MAXIMAL OPERATORS AND CHOQUET COMPONENTS

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It has recently become clear that the natural category in which a Choquet theory can be constructed is the category of Kantorovich spaces ([1]-[4]). The need for constructing such a theory arises in particular from a large collection of problems in analysis and geometry connected with the construction of maximals (in the orderings of Choquet type) of operators (cf., for example, [2]). At the same time the available works on the Choquet operator theory as a rule used two burdensome conditions – the requirement that the cones under consideration be coinitial and the requirement that the maximal operators be completely linear. These conditions naturally give rise to a series of gaps, and this paper is devoted to filling these. In particular, we shall construct below the Choquet boundary for an arbitrary cone in K-space; we shall show that the disjoint complement of the general part of the kernel of maximal operators precisely coincides with the Choquet boundary; we shall also present theorems on balayage, decomposition, and the maximum principle. The results obtained show that the Choquet boundary, the role of which in K-space is played by a certain component, is well suited for the characteristics of maximal operators on the colattice. It is worth noting that in fact this situation (in two particular cases – Radon measure and cone measure), was as a rule investigated earlier (cf., for example, [5], [6]).

Below we shall as a rule use without additional explanation the terminology and results of the theory of Kantorovich spaces ([7], [8]).

Let X be an ordered vector space, and Y a K-space. The symbol $\mathscr{L}^+(X, Y)$ will designate the set of positive linear operators mapping from X to Y. In addition, let H be a cone (convex cone) in X and $T \in \mathscr{L}^+ \cdot (X, Y)$. The positive germ of the operator T on the cone H is the set

$$\mathrm{Spr}\,(T, H) = \{T' \in \mathscr{L}^+(X, Y) : T'h \geqslant Th \ \ (h \in H)\}.$$

An operator T from (X, Y) will be called maximal with respect to the cone H (or H-maximal), if $\mathrm{Spr}\,(T, H) = \{T\}$. We recall that a cone H in X is called coinitial with X if for every x in X the support set $U_X = \{h \in H : h \le x\}$ is nonempty. As is well known [3], an operator T is maximal with respect to a coinitial cone H if and only if $T = q_{H,T}$ where $q_{H,T} : x \mapsto \sup T(U_X)$. In this last situation we also say that H is a supremal generator of X with respect to the operator T.

It is not difficult to see that on a coinitial cone H there always exist H-maximal operators. The converse in general is not true. In fact we have

Proposition 1. There exist maximal operators with values in some (and therefore in any) regular ordered K-space on a given cone H if and only if the set of coboundaries of elements of H is dense in X in the regular topology.*

Proof. $\Rightarrow$ let X_+ be a positive cone in X and $z \in X \ \overline{H + X_+}$. By the separation theorem there exists a form f such that $f(z) \neq 0$ and $f(u) \geq 0$ $(u \in H + X_+)$. It is clear that $f \in \mathscr{L}^+(X, R)$ and, in addition, $f(h) \geq 0$ $(h \in H)$. Let $T : X \to Z$ be an H-maximal operator and $y > 0$, $y \in Z$. We define $T' = T + f \otimes y$, where $f \otimes y : x \mapsto f(x)y$. We have $T' \in \mathscr{L}^+(X, Z)$. Therefore $T'h = Th + f(h)y \geq Th$, i.e., $T' \in \mathrm{Spr}(T, H)$. Thus $f \otimes y = 0$, from which it follows that $f = 0$. This is a contradiction.

*The regular topology in X is the weakest topology in which positive linear forms are continuous. A space X is said to be regularly ordered, if $\mathscr{L}^+(X, R)$ separates points in X. Here and below R is the K-space of real numbers.

