issue is of little interest. In the scalar case, however, a very meaningful theory has been developed.

2.5.6. A Banach space X is called an Asplund space (a weak Asplund space) if each continuous convex function $f: X \rightarrow \mathbb{R}^*$ with an open effective domain $\text{dom } f$ is Frechet differentiable (respectively, Gateaux differentiable) on a dense subset G_f of the set $\text{dom } f$.

The classical Mazur theorem holds in this case.

2.5.7. THEOREM. Any separable Banach space is an Asplund space.

Among the more recent results, we list the following.

2.5.8. THEOREM. A Banach space is an Asplund space if and only if any separable subspace has a separable dual.

2.5.9. The notion of cyclic monotonicity was introduced by Rockafellar [85, 228], who also proved Theorem 2.5.3 in the scalar case $E = \mathbb{R}$. The theorem in the form given here was derived in [41]. Theorem 2.5.5 was proved in [228] (see also [182-184]).

2.5.10. The notion of monotonicity plays an important role in nonlinear functional analysis and, in particular, in the theory of nonlinear differential equations. Numerous applications of monotonicity will be found in [83, 101, 109].

2.5.11. The study of Asplund spaces was begun in [106]. The property of a Banach space to be an Asplund space is an isomorphism invariant and is closely related with the geometry of Banach spaces and, in particular, with Krein-Mil'man and Radon-Nikodym properties (see [137, 138, 216, 238]).

3.1. Clarke's Tangent and Normal Cones, Calculation Method.

Let X be a t.v.s., F an arbitrary subset in X , $x \in \bar{F}$. The symbol \mathfrak{B}_x^X (or simply \mathfrak{B}_x) will denote the neighborhood filter of the point $x \in X$.

3.1.1. The tangent (Clarke) cone to the set F at the point x is the set

$$T(F; x) = \bigcap_{V \in \mathfrak{B}_0} \bigcup_{\substack{U \in \mathfrak{B}_x \\ \varepsilon > 0}} \bigcap_{\substack{x' \in U \cap F \\ t \in (0, \varepsilon)}} [t^{-1}(F - x') + V].$$

In other words, $h \in T(F; x)$ if and only if for any $V \in \mathfrak{B}_h$ there are $U \in \mathfrak{B}_x$ and $\varepsilon > 0$ such that $(x' + tV) \cap F \neq \emptyset$ for any $t \in (0, \varepsilon)$ and $x' \in U \cap F$. If $x \notin \bar{F}$, we take $T(F; x) = \emptyset$.

3.1.2. THEOREM. The set $T(F; x)$ is a convex closed cone for any F and x . If F is a convex set and $x \in F$, then $T(F; x)$ coincides with the closure of the cone of feasible directions

$$Fd_x(F) = \{h \in X: \exists \varepsilon > 0, x + [0, \varepsilon)h \subset F\}.$$

This opens a possibility for studying nonconvex problems by the methods of traditional convex analysis. It is remarkable that convexity here arises automatically without any a priori assumptions.

3.1.3. Let E be a topological K -space. The E -normal cone to the set F at the point $x \in \bar{F}$ is the set

$$N_E(F; x) = \{T \in \mathcal{L}(X, E): Th \leq 0, h \in T(F; x)\}.$$

If $x \notin \bar{F}$, then we take $N_E(F; x) = \mathcal{L}(X, E) \cup \{+\infty\}$, where $+\infty$ is the operator from X to E identically equal to $+\infty$.

The introduction of an E -valued tangent cone makes it possible to handle the case of vector operators with values in a K -space.

3.1.4. The subdifferential of the mapping $f: X \rightarrow \bar{E}$ at the point x , $f(x) \in E$, is the set

$$\partial f(x) = \{T \in \mathcal{L}(X, E): (T, I_E) \in N_E(\text{epi } f; (x, f(x)))\}.$$

The operator $f^0(x): X \rightarrow \bar{E}$ defined by the relationship

$$f^0(x)h = \inf \{k \in E: (h, k) \in T(\text{epi } f; (x, f(x)))\}$$

is called a generalized directional derivative.

Let us now describe a general method for calculating tangent and normal cones (and thus also subdifferentials), based on the notion of general position, although different from the methods of Chap. II.

3.1.5. Let F, F_1, \dots, F_n be arbitrary sets and K, K_1, \dots, K_n cones in the topological vector space X . Consider the point $x \in X$ and let the following conditions hold:

- (1) $F = F_1 \cap \dots \cap F_n, K \supset K_1 \cap \dots \cap K_n$;
- (2) $K \subset T(F; x)$ and $\bar{K}_j \supset T(F_j; x), j = 1, \dots, n$;
- (3) the cones K_1, \dots, K_n are in general position.

Then the following formulas apply:

$$\begin{aligned} T(F; x) &\supset T(F_1; x) \cap \dots \cap T(F_n; x), \\ N_E(F; x) &\subset N_E(F_1; x) + \dots + N_E(F_n; x), \end{aligned}$$

where the right-hand side of the last inclusion is closed in the pointwise convergence topology of the space $\mathcal{L}(X, E)$.

3.1.6. The cones K_j in 3.1.5 are called regularizing, and the condition $\bar{K} \supset T(F; x)$ is called K -regularity of the set F at the point x .

3.1.7. Let the topological vector spaces X and Y , the set $F \subset X$, the point $x \in X$, and the operator $\Lambda \in \mathcal{L}(X, Y)$ satisfy the following condition: for every $V \in \mathfrak{B}_x$ there is a neighborhood $U \in \mathfrak{B}_{\Lambda x}$ such that

$$\overline{\Lambda[V \cap F]} \supset U \cap \Lambda[F].$$

If, moreover, the operator Λ is open, then

$$\begin{aligned} T(\Lambda[F]; \Lambda x) &\supset \Lambda[T(F; x)], \\ N_E(\Lambda[F]; \Lambda x) &\subset \{T \in \mathcal{L}(Y, E) : T \circ \Lambda \in N_E(F; x)\}. \end{aligned}$$

Propositions 3.1.5 and 3.1.7 constitute the basis of the calculation method. Let the mapping $\psi: \prod_{j=1}^n \mathcal{L}^X \rightarrow \mathcal{L}^Y$ be a finite combination of linear continuous transformations and the intersection operation, also let $\{y\} = \psi(\{x_1\}, \dots, \{x_n\})$. Then, by induction, applying the above propositions, we obtain the formulas

$$\begin{aligned} T(\psi(F_1, \dots, F_n); y) &\supset \psi(T(F_1; x_1), \dots, T(F_n; x_n)), \\ N_E(\psi(F_1, \dots, F_n); y) &\subset \psi^*(N_E(F_1; x_1), \dots, N_E(F_n; x_n)), \end{aligned}$$

where the right-hand side of the last inclusion is closed in the pointwise convergence topology of the space $\mathcal{L}(Y, E)$. The mapping ψ^* is uniquely related to ψ and it can be obtained without difficulty from the formulas of Proposition 3.1.7. The induction steps which apply Proposition 3.1.5 require regularity of the corresponding sets $\bar{K}_j \supset T(F_j'; x_j')$.

