SIMPLICIAL CONES AND DIRICHLET OPERATORS IN K SPACES

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0^0. The present work contains a presentation of that fragment of Choquet theory in ordered vector spaces which is concerned with the problem of uniqueness of sweeping and the solvability of the Dirichlet problem on the corresponding Choquet boundary. The basic aim of the work is to show that the indicated fragment of the theory is connected with very strong properties of the structure of the order in the corresponding classes of functions. Generally speaking, the last concept is not new. For so-called admissible subspaces of spaces of continuous functions, the corresponding relations are presented in the works of Choquet, Meier, Bauer, M. G. Krein, et al. (see, for example, [1-5]). However, the existing results are inadequate even for the analysis of well-known problems of the reconstruction of harmonic functions with measurable boundary values.

To achieve the stated aim, three basic facts are established in this work.

1. The property of uniqueness of sweeping does not depend on the range of values of the considered operators and (by the Minkowski duality) is determined by the intrinsic properties of the original cone (this fact can be considered as the solution of one of the Choquet problems posed in [6]).
2. In simplicial cones the maximal operators are factorable by a Dirichlet-type operator; furthermore, a broad class of cones, the so-called standard cones, exist for which the sweeping of any operator is obtained by the superposition of this operator with a Dirichlet operator.
3. Standard cones, connected with subspaces which are K lineals (in the induced orders), generate many solvable Dirichlet problems; furthermore, the solvability of the Dirichlet problem, even in one Shilov projector, implies the standardicity of the cone and the boundedness of the projector.

It can be noted that the given part of the results cited contains new information even for the classical cases. This is achieved chiefly at the expense of the attraction of a wider set of cones and Boolean algebras, and not only at the expense of bearing the technical nature of the passage to operators which act in arbitrary K spaces.

1^0. Let us introduce some auxiliary information from Choquet operator theory (see, in particular, [7]). See [8, 9] for information relative to the properties of ordered vector spaces which will be used.

Let X and Y be ordered vector spaces and let H be a cone (= convex cone) in X. The cone H induces an ordering $\underset{H}{>}$ in the set $\mathscr{L}^+(X, Y)$ of positive linear operators which act from X into Y; namely, $T_1 \quad T_2$ means that $T_1 h \geq T_2 h$ for all $h \in H$. If X is a K lineal and if P(H) is the least upper lattice spanned in H, then the ordering $>$ is called the Choquet ordering. The left indicator $\mathrm{Spr}(T, H) = \{T' \in \mathscr{L}_+^{P(H)}(X, Y) : T' \underset{H}{>} T\}$ is called the positive sprout of the operator T in the cone H. The operator T is called maximal with respect to H if $\mathrm{Spr}(T, H) = \{T\}$. If Y is a K space and the cone H is coinitial to X, i.e., $H + X_+ = X$, where X_+ is the cone of positive elements in X, then T is maximal if and only if $T = q_{H,T}$. Here the increasing superposition operator $q_{H,T}$ is defined by the relation $q_{H,T} : x \to \sup T\langle U_X^H \rangle$, where $T\langle A \rangle$ is the image of A under the mapping T and $U_X^H = \{h \in H : h \leq x\}$ is the supporting H-convex set. Furthermore,

$$q_{H,T}(x) = \inf \{T'x : T' \in \mathrm{Spr}(T, H)\}.$$

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If E is the imbedding of X in the K space Z, then the operator $q_{H,E}$ is called the H-convex hull and is denoted by co_H. We note that the sum of two operators, maximal in the Choquet ordering, is maximal.

The symbol $\mathscr{L}(X, Y)$ denotes the space of regular operators, i.e., $\mathscr{L}^+(X, Y) - \mathscr{L}^+(X, Y)$. In what follows we will always assume that the ordering of the spaces X encountered is regular, i.e., that X and $\mathscr{L}(X, R)$ are reduced to duality by the canonical form $(x, f) \to f(x)$, and that a positive cone in X is closed in some topology consistent with this duality.

Let Z be some K space and let $T_0 \in \mathscr{L}^+(X, Z)$. An operator $T \in \mathscr{L}^+(Z, Y)$ is called T_0-maximal if the operator TT_0 is maximal relative to H. In the case when T_0 is an imbedding of X in Z, T_0-maximal operators are called supermaximal.

The least upper bound (in the Boolean algebra of projectors of Z) of the T_0-maximal projectors is called the Choquet projector and is denoted by $P_{Ch(H,T_0)}$ or $P_{Ch(T_0)}$. The component in which the projection operator $P_{Ch(H,T_0)}$ is defined is called the Choquet component or the Choquet boundary and is denoted by $Ch(H, T_0)$. In the case when T_0 is an imbedding, then the notation P_{Ch} and $Ch(H, X, Z)$ are used.