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$\Leftarrow$ let $\overline{H + X_+} = X$, and Z be a regularly ordered K-space. Let $T \in \mathcal{L}^+(X, Z)$ be the null operator $T: x \mapsto 0$. We choose $T' \in \mathrm{Spr}(T, H)$ and suppose that $T'x \neq 0$ for some $x \in X$. There exists a form $f \in \mathcal{L}^+(Z, R)$ such that $f(T'x) \neq 0$. We consider the form $(T')^* f: x \mapsto f(T'x)$. We have $(T')^* f \in \mathcal{L}^+(X, R)$, so that $(T')^* f(h) \geq 0$ for all $h \in H$. Consequently, $(T')^* f(x) \geq 0$ $(x \in X)$, i.e., $(T')^* f = 0$. Thus $T' = T$. This proves the proposition.

Therefore, maximal operators even exist, for example, on certain finite dimensional subspaces in the space L^p, although it is clear that finite coinitial cones in L^p (for $p < +\infty$) do not exist.

Maximal operators have a series of important properties. In particular, the problem of test functions is solvable in terms of maximality (cf., for example, [2]). In fact we have

THEOREM 1. The following assertions are equivalent:

(1) The operator T is maximal with respect to the cone on H.

(2) Any simply bounded chain (T_α) of operators from $\mathcal{L}^+(X, Y)$, such that $\underline{\lim}_\alpha T_\alpha h \geq Th$ $(h \in H)$, satisfies $T_\alpha x \xrightarrow{(o)} Tx$ $(x \in X)$.

Proof. It is only necessary to prove the implication (1) $\Rightarrow$ (2). We define $q(x) = \underline{\lim}_\alpha T_\alpha x$. By assumption $q: X \to Y$. It is clear that q is a superlinear operator, i.e.,

$$q(x) = \inf \{Ax: Az \geqslant q(z) \quad (z \in X)\}.$$

It is not difficult to see that $(Az \geq q(z)\ (z \in X)) \Rightarrow A \in \mathrm{Spr}(T, H) \Rightarrow A = T$, so that $q = T$. Therefore $Tx = \underline{\lim}_\alpha T_\alpha x = -\underline{\lim}_\alpha T_\alpha(-x) = \overline{\lim}_\alpha T_\alpha x$, as we desire to prove.

We shall find it useful to have one more

Proposition 2. Let T be an H-maximal operator and $0 \leq T' \leq T$. Then T' is also H-maximal.

Proof. In fact, let $T'' \in \mathrm{Spr}(T', H)$. We define $\tilde{T} = T + T'' - T'$. It is clear that $\tilde{T} \in \mathcal{L}^+(X, Y)$. Therefore for $h \in H$ we have $\tilde{T}h = Th + (T'' - T')h \geq Th$. Thus $\tilde{T} \in \mathrm{Spr}(T, H)$ and so $\tilde{T} = T$. From this it follows that $T'' = T'$.

Below, without making special mention of it, we shall assume that there exist maximal operators on any cones being considered, and we shall also assume that all spaces considered below are regularly ordered.

The maximal operators studied will as a rule be introduced by means of ascending to some K-space Z. In fact: X will be considered as a subspace (with the induced order) in Z and we shall consider the communative diagram.

$$\begin{array}{ccc} H \subset X & \xrightarrow{T} & Y \\ \cap & \nearrow_{P} & \\ Z & & \end{array}$$

where $P \in \mathcal{L}^+(Z, Y)$. In this situation the operator P will be called H-overmaximal, if T is maximal with respect to H. The role of the Choquet boundary in this situation clearly must be played by a component in Z.

THEOREM 2. In the Boolean algebra of projections in Z there exists a largest H-overmaximal projector.

Proof. Let M be the set of H-overmaximal projections. We note that $M \neq \phi$. We define $P_{Ch} = \sup M$. We wish to show that $P_{Ch} \in M$.

Let $A \in \mathcal{L}^+(X, Z)$ and $Ah \geq P_{Ch}h$ $(h \in H)$. For any projector P from M we have $PAh \geq PP_{Ch}h = Ph$ $(h \in H)$, i.e., $PAx = Px$ for $x \in X$. In particular for $x \geq 0$ we have

$$Ax \geqslant P_{Ch}Ax = (\sup_{P \in M} P) Ax = \sup_{P \in M} (PAx) = \sup_{P \in M} Px = P_{Ch}x.$$

Therefore $(A - P_{Ch})x \geq 0$ for all $x \in H + X_+$, i.e., A is the restriction of P_{Ch} to X.