3.1.8. The tangent cone $T(F; x)$ and also the related notions of subdifferential and generalized directional derivative for a scalar function were introduced by Clarke [127] by a different method. The above definition of the cone $T(F; x)$ and the proof of its convexity in the general case are due to Rockafellar [231, 233, 234, 235]. The Clarke tangent cone in pseudotopological vector spaces was considered in [43]. On various properties of the cone $T(F; x)$, see also [161, 234, 246].

3.1.9. Subdifferentials of vector-valued mappings were initially introduced for compact Lipschitzian [243, 244] and locally Lipschitzian [40] operators, and later for general operators [42, 246, 247]. Other types of local approximations for operators with values in o.t.v.s. were considered in [126, 191, 209, 210, 211, 214]. The idea of regularizing cones arose, apparently, in [42] and in [246, 247], where the cones $R^1(F; x)$ and $Q^1(F; x)$ respectively were defined (see Sec. 3.2). Note that Rockafellar's approach selects the interior as the regularizing cone, and the general position is realized for one of the reasons noted in Proposition 1.1.2 [231, 233].

3.2. Subdifferential of a Composition

Let us now give a number of rules for calculating tangent and normal cones.

We first introduce two types of regularizing cones.

3.2.1. Let $G \subset X \times Y$ and $z \in \bar{G}$. By definition, let

$$R^1(G; z) = \bigcap_{V \in \mathfrak{B}_0^Y} \bigcup_{U \in \mathfrak{B}_z, \varepsilon > 0} \bigcap_{\substack{z' \in U \cap G \\ t \in (0, \varepsilon)}} [t^{-1}(G - z') + \{0\} \times V],$$

$$Q^1(G; z) = \bigcap_{V \in \mathfrak{B}_0^Y} \bigcup_{\substack{U \in \mathfrak{B}_z, \varepsilon > 0 \\ W \in \mathfrak{B}_0^X}} \bigcap_{\substack{z' \in U \cap G, t \in (0, \varepsilon) \\ h \in W}} [t^{-1}(G - z') + \{h\} \times V].$$

The cones $Q^2(G; z)$ and $R^2(G; z)$ are similarly defined, and $\Phi^2(G; (x, y)) = [\Phi^1(G; (y, x))]^{-1}$, where Φ stands for R or Q . We can show that $\Phi^1(G; z)$ is a convex cone for any G and $z \in \bar{G}$. If $z \notin \bar{G}$, then we set $\Phi(G; z) = \emptyset$. In what follows, instead of $\Phi(G; z)$ -regularity, we simply write Φ -regularity.

Let X, Y, Z be t.v.s., $u = (x, y, z) \in X \times Y \times Z$, $u_2 = (x, y)$, $u_1 = (y, z)$. Consider the correspondences $F_2 \subset X \times Y$ and $F_1 \subset Y \times Z$.

3.2.2. THEOREM. Let the correspondences F_1 and F_2 satisfy the following condition: for any neighborhood $V \in \mathfrak{B}_y$ there are $U \in \mathfrak{B}_x$ and $W \in \mathfrak{B}_z$ such that $\overline{F_1[x'] \cap F_2[z'] \cap V} \neq \emptyset$ for all $(x', z') \in U \times W \cap F_2 \circ F_1$. Moreover, let one of the following assumptions hold:

- (1) the set F_1 is R^1 -regular at the point u_1 for $i = 1, 2$, and the cones $R^2(F_2; u_2) \times Z$ and $X \times R^1(F_1; u_1)$ are in general position;
- (2) the set F_1 is Q^1 -regular at the point u_1 , and the cones $X \times Q^1(F_1; u_1)$ and $T(F_2; u_2) \times Z$ are in general position;
- (3) the set F_2 is Q^2 -regular at the point u_2 and the cones $X \times T(F_1; u_1)$ and $Q^2(F_2; u_2) \times Z$ are in general position.

Then the following formulas hold:

$$T(F_2 \circ F_1; u) \supset T(F_2; u_2) \circ T(F_1; u_1),$$

$$N_E(F_2 \circ F_1; u) \subset N_E(F_2; u_2) \circ N_E(F_1; u_1),$$

and the set in the right-hand part of the last inclusion is closed in the pointwise convergence topology of the space $\mathcal{L}(X \times Z, E)$.

In order to derive the subdifferentiation rule for a superposition, we have to apply Theorem 3.2.1 to epigraphs.

3.2.3. The mapping $f: X \rightarrow \bar{E}$ is called directionally regular at the point x if the condition $f^0(x)h \in E$ implies $(h, f^0(x, h)) \in T(f; x)$ for any $h \in X$.

A mapping f is called Φ -regular at the point x if the set $\text{epi } f$ is Φ -regular at the point $(x, f(x))$.

3.2.4. THEOREM. Let X be a t.v.s., Y an o.t.v.s., E a topological K -space $f: X \rightarrow \bar{Y}$ and $g: Y \rightarrow \bar{E}$, where g is an increasing operator and the restriction of f to $\text{dom } f$ is continuous at the point x , $g \circ f(x) \in E$. Let g be Q^1 -regular at the point $y = f(x)$, and let the cones $T(f; x) \times E$ and $X \times Q^1(g; y)$ be in general position. Then

$$\partial(g \circ f)(x) \subset \bigcup_{\Lambda \in \partial g(y)} \{T \in \mathcal{L}(X, E) : (T, \Lambda) \in N_E(f; x)\}.$$

If, moreover, f is directionally regular at the point x , then

$$(g \circ f)^0(x) \leq g^0(f(x)) \circ f^0(x),$$

$$\partial(g \circ f)(x) \subset \bigcup_{\Lambda \in \partial g(y)} \partial(\Lambda \circ f^0(x)).$$

Also, the set in the right-hand part of the last inclusion is closed.

3.3. Subdifferential of a Sum and a Maximum

We will demonstrate how the proposed method can be applied to subdifferentiation of sums and maxima. Let F_1, \dots, F_n be correspondences from t.v.s. X to t.v.s. Y . Let

$$F_1 + \dots + F_n = \left\{ (x, y) \in X \times Y : y = \sum_{i=1}^n y_i, (x, y_i) \in F_i, i=1, \dots, n \right\},$$

$$F_1 + \dots + F_n = (F_1^{-1} + \dots + F_n^{-1})^{-1}.$$

If the mappings $\sigma_n: (X \times Y)^n \rightarrow X^n \times Y^n$ and $\Delta: X^n \times Y^n \rightarrow X \times Y$ are defined by the relationships

$$\sigma_n: ((x_1, y_1), \dots, (x_n, y_n)) \mapsto (x_1, \dots, x_n, y_1, \dots, y_n),$$

$$\Delta: (x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \left(\frac{1}{n} \sum_{i=1}^n x_i, \sum_{i=1}^n y_i \right),$$

we have

$$F_1 + \dots + F_n = \Delta \left[\sigma_n \left(\prod_{i=1}^n F_i \right) \cap \Delta_n(X) \times Y^n \right].$$

The technique suggested at the end of Sec. 3.1 leads to the following result.

