

SUPREMAL GENERATORS AND THE CONVERGENCE
OF NONEXPANDING OPERATORS

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We establish the necessary and sufficient criteria for elements on which sequences of nonexpanding operators converge when they converge in a given subspace.

1. It was remarked in [1] that the known effect (see [2], [3]) of the definiteness of the convergence of certain sequences of positive operators from their convergence in subspaces also holds for other operators.

The first general results in this direction for nonexpanding operators in spaces of continuous functions were obtained in [4], which was the pretext for this note.

In this paper we describe an elementary construction which makes it possible to obtain various results of this kind starting from the concept of a supremal generator [5]. We retain the fundamental advantages of the previous approach. In particular, the results are local, i.e., we establish the necessary and sufficient criteria for those functions for which sequences converge if they converge in a given subspace (a cone). In addition, as usual, by using generators we can frequently pass from equations to inequalities.

The following description (in part concerning properly nonexpanding operators) is primarily suitable for spaces of continuous functions. The point is that the greatest interest lies in finite generators which, as is well known, occur only in K-linear bounded elements.

Below, as a rule, without specific reference, we use the results of [5, 6] and the terminology of [7].

We recall only one fundamental definition. Let X be a vector subspace of the K-space Y and H a cone (convex cone) in X. An element $y \in Y$ is said to be H-convex if $y = \sup \{h \in H : h \leq y\}$. The cone H is said to be a supremal generator of X with respect to the K-space Y if any element $x \in X$ is H-convex.

2. Thus, let X be some K-linear of bounded elements, order in which is induced from K-space Y, and let H be a cone in X. Let \tilde{H} denote a cone in $X \times X$ on the set $\{(h, -h) : h \in H\}$ and the element $(-1, -1)$ in $Y \times Y$ (here 1 is the identity element in X).

If T is a positive operator from X into a K-space Z (the set of such operators is denoted by $\mathcal{L}^+(X, Z)$ and in addition, $\tilde{T} : \mathcal{L}^+(X \times X, Y \times Y)$, defined by the relation $\tilde{T} : (x, y) \mapsto (Tx, Ty)$, commutes on \tilde{H} with the operation sup, then T is said to be a mixing.

THEOREM 1. Let H be a subspace, where \tilde{H} is a supremal generator of $X \times X$ with respect to the K-space $Y \times Y$ and $T \in \mathcal{L}^+(X, Z)$ is a mixing. Then, for any sequence (T_n) of (regular) operators $T_n : X \rightarrow Z$, $(0) - \lim_n T_n h = Th$ for all $h \in H$ and in addition, the abstract norm $|T_n|$ does not exceed the abstract norm $|T|$ (i.e., $|T_n| \leq |T|$), then $(0) - \lim_n T_n x = Tx$ for all $x \in X$.

Proof. Let f be a normed positive linear functional over X (we assume that X has a topology which is standard for K-lineals of bounded elements). Consider the tensor product

$$W_n = \frac{1}{2} f \otimes (|T| - |T_n|),$$

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i.e., $W_n : x \mapsto \frac{1}{2} f(x) (|T| - |T_n|) 1$. We note that $W_n \in \mathcal{L}^+(X, Z)$. For $x, y \in X$, we now put

$$\begin{aligned} \tilde{T}_n(x, y) &= (T_n^+ x + T_n^- y + W_n(x+y), \\ &T_n^- x + T_n^+ y + W_n(x+y)). \end{aligned}$$

Then $\tilde{T}_n \in \mathcal{L}^+(X \times X, Z \times Z)$, and in addition

$$\tilde{T}_n(h, -h) = (T_n^+ h - T_n^- h, T_n^- h - T_n^+ h) = (T_n h, -T_n h).$$

Also

$$\tilde{T}_n(1, 1) = (|T_n| + 2W_n 1, |T_n| + 2W_n 1) = (T_n, T_n),$$

since $W_n 1 = \frac{1}{2} (T - |T_n|) 1$.

Thus, on a supremal generator we have constructed a sequence of positive linear operators, (o)-convergent on the elements of that generator to an operator which commutes with the operation sup. Consequently, there is (o)-convergence throughout the whole space, i.e., $(o) - \lim_n \tilde{T}_n(x, y) = \tilde{T}(x, y)$ for all $x, y \in X$. In particular, on the elements $(x, -x)$ we have $\tilde{T}_n(x, -x) = (T_n x, -T_n x)$. The theorem is proved.

Note 1. If the cone \tilde{H} is spanned by not more than a denumerable number of generators, (o)-convergence in the theorem can be replaced by (*)-convergence (see [5]).

