BOUNDARY BEHAVIOR OF EXTREMAL OPERATORS

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Introduction

The purpose of the present paper is to identify and study the properties of a special class of operators, the supports of which play a role analogous to the role of extremal subsets of a convex set. The significance of the latter sets in the boundary theory of Choquet [1] is well known. At the first time, even in spaces of continuous functions, the question of the generation of faces associated with a distinguished adapted cone using the Choquet order in the set of Radon measures is unstudied.

It is advisable to carry out the study of the above question in the framework of the theory of Kantorovich space [2, 3]. This situation is explained by two circumstances. First, in the framework of the theory of ordered vector spaces, it is possible to distinguish a universally defined object, viz., the extremal operator (projection) which for a suitable choice of parameters represents most of the known extremal structures: faces of a convex set, faces associated with a cone of functions [4], minimal faces of bounded functionals in Banach spaces, boundary projections in K-spaces, etc. Second, from the point of view of the intrinsic problems of the theory of K-spaces, distinguishing an additional cone in an ordered vector space makes it possible to study the regular operators on the space in more detail. The machinery of maximal operators developed in [5] is well adapted for the analysis of this problem essentially only for a comparatively narrow class of cones, viz., the coinitial cones and closely related ones. The class of extremal operators considered in the present paper is in some sense defined for any cones, and it moreover turns out that the behavior of these operators is in many respects analogous to the behavior of maximal operators.

The principal problem considered in this paper consists in establishing theorems of Choquet type which characterize the extremal operators in terms of the simplest extremal structures: the extremal projections and supports. This problem is completely solved for two situations which are typical in the Choquet theory, viz., in the case of arbitrary completely linear operators and in the case of regular Borel measures on a compact metrizable set.

In Sec. 1, a special variant of the decomposition theorem, a lemma on adopted decompositions, is used to establish the general properties of sets of extremal operators and projections. In particular, we indicate a new approach to the construction of facial structures associated with a cone in a K-space.

In Sec. 2, a variant of the boundary theory for the so-called adapted operators is constructed in order to study the completely linear extremal operators. Using a theorem on the anormality of such operators, cases are established in which extremality of an operator is equivalent to extremality of the projection onto its component of essential positivity.

In Sec. 3 we consider the structure of certain classes of extremal measures on a compact metrizable set. Here the main result states that a measure is extremal if and only if its support contains an extremal subset of full measure. We observe that all the above facts are new even for the case of the usual convex compact sets in a locally convex space.

In what follows, the terminology and facts from Choquet theory in K-spaces will be used without further explanation.

1. The Cone of Extremal Operators

Let H be a (convex) cone in some ordered space X. For convenience in the sequel, we assume that H is an upper lattice in the K-linear set X. In addition, all the spaces considered will be assumed to be regularly ordered. These assumptions are not essential for many of the results, but they lead to a shortening of the formulations of the results. Let further Y be some K-space. The symbol $\succ$ denotes the Choquet ordering

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generated by the cone H in the set of positive operators $\mathscr{L}_+(X, Y)$. We pick some component $\mathscr{L}$ in the K-space of regular operators $\mathscr{L}(X, Y)$.

An operator $T\in\mathscr{L}_+(X, Y)$ is said to be extremal (in $\mathscr{L}$ with respect to the cone H) if for every $S\in\mathscr{L}_+$ such that $S\succ T$, $N(S) \supset N(T)$ holds, where as usual, $N(T) = \{x \in X : T|x| = 0\}$ is the null lattice of T. The set of all extremal operators is denoted by $\mathfrak{E}(H, \mathscr{L})$ or simply $\mathfrak{E}$, if there is no doubt about which H and $\mathscr{L}$ are meant.

We recall that a set U in a K-space Z is called upper regular if for every subset $U_0 \subset U$ bounded from above in Z, $\sup U_0 \in U$ is satisfied. Lower regular sets are defined analogously. A set is called regular if it is upper and lower regular.

The main result of this paragraph is the following theorem.

