# BLASCHKE STRUCTURES IN THE PROGRAMMING OF ISOPERIMETRIC PROBLEMS

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UDC 513.9

The paper deals with questions of programming problems of the isoperimetric type in the geometry of convex surfaces using vector structures generated by the addition of Blaschke sets.

1. The analysis of extremal problems, in particular the geometric problems of the isoperimetric type, is intimately connected with the choice of parametrization, i.e., the vector space in which the original problems are embedded. Such a choice to a large extent determines the class of problems which yield to solution as well as the quality of the optimality criteria, i.e., the Euler–Lagrange equations.

For convex surfaces the most natural vector parameterizations are the Minkowski structures, generated by the identification of a convex surface and its support function, and the Blaschke structures, generated by the identification of a convex surface and its surface function. The first identification embeds a geometric extremal problem in the space of continuous functions on the sphere, while the second embedding is in the space of Borel measures on this sphere.

Methods of using Minkowski structures and the class of problems which thus yield to solution are described in adequate detail in [1]. Unfortunately, a number of interesting geometric problems, including the classical isoperimetric problem, apart from the plane case, are not problems of convex programming with the addition of Minkowski sets, leading to a defect in the Euler–Lagrange equations; namely, their solutions need not also be solutions to the original problems.

An idea for overcoming this difficulty was advanced by Blaschke [2], see also [3] and [4]. Specifically, he drew attention to the fact that by adding together, not the surfaces, but their surface functions, one can convert an isoperimetric problem into a convex one. In itself, this remark does not permit the extremal problem to be solved. In order to formalize Blaschke's idea, one must describe the requisite dual pair of spaces, obtain a representation of the cones dual to cones of possible directions, and, most complex of all, describe the subdifferential volume (in contradistinction to the case of the Minkowski structure, the volume in the Blaschke structure is not a homogeneous polynomial).

In the present note we realize the foregoing program and adduce the necessary examples illustrating the salient features of programming extremal problems in Blaschke structures. In particular, we dissect the generalized isoperimetric problem and the Lindelöf problem.

2. Let $R^n$ be an n-dimensional number space with Euclidean norm $|\cdot|$. We denote by $\mathfrak{V}_n$ the cone of convex compacta in $R^n$ and, by $\mathfrak{V}O_n$, the cone of convex bodies of $\mathfrak{J}_n$. It is convenient to use the standard Minkowski duality and to identify element $\mathfrak{x}$ of $\mathfrak{V}_n$ with its support function

$$\bar{\mathfrak{x}}:\ y \mapsto \max_{x \in \mathfrak{x}} (x, y),$$

and the latter with its trace on the sphere $Z_n = \{x \in R^n: |x| = 1\}$.

The structure induced in $\mathfrak{V}_n$ by the lattice $C(Z_n)$ of continuous functions on $Z_n$ is called a Minkowski structure.

Institute of Mathematics, Siberian Branch, Academy of Sciences of the USSR. Translated from Matematicheskie Zametki, Vol. 14, No. 5, pp. 745-754, November, 1973. Original article submitted October 2, 1972.

In addition to the identification of support functions with compacta, we shall use the identification of the class $\{\mathfrak{x}\}$ of convex bodies coinciding with $\mathfrak{x}$ to within parallel transfer with the corresponding surface function $\mu(\mathfrak{x})$. In the sequel, the sign $\{\ \}$ will be summarily dropped from the notation of a class. The class $\mu(\mathfrak{x}) + \mu(\mathfrak{y})$ is called the Blaschke sum of $\mathfrak{x}$ and $\mathfrak{y}$, and is denoted by $\mathfrak{x} \# \mathfrak{y}$. As a rule, all identifications mentioned are emphasized by the use of identical notation for the corresponding objects (since this does not lead to any ambiguity).