The projector P_{Ch}, defined by Theorem 2, is called the Choquet projector, and the corresponding component Ch is the Choquet component (or the Choquet boundary). The notations $P_{Ch(H,X,Z)}$ and Ch(H, X, Z) are also used.

Remark. Let P(H) be the least colattice generated by H (here X is a K-lineal). It is clear that the restriction $P|_X$ of the projector $P \in \mathscr{L}^+(Z, Z)$ satisfies $\mathrm{Spr}(P|_X, H) = \mathrm{Spr}(P|_X, P(H))$. Therefore in the case when H is coinitial with X, the definition of the Choquet boundary presented above coincides with that introduced in [4].

It is not difficult to see that a sum of H-maximal operators need not be H-maximal. However, if H is a colattice then the situation is different. In fact we have

Proposition 3. Let X be a K-lineal and H be a colattice coinitial with X. Then

$$q_{H,T_1}+q_{H,T_2}=q_{H,T_1+T_2}$$

for $T_1, T_2 \in \mathscr{L}^+(X, Y)$.

Proof. The relation $q_{H,T_1} + q_{H,T_2} \geq q_{H,T_1+T_2}$ is obvious. In addition, for $x \in X$ we have

$$(q_{H,T_1}+q_{H,T_2})x=\sup_{h_1\in U_x}\sup_{h_2\in U_x}(T_1h_1+T_2h_2)\leqslant\sup_{h_1\in U_x}\sup_{h_2\in U_x}(T_1+T_2)(h_1\vee h_2)\leqslant q_{H,T_1+T_2}(x).$$

These propositions indeed indicate the fundamental role of colattices in Choquet theory. The question arises: how do things change for the case of a noncoinitial cone? We shall return to an analysis of this question at the end of the paper within the framework of the general problem of the representation of Choquet ordering.

Now, let $H \subset X \subset Z$ and $\mathfrak{M}(H)$ be the set of H-overmaximal operators defined on Z. We define

$$N_-(T)=\{z\in Z:\ Tz=0\};$$

$$N(T)=\{z\in Z:\ T|z|=0\};$$

$$N_-=\bigcap_{T\in\mathfrak{M}(H)}N_-(T);\quad N=\bigcap_{T\in\mathfrak{M}(H)}N(T).$$

Clearly $N \subset N_- \subset \mathrm{Ch}^d$ (where as usual A^d is the disjunctive complement to A). We have

THEOREM 3. (1) $N_-^d = \mathrm{Ch}$;

(2) If H is coinitial with X or if $\overline{H-H} = X$, then $N^d = \mathrm{Ch}$.

Proof. (1) Let A be an element of $\mathscr{L}^+(X, Z)$ such that $Ah \geq P_{N_-^d}h$ $(h \in H)$ where $P_{N_-^d}$ is a projector on N_-^d and $T: Z \to Y$ is some element of $\mathfrak{M}(H)$. We have $TAh \geq TP_{N_-^d}h$ $(h \in H)$. By Assumption 2 $TP_{N_-^d} \in \mathfrak{M}(H)$. Consequently $T(Ax - P_{N_-^d}x) = 0$ $(x \in X)$. Therefore for $x \in X$ we have $Ax - P_{N_-^d}x \in N_-$, and so $P_{N_-^d}(Ax - P_{N_-^d}x) = 0$. For $x \geq 0$ we have

$$Ax\geqslant P_{N_-^d}Ax=P_{N_-^d}P_{N_-^d}x=P_{N_-^d}x,$$

so that $(A - P_{N_-^d})z \geq 0$ for $z \in H + X_+$, i.e., $P_{N_-^d} \in \mathfrak{M}(H)$. According to Theorem 2, $P_{N_-^d} \leq P_{\mathrm{Ch}}$, i.e., $N_-^d \subset \mathrm{Ch}$. On the other hand since $N_- \subset \mathrm{Ch}^d$, it follows that $N_-^d \supset \mathrm{Ch}^{dd} = \mathrm{Ch}$.