3.3.1. THEOREM. Let the correspondences F_1, \dots, F_n at the point (x, y_1, \dots, y_n) satisfy the following condition: for any $V_i \in \mathfrak{B}_{y_i}^Y$ there are $U \in \mathfrak{B}_x^X$ and $V \in \mathfrak{B}_y^Y, y = y_1 + \dots + y_n$, such that

$$\overline{V_1 \cap F_1[x'] + \dots + V_n \cap F_n[x']} \supset F[x'] \cap V$$

for all $x' \in U$. Let the cones K_1, \dots, K_n in $X \times Y$ be such that either $K_i = R^1(F_i; (x, y_i))$ for all i or $K_1 = T(F_1; (x, y_1))$ and $K_i = Q^1(F_i; (x, y_i))$ for $i \geq 2$. Assume that F_1 is K_1 -regular at the point (x, y_1) for all i , and the cones $\sigma_n \left(\prod_{i=1}^n K_i \right)$ and $\Delta_n(X) \times Y^n$ are in general position. Then

$$T(F; (x, y)) \supset T(F_1; (x, y_1)) + \dots + T(F_n; (x, y_n)),$$

$$N_E(F; (x, y)) \subset N_E(F_1; (x, y_1)) + \dots + N_E(F_n; (x, y_n)),$$

and the set in the right-hand side of the last inclusion is closed.

The formula for the subdifferential of a sum can be obtained by applying this theorem to epigraphs and noting that

$$\text{epi}(f_1 + \dots + f_n) = \text{epi } f_1 + \dots + \text{epi } f_n.$$

Consider the mapping $f_1, \dots, f_n: X \rightarrow \bar{E}$. If the mapping $\Phi: X \rightarrow (E^n)'$ has the form

$$\Phi: x \mapsto \begin{cases} (f_1(x), \dots, f_n(x)), & \text{if } f_i(x) \in E, i=1, \dots, n, \\ -\infty, & \text{if } f_1(x) = \dots = f_n(x) = -\infty, \\ +\infty, & \text{otherwise,} \end{cases}$$

then $f = f_1 \vee \dots \vee f_n = \varepsilon_n \circ \Phi$, where $\varepsilon_n: E^n \rightarrow E$ is a canonical operator.

The rule for subdifferentiation of a maximum is obtained from Theorems 3.2.2 and 3.3.1 and from the form of the support set $\partial \varepsilon_n(z)$.

3.3.2. THEOREM. Let the mappings $f_1, \dots, f_n: X \rightarrow \bar{E}$ be finite at the point x and satisfy the following condition: for any $V_i \in \mathfrak{B}_{f_i(x)}$ $i=1, \dots, n$, there are $U \in \mathfrak{B}_x$ and $V \in \mathfrak{B}_{f(x)}$ such that if $(x', y') \in U \times V \cap \text{epi } f$ then $y' = y_1' \vee \dots \vee y_n'$ for some $y_i' \in V_i, y_i' \geq f_i(x'), i=1, \dots, n$. Assume that the cones K_1, \dots, K_n are such that either $K_i = R^1(f_i; x)$ for all i or $K_1 = T(f_1; x)$ and $K_i = Q^1(f_i; x)$ for $i \geq 2$. Suppose that f_1 is K_1 -regular at the point x for all i , and the cones $\sigma_n \left(\prod_{i=1}^n K_i \right)$ and $\Delta_n(X) \times E^n$ are in general position. Then

$$(f_1 \vee \dots \vee f_n)^0(x)h \leq \sup_{(\alpha_1, \dots, \alpha_n) \in \Gamma(x)} (\alpha_1 \circ f_1^0(x)h + \dots + \alpha_n \circ f_n^0(x)h), \quad h \in X,$$

$$\partial(f_1 \vee \dots \vee f_n)(x) \subset \bigcup_{(\alpha_1, \dots, \alpha_n) \in \Gamma(x)} \{\partial(\alpha_1 \circ f_1^0(x)) + \dots + \partial(\alpha_n \circ f_n^0(x))\},$$

where

$$\Gamma(x) = \left\{ (\alpha_1, \dots, \alpha_n) \in \mathcal{L}^+(E)^n : \sum_{i=1}^n \alpha_i = I_E, \right. \\ \left. \sum_{i=1}^n \alpha_i \circ f_i(x) = \sup_{i=1, \dots, n} f_i(x) \right\}.$$

3.3.3. The results of Secs. 3.2 and 3.3 in their generality are new even for the scalar case $E = \mathbb{R}$. Various versions of the formulas for the subdifferentials of a sum, a maximum, and a composition were derived in [40, 42, 127, 130, 157, 161, 217, 231, 233, 235, 243, 244, 246, 247]. The unique scheme proposed in Sec. 3.1 can be applied to derive all the basic subdifferentiation formulas. The only exceptions are the infinite operations: the maximum and the sum of an infinite family of mappings, integral operators. These topics are of independent interest, and are not treated here. The reader is referred to [3, 17, 22, 27, 51, 63, 81, 86, 124, 132, 160, 224, 229, 231, 241, 245].

3.4. Ioffe Fans. Ekeland's Variational Principle

The subdifferentiation technique presented in Secs. 3.1-3.3 can be applied to derive the necessary conditions for an extremum in multiobjective problems with nonsmooth inequality constraints. This technique, however, is not particularly useful for the case of nonsmooth equality constraints. An appropriate notion of subdifferential for this case was discovered by Ioffe [169]. This notion generalizes the Clarke subdifferential to vector-valued mappings without assuming ordered spaces.

3.4.1. Let X and Y be l.c.s. The correspondence \mathcal{A} from X to Y is called a fan if for any $x, x_1, x_2 \in X$ and $\lambda > 0$ the following conditions are satisfied:

- $\mathcal{A}(x)$ is a nonempty bounded closed convex set;
- $0 \in \mathcal{A}(0)$, $\lambda \mathcal{A}(x) = \mathcal{A}(\lambda x)$;
- $\mathcal{A}(x_1 + x_2) \subset \overline{\mathcal{A}(x_1) + \mathcal{A}(x_2)}$.

The function $s: X \times Y' \rightarrow \mathbb{R}$ defined by the equality

$$s(y', x) = \sup \{ \langle y, y' \rangle : y \in \mathcal{A}(x) \}$$

is called the support function of \mathcal{A} . The support function is sublinear in each of its variables.

