ON AN INEQUALITY OF BIEBERBACH

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A method for analyzing certain extremal problems in the theory of convex surfaces was proposed in [1]. The set $\mathfrak{V}_n$ of convex compact subsets of an n-dimensional space R^n equipped with Minkowski operations was considered as a cone in the vector space of A. G. Pinsker and extremal problems were analyzed by using mathematical programming. In particular, for the problem of maximizing a mixed volume with linear constraints it was shown how the solution of the problem produces an inequality connecting the function being maximized and the constraints. In this note, a generalized inequality of Bieberbach is obtained by a similar method. The derivation given here is based on an inequality of A. D. Aleksandrov.

We agree to denote a convex compactum and its support function by the same symbol. Let z_n be the unit sphere with center at the origin in the space R^n; the sphere of directions (the surface of the unit sphere) is denoted by Z_n. We use the usual notation for mixed volumes $V_{m,k}$ and mixed surface functions $\mu_{m,k}$, i.e.,

$$V_{m,k}(A, x, B) = V(A_1, \ldots, A_{n-m}, \underbrace{x, \ldots, x}_{m-k}, \underbrace{B, \ldots, B}_{k})$$
$$= \frac{1}{n}\int_{Z_n} x\,d\mu(A_1, \ldots, A_{n-m}, \underbrace{x, \ldots, x}_{m-k-1}, \underbrace{B, \ldots, B}_{k}) = \frac{1}{n}\int_{Z_n} x\,d\mu_{m,k}(A, x, B).$$

We let d(x) represent the diameter $x \in \mathfrak{V}_n$, i.e.,

$$d(x) = \max_{u \in Z_n}(x(u) + x(-u)) = \max_{u \in Z_n} b(x, u).$$

Since b(x, u) (the width of x in the direction u) is a linear functional in Pinsker space, d(x) is a sublinear functional. We define the set $U_x = \{u \in Z_n\colon b(x, u) = d(x)\}$ and the cone of feasible directions

$$\mathfrak{V}_{n,x} = \{g \in C(Z_n)\colon \exists \alpha_0 > 0\colon x + \alpha g \in \mathfrak{V}_n;\ (0 \leqslant \alpha \leqslant \alpha_0)\};$$

here C(Q) is the space of continuous functions on the compactum Q.

We shall now consider the problem of maximizing a mixed volume on the set of convex surfaces of given diameter.

Problem 1. We seek an $x \in \mathfrak{V}_n$ from the following conditions: 1) $d(x) = d(\bar{x})$; 2) $V_{m,k}(A, x, B)$ attains a maximum. (The convex compacta $\bar{x}, A_1, \ldots, A_{n-m}, B$ are assumed to be solid.)

THEOREM. The following assertions are equivalent:

1) the convex compactum $\bar{x}$ is a solution of Problem 1;

2) for every convex compactum $x \in \mathfrak{V}_n$, the inequality

$$V_{m,k}(A, x, B)\,d^{m-k}(\bar{x}) - V_{m,k}(A, \bar{x}, B)\,d^{m-k}(x) \leqslant 0$$

is satisfied. (The equality sign holds only for solutions* of Problem 1);

3) for every function g in the cone of feasible directions $\mathfrak{V}_{n,\bar{x}}$, there is satisfied the inequality

$$\max_{u \in U_{\bar{x}}} b(\bar{x}, u)\int_{Z_n} g\,d\mu_{m,k}(A, \bar{x}, B) \leqslant \max_{u \in U_{\bar{x}}} b(g, u)\int_{Z_n} \bar{x}\,d\mu_{m,k}(A\ \bar{x}, B).$$

*More exactly, for solids that are homothetic solutions.

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Proof. Consider Problem 2: find $x \in \mathfrak{B}_n$ such that 1) $d(x) \le d(\bar{x})$; 2) $G(x) = V_{m,k}^{\frac{m-k-1}{m-k}}(A, \bar{x}, B)\, V_{m,k}^{\frac{1}{m-k}} \cdot (A, x, B)$ attains a maximum.

Thus, every solution of Problem 2 is a solution of Problem 1, and vice versa. On the other hand, Problem 2 is a problem in concave programing, since $\mathfrak{B}_n$ is a convex closed cone in the space $C(Z_n)$, and $d(x)$ and $G(x)$ are, respectively, sublinear and superlinear functionals of this cone. It is clear that the condition of Slater is satisfied in this problem. According to the Kuhn–Tucker theorem, $\bar{x}$ is a solution if and only if there exists a positive $\bar{\alpha}$ such that the function

$$\Psi(x) = G(x) + \bar{\alpha}(d(\bar{x}) - d(x)) \tag{1}$$

attains a maximum on $\mathfrak{B}_n$ at the point $\bar{x}$. A necessary and sufficient condition for maximality, by virtue of the concavity of $\Psi(x)$, can be written in the form of inequalities in derivatives with respect to feasible directions:

$$\Psi'_{\bar{x}}(g) \leqslant 0 \qquad (g \in \mathfrak{B}_{n,\bar{x}}), \tag{2}$$

$$\Psi'_{\bar{x}}(\bar{x}) = 0. \tag{3}$$

Since the diameter $d(x)$ is sublinear, its derivative with respect to directions can be written (see [2]) in the form

$$d_x'(g) = \max_{u \in U_x} b(g, u).$$

Using the last formula and the formulas for differentiation of mixed volumes given in [1], we find that (3) is equivalent to the condition $\bar{\alpha} = V_{m,k}(A, \bar{x}, B)/d(\bar{x})$. Considering (1) and (2), we obtain the required equivalences.

COROLLARY 1. Let $\bar{x}$, $A_1, \ldots, A_{n-m}$, B be centrally symmetric convex surfaces, with

$$\mu_{m,k}(A, \bar{x}, B)(Z_n \setminus U_{\bar{x}}) = 0.$$

Then $\bar{x}$ is a solution of Problem 1.

COROLLARY 2. Let $A_1, \ldots, A_{n-m}$, B be centrally symmetric convex surfaces. Then for every convex surface x, we have the inequality

$$V_{m,k}(A, x, B)\, d^{m-k}(z_n) - V_{m,k}(A, z_n, B)\, d^{m-k}(x) \leqslant 0,$$

where equality holds only for solutions of Problem 1.

COROLLARY 3. We have Bieberbach's inequality (see [3])

$$V(x) \leqslant 2^{-n} V(z_n)\, d^n(x),$$

where equality holds only for a sphere.

Remark. Consider the convex Y_n symmetric with respect to the origin bounding the compactum y_n, and let $d_{y_n}(x) = \max_{u \in Y_n} b(x, u)$ be the diameter of x in the metric generated by the normalizing function y_n. Using the natural isomorphism of $C(Y_n)$ and $C(Z_n)$, we can easily apply the foregoing reasoning to the case of the diameter d_{y_n}. In particular, for centrally symmetric convex surfaces $A_1, \ldots, A_{n-m}$, B and for every convex surface x, we obtain the inequality

$$V_{m,k}(A, x, B)\, d_{y_n}^{m-k}(y_n) - V_{m,k}(A, y_n, B)\, d_{y_n}^{m-k}(x) \leqslant 0.$$

LITERATURE CITED

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