SUBDIFFERENTIALS OF CONVEX OPERATORS

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0°. Let X be a vector space, Y a K-space, and $\mathfrak{A}$ a weakly orderwise bounded set in space L(X, Y) of linear operators, acting from X into Y. This means that the sublinear operator

$$P_{\mathfrak{A}}: x \to \sup \{Ax : A \in \mathfrak{A}\}$$

is defined in the entire space X. The set

$$\operatorname{cop}(\mathfrak{A}) = \{A \in L(X, Y) : Ax \leqslant P_{\mathfrak{A}} x \quad (x \in X)\}$$

is called the support hull of the set $\mathfrak{A}$. Clearly, $\mathfrak{A} = \operatorname{cop}(\mathfrak{A})$ if and only if $\mathfrak{A}$ is the support set of an everywhere defined sublinear operator $P_{\mathfrak{A}}$, i.e., is its subdifferential (at zero) $\partial(P_{\mathfrak{A}})$.

A principal topic in theory of sublinear (and arbitrary convex) operators is the problem of explicitly describing the natural Minkowski duality, i.e., the problem of interior characterization of support hulls. Closely connected with this is the problem of evaluating the subdifferentials of various composite convex operators. These topics have been examined in a number of articles (see, e.g., [1-5]). Yet the list of evaluated subdifferentials is quite small. Even less is known about the factoring of the operator $\mathfrak{A} \to \operatorname{cop}(\mathfrak{A})$ of closure in Moore's sense, i.e., about its representation as the composition of a similar algebraic operator and a closure in the sense of some topology.

Our present approach rests on the fact that, to examine any sublinear (or convex) operators, we need to discover how to handle just a "single" canonical operator, in terms of which the action of any operator can be transmitted, in such a way that the residue is linear (or affine). This canonical operator is defined by a set of projectors onto a prime regular subspace, the aim being to construct, e.g., the required factorings, with the aid of these projectors.

As examples of the application of our approach, we evaluate the subdifferentials of the maximum of convex operators; and we perform natural factorings of the support hull operator in the case of discrete K-spaces, and for "compactly generated" sets in the case of operators, acting on the bases of K-spaces of bounded elements.

To simplify the treatment, we primarily consider the vector version of the theory, i.e., no continuity (however defined) constraints are imposed on the linear operators employed. From the point of view of applications, such a restriction is not in fact essential, since support operators are, as a rule, automatically continuous, in the sense in which the initial operator is continuous.

1°. Let Y be a K-space, $\mathfrak{A}$ a set, and $\Delta_{\mathfrak{A}}: Y \to Y^{\mathfrak{A}}$ the imbedding of Y in the diagonal of the space $Y^{\mathfrak{A}}$, i.e., $\Delta_{\mathfrak{A}} y = (y)_{A \in \mathfrak{A}}$. We denote by $(Y^{\mathfrak{A}})_\infty$ the basis of the K-space $Y^{\mathfrak{A}}$ defined by the relation

$$(Y^{\mathfrak{A}})_\infty = \left(\Delta_{\mathfrak{A}} \langle Y \rangle + Y_+^{\mathfrak{A}}\right) \cap \left(\Delta_{\mathfrak{A}} \langle Y \rangle - Y_+^{\mathfrak{A}}\right).$$

Notice that, given any subset $\mathfrak{A}'$ in $\mathfrak{A}$, the space $(Y^{\mathfrak{A}'})_\infty$ admits natural mapping with a component $\operatorname{Pr}_{\mathfrak{A}'} \langle (Y^{\mathfrak{A}})_\infty \rangle$, where the projector $\operatorname{Pr}_{\mathfrak{A}'}$ is given by

$$(\operatorname{Pr}_{\mathfrak{A}'}((y_A)_{A \in \mathfrak{A}}))_{A'} = \begin{cases} y_{A'}, & A' \in \mathfrak{A}', \\ 0, & A' \notin \mathfrak{A}'. \end{cases}$$