(2) If $\overline{H-H} = X$, then the proof of the relation $N^d = \mathrm{Ch}$ can be carried out as in (1). We turn to the case when H is coinitial with X. Let $D = \{z \in Z : \exists x \in X : |z| \leq x - \mathrm{co}_H x\}$ where $\mathrm{co}_H x = \sup U_x$, i.e., the H-convex hull of x. If $T \in \mathfrak{M}(H)$, then as noted above $Tx = q_{H,T}x$. In addition $q_{H,T}(x) \leq T\,\mathrm{co}_H x \leq Tx$. Thus $T(x - \mathrm{co}_H x) = 0$, and therefore $D \subset N$. For the projector P_{D^d} on D^d we have $P_{D^d}x = 0$ for $x \in D$, i.e., $P_{D^d} \in \mathfrak{M}(H)$ and therefore by Theorem 2 $D^d \subset \mathrm{Ch}$. Thus $\mathrm{Ch} \supset D^d \supset N^d \supset \mathrm{Ch}^{dd} = \mathrm{Ch}$, i.e., $D^d = N^d = N_-^d = \mathrm{Ch}$.

COROLLARY 1. Let T be a completely linear H-overmaximal operator, where H is coinitial with X or $\overline{H-H} = X$. Then the component $N(T)^d$ of essential positivity of T is contained in the Choquet boundary.

Proof. We have $N(T) \supset N$, i.e., $N(T)^d \subset N^d = \mathrm{Ch}$.

COROLLARY 2. If H is coinitial with X, then N is a fundamental of Ch^d. In particular, the restriction of a maximal operator to the disjoint complement of the Choquet boundary is a normal.

Remark. Theorem 3 and its corollary show that the Choquet component plays the same role for operators that the usual Choquet boundary plays for a maximal measure.

THEOREM 4. The set N_- is a component if and only if for every H-overmaximal operator T the formula $TP_{\mathrm{Ch}^d} = 0$ holds.

Proof. $\Rightarrow$ we have $N_- = N_-^{dd} = \mathrm{Ch}^d$, for $z \in Z$ we have $P_{\mathrm{Ch}^d}z \in N_-$.

$\Leftarrow$ if $z \in Ch^d$, then $Tz = TP_{Ch^d} z = 0$. Therefore $Ch^d \subset N_- \subset Ch^d$.

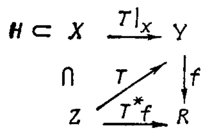
COROLLARY. Suppose that the cone H is coinitial with the K-lineal X and that the positive forms on Z are completely linear. Then N_- is a component.

Proof. Let $T \in \mathscr{L}^+(Z, Y)$ be a H-overmaximal operator. It is clear that $T(x - co_H x) = 0$ for all $x \in X$. Let f be a form from $\mathscr{L}^+(Y, R)$. Then the form $T^*f : z \mapsto f(Tz)$ satisfies the condition

$$T^*f(x) = T^*f(co_H x) = T^*f(co_{P(H)} x) = \sup_{h \leqslant x,\, h \in P(H)} T^*f(h),$$

since T^*f is completely linear and the set $\{h \in P(H) : h \leq x\}$ filtrates from the right. Therefore the form $T^*fP(H)$ is overmaximal. For x from Ch^d (where $Ch = Ch(H, X, Z) = Ch(P(H), X, Z)$) Corollary 1 to Theorem 3 implies that $T^*f(x) = 0$. Since Y is regularly ordered, it follows that $Tx = 0$. Thus for $T \in \mathfrak{M}(H)$ we have $TP_{Ch^d} = 0$. Therefore by Theorem 4 the set N_- is a component.