Let $\mathfrak{A}_n$ be the set of all nondegenerate, invariant with respect to shifts, positive Borel measures on $Z_n$ (we recall that we understand by the invariance of a measure with respect to shifts its orthogonality of one-points sets in $R^n$). We denote by $[\mathfrak{A}_n]$ the linear hull of $\mathfrak{A}_n$ in the adjoint space $C'(Z_n)$. Thus, $[\mathfrak{A}_n]$ is made up of the measures which are invariant with respect to shifts. Further, we denote by $[\widetilde{\mathfrak{B}O}_n]$ the factor space $C(Z_n)$ by subspace $R^n$. Thus, the elements of $[\widetilde{\mathfrak{B}O}_n]$ are the classes of functions differing by a trace on the sphere of linear functionals over $R^n$ (in other words, by a point). We shall furnish space $[\mathfrak{A}_n]$ with the weak topology and space $[\widetilde{\mathfrak{B}O}_n]$ with the factor topology of the weakened topology in $C(Z_n)$. It is not hard to verify that spaces $[\mathfrak{A}_n]$ and $[\widetilde{\mathfrak{B}O}_n]$ can be brought into a duality by means of the bilinear form

$$\langle \mu, f \rangle = \frac{1}{n} \int_{Z_n} f \, d\mu,$$

coinciding on set $\mathfrak{A}_n \times \widetilde{\mathfrak{B}O}_n$ with mixed volume $V_1(\cdot, \cdot)$ (see [5]).

The structure induced by $[\mathfrak{A}_n]$ in $\mathfrak{A}_n$ (and also in $\mathfrak{B}O_n$), is called a Blaschke structure (in the cone of convex bodies).

There is no difficulty in describing the dual cones in the given structure. Indeed, the following simple assertions are true.

Assertion 1. The dual cone $\mathfrak{A}_n^*$ is the cone of positive elements in $[\widetilde{\mathfrak{B}O}_n]$.

Assertion 2. Let $\bar{\mathfrak{x}} \in \mathfrak{A}_n$. For the cone $\mathfrak{A}_{n,\bar{\mathfrak{x}}}^*$, dual to the cone of possible directions at point $\bar{\mathfrak{x}}$, we have the representation

$$\mathfrak{A}_{n,\bar{\mathfrak{x}}}^* = \{f \in \mathfrak{A}_n^* : \langle \bar{\mathfrak{x}}, f \rangle = 0\}.$$

Less trivial is

Assertion 3. Let $\bar{\mathfrak{x}} \in \mathfrak{A}_n$ and $\mathfrak{y} \in \mathfrak{B}O_n$. If $\mathfrak{y} - \bar{\mathfrak{x}} \in \mathfrak{A}_{n,\bar{\mathfrak{x}}}^*$, then $\mathfrak{y} = \bar{\mathfrak{x}}$.

This assertion is obtained, for example, from inherently interesting Theorem 1. Before formulating it, we agree, with a view to economizing space, to use the following notation:

$$p \colon \mathfrak{x} \mapsto \langle \mathfrak{x}, \mathfrak{x} \rangle^{\frac{1}{n}};$$
$$\hat{p} \colon \mathfrak{x} \mapsto \langle \mathfrak{x}, \mathfrak{x} \rangle^{\frac{n-1}{n}}.$$

We mention that, by virtue of the accepted notation, the volume $V(\mathfrak{x})$ of body $\mathfrak{x}$ is written in the form $\langle \mathfrak{x}, \mathfrak{x} \rangle$ while the Minkowski inequality [5] is rewritten in the form

$$\langle \mathfrak{y}, \mathfrak{x} \rangle \geqslant \hat{p}(\mathfrak{y}) \, p(\mathfrak{x}). \tag{1}$$

THEOREM 1. Let $\bar{\mathfrak{x}}, \mathfrak{y} \in \mathfrak{B}O_n$ and let $\mathfrak{B}_{n,\bar{\mathfrak{x}}}^*$ be the cone adjoint (in $C'(Z_n)$) to the cone of possible directions at point $\bar{\mathfrak{x}}$. If $\mu(\mathfrak{y}) - \mu(\bar{\mathfrak{x}}) \in \mathfrak{B}_{n,\bar{\mathfrak{x}}}^*$, then $\mathfrak{y} = \bar{\mathfrak{x}}$.