In the space $(Y^{\mathfrak{A}})_\infty$, the canonical sublinear operator $\varepsilon_{\mathfrak{A}}: (Y^{\mathfrak{A}})_\infty \to Y$ acts in accordance with the rule

$$\varepsilon_{\mathfrak{A}}: (y_A)_{A \in \mathfrak{A}} \to \sup \{y_A : A \in \mathfrak{A}\}.$$

In the case when $\mathfrak{A}$ is a weakly orderwise bounded subset in the space L(X, Y), there arises the natural linear operator $[\mathfrak{A}]: X \to (Y^{\mathfrak{A}})_\infty$, given by the relation

$$[\mathfrak{A}]: x \to (Ax)_{A \in \mathfrak{A}}.$$

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We introduce some notation. We denote by I_Y the identity mapping of the space Y into itself. By $\mathscr{L}_+(X, Y)$, where X and Y are ordered vector spaces, we denote as usual the set of positive linear operators in L(X, Y).

The point of introducing the above constructions can be seen from the following propositions.

Proposition 1. Let $P: X \to Y$ be a sublinear operator, where $\partial(P) = \text{cop}(\mathfrak{A})$. Then $P = \varepsilon_{\mathfrak{A}} \circ [\mathfrak{A}]$.

Proposition 2. We have the representations

$$\partial(\varepsilon_{\mathfrak{A}}) = \{\alpha \in \mathscr{L}_+((Y^{\mathfrak{A}})_\infty, Y) : \alpha \circ \Delta_{\mathfrak{A}} = I_Y\};$$
$$\text{cop}(\mathfrak{A}) = \partial(\varepsilon_{\mathfrak{A}}) \circ [\mathfrak{A}].$$

Proposition 3. If Z is a K-space and $P: Y \to Z$ is an increasing sublinear operator, then

$$\partial(P \circ \varepsilon_{\mathfrak{A}}) = \{A \in \mathscr{L}_+((Y^{\mathfrak{A}})_\infty, Z) : A \circ \Delta_{\mathfrak{A}} \in \partial(P)\}.$$

These propositions can be proved by direct calculation. As an example, let us prove Proposition 3.

Let $Ay \leqslant (P \circ \varepsilon_{\mathfrak{A}})y$ for all $y \in (Y^{\mathfrak{A}})_\infty$. If $y \le 0$, then $\varepsilon_{\mathfrak{A}} y \leqslant 0$, i.e., $A \in \mathscr{L}_+((Y^{\mathfrak{A}})_\infty, Z)$. If $y = \Delta_{\mathfrak{A}} x$, then

$$(A \circ \Delta_{\mathfrak{A}})x = Ay \leqslant (P \circ \varepsilon_{\mathfrak{A}})\Delta_{\mathfrak{A}} x = Px,$$

so that $A \circ \Delta_{\mathfrak{A}} \in \partial(P)$.

If it is known that A is a positive operator, and that $A \circ \Delta_{\mathfrak{A}} \in \partial(P)$, then we have, for $y \in (Y^{\mathfrak{A}})_\infty$

$$Ay \leqslant (A \circ \Delta_{\mathfrak{A}}) \varepsilon_{\mathfrak{A}} y \leqslant P(\varepsilon_{\mathfrak{A}} y) = (P \circ \varepsilon_{\mathfrak{A}})y,$$

which is what we wanted to prove.

Note. Proposition 2 should actually be regarded as the operator analog of the Choquet theorems on integral representations. For, if Y = R, then $(Y^{\mathfrak{A}})_\infty = l_\infty(\mathfrak{A})$. Hence, if f_0 is a linear function in X, and $\mathfrak{A}$ is a weakly bounded set of such functionals, then Proposition 2 means that $f_0(x) \leqslant \sup\{f(x) : f \in \mathfrak{A}\}$ for all $x \in X$, if and only if

$$f_0(x) = \int_{\mathfrak{A}} f(x)\, d\alpha \quad (x \in X)$$

for some finitely additive probability measure in the algebra of subsets $\mathfrak{A}$.

Now let τ be a separable linear topology in the space Y. A τ-operator topology arises in each space L(X, Y), induced by imbedding into the space Y^X, equipped with a Tikhonov topology.

The topology τ is called admissible, if the support set of any sublinear operator is compact in the τ-operator topology.