Remark. In the proof of the last corollory essential use was made of the commutativity of the diagram



Similar diagrams allow us to obtain other operator analogs of theorems from Choquet theory. For example if Q is a metrizable convex compact and locally convex space, C(Q) the space of continuous functions on Q, and Y is some K-space with essentially positive completely linear functionals f, then the operator $T : C(Q) \to Y$ is maximal with respect to the cone of continuous convex functions if and only if the Baire measure T^*f is concentrated on the set of extreme points of Q. It is not difficult to think of examples of applications of the above diagram.

We now turn to the furthest singularities of the Choquet boundary in the coinitial case. We first establish the principle of balayage.

THEOREM 5. Let H be coinitial of X. For every operator $T \in \mathscr{L}^+(X, Y)$ there exists an operator $\tilde{T} \in Spr(T, H)$ such that $Spr(\tilde{T}, H) = Spr(\tilde{T}, H - H)$.

Proof. Let S be the set of restrictions of operators from $Spr(T, H)$ to $H - H$. We shall say that $T_1 \succ T_2$ for $T_1, T_2 \in S$ if for $T_1 h \geq T_2 h (h \in H)$. Clearly $\succ$ is an order relation in S. Let Λ be an arbitrary chain in S. The chain Λ is obviously directed. We consider the chain $(T'h)_{T' \in \Lambda}$ where $h \in H$. Since for $g \in U_{-h}$ the inequality $T'h \leq -Tg (T' \in \Lambda)$ holds, it follows that this chain is increasing and bounded above. We define $T_0 h = (o) - \lim_{T' \in \Lambda} T'h$ and extend T_0 in the natural way to the space $H - H$. Since $H - H$ is coinitial with X, it follows from a theorem of Kantorovich that there exists an extension of T_0 to X. Therefore, the chain Λ is bounded above by the restriction of T_0, and therefore by Zorn's lemma there exists in S a maximal element, which is a restriction to $H - H$ of an operator $\tilde{T}$ from $Spr(T, H)$. Clearly the operator $\tilde{T}$ is the desired operator. This proves the theorem.

We require the one additional concept of boundary projector. A projector P in Z is called H-bounding if for any h from H satisfying $Ph \leq 0$, we have $h \leq 0$.

Remark. The concept of H-bounding projector clearly comes from the concept of Shilov boundary. One must keep in mind however that a projector on a Shilov boundary has stronger properties than bounding (cf., for example, [9]).

Proposition 4. Suppose that the cone H is coinitial with X, where $\overline{H - H} = X$. Then the following assertions are equivalent:

(1) The projector P is H-bounding.

(2) For any H-maximal operator $T : X \to Y$ the following implication holds

$$\forall x \in X (Px \leqslant 0 \Rightarrow Tx \leqslant 0).$$

Proof. (1) $\Rightarrow$ (2). Let $Px \leq 0$ and $h \in U_X$. Then $Ph \leq 0$, while the H-boundedness of P implies that $h \leq 0$. Thus $Tx = \sup T(U_X) \leq 0$.

(2) $\Rightarrow$ (1). Let P^d be the conjugate projector to P. By Theorem 5 there exists an operator $\tilde{T}$ from the germ of the restriction of P^d to X satisfying $Spr(\tilde{T}, H) = Spr(\tilde{T}, H - H)$. Let $(T' \in Spr(\tilde{T}, H - H))$.

Suppose that $(T' - \tilde{T})x \neq 0$ for some $x \in X$. There exists a form $f \in \mathscr{L}^+(Y, R)$ such that $f(T' - \tilde{T})(x) \neq 0$. The form $\hat{f}: x \mapsto f(\tilde{T} - T')x$ is clearly regular, so that $\hat{f}(z) = 0$ for all $z \in H - H$. Thus $T' = \tilde{T}$, i.e., $\tilde{T}$ is H-maximal. Let $Ph \leq 0$. Then $h = Ph + P^d h \leq P^d h \leq \tilde{T}h \leq 0$. This proves the proposition.