3.4.2. Now let X and Y be normed spaces. A fan \mathcal{A} from X to Y is called symmetric if $-\mathcal{A}(x) = \mathcal{A}(-x)$ for all $x \in X$ and bounded if there is a number $k > 0$ such that

$$\| \mathcal{A}(x) \| = \sup \{ \| y \| : y \in \mathcal{A}(x) \} \leq k \| x \|, \quad x \in X.$$

The fan \mathcal{A} is additionally called regular if $\mathcal{A}(x)$ is strongly compact for all $x \in X$ and

$$\inf \{ \| y \| : y \in \mathcal{A}(x), \| x \| \geq 1 \} > 0.$$

3.4.3. Let f be a Lipschitz mapping of some neighborhood of the point $x_0 \in X$ to Y . A symmetric bounded fan $\mathcal{A} \subset X \times Y$ is called a predifferential of f at the point x_0 if

$$\lim_{\substack{x \rightarrow x_0 \\ h \rightarrow 0}} \frac{1}{\|h\|} \inf \{ \|f(x+h) - f(x) - y\| : y \in \mathcal{A}(x) \} = 0.$$

3.4.4. Every Lipschitz mapping has a predifferential. Indeed, the function

$$f^0(x_0; y', h) = \inf_{\varepsilon > 0} \sup_{\substack{\|x+th-x_0\| < \varepsilon \\ \|x-x_0\| < \varepsilon}} \left\{ \frac{1}{t} \langle f(x+th) - f(x), y' \rangle \right\}$$

is a support function of some fan $Df(x_0)$. This fan is a predifferential of f at the point x_0 . The fan $Df(x_0)$ is called the generalized differential of f at the point x_0 .

The basic facts of classical differential calculus are preserved for Lipschitz mappings. As an example, consider the following theorem.

3.4.5. THEOREM. Let X and Y be Banach spaces and f a Lipschitz mapping of the neighborhood U of the point $x_0 \in X$ to Y . Suppose that there exists a regular predifferential of the mapping f at the point x_0 . Then there exist $\delta > 0$ and a Lipschitz mapping g from the neighborhood of the point $f(x_0)$ to X such that $g \circ f(x) = x$ for $\|x - x_0\| < \delta$.

Theorem 3.4.5 (and also many other propositions on nonsmooth functions) is based on Ekeland's well-known variational principle. Numerous applications of this principle to nonsmooth extremal problems are surveyed in [143]. The next subsection formulates Ekeland's principle.

3.4.6. THEOREM. Let (X, d) be a complete metric space and $f: X \rightarrow \mathbb{R}$ a lower semicontinuous function bounded from below which is not identically equal to $+\infty$. Let $\varepsilon > 0$ and $x \in X$ be such that $f(x) \leq \inf_{x' \in X} f(x') + \varepsilon$. Then there is a point $y \in X$ satisfying the following conditions:

- a) $f(x) \geq f(y)$;
- b) $d(x, y) \leq \varepsilon$;
- c) $f(z) > f(y) - \varepsilon d(z, y) \quad \forall z \neq y$.

3.4.7. Definition 3.4.1 is a particular case of Ioffe's fan; Ioffe also proved Theorem 3.4.5 [107, 168, 169]. The generalized differential $Df(x_0)$ in the finite-dimensional case was introduced in a different way in [131, 217]; the finite-dimensional version of the Theorem 3.4.5 was also established by these authors. The implicit function theorem for Lipschitz mappings in the finite-dimensional case is given in [74, 131, 157, 217]. A related notion of derived set and the associated inverse function theorem in a finite-dimensional space were proposed in [9, 253, 254].

Similar topics in Banach spaces were also considered in [145].

3.4.8. Ekeland's principle is traceable to the ideas of Bishop and Phelps [114]. The subsequent stages of its development are presented in [120, 122]. Theorem 3.4.6 and other versions of the principle were established by Ekeland [139-143]. In [118] Ekeland's principle was restated as a general proposition about ordered sets.

3.5. Other Conceptions of Local Approximation

Despite its obvious advantages, Clarke's conception is not always the most convenient and effective. There are obvious examples in which the behavior of a function or a set near some point is better described by other approximation techniques. Other approximation methods can be used, alongside Clarke's approximation, to obtain additional information or they may prove to be preferable for certain classes of sets and functions. For example, it is often

more convenient to work with directional derivatives of a function. Locally convex and quasidifferentiable functions are considered in [29, 81]: these are functions whose directional derivative exists and is a convex function (in a certain sense). This approach was further developed in [20].

3.5.1. The Quasidifferential. Let X be a Banach space, X' its adjoint space. The function f defined on an open set $U \subset X$ is said to be quasidifferentiable at the point $x \in U$ if in any direction $h \in X$ exists the derivative $f'(x)h = \lim_{t \downarrow 0} t^{-1} [f(x+th) - f(x)]$, and the functional $f'(x)$ is representable as a difference of two sublinear functionals. The quasidifferential $\partial f(x)$ of the function f at the point x is an element of the space of convex sets $V(X')$ in X' corresponding to the functional $f'(x)$ under the isomorphism

$$(A_1, A_2) \mapsto s, \quad s(x) = \sup_{\mu \in A_1} \mu(x') - \sup_{\nu \in A_2} \nu(x'),$$

where A_1 and A_2 are convex weakly compact sets in X' (see [56]).

If $(\underline{\partial}f(x), \overline{\partial}f(x))$ is a pair from $\partial f(x)$, then we write $\partial f(x) = (\underline{\partial}f(x), \overline{\partial}f(x))$. A quasidifferential calculus was developed in [20], where necessary conditions for an extremum were obtained in quasidifferential terms.

Directional differentiability of various functions of maximum and minimum was also investigated in [17, 19, 22].

3.5.2. First-Order Convex Approximation. Consider the set Ω in a t.v.s. X . A convex set $F \subset X$ is called a first-order convex approximation to Ω if it satisfies the following conditions:

a) $0 \in F$ and $F \neq \{0\}$,

b) if $\{x_1, \dots, x_n\}$ is an arbitrary finite subset in F and U is an arbitrary neighborhood of 0 in X , then there is a number $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ there is a continuous mapping $\varphi_\varepsilon: \mathbb{R}^n \rightarrow X$ satisfying the relationship

$$\varphi_\varepsilon(a) = \varphi_\varepsilon(a_1, \dots, a_n) \in \left[\varepsilon \left(\sum_{i=1}^n a_i x_i + U \right) \right] \cap \Omega$$

for all $a \in \mathbb{R}^n$.

Using the notion of first-order convex approximation, Neustadt developed an abstract variational theory [209, 210, 211].