Proposition 4. The topology in Y is admissible if and only if ordered intervals in Y are compact in this topology.

Proof. If the intervals are compact, then, by Tikhonov's theorem, the subdifferential $\partial(\varepsilon_{\mathfrak{A}})$ is compact for any set $\mathfrak{A}$. Since, by Proposition 2, for any sublinear operator P we have

$$\partial(P) = \partial(\varepsilon_{\partial(P)}) \circ [\partial(P)],$$

then any subdifferential is compact; this proves the sufficiency. The necessity follows from the fact that the interval [0, y] is the image of the support set ∂(P), where $P: z \to z_+$, under the continuous mapping $A \to Ay$.

Note. Naturally, more refined compactness theorems can be obtained for subclasses of sets of sublinear operators [1]. However, for our future purposes, Proposition 4 is adequate; it shows that, in general, inadmissible topologies have to be brought in in the cop operator factoring problem.

2°. We use the treatment of the previous section for evaluating subdifferentials of the maximum of sublinear operators, and the composition of such operators.

THEOREM 1. Let $P_1, \ldots, P_n : X \to Y$ be sublinear operators and $P: x \to P_1 x \vee \ldots \vee P_n x$. Then,

$$\partial(P) = \bigcup_{\substack{\alpha_1 + \ldots + \alpha_n = I_Y \\ \alpha_1, \ldots, \alpha_n \geqslant 0}} (\alpha_1 \circ \partial(P_1) + \ldots + \alpha_n \circ \partial(P_n)).$$

Proof. Let $\mathfrak{A}$ be the direct sum of sets $\partial(P_1), \ldots, \partial(P_n)$. We have the obvious representation $P = \varepsilon_{\mathfrak{A}} \circ [\mathfrak{A}]$. Hence, $A \in \partial(P)$ if and only if there exists $\alpha \in \partial(\varepsilon_{\mathfrak{A}})$ for which

$$A = \alpha \circ [\mathfrak{A}] = \alpha \circ \Pr_{\partial(P_1)} \circ [\mathfrak{A}] + \ldots + \alpha \circ \Pr_{\partial(P_n)} \circ [\mathfrak{A}].$$

We put $\alpha_k = \alpha \circ \Pr_{\partial(P_k)} \circ \Delta_{\mathfrak{A}}$. Clearly, $\alpha_1, \ldots, \alpha_n \geq 0$, and moreover, $\alpha_1 + \ldots + \alpha_n = I_Y$. Using the above-mentioned possibility of identifying the spaces $(Y^{\partial(P_k)})_\infty$ and $\Pr_{\partial(P_k)}\langle(Y^{\mathfrak{A}})_\infty\rangle$, and also Proposition 2, we find that the operator $\alpha \cdot \Pr_{\partial(P_k)} \cdot [\mathfrak{A}]$ is a support to the sublinear operator $\alpha_k \circ P_k$.

Let us now show that, for any operator $\alpha : Y \to Y$, such that $0 \leq \alpha \leq I_Y$, and any sublinear operator P, we have the equation $\partial(\alpha \circ P) = \alpha \circ \partial(P)$.

Notice first that any two operators $\alpha_1, \alpha_2 : Y \to Y$, such that $0 \leq \alpha_1, \alpha_2 \leq I_Y$, commute. To prove this, it is sufficient to realize Y as a basis in the K-space of expanded continuous functions $C_\infty(\mathfrak{B})$ in some extremal compactum $\mathfrak{B}$, and to observe that the operators α_1 and α_2 admit unique continuations into the space $C_\infty(\mathfrak{B})$. These latter commute in the subspace of bounded elements, and hence everywhere.

It is easily seen from the above remark that we only need to consider the case when $\mathrm{Ker}(\alpha) = 0$. In this case, the space $\alpha\langle Y \rangle$ is a basis in Y. For, if $z = \alpha x \geq 0$ and $0 \leq y \leq z$, then an operator $0 \leq \beta \leq I_Y$ exists, for which $y = \beta z$, since the interval $[0, z]$ is the same as the set $\partial(P_0)z$, where $P_0 : z \to z_+$.