Proof. We have that $\mathfrak{y}$ T-precedes $\bar{\mathfrak{x}}$, with $V(\bar{\mathfrak{x}}) = V_1(\mathfrak{y}, \bar{\mathfrak{x}})$ (see [1]). From (1) we have $\hat{p}(\bar{\mathfrak{x}}) \geqslant \hat{p}(\bar{\mathfrak{x}}) \, p(\bar{\mathfrak{x}})$, i.e., $p(\bar{\mathfrak{x}}) \geqslant p(\mathfrak{y})$. In addition,

$$\langle \bar{\mathfrak{x}}, \mathfrak{y} \rangle \geqslant \hat{p}(\bar{\mathfrak{x}}) \, p(\mathfrak{y}) \geqslant \langle \mathfrak{y}, \mathfrak{y} \rangle \geqslant \langle \bar{\mathfrak{x}}, \mathfrak{y} \rangle$$

(the relationship $\langle \mathfrak{y}, \mathfrak{y} \rangle \geqslant \langle \bar{\mathfrak{x}}, \mathfrak{y} \rangle$ follows from the fact that $\mathfrak{y}$ T-precedes $\bar{\mathfrak{x}}$). Since equality is the case in the Minkowski inequality, we have that $\mathfrak{y} = \bar{\mathfrak{x}}$.

As an independent application of Theorem 1 we provide a sharpening of Theorem 4.4.1 of [1].

THEOREM 2. Admissible body $\bar{\mathfrak{x}}$ is a solution to the generalized Urysohn problem: (a) $V_1(\mathfrak{y}_j, \mathfrak{x}) \leqslant b_j$ $(j = 1, \ldots, m)$, (b) $V(\mathfrak{x})$ attains a maximum if and only if numbers $\bar{\alpha}_1, \ldots, \bar{\alpha}_m \in R_+$, with the property of supplementary nonrigidity, can be found such that $\bar{\mathfrak{x}} = \bar{\alpha}_1 \mathfrak{y}_1 \# \bar{\alpha}_2 \mathfrak{y}_2 \# \ldots \# \bar{\alpha}_m \mathfrak{y}_m$.

Proof. Indeed, the Euler–Lagrange equation has the form

$$\bar{\alpha}_1\mathfrak{y}_1 \# \dots \# \bar{\alpha}_m\mathfrak{y}_m - \bar{\mathfrak{x}} \in \mathfrak{V}^*_{n,\bar{\mathfrak{x}}}.$$

There is an important theorem, apparently known to Minkowski and mentioned, for n = 3, in [4].

THEOREM 3. Functional $\hat{p}$, defined on cone $\mathfrak{A}_n$, is superlinear.

Proof. Indeed, by virtue of (1),

$$\hat{p}(\bar{\mathfrak{x}}) = \inf_{\mathfrak{y} \in \mathfrak{V}O_n} \langle \bar{\mathfrak{x}}, \mathfrak{y}\rangle\, p^{-1}(\mathfrak{y}),$$

i.e., $\hat{p}$ is a pointwise infimum of the family of support linear functionals $\mathfrak{z} \mapsto \langle \mathfrak{z}, \mathfrak{y}\rangle p^{-1}(\mathfrak{y})$ $(\mathfrak{y} \in \mathfrak{V}O_n)$.

COROLLARY 1 (the Herglotz Theorem). Function p, defined on convex set $\mathfrak{A}_n$ is concave

Remark 1. Since the area of surface $\mathfrak{x}$ is written in the form $n\langle \mathfrak{x}, \mathfrak{z}_n\rangle$, where $\mathfrak{z}_n = \{x \in R^n : |x| \leqslant 1\}$ is the unit sphere, the isoperimetric problem in a Blaschke structure then reduces to a convex programming problem. Thus, since the dual cones are described, the matter amounts to finding the support set of the derivative volume in the Blaschke structure.