THEOREM 1.1. The set $\mathfrak{E}$ is an upper regular cone in the K-space $\mathscr{L}(X, Y)$

Before proving this theorem, we establish a variant of the decomposition theorem which is of independent interest, and which we will also need in what follows. We first introduce a definition.

Let $[\omega_0, \omega_1]$ be a segment $\{\omega \in \Omega : \omega_0 \leq \omega \leq \omega_1\}$ of some completely ordered set Ω. A map $\omega \rightarrow T(\omega)$ of the set $[\omega_0, \omega_1]$ into a K-space is said to be adapted if the map is increasing and moreover, for every limit ordinal $\omega \in [\omega_0, \omega_1]$, we have $T(\omega) = \sup\{T(\omega') : \omega_0 \leq \omega' < \omega\}$.

Lemma on Adapted Decompositions. Let $\omega \rightarrow T(\omega)$ be an adapted family of operators in $\mathscr{L}_+(X, Y)$ and let the operator $S\in\mathscr{L}_+(X, Y)$ be such that $S\succ T(\omega_1)$. Then there exists an adapted family $\omega \rightarrow S(\omega)$ of operators in $\mathscr{L}_+(X, Y)$ such that

$$S\geqslant S(\omega),\ S(\omega)\succ T(\omega),\ S-S(\omega)\succ T(\omega_1)-T(\omega) \tag{1.2}$$

for all $\omega \in [\omega_0, \omega_1]$.

Proof. We will carry out the required construction inductively. Since $T(\omega_1) \geq T(\omega_0)$, there exists by the decomposition theorem an operator $S_0\in\mathscr{L}_+(X, Y)$ such that $S_0 \succ T(\omega_0)$ and $S - S_0 \succ T(\omega_1) - T(\omega_0)$, where $S \geq S_0$. Put $S(\omega_0) = S_0$.

Assume now that the map $\omega \rightarrow S(\omega)$ is defined in the segment $[\omega_0, \omega)$ and has the required properties there. If ω is a limit ordinal, we put $S(\omega) = \sup\{S(\omega') : \omega' \in [\omega_0, \omega)\}$. Since for all $x \in X$ the equalities

$$S(\omega)x=(o)-\lim_{\omega'} S(\omega')x,$$

$$T(\omega)x=(o)-\lim_{\omega'} T(\omega')x$$

hold, the map $\omega \rightarrow S(\omega)$ satisfies (1.2) on $[\omega_0, \omega]$. Assume now that the number ω is not a limit ordinal, and let ω' be the immediate predecessor, $\omega' = \sup[\omega_0, \omega)$. Relations (1.2) hold for the element ω', with $T(\omega_1) - T(\omega') \geq T(\omega) - T(\omega')$. Thus, by the decomposition theorem there exists an operator $S_\omega\in\mathscr{L}_+(X, Y)$ such that

$$S-S(\omega')\geqslant S_\omega,\ S_\omega\succ T(\omega)-T(\omega'),$$
$$S-S(\omega')-S_\omega\succ T(\omega_1)-T(\omega).$$

Put $S(\omega) = S(\omega') + S_\omega$. It is obvious that the operator $S(\omega)$ satisfies (1.2). The lemma is proved.

Proof of Theorem 1.1. We first establish that $\mathfrak{E}$ is a cone. Since the relation $\alpha\mathfrak{E}\subset\mathfrak{E}$ for all $\alpha \in R_+\setminus\{0\}$ is obvious, we must check the inclusion $\mathfrak{E}+\mathfrak{E}\subset\mathfrak{E}$. Let $T_1, T_2\in\mathfrak{E}$. If $S \succ T_1 + T_2$, then by the decomposition theorem there exist operators $S_1, S_2\in\mathscr{L}_+(X, Y)$ such that $S_1 + S_2 = S$ and moreover, $S_1 \succ T_1$ and $S_2 \succ T_2$. If $x \in N(T_1 + T_2)$ then $x \in N(S_1) \cap N(S_2) \subset N(S_1 + S_2) = N(S)$. Thus, $T_1+T_2\in\mathfrak{E}$.