Let $\alpha^{-1} : \alpha\langle Y \rangle \to Y$ be the operator, inverse to α. If $\alpha y > 0$, and it is not true that $y > 0$, then we have $\Pr y < 0$ for some projector Pr. Hence $\alpha \Pr y \leq 0 \leq \Pr \alpha y = \alpha \Pr y$, i.e., $\Pr y = 0$. This contradiction implies that the operator α^{-1} is positive.

Now, if $A \in \partial(\alpha \circ P)$, then $A\langle X \rangle \subset \alpha\langle Y \rangle$, and hence the operator $\alpha^{-1} \circ A$ is defined, which is a support to P, i.e., $A \in \alpha \circ \partial(P)$. The inclusion $\alpha \circ \partial(P) \subset \partial(\alpha \circ P)$ is obvious. QED.

COROLLARY. Let X be a vector lattice and $T \in \mathscr{L}_+(X, Y)$. The following assertions are equivalent:

(a) T is a lattice homomorphism of X into Y;

(b) given any operator $T' : X \to Y$ such that $0 \leq T' \leq T$, there exists $\alpha : Y \to Y$, $0 \leq \alpha \leq T$, for which $T' = \alpha \circ T$.

Proof. Consider the sublinear operators $P_1, P_2 : X^2 \to Y$, acting in accordance with the expressions

$$P_1(x_1, x_2) = T(x_1 \vee x_2); \quad P_2(x_1, x_2) = Tx_1 \vee Tx_2.$$

Assertion (a) is equivalent to the fact that $\partial(P_1) = \partial(P_2)$.

THEOREM 2. Let $P_2 : Y \to Z$ be an increasing sublinear operator and $P_1 : X \to Y$ a sublinear operator. Then,

$$\partial(P_2 \circ P_1) = \{A \circ [\partial(P_1)] : A \circ \Delta_{\partial(P_1)} \in \partial(P_2), \quad A \in \mathscr{L}_+((Y^{\partial(P_1)})_\infty, Z)\}.$$

Here, if $\partial(P_1) = \mathrm{cop}(\mathfrak{A}_1)$ and $\partial(P_2) = \mathrm{cop}(\mathfrak{A}_2)$, then

$$\partial(P_2 \circ P_1) = \{A \circ [\mathfrak{A}_1] : \exists \alpha_2 \in \partial(\varepsilon_{\mathfrak{A}_2}) \, A \circ \Delta_{\mathfrak{A}_1} = \alpha_2 \circ [\mathfrak{A}_2]\}.$$

Proof. We can write

$$P_2 \circ P_1 = (P_2 \circ \varepsilon_{\mathfrak{A}_1}) \circ [\mathfrak{A}_1].$$

Consequently, using Propositions 2 and 3, we have

$$\partial(P_2 \circ P_1) = \partial(P_2 \circ \varepsilon_{\mathfrak{A}_1} \circ [\mathfrak{A}_1]) = \partial(P_2 \circ \varepsilon_{\mathfrak{A}_1}) \circ [\mathfrak{A}_1] = \{A \circ [\mathfrak{A}_1] : A \circ \Delta_{\mathfrak{A}_1} \in$$
$$\in \partial(P_2)\} = \{A \circ [\mathfrak{A}_1] : \exists \alpha_2 \in \partial(\varepsilon_{\mathfrak{A}_2}) \, A \circ \Delta_{\mathfrak{A}_1} = \alpha_2 \circ [\mathfrak{A}_2]\},$$

QED.

Note. If $Z = R$, while there is an admissible topology in Y, and the continuous sublinear functional $P_2 : Y \to R$ is (o)-continuous, then $\partial(P_2 \circ P_1) = \partial(P_2) \circ \partial(P_1)$. But it should not be assumed that a similar identity holds for any spaces Z, having admissible topologies.

3°. The constructions described in the previous sections may also be used for investigating any convex operators. We illustrate the scheme of the constructions by taking the example of evaluating the subdifferential of the maximum of convex operators (clearance theorems).