3.5.3. Tents. Let Ω be an arbitrary subset in \mathbb{R}^n . A convex cone $K \subset \mathbb{R}^n$ is called a tent of the set Ω at the point x_0 if there is a smooth mapping φ from some neighborhood of the point x_0 to \mathbb{R}^n such that the following conditions are satisfied:

a) $\varphi(x) = x + r(x)$, and $\lim_{x \rightarrow x_0} \frac{|r(x)|}{|x - x_0|} = 0$;

b) $\varphi(x) \in \Omega$ for $x \in U \cap (x_0 + K)$, where U is a ball centered at x_0 .

The cone K is called a local tent of the set Ω at the point x_0 if for any point $x' \in \text{eri } K$ there is a cone $L \subset K$ which is a tent of the set Ω at the point x' and is such that $x' \in \text{eri } L$ and $L - L = K - K$.

3.5.4. LMO-Approximation. This is a certain modification of the fine convex approximation introduced in [65] (see [165-167]).

Let f be a real function defined in some neighborhood U of the point x_0 in a normed space X . The function $\varphi: U \times X \rightarrow R$ is called an LMO-approximation of f at the point x_0 if the following conditions are satisfied:

- a) $\varphi(x, 0) = f(x), x \in U$;
- b) the function $h \rightarrow \varphi(x, h)$ is convex and continuous for all $x \in U$;
- c) $\liminf_{x \rightarrow x_0, h \rightarrow 0} \|h\|^{-1} [\varphi(x, h) - f(x+h)] \geq 0$.

The LMO-approximation is one of the most powerful and elegant techniques for the analysis of extremal problems. While other forms of one-sided approximation are aimed at derivation of necessary conditions of first order, the LMO-approximation provides necessary and sufficient conditions of higher orders [65, 66].

3.5.5. Let us note some other types of local approximations of nondifferentiable functions. The notion of almost-gradient was introduced in [99] for the class of almost-differentiable functions. The set of almost-gradients of such a function at some point is a closed set whose convex hull coincides with the Clarke subdifferential (see also [16, 100]). A somewhat more general notion than the Clarke subdifferential is considered in [9, 253].

Related notions of subgradient were introduced in [38, 111, 112]. The notions of weakly convex function and its quasigradient [39] fall in the same class of ideas. We finally mention the notion of upper convex approximation, and also upper and lower directional derivatives [214].

CHAPTER IV

SOME GENERAL METHODS OF ANALYSIS OF EXTREMAL PROBLEMS

4.1. Vector Programs. Generalized Solutions

The theory of extremal problems is the traditional field of application of local convex analysis. This tradition goes back to the classical work of Kantorovich [30], Karush [171], and Kuhn and Tucker [182] (see also [181]).

One of the most topical branches of the theory of extremal problems is vector programming (multiobjective optimization or multicriterial decision making), i.e., the theory of problems with vector-valued objective functions.

A characteristic feature of vector programs is that different conflicting aims (interests) intertwine into a single complex objective such that, as a rule, it is no longer possible to separate out individual objectives, ignoring all the rest, unless the original statement of the problem is changed radically.

The existence of an ideal unattainable objective is a qualitatively new difficulty which is not encountered in scalar problems. This leads to a number of specific topics: what constitutes a solution of a vector program; how to ensure consistency of the various objectives; is it in principle possible to reconcile the conflicting interest, etc. The formulation of these questions is of general scientific nature, including philosophical, socioeconomical, psychological, and other aspects. The gnosiology of these issues is apparently closely related with Bohr's complementarity principle [7].

Multiobjective optimization originated in economics and its formulation as a mathematical discipline is primarily associated with Pareto [213]. The concepts and methods of

vector optimization began to crystallize in the framework of utility theory, social welfare theory, and game theory. The 1950s represented the beginning of a new stage in the development of multiobjective optimization problems: in that period they were incorporated in general mathematical programming (see, e.g., [175, 182]). A comprehensive survey of vector optimization from 1776 to 1960 will be found in [237]. Subsequent development of this subject is reflected in [8, 103, 199, 200, 201, 256]. A comprehensive bibliography was collected, in particular, in [103] and [201]. Some techniques for the analysis of vector programs based on subdifferential calculus are also presented in these publications: they include the notion of generalized solution, the method of penalty functions, and the scalarization method. We do not touch here on the applications of local convex analysis in optimal control theory (on this topic, see, in particular, [2, 5, 9, 12-14, 29, 84, 101, 102, 109, 128, 129, 132, 183, 211, 224, 225, 227, 232, 253]).

Let X be a t.v.s., E a topological K -space, $F \subset X$, $f: X \rightarrow E$, and consider the program

$$x \in F, \quad f(x) \rightarrow \inf \quad (P)$$

4.1.1. The set $\{x_1, \dots, x_n\} \subset F$ is called a generalized local optimum of the program (P) if there is a neighborhood U of 0 such that $f(x_1) \wedge \dots \wedge f(x_n) \leq f(x_1') \wedge \dots \wedge f(x_n')$ for all $x_i' \in (x_i + U) \cap F$, $i=1, \dots, n$. The set $U \subset F$ is called a generalized ε -optimum of the program (P) if $\inf\{f(x): x \in U\} \geq \inf\{f(x): x \in F\} - \varepsilon$.

We will only consider the case of finite $U = \{x_1, \dots, x_n\}$.

4.1.2. Consider the operator $\varphi: X^n \rightarrow E$ defined by the formula

$$\varphi(x_1, \dots, x_n) = \alpha_1 \circ f(x_1) + \dots + \alpha_n \circ f(x_n);$$

where $\alpha_1, \dots, \alpha_n \in \mathcal{L}^+(E)$ are such that $(\alpha_1, \dots, \alpha_n) \in \partial \varepsilon_n(f(x_1), \dots, f(x_n)) \circ (-I_{X^n})$. It is easily seen that the set U is a generalized solution of the program (P) if and only if the element $(x_1, \dots, x_n) \in X^n$ is a (respectively ε - or local) optimum of the program

$$\xi \in F^n, \quad \varphi(\xi) \rightarrow \inf.$$

Let us now state the corresponding results.

4.1.3. THEOREM. Let $f: X \rightarrow E$ be a continuous convex operator, F a convex set. The set $\{x_1, \dots, x_n\}$ is a generalized ε -optimum of the program (P) if and only if the following system of conditions is consistent:

$$\begin{aligned} \alpha_1, \dots, \alpha_n \in \mathcal{L}^+(E), \quad \alpha_1 + \dots + \alpha_n = I_E; \\ \sum_{i=1}^n \alpha_i \circ f(x_i) = f(x_1) \wedge \dots \wedge f(x_n); \\ \varepsilon_1 \geq 0, \quad \varepsilon_2 \geq 0, \quad \varepsilon_1 + \varepsilon_2 = \varepsilon; \\ 0 \in \partial_{\varepsilon_1}(\alpha_i \circ f)(x_i) + \partial_{\varepsilon_2}(\delta_E(M))(x_i), \quad i=1, \dots, n, \end{aligned}$$

where $\delta_E(F)$ is the indicator operator of the set F .