The Choquet projector is T_0-maximal and, in addition, the Choquet boundary coincides with the disjoint complement of the common part of the kernels (or the common part of the null lineals) of the T_0-maximal operators. We recall that the null lineal N(T) of an operator T is the set $\{z \in Z : |z| \in \operatorname{Ker}(T)\}$ and $\operatorname{Ker}(T) = T^{-1}(0)$. In particular, a completely linear operator is T_0-maximal in the Choquet ordering if and only if the component of its essential positive (i.e., the disjoint complement $N(T)^d$ of its null lineal) is contained in the Choquet component.

For a coinitial cone H in X there is a lemma on sweeping, i.e., for every operator $T \in \mathscr{L}^+(X, Y)$, there exists an operator $\tilde{T} \in \operatorname{Spr}(T, H)$ such that $\operatorname{Spr}(\tilde{T}, H) = \operatorname{Spr}(\tilde{T}, H - H)$.

Any mapping of $\mathscr{L}^+(X, Y)$ into the set of maximal operators such that $\psi_H^Y(T) \underset{H}{\succ} T$ is called the sweeping ψ_H^Y generated by H. Thus a sweeping exists for the space $\overline{H - H}$ in a coinitial cone H in X.

Let H be a cone in the K space Z. A projector P in Z is Shilov if $Ph \le 0$ implies that $h \le 0$ (for an element $h \in H$).

To conclude this section we give the following statement which is useful in what follows. If T is a positive operator which acts in a K space Y, then $\sup T\langle U\rangle = \sup T\langle \bar{U}\rangle$ for a convex set U which is bounded above.

In fact, let $z \ge T\langle U\rangle$ and $x \in \bar{U}$. Let us find a mesh $(x_\alpha) \subset U$ such that $x_\alpha \to x$. Then, $f(Tx_\alpha) \to f(Tx)$ for every positive form $f \in \mathscr{L}^+(Y, R)$ (since one and the same set serves as the closures of U in all topologies consistent with the duality). Hence, $f(z) \ge f(Tx)$. The cone of positive elements in Y being closed implies that $z \ge Tx$, and thus $\sup T\langle U\rangle \ge \sup T\langle \bar{U}\rangle$. On the other hand, since $\bar{U} \supset U$, it follows that $\sup T\langle \bar{U}\rangle \ge \sup T\langle U\rangle$.

2^0. In this section we clarify the conditions of uniqueness of sweeping and the simplest peculiarities of the structure of maximal operators in this case.

THEOREM 1 (The Filtration of the Sprout). Let H be a coinitial reproducing cone in X and let Y be some K space. The sprout Spr (T, H) of the operator $T \in \mathscr{L}^+(X, Y)$ is filtered to the right in the ordering $\underset{H}{\succ}$ if and only if the operator $q_{H,T}$ is additive in the cone −H.

Proof. Sufficiency. Let $h_1, h_2, g_1, g_2 \in H$ be such that $h_1 - h_2 = g_1 - g_2$. From the additivity of $q_{H,T}$ in −H, we have

$$q_{H,T}(-h_2) - q_{H,T}(-h_1) = q_{H,T}(-g_2) - q_{H,T}(-g_1).$$

Thus, it is reasonable to define the linear operator:

$$\tilde{T}: h_1 - h_2 \to q_{H,T}(-h_2) - q_{H,T}(-h_1).$$

If $h_1 - h_2 \ge 0$, then $\tilde{T}h_1 \ge \tilde{T}h_2$ by the increase of the operator $q_{H,T}$. Thus, $\tilde{T} \in \mathscr{L}^+(X, Y)$. Now let $T' \in \operatorname{Spr}(T, H)$. For elements $h \in -H$ and $h' \in U_h^H$, we have $Th' \le T'h' \le T'h$. Hence, it is possible to pass to the supremum of the left-hand side of the inequality, which gives $\tilde{T}h = \sup T\langle U_h^H\rangle \le T'h$. Hence, $\tilde{T} \in \operatorname{Spr}(T', H)$.

Necessity. Let $\tilde{T}$ be a maximal operator from Spr (T, H). For an element −h, where $h \in H$, we have

$$\tilde{T}(-h) \leqslant T'(-h) \quad (T' \in \operatorname{Spr}(T, H)).$$

Thus

$$q_{H,T}(-h) \leqslant \tilde{T}(-h) \leqslant \inf_{T' \in \operatorname{Spr}(T,H)} T'(-h) = q_{H,T}(-h).$$

And so, $q_{H,T}(-h) = \tilde{T}(-h)$, QED.