Note 2. The proof remains valid if we require only that all the elements of the form $(x, -x)$, where $x \in X$, are H-convex.

From this result, in particular, we can obtain a number of theorems on convergence in spaces of measurable functions of the type described in [1], [5] for positive operators. To illustrate we give one example of such a theorem.

Consider the space $C(Q)$ of continuous functions on a compact (for convenience, metric) space Q . Suppose a positive Borel measure μ is given for Q and let $S(Q)$ denote the corresponding space of measurable functions. We shall assume that the compact Q is realized in the conjugate space $C^*(Q)$, i.e., we identify each point $x \in Q$ with Dirac measure $\varepsilon_x : f \mapsto f(x)$. Put $\hat{Q} = Q \cup (-Q)$. Suppose that for any Borel set $e \subset \hat{Q}$ we have $\hat{\mu}(e) = \mu(e \cap Q) + \mu(-e \cap Q)$ and let $S(\hat{Q})$ be the corresponding space of measurable functions. Now we stretch the cone \hat{H} in $C(\hat{Q})$ to the function -1 ($1 : x \mapsto 1, x \in \hat{Q}$) and the set

$$\{\hat{h} \in C(\hat{Q}) : h \in H, \hat{h}(\varepsilon_x) = h(x), \hat{h}(-\varepsilon_x) = -h(x)\},$$

where H is a subspace of $C(Q)$.

We say that the subspace H in $C(Q)$ is the dual generator (with respect to $S(Q)$) if the Shoke boundary of the cone \hat{H} contains the set Q' of complete measure (i.e., $\hat{\mu}(Q') = \hat{\mu}(\hat{Q}) = 2\mu(Q)$). It is easy to see that \hat{H} is thus the supremal generator of $C(\hat{Q})$ with respect to $S(\hat{Q})$ and thus with respect to the K-space Z normally embedded in $S(\hat{Q})$ and containing $C(\hat{Q})$. Thus, in this case Theorem 1 has the following appearance.

THEOREM 2. Let Z_1, Z_2 be Banach spaces contained in $S(Q)$ and containing $C(Q)$, where $C(Q)$ is dense in Z_1 and Z_2 is a KB-space, normally embedded in $S(Q)$. Further, let H be the dual generator, $T \in \mathcal{L}^+(Z_1, Z_2)$ a mixing and the sequence (T_n) of regular operators from Z_1 to Z_2 such that $|T_n| \leq |T|$, and in addition, $\|T_n h - T h\| \rightarrow 0$ for all $h \in H$. Then $\|T_n x - T x\| \rightarrow 0$ for all $x \in Z_1$.

To prove the theorem we only have to note that in a KB-space (*)-convergence coincides with convergence in norm, and that $\|T_n\| \leq \|T\|$, and then refer to [5].

In a certain sense, Theorem 1 can be inverted. In particular,

THEOREM 3. The following assertions are equivalent:

(1) \tilde{H} is a supremal generator of the space $X \times X$ with respect to the K-space $Y \times Y$.

(2) Each element of the form $(x, -x)$, where $x \in X$, is \tilde{H} -convex.

(3) For any sequence of operators (T_n) from X to Y , such that $|T_n| \leq 1$ and $(o) - \lim_n T_n h = h$ for all $h \in H$, it follows that $(o) - \lim_n T_n x = x$ for all $x \in X$.

(4) For any operator T such that the abstract norm $\|T\| \leq 1$ and $Th = h$ for $h \in H$, it follows that $T = E$.*

Proof. We only need to verify the implication (4) \Rightarrow (1). We have to verify that the positive shoot of the operator \widehat{E} of the identity embedding of $X \times X$ in the K -space $Y \times Y$ coincides with \widehat{E} . Thus, let $T \in \mathcal{L}^+(X \times X, Y \times Y)$; $T(h, -h) = (h, -h)$ for $h \in H$ and also $T(1, 1) \leq (1, 1)$. Let $Pr_1(Pr_2)$ denote the projector of $Y \times Y$ on the first (second) factor and put

$$\begin{aligned} T_1^1 &: x \mapsto Pr_1 [T(x, 0)]; & T_2^1 &: x \mapsto Pr_2 [T(x, 0)], \\ T_1^2 &: y \mapsto Pr_1 [T(0, y)]; & T_2^2 &: y \mapsto Pr_2 [T(0, y)]. \end{aligned}$$

Then for $h \in H$ we have

$$\begin{aligned} (T_1^1 - T_2^1)h &= Pr_1 [T(h, 0)] + Pr_1 [T(0, -h)] = Pr_1 [T(h, -h)] = h, \\ (T_2^2 - T_1^2)h &= -Pr_2 [T(0, -h)] - Pr_2 [T(h, 0)] = -Pr_2 [T(h, -h)] = h. \end{aligned}$$