Let $U \subset X$ be a convex set, absorbent with respect to the point $x_0 \in U$, in the vector space X, and let $F : U \to Y$ be a convex operator. The sublinear operator $F'(x_0) : X \to Y$ is defined, i.e., the directional derivative of the operator F. Recall that the set $\partial(F'(x_0))$ is called the subdifferential of the mapping F at the point x_0 and is denoted by $\partial_{x_0}(F)$.

Proposition 5. For the sublinear operator $P: X \to Y$, we have $\partial_{x_0}(P) = \{A \in \partial(P) : Ax_0 = Px_0\}$.

Proposition 6. Let $A \in L(X, Y)$ and $A_y : x \to Ax + y$, where $y \in Y$. Given any convex operator $G: Y \to Z$, defined in a convex set, absorbent with respect to the point $A_y x_0$, we have

$$\partial_{x_0}(G \circ A_y) = \partial_{A_y x_0}(G) \circ A.$$

Both propositions can be proved by simple calculation.

Proposition 7. For the operator $F: U \subset X \to Y$ and $x_0 \in U$, we put

$$F_\varepsilon : x \to \sup\{Ax - Ax_0 + Fx_0 : A \in \partial_{x_0}(F)\} \quad (x \in X).$$

Given any K-space Z, and any increasing convex operator $G: Y \to Z$, defined in a set, absorbent with respect to the point Fx_0, and (o)-continuous at this point, we have

$$\partial_{x_0}(G \circ F) = \partial_{x_0}(G \circ F_\varepsilon).$$

Here, we can write

$$\partial_{x_0}(G \circ F) = \{A \circ [\partial_{x_0}(F)] : A \circ \Delta_{\partial_{x_0}(F)} \in \partial_{Fx_0}(G); \quad A \in \mathscr{L}_+((Y^{\partial_{x_0}(F)})_\infty, Z)\}.$$

Proof. Notice first that

$$F_\varepsilon = \varepsilon_{\mathfrak{A}} \bullet [\mathfrak{A}]_y; \quad \mathfrak{A} = \partial_{x_0}(F); \quad y = \Delta_{\mathfrak{A}} F x_0 - [\mathfrak{A}] x_0.$$

Using Propositions 2, 5, and 6, we get

$$\partial_{x_0}(F_\varepsilon) = \partial_{[\mathfrak{A}]_y x_0}(\varepsilon_{\mathfrak{A}}) \circ [\mathfrak{A}] = \{\alpha \circ [\mathfrak{A}] : \alpha \circ [\mathfrak{A}]_y x_0 = \varepsilon_{\mathfrak{A}} \circ [\mathfrak{A}]_y x_0;$$
$$\alpha \circ \Delta_{\mathfrak{A}} = I_Y\} = [\mathfrak{A}] = \partial_{x_0}(F),$$

since $[\mathfrak{A}]_y x_0 = \Delta_{\mathfrak{A}} F x_0$ and $\varepsilon_{\mathfrak{A}} \circ [\mathfrak{A}]_y x_0 = F x_0$.

Hence, using Proposition 2 of Sec. 8 of [1], which holds in our present situations, we obtain

$$(G \circ F)'(x_0) = G'(Fx_0) \circ F'(x_0);$$
$$(G \circ F_\varepsilon)'(x_0) = G'(F_\varepsilon x_0) \circ F'_\varepsilon(x_0),$$

which prove the first part of our proposition.

To prove the second part, we have to verify, using the method of Proposition 3, that, given any set $\mathfrak{A}$ and a point $y_0 \in (Y^{\mathfrak{A}})_\infty$ such that $Fx_0 = \varepsilon_{\mathfrak{A}} y_0$, we have

$$\partial_{y_0}(G \circ \varepsilon_{\mathfrak{A}}) = \{A \in \mathscr{L}_+((Y^{\mathfrak{A}})_\infty, Z) : A \circ \Delta_{\mathfrak{A}} \in \partial_{Fx_0}(G)\}$$

[this representation also holds without the assumption of (o)-continuity]. The proof is completed in accordance with the scheme of Theorem 2.

Using the last proposition and Theorem 1, we can prove the following theorem.