4.1.4. THEOREM. Let the mapping $f: X \rightarrow E$ be Lipschitzian at the points x_1, \dots, x_n , and let F be an arbitrary subset in X . If the set $\{x_1, \dots, x_n\}$ is a generalized local optimum of the program (P), then there exist $\alpha_1, \dots, \alpha_n \in \mathcal{L}^+(E)$ such that

$$\begin{aligned} \alpha_1 + \dots + \alpha_n = I_E; \\ \sum_{i=1}^n \alpha_i \circ f(x_i) = f(x_1) \wedge \dots \wedge f(x_n); \\ 0 \in \alpha_i \circ \partial f(x_i) + N_E(F; x_i), \quad i=1, \dots, n. \end{aligned}$$

4.1.5. On ideal solutions in vector programs, see, in particular, [1, 57, 220, 260, 261] and also [199-201, 257]. The notion of generalized solution was introduced in [57]. Generalized solutions play a special role for vector programs. Indeed, a generalized solution always exists, whereas an ideal optimum often does not exist.

More general propositions than Theorems 4.1.3 and 4.1.4 were derived in [54] and [43], respectively.

4.2. Method of Penalty Functions

The method of penalty functions is one of the most effective tools of extremum analysis. Abstracting from concrete difficulties associated with the use of penalty functions (e.g., in numerical solution), this method may be regarded as highly universal in the sense that it reduces any constrained extremal problem to an unconstrained problem or to a problem with fewer constraints. At the same time, its universality premises an exceedingly wide class of relevant functions. We are forced to admit very ill-behaved functions, such as nondifferentiable functions and functions with infinite values. Subdifferential calculus makes it possible to overcome these difficulties. As an illustration, let us consider one specific derivation of necessary conditions for an extremum.

4.2.1. Let X be a t.v.s., E a topological K -space, $f, g: X \rightarrow E$, and consider the following program:

$$g(x) \geq 0, \quad f(x) \rightarrow \inf. \quad (P')$$

The mapping $\varphi: X \rightarrow E$, $\varphi: x' \mapsto [f(x') - f(x)] \vee g(x')$ is called an Ioffe penalty of this program [1].

In what follows, we assume for simplicity that all the relevant operators f, g are everywhere defined and continuous.

4.2.2. Let $\epsilon \in E^+$. A feasible point x is called ϵ -optimal for (P') if $f(x) \leq f(x') + \epsilon$ for all $x' \in X$ such that $g(x') \in E^+$. Clearly x is ϵ -optimal for (P') if and only if x is ϵ -optimal in the unconstrained problem $\varphi(x) \rightarrow \inf$.

4.2.3. THEOREM. Let f and g be convex operators such that for some $x_0 \in X$ the element $-g(x_0)$ is the unit in E . Then a feasible point x is ϵ -optimal if and only if the following system of conditions is consistent:

$$\begin{aligned} \alpha, \beta \in \mathcal{L}^+(E); \quad \alpha + \beta = I_E; \quad \text{Ker}(\alpha) = \{0\}; \\ \epsilon_1 \geq 0, \quad \epsilon_2 \geq 0, \quad \epsilon_1 + \epsilon_2 \leq \alpha \epsilon + \beta \circ g(x); \\ 0 \in \partial_{\epsilon_1}(\alpha \circ f)(x) + \partial_{\epsilon_2}(\beta \circ g)(x). \end{aligned}$$

The proof follows immediately by applying the ϵ -subdifferentiation formulas and 4.1.2.

Let us now consider the case of nonconvex f and g .

4.2.4. A feasible point x is called locally optimal for (P') if there is $U \in \mathfrak{B}_x$ such that $f(x) \leq f(x')$ for all $x' \in U, g(x') \in E^+$.

As in 4.1.2, any locally optimal point of the program (P') is an unconstrained (locally) optimal point for the Ioffe penalty. The following theorem thus again follows directly from subdifferentiation formulas.

4.2.5. THEOREM. Let f and $g: X \rightarrow E$ be R^1 -regular mappings, and let $R^1(f; x)$ and $R^1(g; x)$ be in general position. Then if x is locally optimal for (P') there are operators $\alpha, \beta \in \mathcal{L}^+(E)$ such that

$$\alpha + \beta = I_E, \quad \beta \circ g(x) = 0, \\ 0 \in \partial(\alpha \circ f^0(x)) + \partial(\beta \circ g^0(x)).$$

If, moreover, for some $h \in X$ the element $-[g(x) + g^0(x)h]$ is the unit in E , then $\text{Ker}(\alpha) = \{0\}$.

4.2.6. The method of penalty functions has been known in the theory of extremal problems since the 1940s [94]. On its role in the development of computer algorithms, see [94]. Theorem 4.1.3 was established in [52, 54], and a somewhat weaker form of Theorem 4.1.5 in [43].

4.3. Scalarization Method

Scalarization is a technique whereby vector constraints or objectives are replaced by scalar constructs. We will describe the scheme of the scalarization method in the simplest case of continuous convex and nonconvex Lipschitzian operators.

4.3.1. The operator $f: X \rightarrow E$ is called Lipschitzian at the point $x \in X$ if there are a neighborhood $U \in \mathfrak{B}_x$ and a continuous sublinear operator $p: X \rightarrow E$ such that

$$-p(x'' - x') \leq f(x') - f(x'') \leq p(x' - x'')$$

for all $x', x'' \in U$.

For simplicity, we will consider only continuous (everywhere defined) convex operators and nonconvex Lipschitzian mappings. In this way, we will avoid certain technical details which have no immediate bearing on the scalarization method.

4.3.2. Let X_1 and X be t.v.s., E a K -space of bounded elements, Y an o.t.v.s., so that $\text{int } Y^+ \neq \emptyset$, and consider the program

$$g(x) \leq 0, \quad h(x) = 0, \quad f(x) \rightarrow \inf, \quad (Q)$$

where $f: X \rightarrow E$ and $g: X \rightarrow Y$ are continuous convex operators and $h \in \mathcal{L}(X, X_1)$ is an open operator. This program is called Slater regular if $-g(x_0) \in \text{int } Y^+$ for some feasible $x_0 \in X$.

4.3.3. Let us realize Y as a dense subspace in the space of continuous functions $C(Q)$ on some compactum Q so that the element $-g(x_0)$ is associated with a function identically equal to 1. Consider the operator $\varepsilon \otimes e: Y \rightarrow E$,

$$(\varepsilon \otimes e): y \rightarrow [\sup_{q \in Q} y(q)] \cdot e,$$

where $e \in E^+$, $e \neq 0$. Then the program (Q) is equivalent to the program

$$(\varepsilon \otimes e) \circ g \leq 0, \quad h(x) = 0, \quad f(x) \rightarrow \inf$$

with scalarized inequality constraint.