Let $H \subset Z$ and H be coinitial with Z. We shall say that the ascending condition holds for an operator $T \in \mathscr{L}^+(Z, Y)$ if $\mathrm{Spr}(T, H) = \mathrm{Spr}(T, P_b(H))$, where $P_b(H)$ is the least conditionally complete colattice generated by H (i.e., the set of elements $z \in Z$ allowing the representation $z = \mathrm{co}_H z$). We remark that the ascending condition is clearly satisfied for any completely linear operator on a colattice.

Proposition 5. Suppose that the operator T satisfies the ascending condition and $TP_{Ch}d = 0$. Then T is H-overmaximal.

Proof. Let $T' \in \mathscr{L}^+(X, Y)$ and $T'h \geq Th$ $(h \in H)$. We consider a monotone extension $\tilde{T}$ of the operator T'. The ascending condition implies that for $x \in X$ we have $\tilde{T}x \geq \tilde{T}\mathrm{co}_H x \geq T\mathrm{co}_H x$. And since $T(x - \mathrm{co}_H x) = TP_{Ch}d(x - \mathrm{co}_H x) = 0$, it follows that $\tilde{T}x \geq Tx$ $(x \in X)$, i.e., $T' = T$.

Remark. It is worth noting that an operator T satisfying on the coinitial cone H the condition $TP_{Ch}d = 0$, is pseudo-overmaximal in the sense that $T(x - \mathrm{co}_H x) = 0$ for all $x \in X$. Thus Proposition 5 gives an indication in the usual sense of the overmaximality of a pseudo-overmaximal operator. We emphasize that pseudo-overmaximal operators possess a series of properties analogous to the properties of overmaximal operators. For example, let $\hat{\mathfrak{M}}(H)$ be the set of pseudomaximal operators acting on Z, and

$$\hat{N} = \bigcap_{T \in \hat{\mathfrak{M}}(H)} N(T); \quad \hat{N}_- = \bigcap_{T \in \hat{\mathfrak{M}}(H)} N_-(T).$$

Then, reasoning as in Theorem 3, one can show that $\hat{N}^d = \hat{N}_-^d = \mathrm{Ch}$. On the other hand, the set $\hat{B}_Y(H)$ of pseudo-upperbounded operators with values in Y, i.e., $\hat{B}_Y(H) = \{T: Z \to Y: |T| \in \hat{\mathfrak{M}}(H)\}$ is a component. At the same time the set $B_Y(H)$ of upperbounded operators, i.e., $B_Y(H) = \{T: Z \to V: |T| \in \mathfrak{M}(H)\}$ is in general not even convex.

From Propositions 4 and 5 we obtain, in particular, the following assertions concerning the equivalence of the maximum principle and Choquet's theorem in K-space.

THEOREM 6. Let H be coinitial with the K-space X, where $\overline{P_b(H) - P_b(H)} = X$. Then the following assertions are equivalent

(1) A Choquet projector is H-bounding.

(2) An operator T is $P_b(H)$-maximal if and only if $TP_{Ch}d = 0$.

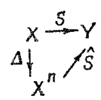
Proof. (1) ⇒ (2). Let $x \in P_b(H)$ and $P_{Ch}x \leq 0$. Then for $h \in U_x$ we have $P_{Ch}h \leq P_{Ch}x \leq 0$, i.e., $h \leq 0$. Consequently, $x = \mathrm{co}_H x \leq 0$. Therefore P_{Ch} is $P_b(H)$-bounding. It then follows from Proposition 4, that for a $P_b(H)$-maximal operator T the condition $Tx = 0$ holds, if $x \in \mathrm{Ch}^d$. The converse assertion is contained in Proposition 5.