We set

$$S_{\bar{\mathfrak{x}}} = \{f \in [\widetilde{\mathfrak{V}}O_n]\colon \langle \mathfrak{z}, f\rangle \geqslant \hat{p}(\mathfrak{z})\, p(\bar{\mathfrak{x}}) \quad (\mathfrak{z} \in \mathfrak{A}_n),$$
$$\langle \bar{\mathfrak{x}}, f\rangle = \langle \bar{\mathfrak{x}}, \bar{\mathfrak{x}}\rangle\}.$$

Then we have

THEOREM 4. Function $f$ occurs in $S_{\bar{\mathfrak{x}}}$ if and only if $f \geq \bar{\mathfrak{x}}$ and, moreover, $f(z) = \bar{\mathfrak{x}}(z)$ for all z of the carrier $s(\bar{\mathfrak{x}})$.

Proof. Sufficiency. We have

$$\langle \mathfrak{z}, f\rangle \geqslant \langle \mathfrak{z}, \bar{\mathfrak{x}}\rangle \geqslant \hat{p}(\mathfrak{z})\, p(\bar{\mathfrak{x}}) \qquad (\mathfrak{z} \in \mathfrak{A}_n)$$

and, moreover,

$$\langle \bar{\mathfrak{x}}, f\rangle = \frac{1}{n}\int_{Z_n} f\, d\bar{\mathfrak{x}} = \frac{1}{n}\int_{s(\bar{\mathfrak{x}})} f\, d\bar{\mathfrak{x}} = \frac{1}{n}\int_{s(\bar{\mathfrak{x}})} \bar{\mathfrak{x}}\, d\bar{\mathfrak{x}} = \langle \bar{\mathfrak{x}}, \bar{\mathfrak{x}}\rangle.$$

Necessity. With no loss of generality we can assume that $p(\bar{\mathfrak{x}}) = 1$. We set

$$S = \{\mathfrak{y} \in \widetilde{\mathfrak{V}}O_n\colon\ p(\mathfrak{y}) = 1\};$$
$$\widetilde{S} = S + \mathfrak{A}_n^*.$$

We verify that $\widetilde{S}$ is a closed convex set. We first note that if $u \in \widetilde{S}$ and $\alpha \geq 1$ then $\alpha u \in \widetilde{S}$. Indeed, if $u = \mathfrak{y} + g$ $(\mathfrak{y} \in S,\ g \in \mathfrak{A}_n^*)$, then $\alpha u = \mathfrak{y} + (\alpha - 1)\mathfrak{y} + \alpha g$. Now, let $\alpha_1 + \alpha_2 = 1$; $\alpha_1, \alpha_2 \in R_+$; $u_1 = \mathfrak{y}_1 + g_1$; $u_2 = \mathfrak{y}_2 + g_2$, where $\mathfrak{y}_1, \mathfrak{y}_2 \in S$ and $g_1$, $g_2 \in \mathfrak{A}_n^*$. The element $p^{-1}(\alpha_1\mathfrak{y}_1 + \alpha_2\mathfrak{y}_2)\ (\alpha_1\mathfrak{y}_1 + \alpha_2\mathfrak{y}_2)$ occurs in S. With this, by the Brunn–Minkowski Theorem,

$$p(\alpha_1\mathfrak{y}_1 + \alpha_2\mathfrak{y}_2) \geqslant \alpha_1 p(\mathfrak{y}_1) + \alpha_2 p(\mathfrak{y}_2) = 1.$$

And, since

$$\alpha_1 u_1 + \alpha_2 u_2 = p(\alpha_1\mathfrak{y}_1 + \alpha_2\mathfrak{y}_2)\left(\frac{\alpha_1\mathfrak{y}_1 + \alpha_2\mathfrak{y}_2}{p(\alpha_1\mathfrak{y}_1 + \alpha_2\mathfrak{y}_2)} + \frac{\alpha_1 g_1 + \alpha_2 g_2}{p(\alpha_1\mathfrak{y}_1 + \alpha_2\mathfrak{y}_2)}\right),$$

then, by what was proven earlier, $\alpha_1 u_1 + \alpha_2 u_2 \in \widetilde{S}$.