We now prove that $\mathfrak{E}$ is an upper lattice. If $T_1, T_2\in\mathfrak{E}$ and $S\succ T_1 \vee T_2$, then $S + T_1 + T_2 - T_1 \vee T_2 \succ T_1 + T_2$. Thus by what has been established above, $N(T_1 + T_2) \subset N(S)$. Furthermore, it is obvious that $N(T_1 \vee T_2) \subset N(T_1 + T_2)$.

We now turn to the proof that $\mathfrak{E}$ is regular. Let $\{T_\omega\}_{\omega\in\Omega}$ be an upper bounded family of elements in $\mathfrak{E}$, and $T = \sup\{T_\omega : \omega \in \Omega\}$. We will assume without loss of generality that the set Ω is completely ordered and has a smallest element ω_0. We adjoin to Ω a largest element ω_1. It is clear that $\Omega \cup \{\omega_1\} = [\omega_0, \omega_1]$. Now define a map $\omega \rightarrow T(\omega)$ of the segment $[\omega_0, \omega_1]$ into $\mathscr{L}_+(X, Y)$ as follows. Put $T(\omega_0) = T_{\omega_0}$. If ω is a limit ordinal, we put $T(\omega) = \sup\{T(\omega') : \omega' \in [\omega_0, \omega)\}$. If ω is not a limit ordinal and ω' immediately precedes ω, then we put $T(\omega) = T(\omega') \vee T_\omega$. It is clear that the map $\omega \rightarrow T(\omega)$ is adapted, and moreover, $T(\omega_1) = T$. We now prove that all the operators $T(\omega)$ are extremal.

Assume that $T(\omega') \in \mathfrak{E}$ for all $\omega' \in [\omega_0, \omega)$. If ω is not a limit ordinal, the operator $T(\omega)$ is extremal by what was proved above. Consider the case when ω is a limit ordinal.

Take $S \succ T(\omega)$. By the lemma on adapted decompositions, there exists an adapted family $\omega \to S(\omega)$ defined on the segment $[\omega_0, \omega]$ and satisfying (1.2) there. If $T(\omega)x = 0$ for some $x \in X_+$, then $T(\omega')x = 0$ and $S(\omega')x = 0$ for $\omega' \in [\omega_0, \omega)$. It follows that $S(\omega)x = 0$. Furthermore, $S - S(\omega) \succ T(\omega) - T(\omega)$. Since $S - S(\omega) + T(\omega_0) \succ T(\omega_0)$ holds, we have $(S - S(\omega))x = 0$. We obtain finally that $N(S) \supset N(T(\omega))$. Thus, $T(\omega) \in \mathfrak{E}$. It remains to invoke the principle of transfinite induction.

Remark 1.3. The cone $\mathfrak{E}$ need not be a lower regular set. Moreover, this cone is generally speaking not even a lower lattice. We observe that the simplest condition for it to be a lattice which does not depend on the cone H is that X be normally imbeddable into the K-space $\mathscr{L}(\mathscr{L}(X, Y), Y)$. The last property is evidently equivalent to X being a K-space for which all the operators in $\mathscr{L}_+(X, Y)$ are completely linear, with the operators $T_1, T_2 \in \mathscr{L}_+(X, Y)$ disjoint if and only if the components $N(T_1)^d$ and $N(T_2)^d$ of essential positivity are disjoint. These properties are certainly satisfied if the positive forms on the K-space X are completely linear and $Y = R$. For $Y \neq R$, it is generally speaking not possible to ensure the above properties. At the same time, in many cases one is successful in choosing a natural sublattice in $\mathfrak{E}$. We remark here that $\mathfrak{E}$ may even fail to be a normal set.