(2) ⇒ (1). If $P_{Ch}x \leq 0$, then for an H-maximal operator T (since an H-maximal operator is clearly $P_b(H)$-maximal) we have $Tx = TP_{Ch}dx + TP_{Ch}x \leq 0$. Consequently, by Proposition 4 the projector P_{Ch} is H-bounded. We now turn to the question of the maximality of a sum of maximal operators. It is clear that this question is a particular case of the general problem on the representation of the Choquet ordering for operators (cf., [2]).

We introduce some definitions necessary for what follows. Let X be an Archimedean K-lineal, X^n the corresponding power of X, and $\Delta: x \mapsto (x, \ldots, x)$ be the diagonal operator $\Delta: X \to X^n$. The symbol Δ^* will designate the operator from X^n into X defined by the formula $\Delta^*(x_1, \ldots, x_n) = x_1 \vee \ldots \vee x_n$.

The operator $\check{S}: X^n \to Y$, defined by the relation $\check{S} = S\Delta^*$, where $S \in \mathscr{L}^+(X, Y)$, is called the superdecomposition of the operator S.

The operator $\hat{S}$ from $\mathscr{L}^+(X^n, Y)$ is called a decomposition of the operator S, if the following diagram commutes



Proposition 6. (1) A linear operator A supports a superdecomposition S (a superdecomposition of S (i.e., $Ax \leq \check{S}x$ $(x \in X^n)$)) if and only if A is a decomposition of S.

(2) For any $x \in X^n$ there exists a decomposition S such that for $\check{S}x = \hat{S}x$.

(3) The following representation is valid

$$\check{S}x = \sup \{\hat{S}x : \hat{S}\Delta = S\} \quad (x \in X^n).$$

Proof. (1). Let $Ax \leq Sx (x \in X^n)$. We have for $A\Delta x \leq S\Delta^* \Delta x = Sx$, i.e., $A\Delta = S$. In addition, if $x \leq 0$, $x \in X^n$, then $\Delta^* x \leq 0$, i.e., $Ax \leq 0$ and therefore $A \in \mathscr{L}^+(X^n, Y)$. Thus, A is a decomposition of S. On the other hand, let $A\Delta = S$. For $x \in X^n$ we have $Ax \leq A\Delta\Delta^* x = S\Delta^* x = \check{S}x$, i.e., A supports $\check{S}$.

(2). It is sufficient to verify that for any h_1, $h_2 \in X$ there are operators S_1, $S_2 \in \mathscr{L}^+(X, Y)$ such that $S_1 + S_2 = S$, and in addition $S_1(h_1) + S_2(h_2) = S(h_1 \vee h_2)$. We define $\tilde{h}_1 = h_1 - h_1 \wedge h_2$, $\tilde{h}_2 = h_2 - h_1 \wedge h_2$. It is clear that $\tilde{h}_1 \geq 0$, $\tilde{h}_2 \geq 0$. In addition

$$\tilde{h}_1 \wedge \tilde{h}_2 = (h_1 - h_1 \wedge h_2) \wedge (h_2 - h_1 \wedge h_2) = h_1 \wedge h_2 - h_1 \wedge h_2 = 0,$$

i.e., $\tilde{h}_1$ and $\tilde{h}_2$ are disjunctive. We consider the Dedekind completion $\hat{X}$ of the space X, and let P_1 be a projector on $\{\tilde{h}_1\}^d$ while P_2 is a projector on $\{\tilde{h}_1\}^{dd}$. It is clear that $P_1(\tilde{h}_2) = \tilde{h}_2$ and $P_2\tilde{h}_1 = \tilde{h}_1$. By a theorem of Kantorovich there exists a monotone extension $\tilde{S}$ of the operator S to $\hat{X}$. Let S_1 be the restriction to X of the operator $\tilde{S}P_1$ and let S_2 be its restriction to $\tilde{S}P_2$. It is clear that $S_1 + S_2 = S$. In addition

$$S(h_1 \vee h_2) = S(h_1 \vee h_2) - S(h_1 \wedge h_2) + S(h_1 \wedge h_2) = S(\tilde{h}_1 \vee \tilde{h}_2)$$

$$+ S(h_1 \wedge h_2) = S_1(\tilde{h}_1) + S_2(\tilde{h}_2) + S_1(h_1 \wedge h_2) + S_2(h_1 \wedge h_2) = S_1(h_1) + S_2(h_2).$$

(3). This follows from (1) and (2).

