ON CONVERGENCE TO THE DIRAC MEASURE AND TO THE IDENTITY OPERATOR

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It was shown [1] that, for a sequence of positive measures defined on an interval, the weak convergence to the Dirac measure on the set of second-degree trinomials implies the weak convergence of this sequence to the Dirac measure. Similarly, for a sequence of positive linear operators (T_n), the uniform convergence $T_n f \to f$, for every second-degree trinomial f, implies the convergence of (T_n) to the identity operator topology. Some papers which explain the role of a finite-dimensional subspace in these theorems have appeared recently. This note establishes a connection between the above (and somewhat more subtle) convergence phenomena and the construction of supremal generation.

Let X be a compact topological space, and C(X) the space of all continuous functions on X. We introduce the Chebyshev norm and the natural ordering by means of the cone of nonnegative functions in C(X). The Dirac measure at a point $z \in X$, i.e., the functional $f \mapsto f(z)$, will be denoted by ε_z.

We say that a closed, convex cone H supremally generates C(X) if for every function $f \in C(X)$ there exists a subset A of H such that $f = \sup A$, where sup is taken pointwise.

Theorem 1. A cone H supremally generates C(X) if and only if for every $z \in X$ the only positive Radon measure μ such that $\mu(h) \geq h(z)$ for all $h \in H$ is ε_z.

Proof. Necessity. Suppose $\mu(h) \geq h(z)$ for all $h \in H$. Then, since μ is positive, $\mu(f) \geq \mu(h) \geq h(z)$ for all $h \in H$ such that $h \leq f$. Also $f = \sup\{h \in H : h \leq f\}$. Consequently, $\mu(f) \geq f(z)$ for all $f \in C(X)$, which means that $\mu = \varepsilon_z$.

Sufficiency. We first show that H contains a strongly negative element f_H. Suppose there is no such element. We apply the separation theorem to find a nonzero positive measure μ such that $\mu(h) \geq 0$ for all $h \in H$. Then $(\mu + \varepsilon_z)(h) \geq h(z)$ for every $h \in H$, a contradiction.

Next, assume that C(X) is not supremally generated by H. This means that there is a function $f_0 \in C(X)$ and a point $z \in X$ such that $f_0(z) > \sup\{h(z) : h \in H;\ h \geq f_0\}$. We consider the functional $p_z : C(X) \to R$ given by $p_z : f \mapsto \sup\{h(z) : h \in H;\ h \leq f\}$. A straightforward verification shows that the functional p_z is concave, positively homogeneous, and monotone, i.e., $f_1 \leq f_2 \Rightarrow p_z(f_1) \leq p_z(f_2)$. In order to show that p_z is superlinear [2] it suffices to prove its continuity.

We can find a number $\alpha > 0$ such that $1 \leq -\alpha f_H$ ($1 \in C(X)$ is the constant function equal to one on X). Therefore, we have $\alpha f_H \|f\| \leq f \leq -\alpha f_H \|f\|$. Consequently, $\alpha\|f\| p_z(f_H) \leq p_z(f) \leq \alpha\|f\| p_z(-f_H)$. By superadditivity of p_z, we have

$$0 = p_z(f_H - f_H) \geqslant p_z(f_H) + p_z(-f_H) = f_H(z) + p_z(-f_H).$$

Therefore $p_z(-f_H) \leq -f_H(z)$ and $|p_z(f)| \leq -\alpha f_H(z)\|f\|$ $(f \in C(X))$.

The last inequality proves continuity of p_z. By the theorem of Hörmander [2], there exists a Radon measure μ such that $\mu(f) \geq p_z(f)$ for all $f \in C(X)$. Also $\mu(f_0) = p_z(f_0) < f_0(z)$. Hence $\mu \neq \varepsilon_z$. On the other hand, μ is positive since $f \geq 0 \Rightarrow p_z(f) \geq 0$. For all $h \in H$, $\mu(h) \geq p_z(h) = h(z)$. This is a contradiction and it proves the theorem.

Theorem 2. C(X) is supremally generated by a cone H if and only if for every $z \in X$ and every sequence (μ_n) of positive measures such that $\lim_n \mu_n(h) \geq h(z)$, for every $h \in H$, the sequence (μ_n) is weakly convergent to ε_z.