Thus, let X be a K-space and $Y = X$. We denote by $\mathbf{B}(X)$ a base of the K-space X, i.e., a complete Boolean algebra of projections in X. We assume concerning the component $\mathscr{L}$ in $\mathscr{L}(X, X)$ that $\mathbf{B}(X) \subset \mathscr{L}$. The symbol $\mathrm{Ext}(H, \mathscr{L})$ denotes the set of extremal projections in $\mathbf{B}(X) \cap \mathfrak{E}(H, \mathscr{L})$.

THEOREM 1.4. The set $\mathrm{Ext}(H, \mathscr{L})$ is an upper regular sublattice of the base $\mathbf{B}(X)$. If in addition the component $\mathscr{L}$ consists of completely linear operators, then $\mathrm{Ext}(H, \mathscr{L})$ is a regular sublattice in $\mathbf{B}(X)$.

Proof. We first verify that $\mathrm{Ext}(H, \mathscr{L})$ is a lower lattice. For this, we first observe that $P \in \mathrm{Ext}(H, \mathscr{L})$ if and only if for every $S \in \mathscr{L}_+$ such that $S \succ P$, we have $SP = S$. Indeed, if this condition is satisfied, then $N(S) \supset N(P)$. On the other hand, if we know that $N(S) \supset N(P)$, then $Sx = SP^dx + SPx = SPx$ since $P^dx \in N(P)$.

Thus, let $P_1, P_2 \in \mathrm{Ext}(H, \mathscr{L})$ and $S \succ P_1 \wedge P_2$. Since the projection $P_1 \vee P_2 \in \mathrm{Ext}(H, \mathscr{L})$ by Theorem 1.1, we have $SP_1 = S$, $SP_2 = S$, and $S(P_1 \vee P_2) = S$. It follows that $S(P_1 \wedge P_2) = S(P_1 + P_2 - P_1 \vee P_2) = S$. Thus $P_1 \wedge P_2 \in \mathrm{Ext}(H, \mathscr{L})$.

We turn to the proof of the second part of the theorem. Let $\{P_\omega\}$ be a family of extremal operators and $P = \inf\{P_\omega\}$. Without loss of generality, we may assume that the family $\{P_\omega\}$ is filtered downward. If $S \succ P$, then $S + P_\omega - P \succ P_\omega$, so that $SP_\omega = S$. Consequently, $S = \inf\{SP_\omega\} = S \inf\{P_\omega\} = SP$, since by hypothesis the operator S is completely linear. The rest of the theorem is contained in Theorem 1.1.

Remark 1.5. The last theorem can serve as a basis for various topological considerations in the context of extremal projections. In fact, there is a natural closure operator on the base $\mathbf{B}(X)$. The case is particularly interesting when X (a discrete K-space) is a base in a product of lines R^Q and the component $\mathscr{L}$ is contained in the component of completely linear operators. We decree that the closed sets in Q are precisely the ones which correspond to projections in $\mathrm{Ext}(H, \mathscr{L})$. The corresponding topology in Q is called the extremal or strongest facial topology. It is obvious that this topology coincides with the right interval topology of some uniquely determined pre-order relation on Q. In Choquet theory and convex analysis, one is often interested in the facial structure of a convex set. The latter, for corresponding X, H, and $\mathscr{L}$, is precisely the factor set of Q by the equivalence relation associated with the above pre-order.

2. Anormality of Extremal Operators

It is easy to see that the class of objects introduced in the preceding paragraph encompasses many extremal structures, e.g., extremal subsets of a convex set, measures whose supports are faces in the sense of Mayer [4], boundary operators in the sense of Choquet, etc. At the same time, many important objects, e.g., the Choquet boundary or the face of a cone in the sense of Mayer, are not extremal operators in the literal sense of the word. The point is that in the above cases, one is interested in operators defined on a space larger than X, and which reduce to extremal operators on X. The latter situation is technically convenient in many ways.