A cone H is called simplicial if it generates a unique sweeping ψ_H^Y [in the space $\mathscr{L}^+(X, Y)$] for each K space Y. In what follows, for convenience a sweeping is denoted by the symbol ψ_H.

THEOREM 2. The following statements are equivalent.

1. The cone H is simplicial.
2. The sprout of any positive form is filtered to the right.
3. For any $h_1, h_2 \in -H$,

$$\overline{U_{h_1}^H + U_{h_2}^H - X_+} \supset U_{h_1+h_2}^H.$$

Proof. $1 \Rightarrow 2$. Obvious.

$2 \Rightarrow 3$. For some $h_1, h_2 \in -H$, let an element $h \in U_{h_1+h_2}^H$ be found such that $h \overline{\in} \overline{U_{h_1}^H + U_{h_2}^H - X_+}$. Let us find a form $f \in \mathscr{L}(X, R)$ for which

$$f(h) > \sup f\langle U_{h_1}^H + U_{h_2}^H - X_+\rangle.$$

It is clear that the form f is positive; moreover,

$$q_{H,f}(h_1 + h_2) \geqslant f(h) > \sup f\langle U_{h_1}^H + U_{h_2}^H\rangle = q_{H,f}(h_1) + q_{H,f}(h_2).$$

Thus, $q_{H,f}$ is not additive on $-H$, which contradicts Theorem 1 since the sprout Spr (f, H) is filtered to the right.

$3 \Rightarrow 1$. Let $T \in \mathscr{L}^+(X, Y)$ and $h_1, h_2 \in -H$. We have

$$q_{H,T}(h_1 + h_2) \geqslant q_{H,T}(h_1) + q_{H,T}(h_2) = \sup T\langle U_{h_1}^H + U_{h_2}^H\rangle$$

$$= \sup T\langle U_{h_1}^H + U_{h_2}^H - X_+\rangle = \sup T\langle \overline{U_{h_1}^H + U_{h_2}^H - X_+}\rangle \geqslant \sup T\langle U_{h_1+h_2}^H\rangle = q_{H,T}(h_1 + h_2).$$

Thus, the operator $q_{H,T}$ is additive on $-H$ and the sprout Spr (T, H) is filtered to the right. The last property is obviously equivalent to the uniqueness of the sweeping.

Proposition 1. If a simplicial cone H is an upper lattice, then the sweeping operator ψ_H is additive.

Proof. We obviously have that $\psi_H(T_1) + \psi_H(T_2) \in \operatorname{Spr}(T_1 + T_2, H)$. Since the operators $\psi_H(T_1)$ and $\psi_H(T_2)$ are maximal, and H is an upper lattice, the operator $\psi_H(T_1) + \psi_H(T_2)$ is maximal. Hence, $\psi_H(\psi_H(T_1) + \psi_H(T_2)) = \psi_H(T_1) + \psi_H(T_2) = \psi_H(T_1 + T_2)$ since H is simplicial.

Namely, the peculiar role of simplicial upper lattices is connected with this proposition. However, it is not necessary to assume that nonlattice simplicial cones occur.

Example 1. Let Q be a convex compactum in a locally convex space and let A be the subspace of continuous affine functions in the space C(Q) of functions continuous on Q. The cone $-P(A)$ is simplicial in C(Q). In fact, by Theorem 2 it is sufficient to verify that for each positive Radon measure μ, there exists a unique $(-P(A))$-maximal measure from the sprout μ. Such a measure obviously has the form $\mu(1)\varepsilon_{x_\mu}$, where ε_z is the Dirac measure, and x_μ is the barycenter of the probability measure $\mu/\mu(1)$.

Now we turn to a canonical method of construction of simplicial cones. First, we recall the following definition.

A subspace H in a K lineal X has the Riesz interpolation property if for any elements $h_1, h_2, g_1, g_2 \in H$ such that $h_1 \vee h_2 \leq g_1 \wedge g_2$, there exists an element $h \in H$ for which $h_1 \vee h_2 \leq h \leq g_1 \wedge g_2$.