THEOREM 3. Let $F_1, \ldots, F_n : U \subset X \to Y$ be convex operators in a set, absorbent with respect to the point x_0. Then,

$$\partial_{x_0}(F_1 \vee \ldots \vee F_n) = \bigcup_{(\alpha_1, \ldots, \alpha_n) \in \Gamma(x_0)} (\alpha_1 \circ \partial_{x_0}(F_1) + \ldots + \alpha_n \circ \partial_{x_0}(F_n)),$$

where the union is taken with respect to the following set:

$$\Gamma(x_0) = \Big\{(\alpha_1, \ldots, \alpha_n) : \alpha_1, \ldots, \alpha_n \geqslant 0,$$
$$\sum_{k=1}^n \alpha_k = I_Y, \qquad \sum_{k=1}^n \alpha_k \circ F_k x_0 = (F_1 \vee \ldots \vee F_n) x_0\Big\}.$$

4°. We now use the scheme described above for the description of the subdifferentials of sublinear operators (in view of the earlier remarks, this is the general case). We require the following definitions.

We say that a weakly orderwise bounded set $\mathfrak{A}$ in space L(X, Y) is operator-convex, if, given any two elements $A_1, A_2 \in \mathfrak{A}$ and operators $\alpha_1, \alpha_2 \in \mathscr{L}_+(Y, Y)$, $\alpha_1 + \alpha_2 = I_Y$, we have $\alpha_1 \circ A_1 + \alpha_2 \circ A_2 \in \mathfrak{A}$.

The set $\mathfrak{A}$ is called strongly operator-convex if, given any (o)-summable family $(\alpha_\xi)_{\xi \in \Xi}$ of operators $\alpha_\xi : Y \to Y$ such that $0 \le \alpha_\xi \le I_Y$, and moreover, $\sum_{\xi \in \Xi} \alpha_\xi = I_Y$, and any family $(A_\xi)_{\xi \in \Xi}$ of operators of $\mathfrak{A}$, we have $\sum_{\xi \in \Xi} \alpha_\xi \circ A_\xi \in \mathfrak{A}$.

Proposition 8. The following assertions are valid:

(a) given any sublinear operator, its support set is strongly operator-convex;

(b) given any weakly orderwise bounded set $\mathfrak{A}$, a minimal operator-convex set $(\mathfrak{A})$, containing $\mathfrak{A}$, exists, where

$$\operatorname{op}(\mathfrak{A}) = \left\{ \sum_{k=1}^{n} \alpha_k \circ A_k : A_k \in \mathfrak{A},\ 0 \leqslant \alpha_k \leqslant I_Y,\ \sum_{k=1}^{n} \alpha_k = I_Y \right\};$$

(c) given any weakly orderwise bounded set $\mathfrak{A}$, a minimal strongly operator-convex set $\operatorname{stop}(\mathfrak{A})$, containing $\mathfrak{A}$, exists, where

$$\operatorname{stop}(\mathfrak{A}) = \left\{ \sum_{\xi\in\Xi} \alpha_\xi \circ A_\xi : A_\xi \in \mathfrak{A},\ 0 \leqslant \alpha_\xi \leqslant I_Y,\ \sum_{\xi\in\Xi} \alpha_\xi = I_Y \right\}.$$

Proof. Let us prove, e.g., (c). The existence of the set $\operatorname{stop}(\mathfrak{A})$ follows from (a), in view of the obvious inclusion $\operatorname{stop}(\mathfrak{A}) \subset \operatorname{cop}(\mathfrak{A})$. To prove the second part of the assertion, we have to take families $(\beta_\gamma)_{\gamma\in\Gamma}$, $(\alpha_\xi^\gamma)_{\xi\in\Xi(\gamma)}$, $(A_\xi^\gamma)_{\xi\in\Xi(\gamma)}$, such that

$$0 \leqslant \beta_\gamma,\quad \alpha_\xi^\gamma \leqslant I_Y;\quad A_\xi^\gamma \in \mathfrak{A};$$

$$\sum_{\xi\in\Xi(\gamma)} \alpha_\xi^\gamma = I_Y;\quad \sum_{\gamma\in\Gamma} \beta_\gamma = I_Y;\quad \sum_{\xi\in\Xi(\gamma)} \alpha_\xi^\gamma \circ A_\xi^\gamma = A_\gamma \in \mathfrak{A}.$$