Thus, in accordance with the above we consider some K-space Z and fix a positive operator $T_0 \in \mathscr{L}_+(X, Z)$. As usual in Choquet theory, the study of the extremal operators is carried out by means of lifting to the "test" K-space Z. Indeed, if H is an upper lattice in X and $\mathscr{L}$ is a component in $\mathscr{L}(X, Y)$, the operator $T \in \mathscr{L}_+(Z, Y)$ is said to be T_0-extremal (in $\mathscr{L}$ with respect to the cone H) if the composition TT_0 lies in $\mathfrak{E}(H, \mathscr{L})$. The set of T_0-extremal operators is denoted by $\mathfrak{E}(T_0)$, or more fully, by $\mathfrak{E}(T_0, H, \mathscr{L})$.

In order to study the properties of the operators of the class $\mathfrak{C}(T_0)$, we introduce special components of operators constructed in a rather simple way, the union of which covers the set $\mathfrak{C}(T_0)$. These operators are also of independent interest, since in all probability they are the largest class of objects for which a boundary theory is possible.

An operator $T \in \mathscr{L}_+(Z, Y)$ is said to be (T_0, H_0)-adapted (with respect to the cone H in X), or simply T_0-adapted, if for every $S \in \mathscr{L}_+(X, Y)$ such that $S \succ TT_0$, we have $Sh \geq TT_0h$ for all $h \in H_0$. Here H_0 is some cone in X. The set of all operators T such that $|T|$ is T_0-adapted is denoted by $\mathfrak{B}_{H_0}(T_0, Y)$.

The operators of the class $\mathfrak{B}_{H_0}(T_0, Y)$ are constructed analogously to the elements of the component $\mathfrak{B}(T_0, Y)$ of boundary operators in the sense of Choquet. We remark that $\mathfrak{B}(T_0, Y) = \mathfrak{B}_X(T_0, Y)$. We restrict ourselves here to listing the properties needed in the sequel.

Proposition 2.1. The set $\mathfrak{B}_{H_0}(T_0, Y)$ is nonempty if and only if $\overline{H + X_+} \supset H_0$.

In the following, we will consider only the case when $\mathfrak{B}_{H_0}(T_0, Y) \neq \varnothing$, in the absence of explicit mention to the contrary.

THEOREM 2.2. The set $\mathfrak{B}_{H_0}(T_0, Y)$ is a component in the K-space $\mathscr{L}(Z, Y)$.

Remark 2.3. The proofs of Proposition 2.1 and Theorem 2.2 can be carried out in the same way as the proofs of the analogous facts for maximal operators. Moreover, in checking that the set $\mathfrak{B}_{H_0}(T_0, Y)$ is regular it is necessary to apply the lemma on adapted decompositions.

By Theorem 2.2, there exists in $\mathfrak{B}_{H_0}(T_0, Z)$ a largest T_0-adapted projection. This projection is called the Choquet projection (with respect to the cone H_0). The component on which this projection acts is called the Choquet component or Choquet boundary (with respect to the cone H_0) and is denoted by $Ch_{H_0}(T_0)$. We remark that for the usual Choquet boundary, $Ch(T_0) = Ch_X(T_0)$ in K-spaces.

The theorem on the anormality of maximal operators is of essential importance in the Choquet theory. An analogous fact is also true for adapted operators. However, its proof is somewhat more complicated.

THEOREM 2.4. The restriction of a T_0-adapted operator to the disjoint complement of the Choquet boundary is anormal. Moreover, the Choquet boundary is the disjoint complement of the common part of the kernels (null lattices) of all T_0-adapted operators defined on the test K-space.

Proof. It is not complicated to check that the intersection of the kernels of all the T_0-adapted operators coincides with the set N, the intersection of the null lattices of the operators. Moreover, $N^d \supset Ch_{H_0}(T_0)$. Thus to complete the proof, if it is necessary only to check that the projection P onto the component N^d is T_0-adapted.