Remark 1. If H has the Riesz interpolation property, then for any elements $f \in P(H)$ and $g \in -P(H)$ such that $f \leq g$, there is an element $h \in H$ for which $f \leq h \leq g$. In fact, first let $h_1 \vee h_2 \leq g_1 \wedge \ldots \wedge g_n \wedge g_{n+1}$ and a $z_1 \in H$ exist such that $h_1 \vee h_2 \leq z_1 \leq g_1 \wedge \ldots \wedge g_n$. We have that $h_1 \vee h_2 \leq z_1 \wedge g_{n+1}$, and hence there exists a $z_2 \in H$ for which $h_1 \vee h_2 \leq z_2 \leq z_1 \wedge g_{n+1} \leq g_1 \wedge \cdots \wedge g_{n+1}$. In a similar way, induction can be carried out according to the length of the elements of the lattice P(H).

Proposition 2. If a coinitial subspace of H has the Riesz interpolation property, then the cone P(H) is simplicial [in $\overline{P(H)-P(H)}$].

Proof. Let $f_1, f_2 \in -P(H)$ and $f \in U_{f_1+f_2}^{P(H)}$. There exists an $h \in H$ such that $f \le h \le f_1 + f_2$. Since obviously, $h - f_1 \in P(H)$, there exists an element $h_2 \in H$ such that $h - f_1 \le h_2 \le f_2$. Let $h_1 = h - h_2$. Then $h_1 \le f_1$ and $h_2 \le f_2$, and moreover, $h_1 + h_2 = h \ge f$. Thus

$$U_{f_1+f_2}^{P(H)} \subset U_{f_1}^{P(H)} + U_{f_2}^{P(H)} - X_+,$$

i.e., by Theorem 2 the cone P(H) is simplicial in $P(H)-P(H)$. Since H is coinitial to $\overline{P(H)-P(H)}$, it follows by a theorem of Kantorovich that any positive operator defined in $P(H)-P(H)$ admits a unique extension with preservation of positivity in $\overline{P(H)-P(H)}$. Thus P(H) is also simplicial in $\overline{P(H)-P(H)}$.

Remark 2. If H – containing constants, the closure, and separating points – is a subspace of the space C(Q), then the property that P(H) is simplicial is equivalent to the Riesz interpolation property in H. In this case the pair (H, C(Q)) is called the Choquet simplex [1].

Let H be a cone in a K space Z. An element $h_1 \in Z$ is called 1-affine if there exists an increasing mesh $(h_\alpha) \subset H$ such that $h_\alpha \uparrow h_1$. An element $h_2 \in H$ is called 2-affine if there exists a decreasing net (h'_α) of 1-affine elements such that $h'_\alpha \downarrow h_2$. The set of 2-affine elements is denoted by ${}^2H$.

Proposition 3. Let H be coinitial to the K space Z and let $X = \overline{P(H)-P(H)}$. If H has the Riesz interpolation property and the projector P is such that $P \le P_{Ch}$, then $P\langle X\rangle \subset P\langle {}^2H\rangle$.

Proof. By the Proposition 2 the element $co_H g$ is 1-affine for every element $g \in -P(H)$. Let us pick an element $x \in X$. Then

$$Px = \inf\{Pg : g \geqslant x,\ g \in -P(H)\} = \inf\{P co_H g : g \geqslant x,\ g \in -P(H)\} = P\inf\{co_H g : g \geqslant x,\ g \in -P(H)\}.$$

The set $\{g \in -P(H) : g \ge x\}$ is filtered by decrease, whence the required statement follows.

This proposition shows that supermaximal operators in a known sense must be factorable by a linear extension of an operator by an H-convex hull with the cone $-P(H)$. This fact is correct. More precisely, we have the following proposition.

Proposition 4. Let H be a simplicial cone in X. An operator $T_0 \in \mathscr{L}^+(X, Z)$ is fixed and $T \in \mathscr{L}^+(Z, Y)$ is some T_0-maximal operator. The following graph is commutative:

$$\begin{array}{ccc} H \subset X & \xrightarrow{T_0} & Z \\ {\scriptstyle \psi_H(T_0)}\downarrow & & \downarrow T \\ Z & \xrightarrow{T} & Y \end{array}.$$

Proof. For the operator $\psi_H(T_0)$ in the cone $-H$, by Theorem 1,

$$\psi_H(T_0)(-h) = q_{H,T_0}(-h) \quad (h \in H).$$

We have $T\psi_H(T_0)(-h) = Tq_{H,T_0}(-h) \ge q_{H,TT_0}(-h)$. Since $q_{H,T_0}(-h) \le T_0(-h)$, it follows that $TT_0(-h) \ge T\psi_H(T_0)(-h) = q_{H,TT_0}(-h)$. The last inequality holds since the operator T is T_0-maximal. Since the operators $T\psi_H(T_0)$ and TT_0 are positive and coincide on a cone, dense in H, they coincide on X. This completely proves the proposition.