4.3.4. Let ε be a positive number. A feasible point x is called Pareto ε -optimal (locally optimal) if for any feasible point x' (from some neighborhood of the point x) such that $f(x') - f(x) \leq -\varepsilon 1$ (respectively, $f(x') - f(x) \in \text{int } E^+ \cup \{0\}$) we have $f(x') - f(x) = -\varepsilon 1$ (respectively, $f(x') = f(x)$).

4.3.5. Suppose that E has been realized as $C(Q)$ and consider the operator $\varepsilon \otimes y: E \rightarrow Y$ defined as in 4.3.3. Then, obviously, the point x is Pareto ε -optimal (locally optimal) in the program (Q) if and only if it is ε -optimal (respectively, locally optimal) in the following program:

$$g(x') \leq 0, \quad h(x') = 0, \quad (\varepsilon \otimes y) \circ [f - f(x)] x' \rightarrow \inf.$$

We now give the corresponding results.

4.3.6. THEOREM. A feasible point x is ε -optimal in a Slater regular program if and only if the following system of conditions is consistent:

$$\begin{aligned} & \forall \mathcal{L}^+(Y, E); \quad \mu \in \mathcal{L}(X_1, E); \\ & \varepsilon_1 \geq 0, \quad \varepsilon_2 \geq 0, \quad \varepsilon_1 + \varepsilon_2 \leq \gamma \circ g(x) + \varepsilon; \\ & 0 \in \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} (\gamma \circ g)(x) + \mu \circ h. \end{aligned}$$

4.3.7. THEOREM. If the point x is Pareto ε -optimal in a Slater regular program, and $0 \leq \varepsilon < 1$, then for some functionals α, β, γ on the spaces E, Y , and X_1 respectively we have

$$\begin{aligned} & \alpha > 0, \quad \beta \geq 0, \quad \varepsilon_1 \geq 0, \quad \varepsilon_2 \geq 0, \\ & \varepsilon_1 + \varepsilon_2 \leq \beta \circ g(x) + \varepsilon; \\ & 0 \in \partial_{\varepsilon_1} (\alpha \circ f)(x) + \partial_{\varepsilon_2} (\beta \circ g)(x) + \gamma \circ h. \end{aligned}$$

Conversely, if the above conditions are satisfied for some feasible point x , and $\alpha(1) = 1$, then x is a Pareto ε -optimal point.

Now let the operators f and g be Lipschitzian at the point x , and Y_1, Y , and E as before. Consider the program

$$x \in F; \quad g(x) \leq 0, \quad f(x) \rightarrow \inf, \quad (Q')$$

where $F \subset X$.

4.3.8. THEOREM. Let f, g, E , and Y satisfy the above-listed conditions, and let g be directionally correct for $h \in T(F; x)$ at the point x . If x is an optimal element of the problem (Q') , there exist $\alpha \in \mathcal{L}^+(E)$ and $\lambda \in \mathcal{L}^+(Y, E)$ such that

$$\begin{aligned} & 0 \in \partial(\alpha \circ f^0(x)) + \partial(\lambda \circ g^0(x)) + N_E(F; x); \\ & \lambda \circ g(x) = 0. \end{aligned}$$

If, moreover, $g^0(x)h + g(x) \in -\text{int } Y^+$ for some $h \in T(F; x)$, then $\text{Ker}(\alpha) = \{0\}$.

4.3.9. THEOREM. Let the conditions of Theorem 4.2.5 hold, and let f be directionally correct for $h \in T(F; x)$. If x is a Pareto optimum in the program (Q') , then there exist continuous positive functionals $\lambda \in Y', \mu \in E'$ such that

$$\begin{aligned} & 0 \in \partial(\mu \circ f^0(x)) + \partial(\lambda \circ g^0(x)) + N_R(F; x), \\ & \lambda \circ g(x) = 0. \end{aligned}$$

If, moreover, $g^0(x)h + g(x) \in -\text{int } Y^+$ for some $h \in T(F; x)$, then $\mu > 0$.

4.3.10. Theorems 4.3.6 and 4.3.7 were established in [52, 54], and Theorems 4.3.8 and 4.3.9 in [43].

Clarke's notions of tangent cone and subdifferential acquired considerable popularity in nonlinear programming. Recent research highlighted the fruitfulness of these notions in abstract mathematical programming and in optimal control [40-43, 104, 108, 127, 130, 134-136, 156-159, 161, 162, 217, 231, 242, 244].

4.4. Dubovitskii-Milyutin Method

The Dubovitskii-Milyutin method proposed in [24, 25] is one of the most popular tools of analysis of extremal problems, alongside the analytical apparatus of subdifferential calculus. We will briefly describe the main aspects of this method.

4.4.1. Stage 1. Reduce the extremal problem to determining an intersection point of a finite system of sets $\Omega_1, \dots, \Omega_n$. Thus, for instance, with the program

$$g(x) \leq 0, \quad h(x) = 0, \quad f(x) \rightarrow \inf,$$

where f, g, h are vector functions on X , we can associate the sets $\Omega_f = \{x' \in X : f(x') \neq f(x)\} \cup \{x\}$, $\Omega_g = \{x \in X : g(x) \leq 0\}$ and $\Omega_h = \{x \in X : h(x) = 0\}$. Then a feasible point x is optimal if and only if x is the unique intersection point of the sets Ω_f, Ω_g and Ω_h .

4.4.2. Stage 2. Construct convex cones K_1, \dots, K_n , which constitute local approximations of the sets $\Omega_1, \dots, \Omega_n$ at the point x . These approximations should satisfy the following condition: if K_1, \dots, K_n are in general position, then the set $\Omega_1 \cap \dots \cap \Omega_n$ can be "entered" along the direction of some vector $0 \neq x_0 \in K_1 \cap \dots \cap K_n$. In other words, if $\Omega_1 \cap \dots \cap \Omega_n = \{x\}$, then K_1, \dots, K_n are not in general position. Different local approximations in this scheme lead to different theories.

4.4.3. Stage 3. This is basic for convex analysis, since the preceding stages are reduced to calculating concrete approximations. In stage 3, the cones K_1, \dots, K_n are defined by continuous linear functionals. This is possible due to the following theorems.

4.4.4. THEOREM. Let K_0, K_1, \dots, K_n be cones in t.v.s., and $\text{int} K_i \neq \emptyset$ for $i = 1, \dots, n$. Then the cones $K_0, \text{int} K_1, \dots, \text{int} K_n$ are not in general position if and only if there exist linear functionals $x_0' \in K_0', \dots, x_n' \in K_n'$ which do not vanish simultaneously such that

$$x_0' + x_1' + \dots + x_n' = 0.$$

Here $K' = \partial \delta_R(K) = \{x' \in X' : \langle x, x' \rangle \leq 0, x \in K\}$.