We now establish the closure of $\widetilde{S}$. Let $f = \lim_m (\mathfrak{y}_m + g_m)$ [where the limit is taken in the factor topology of the weak topology in $C(Z_n)$]. Since there exists $m_0$ such that the following inequality holds when $m \geq m_0$;

$$\mathfrak{y}_m \leqslant \mathfrak{y}_m + g_m \leqslant f + \mathfrak{z}_n,$$

then sequence $\{\mathfrak{y}_m\}$ is bounded, and by virtue of the local compactness of $\mathfrak{V}_n$ we can assume that $\{\mathfrak{y}_m\}$ converges.

We now verify that $f$ occurs in $\widetilde{S}$. In the contrary case, by the separability theorem, we could find an element $\tilde{\mathfrak{y}}$ of $[\mathfrak{A}_n]$ such that

$$\inf_{t \in \widetilde{S}} \langle \tilde{\mathfrak{y}}, t\rangle > \langle \tilde{\mathfrak{y}}, f\rangle.$$

It is obvious that $\tilde{\mathfrak{y}}$ is a positive measure which is invariant with respect to shifts. In addition,

$$\inf_{t \in \tilde{S}} \langle \bar{\mathfrak{x}}, t \rangle = \langle \bar{\mathfrak{x}}, f \rangle.$$

Indeed, when $t = \mathfrak{y} + g$ we have

$$\langle \bar{\mathfrak{x}}, t \rangle \geqslant \langle \bar{\mathfrak{x}}, \mathfrak{y} \rangle \geqslant \hat{p}(\bar{\mathfrak{x}})\, p(\mathfrak{y}) = 1 = \langle \bar{\mathfrak{x}}, f \rangle.$$

Consequently, transferring if necessary to the measure $\tilde{\mathfrak{y}} + \bar{\mathfrak{x}}$, we can assume that $\hat{\mathfrak{y}} \in \mathfrak{A}_n$. We set $\hat{\mathfrak{y}} = p^{-1}(\tilde{\mathfrak{y}})\tilde{\mathfrak{y}}$ . Then, in view of the hypothesis,

$$1 = \langle \hat{\mathfrak{y}}, \hat{\mathfrak{y}} \rangle > \langle \hat{\mathfrak{y}}, f \rangle \geqslant \hat{p}(\hat{\mathfrak{y}})\, p(\bar{\mathfrak{x}}) = 1.$$

We have obtained a contradiction. Thus, $f \in \tilde{S}$; i.e., we can find an element $\mathfrak{y}$ of $\mathfrak{A}_n$ such that

$$p(\mathfrak{y}) = p(\bar{\mathfrak{x}}) \text{ and } f \geqslant \mathfrak{y}.$$

We have

$$1 = \langle \bar{\mathfrak{x}}, \bar{\mathfrak{x}} \rangle = \langle \bar{\mathfrak{x}}, f \rangle \geqslant \langle \bar{\mathfrak{x}}, \mathfrak{y} \rangle \geqslant \hat{p}(\bar{\mathfrak{x}})\, p(\mathfrak{y}) = 1$$

and, consequently, $\mathfrak{y} = \bar{\mathfrak{x}}$. Since, by hypothesis, $\langle \bar{\mathfrak{x}}, f \rangle = \langle \bar{\mathfrak{x}}, \bar{\mathfrak{x}} \rangle$ and moreover $f \geq \bar{\mathfrak{x}}$, then $f(z) = \bar{\mathfrak{x}}(z)$ for all z of $s(\bar{\mathfrak{x}})$. Theorem 4 is proven.

We now denote by $p_{\bar{\mathfrak{x}}}$ the closure of the derivative of functional $\mathfrak{y} \mapsto \hat{p}(\mathfrak{y})\, p(\bar{\mathfrak{x}})$ at point $\bar{\mathfrak{x}}$.