COROLLARY 1. T_0-maximal operators vanish on $T_0\langle \mathrm{Ker}(\psi_H(T_0))\rangle$.

COROLLARY 2. The Choquet boundary Ch (H, T_0) coincides with $T_0\langle \mathrm{Ker}(\psi_H(T_0))\rangle^d$.

COROLLARY 3. If T is a completely linear operator and if $\mathrm{Ker}(T) \supset T_0\langle \mathrm{Ker}(\psi_H(T_0))\rangle$, then T is T_0-maximal in the Choquet ordering.

The auxiliary information collected in Propositions 1-4 shows that the operator $\psi_H(T_0)$ plays the role of an extension operator with a Choquet boundary. In the following section for the class of cones most important in the applications, the corresponding connections are completely described.

3^0. Let H be a coinitial upper lattice in the K ideal X, where $X = \overline{H-H}$ and the cone H is simplicial.

A cone H is called standard (or simplicial in the sense of Bauer if the operator of sweeping in the forms $\psi_H : \mathscr{L}(X, R) \to \mathscr{L}(X, R)$ is continuous by the allotment of $\mathscr{L}(X, R)$ by the topology $\sigma(\mathscr{L}(X, R), X)$ generated by the duality X and $\mathscr{L}(X, R)$.

Let us consider an element $f \in \mathscr{L}^+(X, R)$ and a point $x \in X_+$. The function $f \to \psi_H f(x)$ is a positive form in $\mathscr{L}$ and, by assumption, this form is generated by an element $\psi_H^* x \in X$. Thus, the adjoint operator ψ_H^* acts from X into X. This operator is called the Dirichlet operator of the standard cone H and is denoted by $\mathscr{D}_H$.

Example 2. Let H be the cone of increasing continuous convex functions on [0, 1]. It is clear that this cone is an upper lattice and $\overline{H - H} = C([0, 1])$. The maximal measures on H are precisely the measures of the form $\alpha\varepsilon_1$, where $\alpha \geq 0$. The cone H is standard and the Dirichlet operator acts according to the formula

$$\mathscr{D}_H : f \to f(1)\mathbf{1}.$$

Example 3. Let us consider the interval [0, 3] and let H be the subspace of C([0, 3]) consisting of the functions affine on [0, 1] and [2, 3], and constant in [1, 2]. Let X be P(H) − P(H). It can be shown that P(H) is a standard cone. Here, the Dirichlet operator on [0, 1] and [2, 3] affinely interpolates a function to the boundary values and leaves it fixed on [1, 2].

Example 4. Let Q be the simplex in R^n spanned by the points $e_0, \ldots, e_n$. The Dirichlet operator $\mathscr{D}$ generated by the cone of continuous convex functions is the operator of affine interpolation in C(Q), i.e.,

$$\mathscr{D}f : \sum_{k=0}^{n} \alpha_k e_k \to \sum_{k=0}^{n} \alpha_k f(e_k),$$

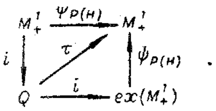
where $\alpha_0, \ldots, \alpha_n \geq 0$, $\alpha_0 + \ldots + \alpha_n = 1$.

Example 5. Let B^n be the unit sphere in R^n and let H be the space of functions continuous in B^n and harmonic inside of the sphere. It is clear that $\overline{P(H) - P(H)} = C(B^n)$ and the Dirichlet operator $\mathscr{D}_{P(H)}$ maps $f \in C(B^n)$ into the Poisson integral of its boundary values on the sphere S^{n-1}.

Example 6. This example is a generalization of Examples 3–5. Let H be a closed subspace in C(Q) which contains constants and separating points, and such that H is a K lineal in the induced order. In this case, the pair (H, C(Q)) is called a Bauer simplex [1]. The pair (H, C(Q)) is a Bauer simplex if and only if the cone P(H) is standard. In fact, a characteristic property of the Bauer simplex is the broad continuity of the mapping:

$$\tau : x \to \text{(maximal probability measure with barycenter x)}.$$

Thus, this statement follows from the commutativity of the diagram



Here τ is the barycentric mapping $\mu \to x_\mu$, the function i is the canonical imbedding of Q in the set M_+^1 of Radon probability measures, ex (M_+^1) is the set of extreme points of M_+^1, and $\psi_{P(H)}$ is an operator of sweeping. It is easy to see that the Dirichlet operator $\mathscr{D}_{P(H)}$ in the given case maps a function from C(Q) into a function from H which is the solution of the Dirichlet problem with boundary values $f|_{Ch}$. Here $Ch = Ch(H, C(Q), R^Q)$.