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Proof. Sufficiency follows from Theorem 1.

Necessity. Let $f \in C(X)$. For an arbitrary $\varepsilon > 0$ choose functions h, $\bar{h} \in H$ such that $f \geq h$; $f(z) < h(z) + (\varepsilon/2)$; $-f \geq \bar{h}$; $-f(z) < \bar{h}(z) + (\varepsilon/2)$. Such functions exist because $C(X)$ is supremally generated by H. Set $\delta(h) = \lim_n \mu_n(h)$, where $h \in H$, and choose $n(\varepsilon)$ so that, for $n \geq n(\varepsilon)$, the following inequalities hold simultaneously: $|\mu_n(h) - \delta(h)| < \varepsilon/2$ and $|\mu_n(\bar{h}) - \delta(\bar{h})| < \varepsilon/2$. Then, since

$$\mu_n(f) - f(z) \geqslant \mu_n(h) - h(z) - \varepsilon/2 \leqslant \mu_n(h) - \delta(h) - \varepsilon/2,$$
$$\mu_n(f) - f(z) \leqslant -\mu_n(\bar{h}) + \bar{h}(z) + \varepsilon/2 \leqslant \mu_n(\bar{h}) + \delta(\bar{h}) + \varepsilon/2,$$

for $n \geq n(\varepsilon)$, we have $|\mu_n(f) - f(z)| \leq \varepsilon$. This proves the theorem.

Theorem 3. Suppose a cone H supremally generates $C(X)$. Then every sequence (T_n) of positive linear operators T_n: $C(X) \to B(X)$ with the property $\lim_n T_n h \geq h$ for every $h \in H$ converges to the identity operator in the operator topology.* If, in addition, H contains a strongly negative element and the compact space X is metrizable, then the above condition is also sufficient.

Proof. Sufficiency is implied by the following reformulation of the theorem of Strassen [3]. Suppose X is a metrizable, compact space and a cone H contains a strongly negative element. $C(X)$ is supremally generated by H if and only if every positive linear operator T: $C(X) \to B(X)$ such that, for all $h \in H$, $Th \geq h$ is equal to the identity operator.

Necessity. H contains a strongly negative element since it supremally generates $C(X)$. Without any loss of generality, we may assume that -1 belongs to H. Let f be an arbitrary function from $C(X)$. Set $\gamma(h) = \lim_n T_n h$, for $h \in H$. Fix $\varepsilon > 0$ and an arbitrary point $z \in X$. Reasoning as in the proof of Theorem 2, we can find functions h_z, $\bar{h}_z \in H$ such that

$$T_n f - f \geqslant T_n b_z - h_z - \frac{\varepsilon}{2} 1 \geqslant T_n h_z - \gamma(h_z) - \frac{\varepsilon}{2} 1,$$
$$T_n f - f \leqslant -T_n \bar{h}_z + \bar{h}_z + \frac{\varepsilon}{2} 1 \leqslant -T_n \bar{h}_z + \gamma(\bar{h}_z) + \frac{\varepsilon}{2} 1$$

in some neighborhood U_z of the point z.

Choose $n(\varepsilon)$ so that for $n \geq n(\varepsilon)$ $\|T_n h_z - \gamma(h_z)\| < \varepsilon/2$ and $\|T_n \bar{h}_z - \gamma(\bar{h}_z)\| < \varepsilon/2$. Then $\sup_{y \in U_z} |(T_n f)(y) - f(y)| \leq \varepsilon$.

The family $\{U_z\}_{z \in X}$ is a covering of X. We choose a finite subcovering to get the result.

We conclude with two remarks. Firstly, $C(X)$ is supremally generated by a finite cone if and only if the topological rank (Fréchet dimension type) of X is finite. In that case, if the topological rank is not greater than n, $C(X)$ is supremally generated by a finite cone spanned by $n + 2$ functions. Secondly, the results presented here hold, with obvious modifications, for finitely supremally generated Banach lattices.

LITERATURE CITED

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*B(X) denotes the space of all bounded functions on X.