THEOREM 5. Let $g \in \mathfrak{A}_{n, \bar{\mathfrak{x}}}$. We have the representation

$$p_{\bar{\mathfrak{x}}}(g) = \langle g, \bar{\mathfrak{x}} \rangle.$$

Proof. In correspondence with the general theorem [6] we have

$$p_{\bar{\mathfrak{x}}}(g) = \inf_{f \in S_{\bar{\mathfrak{x}}}} \langle g, f \rangle.$$

We note that $g = \mathfrak{y} - \bar{\mathfrak{x}}$, where $\mathfrak{y} \in \mathfrak{A}_n$. With this, by Theorem 4 for $f \in S_{\bar{\mathfrak{x}}}$, we have that $f \geq \bar{\mathfrak{x}}$ and, moreover $\langle \bar{\mathfrak{x}}, f \rangle = \langle \bar{\mathfrak{x}}, \bar{\mathfrak{x}} \rangle$ . Consequently,

$$p_{\bar{\mathfrak{x}}}(g) = \inf_{f \in S_{\bar{\mathfrak{x}}}} (\langle \mathfrak{y}, f \rangle - \langle \bar{\mathfrak{x}}, \bar{\mathfrak{x}} \rangle) = \langle \mathfrak{y}, \bar{\mathfrak{x}} \rangle - \langle \bar{\mathfrak{x}}, \bar{\mathfrak{x}} \rangle = \langle g, \bar{\mathfrak{x}} \rangle.$$

Theorem 5 is proven.

We can now turn to examples of programming in Blaschke structures.

**3.** Consider the following problem.

Problem 1 (the generalized isoperimetric problem). We are given bodies $\mathfrak{y}_1, \ldots, \mathfrak{y}_m$ and numbers $b_1, \ldots, b_m$. It is required to find, among the figures satisfying the inequalities $V_1(\mathfrak{x}, \mathfrak{y}_j) \leqslant b_j$ $(j = 1, \ldots, m)$, a body providing a maximum to volume $V(\mathfrak{x})$.

THEOREM 6. An admissible body $\bar{\mathfrak{x}}$ is a solution to the generalized isoperimetric problem if and only if numbers $\bar{\alpha}_1, \ldots, \bar{\alpha}_m \in R_+$, with the property of supplementary nonrigidity, can be found such that

$$\bar{\mathfrak{x}} = \bar{\alpha}_1\mathfrak{y}_1 + \bar{\alpha}_2\mathfrak{y}_2 + \ldots + \bar{\alpha}_m\mathfrak{y}_m.$$

Proof. By virtue of Theorem 3, the problem just posed is equivalent to that of convex programming in $[\mathfrak{A}_n]$. The Lagrange function of this problem has the form

$$\varphi(\mathfrak{x}, \alpha) = V^{\frac{1}{n}}(\bar{\mathfrak{x}})\, V^{\frac{n-1}{n}}(\mathfrak{x}) + \sum_{j=1}^{m} \alpha_j (b_j - V_1(\mathfrak{x}, \mathfrak{y}_j))$$

and is defined on $\mathfrak{A}_n \times R_+^m$. The Slater condition obviously holds and, therefore, it follows from the usual reasoning that the saddle-point inequality $\varphi(\bar{\mathfrak{x}}, \bar{\alpha}) \geqslant \varphi(\mathfrak{x}, \bar{\alpha})$ $(\mathfrak{x} \in \mathfrak{A}_n)$ is equivalent to the condition

$$\sum_{j=1}^{m} \bar{\alpha}_j\mathfrak{y}_j - p_{\bar{\mathfrak{x}}} \in \mathfrak{A}^*_{n, \bar{\mathfrak{x}}}.$$

By assertion 3 and Theorem 5, this also provides what we are trying to prove.

Remark 2. The example of problem 1 makes quite clear the difference in programming in a Blaschke structure and in a Minkowski structure.