Thus, let $S \succ PT_0$, and assume that for some $h \in H_0$ the inequality $Sh \geq PT_0h$ is false. Then there exists a projection P_0 in Z such that $P_0PT_0h > P_0Sh$. Moreover, we have $TP_0S \succ TP_0PT_0$ for every T_0-adapted operator T. By Theorem 2.2, the operator TP_0P is T_0-adapted. Hence $0 \leq TP_0Sh - TP_0PT_0h \leq 0$. Thus $P_0PT_0h - P_0Sh \in N$ and $PP_0PT_0h - PP_0Sh = 0$. In other words, $P_0(PSh - PT_0h) = 0$. Furthermore, $P^dS \succ 0$, so that $P^dSh \geq 0$. Thus $P_0(Sh - PT_0h) \geq 0$. Thin contradiction means that $Sh \geq PT_0h$ for all $h \in H_0$. Thus $N^d \subset Ch_{H_0}(T_0)$ and the theorem is proved.

Using the above theorem, it is possible to give a rather complete description of the T_0-extremal operators in the case when $\mathscr{L} = \mathscr{L}(X, Y)$. The point is that in this case an operator T is T_0-extremal if and only if T is $(T_0, N(TT_0))$-adapted.

THEOREM 2.5. In the case $\mathscr{L} = \mathscr{L}(X, Y)$, the following assertions are valid.

(1) $\mathfrak{C}(T_0)$ is an upper regular cone in $\mathscr{L}(Z, Y)$

(2) The restriction of a T_0-adapted operator T to the disjoint complement of the Choquet boundary $Ch_{N(TT_0)}(T_0)$ is anormal.

(3) The projection onto the Choquet component $Ch_{N(TT_0)}(T_0)$ is T_0-extremal for every completely linear T_0-extremal operator T.

(4) If $T \in \mathfrak{C}(T_0)$ and T is completely linear, the projection P_T onto its component of essential positivity is T_0-extremal.

(5) If the cone H is coinitial in X and T is completely linear, then $T \in \mathfrak{C}(T_0)$ if and only if $P_T \in \mathfrak{C}(T_0)$.

Proof. (1) We verify that $\mathfrak{G}(T_0)$ is an upper lattice. If $T_1, T_2 \in \mathfrak{G}(T_0)$ and $S \succ (T_1 \vee T_2)T_0$, then by the relation $T_1T_0 + T_2T_0 \geq (T_1 \vee T_2)T_0 \geq T_1T_0 \vee T_2T_0$ and Theorem 1.1, we have $N(S) \supset N(T_1T_0 + T_2T_0) \supset N((T_1 \vee T_2)T_0)$. The remaining properties of $\mathfrak{G}(T_0)$ are verified analogously.

(2) This follows from Theorem 2.4.

(3) Let P be the Choquet projection [onto the Choquet component $Ch_{N(TT_0)}(T_0)$] and $S \succ PT_0$. Then by hypothesis, $N(S) \supset N(TT_0)$. Since in turn $N(TT_0) = N(P_TT_0)$ and moreover, $P_T \leq P$, we have $N(S) \supset N(PT_0)$.

(4) By Theorem 2.4 we have $N(T) \supset Ch_{N(TT_0)}(T_0)^d$. Thus the projection P_T is majorized by the Choquet projection.

(5) To prove the sufficiency of this condition, we observe that extremality of the operator $T \in \mathscr{L}_+(X, Y)$ with respect to the coinitial cone H in X is equivalent to the relation $q_{H,T} = q_{H(T),T}$, where $H(T) = H + N(T)$ and $q_{M,S}(x) = \sup\{Sm : m \in M,\ m \leq x\}$. Assume that the projection P_T lies in $\mathfrak{G}(T_0)$. Then for all $x \in X$ we have the following equation:

$$\sup\{P_TT_0h : h \in H,\ h \leqslant x\} = \sup\{P_TT_0h : h + h_0 \leqslant x,\ h \in H,\ h_0 \in N(TT_0)\}.$$

In other words,

$$T\sup\{T_0h : h \in H,\ h \leqslant x\} = T\sup\{T_0h : h + h_0 \leqslant x,\ h \in H,\ h_0 \in N(TT_0)\}.$$

The sets appearing in the sup sign in the left-hand and right-hand sides of the last equality are filtered upward. Thus $q_{H,TT_0} = q_{H(TT_0),TT_0}$, which means that T is T_0-extremal. The theorem is proved.