Also

$$(T_1^1 + T_2^1)1 \leq 1; (T_2^2 + T_1^2)1 \leq 1.$$

Consequently, since $|T_1^1 - T_2^1|1 \leq (T_1^1 + T_2^1)1 \leq 1$ ($i = 1, 2$), we have

$$(T_1^1 - T_2^1)x = x; (T_2^2 - T_1^2)x = x \quad (x \in X).$$

We note now that $1 \leq T_1^1 1 \leq T_1^1 1 + T_2^1 1 \leq 1$. Thus, $T_1^1 = 0$. Similarly, $T_2^2 = 0$. Finally, we obtain $T_1^1 x = x$, $T_2^2 y = y$ ($x, y \in X$). Consequently, $T(x, y) = (T_1^1 x + T_2^1 y, T_2^2 x + T_1^2 y) = (x, y)$. The theorem is proved.

In general it is desirable in this theorem to replace the abstract norm by the ordinary norm. Unfortunately, this cannot be done in general, even when Y is a KB -space with additive norm. This can easily be verified by a suitable example.

We take as Q a segment of the circle $Q = \{x \in \mathbb{R}_+^2, \|x\| = 1\}^\dagger$ and as H the space of the traces of linear functions on Q . It is easy to show that any operator from $C(Q)$ into the space $L(Q)$ of Lebesgue-summable functions on Q , coinciding with the identity on H , and of abstract norm not exceeding 1, is the identity (H is the supremal generator of the space $C(Q) \times C(Q)$ with respect to the K -space $B(Q) \times B(Q)$, where $B(Q)$ is the space of functions bounded on Q , i.e., H is necessarily the dual generator). On the other hand, putting

$$(T1)x = \begin{cases} \frac{1}{4\pi}, & x_1 \geq x_2, \\ \frac{15}{4\pi}, & x_1 < x_2, \end{cases}$$

and $Th = h$ for all $h \in H$, and then extending the operator T in accordance with the Hahn-Banach-Kantorovich theorem on $C(Q)$, we obtain an operator from $C(Q)$ to $L(Q)$, which is not the identity, but which coincides with the identity on H and has unit norm.

However, for spaces of bounded functions, the situation is quite different. We have

PROPOSITION 1. Let T be a linear operator from $C(Q)$ into $B(Q)$. Then $\|T\| \leq 1$ if and only if T is a nonexpanding operator (i.e., $\|T\| \leq 1$).

Proof. In fact $\| |T| 1 \| = \| |T| \|$. On the other hand,

$$\begin{aligned} \| |T| \| &= \sup_{\|f\| \leq 1} \| |Tf| \| = \sup_x \{ \sup_{\|f\| \leq 1} |Tf| (x) \} \\ &= \sup_x \sup_{\|f\| \leq 1} (|Tf| (x)) = \sup_x \sup_{\|f\| \leq 1} |(Tf) (x)| \\ &= \sup_{\|f\| \leq 1} \sup_x |(Tf) (x)| = \sup_{\|f\| \leq 1} \|Tf\| = \|T\|. \end{aligned}$$

Thus we obtain the following:

THEOREM 4. Let H be a subspace of $C(Q)$. The following assertions are equivalent:

- (1) H is the supremal generator of the space $C(Q) \times C(Q)$ with respect to the K -space $B(Q) \times B(Q)$.

* As usual, E is the operator identically embedding X in Y .

$\dagger \mathbb{R}_+^n$ is the nonnegative orthant of the n -dimensional numerical space \mathbb{R}^n , $\|x\|$ is the Euclidean norm of the vector x .

(2) For any sequence (T_n) of nonexpanding operators $T_n: C(Q) \rightarrow B(Q)$ such that $\lim_n T_n h = h$ is uniform for all $h \in H$, it follows that the sequence (T_n) converges (in a strong operator topology) to the identity embedding operator.

(3) For each nonexpanding operator T such that $Th = h$ for all $h \in H$ it follows that $T = E$.

Proof. We have to verify only that (1) \Rightarrow (2). To do this it is sufficient to repeat the discussion of Theorem 1, first noting that the supremal generator in this case is characterized in terms of uniform convergence [5].