The following two examples differ in principle from Example 6.

Example 7. Let H^p, where $1 < p \leq +\infty$ for convenience, be the space of functions f, harmonic inside of the sphere B^n, with boundary values $\tilde{f}$ from $L^p(S^{n-1})$ (see, for example, [10]). Let us consider the set of pairs of functions $\{(f, \tilde{f}) : f \in H^p\}$. Let H denote the corresponding cone in the space $Z = R^{B^n \setminus S^{n-1}} \times L^p(S^{n-1})$. It can be shown that H is coinitial to the space P(H) − P(H). The Choquet boundary Ch (H, P(H) − P(H), Z) is the space $\{0\} \times L^p(S^{n-1})$. Here, P(H) is simplicial in the Bauer sense, and the Dirichlet operator acts according to the formula

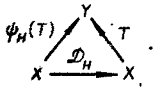
$$\mathscr{D}_{P(H)} : (f, g) \to \text{(Poisson integral of g, g)}.$$

Example 8. For $1 < p \leq +\infty$, this example is a restatement of the previous one (to within an isomorphism). However, it is suitable also for the class H^1 of differences of positive harmonic functions; namely, let us consider the space $Z = R^{B^n \setminus S^{n-1}} \times H^p$ and the cone $H = \{(h, h) : h \in H^p\}$. Then Ch (H, P(H) − P(H), Z) = $\{0\} \times H^p$ and the Dirichlet operator $\mathscr{D}_{P(H)}$ acts according to the formula (f, g) → (g, g).

Now let us turn to the study of standard cones and their connection with the Dirichlet problem. The results presented below, in particular, contain details omitted in some of the examples.

THEOREM 3 (Factorization). 1) The Dirichlet operator is maximal with respect to the cone H;

2) For any K space Y and operator $T \in \mathscr{L}^+(X, Y)$, the following diagram is commutative



Proof. 1) We note first that $\mathscr{D}_H \in \mathscr{L}^+(X, X)$. In fact, if $x \in X_+$ and if $f \in \mathscr{L}^+(X, R)$, then $f(\mathscr{D}_H x) = \psi_H f(x) \geq 0$, which means the positivity of $\mathscr{D}_H x$ (since the ordering of X is regular).

Now let $T' \in \mathrm{Spr}(\mathscr{D}_H, H)$. For the form f and an element $h \in H$, we have $f(T'h) \geq f(\mathscr{D}_H h) = \psi_H f(h)$. Since ψ_H is sweeping, it follows that $f(T'x) = \psi_H f(x) = f(\mathscr{D}_H x)$ for all $x \in X$. Thus, $T' = \mathscr{D}_H$.

2) Let $T \in \mathscr{L}^+(X, Y)$. For the proof we establish that $T\mathscr{D}_H$ is a maximal operator and, moreover, that $T\mathscr{D}_H \in \mathrm{Spr}(T, H)$. Then the statement 2 will follow since H is simplicial.

Let us choose $T' \in \mathscr{L}^+(X, Y)$ such that $T'h \geq T\mathscr{D}_H h$ for all $h \in H$. Then for $f \in \mathscr{L}^+(Y, R)$, we have

$$f(T'h) \geqslant f(T\mathscr{D}_H h) = T^*f(\mathscr{D}_H h) = \psi_H T^*f(h).$$

Hence, from the maximality of the form $\psi_H T^*f$ we have that $(T')^*f = \psi_H T^*f$. This means that $\mathrm{Spr}(T\mathscr{D}_H, H) = \{T\mathscr{D}_H\}$.

Now let $h \in H$, and f, as above, be contained in $\mathscr{L}^+(Y, R)$. Then

$$f(T\mathscr{D}_H h) = T^*f(\mathscr{D}_H h) = \psi_H T^*f(h) \geqslant T^*f(h) = f(Th).$$

In other words, $T\mathscr{D}_H \underset{H}{>} T$. This completely proves the theorem

Remark 3. Theorem 3 establishes a decision property of the Dirichlet operator: each maximal operator is completely determined by its values in the space $\mathscr{D}_H\langle X\rangle$.

We state some consequences of Theorem 3 in the form of propositions.

Proposition 5. For each $h \in -H$ there exists an element $\mathrm{co}_H h = \sup_X U_h^H$. The following representation holds for the Dirichlet operator:

$$\mathscr{D}_H(h_1 - h_2) = \mathrm{co}_H(-h_2) - \mathrm{co}_H(-h_1),$$

where $h_1, h_2 \in H$.