In view of (a), all the families in question are absolutely summable, so that, in view of the complete linearity of the operators of the interval $0 \leq \alpha \leq I_Y$, we get

$$\sum_{\gamma\in\Gamma} \beta_\gamma \circ A_\gamma = \sum_{\gamma\in\Gamma} \beta_\gamma \circ \sum_{\xi\in\Xi(\gamma)} \alpha_\xi^\gamma \circ A_\xi^\gamma = \sum_{\gamma\in\Gamma} \sum_{\xi\in\Xi(\gamma)} \beta_\gamma \circ \alpha_\xi^\gamma \circ A_\xi^\gamma \in \mathfrak{A},$$

since it is clear that $\sum_{\gamma\in\Gamma} \sum_{\xi\in\Xi(\gamma)} \beta_\gamma \circ \alpha_\xi^\gamma = I_Y$. QED.

The following is a key example.

Example. Let Y be a discrete K-space which is a basis in the product of straight lines $R^{\mathfrak{B}}$. Notice that

$$(Y^{\mathfrak{A}})_\infty = \{ y \in R^{\mathfrak{A}\times\mathfrak{B}} : \sup_{A\in\mathfrak{A}} | y(A, \cdot) | \in Y \}.$$

Consider the set $\mathfrak{P} = \{p_A : A \in \mathfrak{A}\}$, where p_A is the coordinate projector $p_A : (y_A)_{A\in\mathfrak{A}} \to y_A$. Obviously, $\varepsilon_{\mathfrak{A}} = P_{\mathfrak{P}}$. Here, the operator $\alpha : (Y^{\mathfrak{A}})_\infty \to Y$ appears in $\partial(\varepsilon_{\mathfrak{A}})$ if and only if

$$\alpha y(B) = \int_{\mathfrak{A}} y(\cdot, B)\, a\mu_B,$$

where μ_B is a finitely additive probability measure in the algebra of subsets $\mathfrak{A}$.

It can be shown that $\alpha_0 \in \operatorname{stop}(\mathfrak{P})$ if and only if numbers α_A^B exist such that

$$0 \leqslant \alpha_A^B \leqslant 1;\quad \sum_{A\in\mathfrak{A}} \alpha_A^B = 1;$$

$$\alpha_0 y(B) = \sum_{A\in\mathfrak{A}} \alpha_A^B y\ (A, B).$$

At the same time, corresponding to elements of $(\mathfrak{P})$ we have families (α_A^B), in which, for all A belonging to the complement of a finite set in $\mathfrak{A}$, we have $\alpha_A^B = 0$.

Under our assumptions on the space Y, we can regard the space L(X, Y), where X is any vector space, as a subspace $L(X, R_{\mathfrak{B}})$. Thereby a simple operator topology arises in L(X, Y), i.e., by definition, the topology induced into L(X, Y) by the weak operator topology of the space $L(X, R_{\mathfrak{B}})$. If Y is a basis in $l_\infty(\mathfrak{B})$, it is meaningful to equip L(X, Y) with a strong operator topology [i.e., an operator topology, generated by the norm in Y, induced from $l_\infty(\mathfrak{B})$].

The following given examples of corresponding factorings of the operator cop:

THEOREM 4. (a) If Y is a basis $R_{\mathfrak{B}}$, then the set of operators is a support set if and only if it is weakly orderwise bounded, operator-convex, and closed in the simple operator topology.

(b) if Y is a basis in $l_\infty(\mathfrak{B})$, then the set of operators is a support set if and only if it is weakly orderwise bounded, strongly operator-convex, and closed in the strong operator topology.