3. Now we study in more detail the case of the space of continuous functions. First we stipulate that for $z \in Q$, the symbol $\tilde{\varepsilon}_z$ denotes the functional $\tilde{\varepsilon}_z: (f, g) \mapsto f(z)$, where $(f, g) \in C(Q) \times C(Q)$. †

THEOREM 5. Let H be a cone in $C(Q)$. The following assertions are equivalent:

(1) \tilde{H} is the supremal generator of the space $C(Q) \times C(Q)$ with respect to the functional $\tilde{\varepsilon}_z$ (i.e., $\tilde{\varepsilon}_z(f, g) = \sup \{ \tilde{\varepsilon}_z(u, v) : (u, v) \in \tilde{H}, (u, v) \leq (f, g) \}$ for all $f, g \in C(Q)$).

(2) For any $f, g \in C(Q)$ and $\varepsilon > 0$, there can be found $\alpha \geq 0$ and a function $h \in H$ such that

$$f \geq h - \alpha 1; \quad g \geq -h - \alpha 1; \quad f(z) < h(z) + \varepsilon - \alpha.$$

(3) For each sequence of measures (μ_n) such that $\|\mu_n\| \leq 1$ and $\lim_n \mu_n(h) \geq h(z)$ for all $h \in H$, the sequence (μ_n) converges weakly to ε_z .

(4) For each measure μ such that $\|\mu\| \leq 1$ and $\mu(h) \geq h(z)$ for all $h \in H$, it follows that $\mu = \varepsilon_z$.

Proof. The equivalence (1) \Leftrightarrow (2) is obvious.

(1) \Rightarrow (3). Put $\tilde{\nu}(f, g) = \nu(f)$; $\nu(f, g) = \nu(g)$, where $\nu \in C^*(Q)$, $\nu \geq 0$. Then $\tilde{\nu}, \nu$ are positive functionals on $C(Q) \times C(Q)$. For elements $h \in H$ we have

$$\tilde{\varepsilon}_z(h, -h) = h(z) \leq \lim_n \mu_n(h) = \lim_n (\mu_n^+ - \mu_n^-)(h) = \lim_n (\tilde{\mu}_n^+ + \tilde{\mu}_n^-)(h, -h)$$

In addition,

$$\lim_n (\tilde{\mu}_n^+ + \tilde{\mu}_n^-)(-1, -1) = \lim_n |\mu_n|(-1) = -\overline{\lim}_n \|\mu_n\| \geq -1 = \tilde{\varepsilon}_z(-1, -1).$$

Since \tilde{H} is the supremal generator with respect to the functional $\tilde{\varepsilon}_z$, the sequence $(\tilde{\mu}_n^+ + \tilde{\mu}_n^-)$ converges weakly to $\tilde{\varepsilon}_z$. In particular,

$$\lim_n \mu_n(f) = \lim_n (\tilde{\mu}_n^+ + \tilde{\mu}_n^-)(f, -f) = \tilde{\varepsilon}_z(f, -f) = f(z).$$

(3) \Rightarrow (4). is obvious.

(4) \Rightarrow (1). We have to verify that the positive shoot of the functional $\tilde{\varepsilon}_z$ on \tilde{H} coincides with $\{\tilde{\varepsilon}_z\}$, i.e., $\mu \geq 0$, $\mu(h, -h) \geq \tilde{\varepsilon}_z(h, -h)$ ($h \in H$) and $\mu(1, 1) \leq 1$, then $\mu = \tilde{\varepsilon}_z$.

Put $\mu_1(f) = \mu(f, 0)$ and $\mu_2(g) = \mu(0, g)$ ($f, g \in C(Q)$). Then, for all $h \in H$, we find that

$$\mu_1(h) - \mu_2(h) = \mu(h, 0) + \mu(0, -h) = \mu(h, h) \geq \tilde{\varepsilon}_z(h, -h) = h(z).$$

Similarly, $\mu_1(1) + \mu_2(1) = \mu(1, 0) + \mu(0, 1) = \mu(1, 1) \leq 1$. Since, in turn, $\|\mu_1 - \mu_2\| = |\mu_1 - \mu_2|(1) \leq (\mu_1 + \mu_2)(1) \leq 1$, by hypothesis, $\mu_1 - \mu_2 = \varepsilon_z$. Thus it follows that $\mu_2 = 0$ and $\mu_1 = \varepsilon_z$. The theorem is proved.

Note 3. In proving that (1) \Rightarrow (3) we actually only used (2) for pairs of the form $(f, -f)$. Thus, (2) can be replaced by the equivalent

(2'). For every $f \in C(Q)$ and number $\varepsilon > 0$, there can be found a function $h \in H$ such that $f(z) - h(z) < \varepsilon - \|f - h\|$.