Proof. Let Z be some K space which contains X (for example, the Dedekind completion of X) and let $E: X \to Z$ be the identical imbedding. By Theorem 3 the following representation holds: $\psi_H(E)x = \mathscr{D}_H x$ $(x \in X)$. On the other hand, by a theorem on the filtration of the sprout for an element $h \in -H$,

$$\psi_H(E)h = q_{H,E}(h) = \sup_Z U_h^H.$$

Since, moreover, the element $\psi_H(E)h = \mathscr{D}_H h$ is contained in X, it follows that $\mathscr{D}_H h$ is the supremum in X of the set U_h^H.

Proposition 6. An operator $T \in \mathscr{L}^+(X, Y)$ is maximal with respect to a standard cone if and only if the null ideal N(T) contains the null ideal of the Dirichlet operator.

Proof. By Theorem 3 and Proposition 4 the maximality of the operator T is equivalent to the condition $Th = T\mathrm{co}_H h$ for $h \in -H$. Since the elements $h - \mathrm{co}_H h$ are contained in $N(\mathscr{D}_H)$, the condition $N(T) \supset N(\mathscr{D}_H)$ implies T is maximal. Conversely, by Theorem 3 for a maximal operator, $T = T\mathscr{D}_H$.

Let us return to simplicial cones connected with the Riesz interpolation property.

Proposition 7. Let H be a coinitial subspace in a K lineal Z, where H is a K lineal in the ordering induced in Z. Then P(H) is a standard cone in the space $X = P(H) - P(H)$.

Proof. Proposition 2 established that P(H) is simplicial in X since it is clear that H has the Riesz interpolation property. Let us verify that P(H) is simplicial in the Bauer sense.

First, let us show that for an element $g \in P(H)$ there exists an element $co_H(-g) = \sup_Z U_{-g}^{P(H)}$, where $co_H(-g) \in H$. Indeed, $-g = h_1 \wedge \ldots \wedge h_n$, where $h_k \in H$. By assumption, an element $g_0 = \inf_H\{h_1, \ldots, h_n\}$ exists. If an element $f = h_1' \vee \ldots \vee h_m'$, where $h_s' \in H$, is such that $f \leq -g$, then $g_0 \geq h_s'$ ($s = 1, \ldots, m$) and thus $g_0 \geq f$. On the other hand, $g_0 \in H \subset P(H)$ and $g_0 \leq -g$, i.e., $g_0 \in U_{-g}^{P(H)}$. Thus, $g_0 = co_H(-g)$.

Now let us take a form $f \in \mathscr{L}^+(X, R)$. Then by a theorem on the filtration of the sprout,

$$\psi_{P(H)}(f)(h_1 - h_2) = q_{P(H),f}(-h_2) - q_{P(H),f}(-h_1)$$
$$= \sup f\langle U_{-h_2}^{P(H)}\rangle - \sup f\langle U_{-h_1}^{P(H)}\rangle = f(co_H(-h_2)) - f(co_H(-h_1))$$

for any $h_1, h_2 \in P(H)$. Thus, the function $f \to \psi_{P(H)}$ is continuous for $x \in X$. This proves the proposition.

Now let us establish a fundamental result on the solvability of the Dirichlet problem.

THEOREM 4. Let H be a subspace in the K space Z, where H is coinitial to the space $X = P(H) - P(H)$ and P_{Ch} is the Choquet projector of the triple (H, X, Z).

1) If H is a K lineal in the ordering induced on Z and if P is a projector in Z, where $P \leq P_{Ch}$, then $P\langle X\rangle = P\langle H\rangle$.

2) If the Shilov projector P is such that $P\langle X\rangle = P\langle H\rangle$, then $P \leq P_{Ch}$ and, moreover, H is a K lineal in the induced order from Z.

Proof. 1) Let us consider the Dirichlet operator $\mathscr{D} = \mathscr{D}_{P(H)}$ defined by Proposition 7. By the Proposition 5, $\mathscr{D}(h_1 - h_2) = co_H(-h_2) - co_H(-h_1)$. The last two elements belong to H, and thus $\mathscr{D}(h_1 - h_2) \in H$. Hence, $\mathscr{D}\langle X\rangle = H$.

The projector P is majorized by the projector P_{Ch} and hence by the supermaximality of the Choquet projector, P is supermaximal. Thus, by a theorem on factorization for $x \in X$, we have $Px = P\mathscr{D}x$. Hence, $P\langle X\rangle = P\langle \mathscr{D}\langle X\rangle\rangle = P\langle H\rangle$.