4.4.5. THEOREM. The cones K_0, K_1, \dots, K_n in a finite-dimensional t.v.s. are not in general position if and only if there exist linear functionals $x_0' \in K_0', \dots, x_n' \in K_n'$ which do not vanish simultaneously such that

$$x_0' + x_1' + \dots + x_n' = 0.$$

The last equality is known as the Euler equation.

4.4.6. Stage 4. Necessary conditions for an extremum are written in the form of a Lagrange multipliers rule. To this end, the functionals x'_i entering the Euler equations are linked with the data of the given extremal problem by means of appropriate local approximations of these data.

If the local approximations are the first-order convex approximation, we obtain Neustadt's theory [209-211].

Note that the tent method is also derivable by the Dubovitskii-Milyutin scheme if tents are used as the local approximation (see Sec. 3.5). The Dubovitskii-Milyutin scheme is based on Theorem 4.4.4, whereas the tent method is based on Theorem 4.4.5. These theorems can be combined into one proposition.

4.4.7. THEOREM. Let the cones K_0, K_1, \dots, K_n in t.v.s. be such that if $\bigcap_{s=1}^k K_{i_s}$ and $K_{i_{k+1}}$ for some i_1, \dots, i_{k+1} are not in general position, then they are separated by some linear continuous functional. Then the cones K_0, \dots, K_n are not in general position if and only if there are $x'_i \in K'_i$ which do not vanish simultaneously and which satisfy the Euler equation.

4.4.8. Various realizations of the Dubovitskii-Milyutin scheme are also given in [39, 104, 126, 150-155].

4.5. The Characteristic of Optimal Paths

4.5.1. Consider a finite-step terminal dynamic problem. Let X_0, \dots, X_n be t.v.s. and G_i a correspondence from X_{i-1} to X_i , $i = 1, \dots, n$. The collection G_1, \dots, G_n specifies

a dynamic family of processes $(G_{i,j})_{i < j \leq n}$, where $G_{i,j}$ is a correspondence from X_i to X_j :

$$\begin{aligned} G_{i,j} &= G_{i+1} \circ \dots \circ G_j, \text{ if } j > i+1, \\ G_{i,i+1} &= G_{i+1}, \text{ } i=0, 1, \dots, n-1. \end{aligned}$$

Clearly, $G_{i,j} \circ G_{j,k} = G_{i,k}$ for all $i < j < k \leq n$.

4.5.2. A path of this family of processes is a combination of elements (x_0, \dots, x_n) such that $x_j \in G_{i,j}[x_i]$ for $i < j \leq n$.

Let E be a topological K -space, $f: X \rightarrow \bar{E}$ and $G_0 \subset X_0$. Consider the program

$$x \in G_{0,n}[G_0], \quad f(x) \rightarrow \inf.$$

4.5.3. Let $\varepsilon \in E^+$. The path (x_0, \dots, x_n) is called ε -optimal if for any path (y_0, \dots, y_n) with $y_0 \in G_0$ we have the inequality $f(x_n) \leq f(y_n) + \varepsilon$. The combination of linear continuous operators (T_0, \dots, T_n) where $T_i \in \mathcal{L}(X_i, E)$ is called a ε -characteristic of the path (x_0, \dots, x_n) if there are positive elements $\varepsilon_1, \dots, \varepsilon_n$ such that $\varepsilon = \varepsilon_1 + \dots + \varepsilon_n$ and for any $0 \leq i < j \leq n$

$$T_i x - T_j y \leq T_i x_i - T_j x_j + \varepsilon_{i+1} + \dots + \varepsilon_n, \quad (x, y) \in G_{i,j}.$$

4.5.4. THEOREM. Let $f: X \rightarrow E$ be a convex operator, G_1, \dots, G_n convex correspondences, and $G_0 = \{x_0\}$. Let the convex sets

$$\begin{aligned} F_1 &= G_0 \times \prod_{i=2}^{n+1} X_i, \dots, F_n = \prod_{i=0}^{n-2} X_i \times G_n \times X_{n+1}, \\ F_{n+1} &= \prod_{i=0}^{n-1} X_i \times \text{epi } f, \quad F_{n+2} = \{x_0\} \times \prod_{i=1}^{n+1} X_i, \end{aligned}$$

where $X_{n+1} = E$, be in general position. Then the path (x_0, \dots, x_n) is ε -optimal if and only if for some $0 \leq \delta \leq \varepsilon$ there is a δ -characteristic $(\alpha_0, \dots, \alpha_n)$ such that $\alpha_n \in \partial_{\varepsilon-\delta} f(x_n)$.

4.5.5. Let us now consider the nonconcave case. A path (x_0, \dots, x_n) is called (locally) optimal if there is a neighborhood U of the point x_n such that for any path (y_0, \dots, y_n) with $y_0 \in G$ and $x_n \in U$ we have the inequality $f(x_n) \leq f(y_n)$.

Consider the cones

$$\begin{aligned} K_1 &= R(G_1; (x_0, x_1)) \times \prod_{i=2}^{n+1} X_i, \dots, K_n = \prod_{i=0}^{n-2} X_i \times R(G_n; (x_{n-1}, x_n)) \times E, \\ K_{n+1} &= \prod_{i=0}^{n-1} X_i \times R^1(f; x_n), \quad K_{n+2} = R(G_0; x_0) \times \prod_{i=1}^{n+1} X_i. \end{aligned}$$

4.5.6. THEOREM. Let the mapping f be R^1 -regular at the point x_n , the set G_0 R -regular at the point x_0 , and the set G_i R -regular at the point (x_{i-1}, x_i) for $i = 1, \dots, n$. Let (x_0, \dots, x_n) be an optimal path, and let the cones K_1, \dots, K_{n+2} be in general position. Then there are operators $\alpha_i \in \mathcal{L}(X_i, E)$ satisfying the following conditions:

$$\begin{aligned} \alpha_0 &\in \mathcal{N}_E(G_0; x_0); \quad \alpha_n \in \partial f(x_n); \\ (\alpha_{i-1}, \alpha_i) &\in \mathcal{N}_E(G_i; (x_{i-1}, x_i)), \quad i = 1, \dots, n. \end{aligned}$$

4.5.7. The dual study of convex processes was begun by Rockafellar and Rubinov (on this subject and the past history, see [76, 77, 85, 91, 222] and also [4, 64, 82]).

Numerous applications to mathematical-economic problems and an extensive bibliography will also be found in these references. The asymptotic properties of the dynamic system of processes were studied in [75-77, 87-91], see also Rubinov's survey in this volume. Some

versions of Theorem 4.5.4 will be found in [44, 45, 77, 90]. Theorem 4.5.6 was derived by Kusraev. Theorems on the characteristic of optimal paths essentially constitute one of the possible forms of the discrete maximum principle [5, 12, 23, 29, 84, 91].

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