In the latter (when $n/\geq 3$), problem 1 is not convex and, as is well known (see [1]), the necessary condition for an extremum for it has the form

$$\mu(\bar{\mathfrak{x}}) = \mu_1\left(\bar{\mathfrak{x}}, \sum_{j=1}^{m} \bar{\alpha}_j \mathfrak{y}_j\right), \tag{2}$$

where $\mu_1(\cdot, \cdot)$ is the corresponding mixed surface function. To extract from (2) the requisite representation of the solution is only possible under the assumption of a priori regularity of the bodies $\bar{\mathfrak{x}}, \mathfrak{y}_1, \ldots, \mathfrak{y}_m$.

As a further typical example, we consider the following problem with an operator constraint on curvature. To save space we limit ourselves to just one constraint of the general form.

Problem 2 (the Lindelöf problem). Let $\mathfrak{x}_0, \mathfrak{y}$ be given bodies. Among the bodies satisfying the conditions

$$\text{(a)}\ \mu(\mathfrak{x}) \leqslant \mu(\mathfrak{x}_0), \quad \text{(b)}\ V_1(\mathfrak{x}, \mathfrak{y}) \leqslant V_1(\bar{\mathfrak{x}}, \mathfrak{y}),$$

it is required to find a figure maximizing the volume.

Remark 3. The programming of a problem with constraints of form (a) directly in a Minkowski structure when $n \geq 3$ is difficult in connection with the nonconvexity of the corresponding admissible region.

THEOREM 7. Admissible body $\bar{\mathfrak{x}}$ is a solution to the Lindelöf problem if we can find $\bar{\alpha} \geq 0$ such that, for

$$\bar{\mathfrak{x}} \geqslant \bar{\alpha}\mathfrak{y} \text{ and } \bar{\mathfrak{x}}(z) = \bar{\alpha}\mathfrak{y}(z)$$

all z of the carrier have measure $\mu(\mathfrak{x}_0) - \mu(\bar{\mathfrak{x}})$.

Proof. The Lagrange function of the corresponding convex programming problem is defined on $\mathfrak{A}_n \times \mathfrak{A}_n^* \times R_+$ and has the form

$$\varphi(\mathfrak{x}, f, R_+) = V^{\frac{1}{n}}(\bar{\mathfrak{x}})\, V^{\frac{n-1}{n}}(\mathfrak{x}) + \alpha\left(V_1(\bar{\mathfrak{x}}, \mathfrak{y}) - V_1(\mathfrak{x}, \mathfrak{y})\right) + \left(\mu(\mathfrak{x}_0) - \mu(\mathfrak{x})\right)(f).$$

By setting $\bar{f} = \bar{\mathfrak{x}} - \bar{\alpha}\mathfrak{y}$, we convince ourselves that at point $(\bar{\mathfrak{x}}, \bar{f}, \bar{\alpha})$ function $\varphi$ has a saddle.

As an example of the use of this theorem we shall show that among the polyhedra having a specified surface area and fixed directions of the exterior normals to the (n − 1)-dimensional faces, the greatest volume is possessed by the polyhedron circumscribed about the sphere (the Lindelöf Theorem [7]). Indeed, in problem 2 we set $\mathfrak{x}_0 = c\,\tilde{\mathfrak{x}}$, where $\tilde{\mathfrak{x}}$ is an arbitrary polyhedron with a carrier at the fixed directions and constant c is sufficiently large. Moreover, let $\mathfrak{y} = \mathfrak{z}_n$. Since the condition $\mu(\mathfrak{x}) \leqslant \mu(\mathfrak{x}_0)$ again means that the admissible class is comprised of those polyhedra with the fixed directions of the (n − 1)-dimensional faces, then the criterion of Theorem 7 denotes optimality of the polyhedron circumscribed about the sphere.

4. The author acknowledges his debt to V. M. Tikhomirov for constructive critical remarks and to S. B. Stechkin for his attention to this paper.

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