Proposition 7. Let S, $T \in \mathscr{L}^+(X, Y)$ and H be a cone in X^n. Then

$$(\forall \hat{T} \exists \hat{S} : \hat{S} \in \mathrm{Spr}(\hat{T}, H)) \Rightarrow (\forall h \in H \; \check{S}h \geqslant \check{T}h).$$

Proof. It follows from assertion (3) of Proposition 6 that

$$\check{T}h = \sup_{\hat{T}\Delta = T} \hat{T}h \leqslant \sup_{\hat{S}\Delta = S} \hat{S}h = \check{S}h.$$

Remark. The first detailed assertions for the case of convex surfaces (more precisely their support functions) was established by Yu. G. Reshetnyak [10].

Those situations when Proposition 7 has a converse play an essential role in Choquet theory. In this case we say that the decomposition theorem is valid.

Proposition 8. Let H be a colattice and suppose that the decomposition theorem holds. Then the set of H-maximal operators is a cone.

THEOREM 7. If H is a subspace or if X is a separable locally convex K-lineal, if Y is a K-space of bounded elements (with the standard topology), and S, T are continuous, then the decomposition theorem holds.

Proof. First let H be a subspace and $\check{S}h \geq \check{T}h$ for all $h \in H$. Let $\hat{T}$ be some decomposition of T. It follows from Proposition 6 that $\hat{T}h \leq \check{S}h$ $(h \in H)$. By the Hahn–Banach–Kantorovich theorem there exists an extension A of the restriction of $\hat{T}$ to H, supporting $\check{S}$. By Proposition 6 A is a decomposition of the operator S. In addition, by construction A belongs to $\mathrm{Spr}(\hat{T}, H)$.

We now consider the second case. We realize Y as the space C(Q) of continuous functions on the extremal compactum of Q. Since S is continuous and X is a locally convex K-lineal, it follows that $\check{S}$ is continuous. We define $\check{S}_t : x \mapsto \check{S}x(t)$, where $t \in Q$. Then $\check{S}_t$ is a sublinear continuous functional so its support set $U_{\check{S}t} = \{l \in (X_n)' : l(x) \geq \check{S}_t(x) \; (x \in X^n)\}$ is convex and $\sigma((X_n)', X^n)$-compact. We consider the mapping $t \mapsto U_{\check{S}t}$. It is well known [11], this mapping is continuous in the Hausdorff topology (generated by $\sigma((X^n)', X^n)$). The mapping $t \mapsto \hat{T}_t + H^*$ where $\hat{T}_t(x) = \hat{T}x(t)$ and $H^* = \{l \in (X^n)' : l(h) \geq 0 \; (h \in H)\}$, is also continuous. In addition, the decomposition theorem for functionals [2] implies that $A_t = U_{\check{S}t} \cap (\hat{T}_t + H^*)$ is nonempty. Clearly the mapping $t \mapsto A_t$ is upper semicontinuous, and it therefore follows from a theorem of Hasumi [12], that there exists a continuous (endowing $(X^n)'$ with the topology $\sigma((X^n)', X)$) selector φ for this mapping. We define $\hat{S}x : t \mapsto \varphi(t)(x)$ $(x \in X^n, t \in Q)$. Then $\hat{S}$ is an operator from X^n into Y, and so $\hat{S}$ supports $\check{S}$, i.e., by Proposition 6 $\hat{S}$ is a decomposition of S. In addition, by construction $\hat{S} \in \mathrm{Spr}(\hat{T}, H)$. This proves the theorem.

Remark. The technique of selectors used above has been discovered by other scientists. At the same time it would seem that the decomposition theorem without additional restriction is not valid.

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