The usefulness of the above theorem is, in particular, that there are finite cones H with the properties indicated in it. We give a simple, but (as, for example the results of [5] show) typical example.

†In this case it is also more convenient to work in the space of pairs $C(Q) \times C(Q)$ and not in its obvious realization.

Let $Q = \{x \in R_+^n; \|x\| = 1\}$. We shall assume that the space R^n is realized as the hypersubspace $\hat{R}^n = \{z \in R^{n+1}; \sum_{k=1}^{n+1} z_k = 0\}$ of the space R^{n+1} . It is easy to see that the cone H on the traces of the coordinate functions of R^{n+1} on \hat{Q} (the image of Q in the given realization of R^n) is such that \tilde{H} is the supremal generator of the space $C(\hat{Q}) \times C(Q)$ with respect to any functional $\tilde{\varepsilon}_z$. We note, incidentally, that the minimal number of such functions is exactly equal to the rank [5] of the compact Q , increased by unity. The following simple propositions make it possible to compare Theorems 4 and 5.

PROPOSITION 2. Let H be a subspace. If \tilde{H} is the supremal generator with respect to the functional $\tilde{\varepsilon}_z$, \hat{H} is the supremal generator with respect to the functional $\varepsilon_z : (f, g) \mapsto g(z)$.

Proof. Let $\mu(h, -h) = \varepsilon_z(h, -h)$, $\mu \geq 0$ and $\mu(1, 1) \leq 1$. Put $\hat{\mu}(f, g) = \mu(g, f)$. Then $\mu(1, 1) \leq 1$ and $\hat{\mu}(h, -h) = \mu(-h, h) = \varepsilon_z(-h, h) = h(z) = \tilde{\varepsilon}_z(h, -h)$.

Since \hat{H} is a generator with respect to $\tilde{\varepsilon}_z$, $\hat{\mu}(f, g) = \tilde{\varepsilon}_z(f, g)$ for all $f, g \in C(Q)$. Thus,

$$\mu(f, g) = \hat{\mu}(g, f) = g(z) = \varepsilon_z(f, g).$$

PROPOSITION 3. The cone N in $C(Q) \times C(Q)$ is the supremal generator of the space $C(Q) \times C(Q)$ with respect to $B(Q) \times B(Q)$ if and only if N is the supremal generator of $C(Q) \times C(Q)$ with respect to all functionals $\tilde{\varepsilon}_z$, where $z \in Q$.

The proof is either direct or we can use the realization of $C(Q) \times C(Q)$ as the space of continuous functions on the compact \hat{Q} defined following Note 2.

COROLLARY. \hat{H} is the supremal generator of $C(Q) \times C(Q)$ with respect to $B(Q) \times B(Q)$ if and only if \tilde{H} is the supremal generator of $C(Q) \times C(Q)$ with respect to any functional $\tilde{\varepsilon}_z$, where $z \in Q$.

Now we can compare Theorems 4 and 5. We do this in the promised local form. We make only one stipulation about notation. If H is a subspace of $C(Q)$, the symbol \underline{H} denotes the conical hull of \tilde{H} in $(1,1)$.

THEOREM 6. The following assertions are equivalent:

- (1) The functions $(f, -f)$ and $(-f, f)$ are \tilde{H} -convex.
- (2) The functions $(f, -f)$ and $(-f, f)$ are \underline{H} -convex.
- (3) For each point $z \in Q$ and any sequence of measures (μ_n) such that $\|\mu_n\| \leq 1$ and $\lim_n \mu_n(h) = h(z)$ for all $h \in H$, the sequence $(\mu_n(f))$ converges to $f(z)$.
- (4) For each point $z \in Q$ and any measure μ such that $\|\mu\| \leq 1$ and $\mu(h) = h(z)$ for all $h \in H$, it follows that $\mu(f) = f(z)$.
- (5) For any sequence of nonexpanding operators (T_n) such that $\lim_n T_n h = h$ uniformly for all $h \in H$, it follows that the sequence $(T_n f)$ converges uniformly to f .
- (6) For any nonexpanding operator T such that $Th = h$ for all $h \in H$, it follows that $Tf = f$.

Proof. Only (2) is significantly new here. Since it is obvious that (1) \Rightarrow (2), it is sufficient to establish that (2) \Rightarrow (5). This follows immediately from the construction of the proof of Theorem 1 and Theorem 4.

Note 4. The equivalence (4) \Leftrightarrow (5) was actually established in [4].

Note 5. The specific attention paid to the Dirac measure and the identity embedding operator is fundamentally traditional. It is easy to see that the results described above, with obvious changes, are valid for any measure with norm not exceeding unity and for any nonexpanding operator.

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