Proof. Let us prove, e.g., (b). Let $\mathfrak{A}$ be weakly orderwise bounded, strongly operator-convex, and closed in the strong operator topology. It has to be shown that $\operatorname{cop}(\mathfrak{A}) \subset \mathfrak{A}$. Let $A \in \operatorname{cop}(\mathfrak{A})$. Then, by

Proposition 2, we have $A=\alpha\circ[\mathfrak{A}]$ for some $\alpha\in\partial(\varepsilon^{\mathfrak{A}})$. A mesh (α_γ) of operators stop $(\mathfrak{P})$ exists in the space $\mathscr{L}_+((Y^{\mathfrak{A}})_\infty, Y)$, such that (α_γ) is convergent to α in the strong operator topology. Since the set $\mathfrak{A}$ is strongly operator-convex, then $\alpha_\gamma\circ[\mathfrak{A}]$ appears in $\mathfrak{A}$, since

$$\alpha_\gamma\circ[\mathfrak{A}]=\sum_{\xi\in\Xi(\gamma)}\beta_\xi^\gamma\circ p_{A_\xi^\gamma}\circ[\mathfrak{A}]=\sum_{\xi\in\Xi(\gamma)}\beta_\xi^\gamma\circ A_\xi^\gamma$$

for some family $0\leqslant\beta_\xi^\gamma\leqslant I_Y$, $\sum_{\xi\in\Xi(\gamma)}\beta_\xi^\gamma=I_Y$. Hence $A=\alpha\circ[\mathfrak{A}]=\lim_\gamma\alpha_\gamma\circ[\mathfrak{A}]\in\mathfrak{A}$ since $\mathfrak{A}$ is closed in the strong operator topology.

The situation is naturally much more complicated in the case of continuous K-spaces. At the same time, it is possible to obtain natural factorings for some classes of operators.

THEOREM 5. Let Y be a basis in the space $C(\mathfrak{B})$ of continuous functions in the extremal compactum $\mathfrak{B}$, and let the weakly orderwise bounded set $\mathfrak{A}$ be compactly generated in L(X, Y), i.e., for some subset $\mathfrak{C}\subset\mathfrak{A}$, compact in the strong operator topology, we have $P_{\mathfrak{C}}=P_{\mathfrak{A}}$. Then, the set $\mathfrak{A}$ is a support set if and only if it is operator-convex and closed in the simple operator topology.

Proof. It only has to be shown that, in the case of an operator-convex closed set $\mathfrak{A}$, we have $\mathfrak{A}\supset\operatorname{cop}(\mathfrak{A})=\operatorname{cop}(\mathfrak{C})$. Let A appear in cop $(\mathfrak{C})$; then, by Proposition 2, $A=\alpha\circ[\mathfrak{C}]$ for some $\alpha\in\partial(\varepsilon_{\mathfrak{C}})$. Notice that, since $\mathfrak{C}$ is compact in the strong operator topology, the element $[\mathfrak{C}]x$, for any $x\in X$, lies in the space $C(\mathfrak{B}\times\mathfrak{C})$. For, if $B_\xi\to B_0$ and $C_\xi\to C_0$, then

$$|C_\xi x(B_\xi)-C_0x(B_0)|\leqslant\|C_\xi x-C_0x\|+|C_0x(B_\xi)-C_0x(B_0)|\to 0.$$

It is therefore sufficient to assume that the operator α is a support to the trace of the canonical operator $\varepsilon_{\mathfrak{C}}$ onto the space $C(\mathfrak{B}\times\mathfrak{C})$. Notice that, for every $B\in\mathfrak{B}$, we have

$$\alpha y(B)=\int_{\mathfrak{B}\times\mathfrak{C}}y\,d\mu_B,$$

where μ_B is a regular Borel measure in $\mathfrak{B}\times\mathfrak{C}$ and a support to the sublinear functional $y\to(\sup_{C\in\mathfrak{C}}y(\cdot,C))(B)$. It is easily shown that μ_B is a probability measure with support in the fiber $\{B\}\times\mathfrak{C}$. It follows directly from this that α is the point of tangency (in the simple operator topology) of the operator-convex hull of the collection of coordinate projectors $\mathfrak{P}=\{p_c: C\in\mathfrak{C}\}$. Hence,

$$\alpha\circ[\mathfrak{C}]=\lim_{\xi\in\Xi}\sum_{k=1}^{n(\xi)}\alpha_k^\xi\circ p_{C_k^\xi}\circ[\mathfrak{C}]=\lim_{\xi\in\Xi}\sum_{k=1}^{n(\xi)}\alpha_k^\xi\circ C_k^\xi\in\mathfrak{A},$$

QED.

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