2) First of all we note that the restriction of the Shilov projector P to H is invertible, where the inverse operator $p^{-1}: P\langle H\rangle \to H$ is positive. In fact, if $Ph_1 = Ph_2$, then $P(h_1 - h_2) = 0$, and hence $h_1 = h_2$. The monotonicity of p^{-1} coincides with the definition of the Shilov projector.

Let us define an operator $\mathscr{D}: X \to X$ by the relation $\mathscr{D}x = p^{-1}Px$ for $x \in X$. We note that $\mathscr{D}$ is a positive operator, where $\mathscr{D}^2 = \mathscr{D}$ and $\mathscr{D}\langle X\rangle = H$.

Let us choose elements $h_1, h_2 \in H$ and consider the element $h_0 = \mathscr{D}(h_1 \wedge h_2)$. Since $\mathscr{D}$ is monotonic, $h_0 \leq \mathscr{D}h_1 = h_1$ and $h_0 \leq \mathscr{D}h_2 = h_2$. If h is some element of H, where $h \leq h_1 \wedge h_2$, then $h = \mathscr{D}h \leq \mathscr{D}(h_1 \wedge h_2) = h_0$. Thus, h_0 is the infimum of $\{h_1, h_2\}$ in the subspace H, and hence H is a K lineal in the ordering induced from Z.

Now let us apply Proposition 7. By this proposition, P(H) is a standard cone and the operator $\mathscr{D}$ introduced above coincides with the Dirichlet operator of the cone P(H). On the other hand, by construction $\mathscr{D}x = p^{-1}Px$ ($x \in X$), and hence, $P\mathscr{D}x = Pp^{-1}Px = Px$. The last relation, by a theorem on factorization, means that P is a supermaximal projector, and hence $P \leq P_{Ch}$. This completely proves the theorem.

Remark 4. The condition in part 2) of the theorem that the projector P be Shilov is essential. Furthermore, cases are known in which many boundary projectors solve the Dirichlet problem relative to a subspace H, but H does not have the Riesz interpolation property (see, for example. II.3.19 in [1]).

Remark 5. Let X be some KB lineal, H be a coinitial closed subspace which is a K lineal in the induced order, and $(P(H) - P(H)) \cap X_+ = X_+$. Then the Dirichlet problem is solvable for all X. More precisely, $\mathscr{D}\langle X\rangle = \mathscr{D}\langle H\rangle = H$, where $\mathscr{D}$ is the Dirichlet operator of the cone P(H).

Remark 6. Theorem 4 contains a proof of the solvability of the Dirichlet problem on the Choquet boundary of a Bauer simplex. In fact, if A is a closed subset of the Choquet boundary and if $f \in C(A)$, then by the Tietze–Urysohn theorem, f admits a continuous extension $\tilde{f}$ to the whole compactum. By Remark 5, a function $h \in H$ exists such that $f = \tilde{f}|_A = h|_A$. Let us note that h coincides with the value of the operator $\mathscr{D}_{P(H)}$ on $\tilde{f}$; namely, in connection with this property, the operator $\mathscr{D}_{P(H)}$ is called a Dirichlet operator.

Remark 7. Let us consider the situation of part 1) of Theorem 4, some K space Z, and operator $T_0 \in \mathscr{L}^+(X, Z_1)$. Let $T \in \mathscr{L}^+(Z, Y)$ be some T_0-maximal operator. By Theorem 2 we have $\psi_{P(H)}(TT_0) = TT_0\mathscr{D}_{P(H)}$ (this representation is made precise by Proposition 4). In particular, for the Choquet projector

$P_{Ch(T_0)}$,

$$P_{Ch(T_0)}T_0\langle X\rangle = P_{Ch(T_0)}T_0\langle H\rangle.$$

In other words, the Dirichlet problem is solvable and in the consideration of arbitrary Boolean algebras, true, even for pairs of spaces transformed by means of T_0.

Remark 8. Theorem 4 for convenience is stated in the assumption that H is coinitial to the space X. This assumption is not essential; namely, if H is an arbitrary subspace which is a K lineal in the induced order relation then H is coinitial to $P(H) - P(H)$. In fact, the element $h_1 \vee \ldots \vee h_n \in P(H)$ is majorized by the element $\sup_H \{h_1, \ldots, h_n\}$ and the element $h_1 \wedge \cdots \wedge h_m \in -P(H)$ is majorized by the element $\inf_H \{h_1, \ldots, h_m\}$.

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