In concluding this section, we cite a characteristic of another class of extremal operators. Namely, we consider the cone $\mathfrak{G}(T_0, H\ \mathfrak{B}(Y))$, where $\mathfrak{B}(Y)$ is a component of boundary operators in the sense of Choquet, $\mathfrak{B}(Y) \subset \mathscr{L}(X, Y)$. We denote this class by $\mathfrak{G}_s(T_0)$. The operators in $\mathfrak{G}_s(T_0)$ are called superextremal [in $\mathscr{L}(X, Y)$ with respect to the cone H]. This terminology is due to the evident inclusion $\mathfrak{G}_s(T_0) \supset \mathfrak{G}(T_0)$. We remark that the last inclusion is in general strict.

Proposition 2.6. Let the operator T be completely linear.

(1) If H is coinitial in X and $T \in \mathfrak{G}_s(T_0)$, then $P_T \in \mathfrak{G}_s(T_0)$.

(2) If the cone H is simplicial and $P_T \in \mathfrak{G}_s(T_0)$, then $T \in \mathfrak{G}_s(T_0)$

Proof. (1) Let $S \in \mathfrak{B}(Y)$ and $S \succ P_TT_0$. Then $TS \succ TP_TT_0 = TT_0$. We have $TSx = Tq_{H,S}(x) = T\sup\{Sh : h \in H, h \leq x\} = q_{H,TS}(x)$ since T is completely linear and the set $\{Sh : h \in H,\ h \leq x\}$ is filtered upwards. Thus, $TS \in \mathfrak{B}(Y)$ and hence $N(TS) \supset N(TT_0) = N(P_TT_0)$, which means that $P_T \in \mathfrak{G}_s(T_0)$.

(2) Since the cone H is simplicial, a unique balayage Ψ_H is defined in $\mathscr{L}_+(X, Y)$, i.e., $\Psi_H(S)$ is the unique maximal operator majorizing S in the Choquet ordering. Repeating the arguments of (1), we convince ourselves that the operator $T\Psi_H(P_TT_0)$ is maximal, so that $\Psi_H(TT_0) = T\Psi_H(P_TT_0)$. Hence if $P_T \in \mathfrak{G}_s(T_0)$, then $N(\Psi_H(TT_0)) \supset N(T\Psi_H(P_TT_0)) \supset N(TP_TT_0) = N(TT_0)$. Thus $T \in \mathfrak{G}_s(T_0)$.

3. Supports of Extremal Measures

In this section, we study the case of extremal Radon measures, which is important in the applications. Part of the results given below follow from the assertion of the preceding section. However, we succeed here in giving a complete boundary description of arbitrary positive extremal functionals. We remark that it is well known how to carry over these results to certain classes of operators [6], and therefore we will not consider such generalizations in what follows.

Thus let H be a coinitial upper lattice in the space C(Q) of continuous functions on the compact metrizable space Q. When we speak of extremality (in one sense or another) of a set in Q, we will mean extremality of the projection given by restriction to this set in the space R^Q with respect to the natural imbedding $E : C(Q) \to R^Q$. Furthermore, the space $C'(Q) = \mathscr{L}(C(Q), R)$ is taken as the component $\mathscr{L}$ if we are talking about measures; if we talk about operators, we take this component to be the space $\mathscr{L}(C(Q), R^Q)$. The positive Radon measures will be identified with the corresponding regular Borel measures without special mention. If μ is a measure, the symbol supp (μ) denotes its support, and Ch (μ) denotes the Choquet boundary $Ch_{N(\mu)}(E)$ constructed with respect to the subspace $N(\mu)$ in C(Q). The symbol Ch(H) denotes the generally known boundary $Ch_{C(Q)}(E)$, and ε_x is the Dirac measure $f \to f(x)$.

THEOREM 3.1. A measure is extremal if and only if its support contains an extremal set of full measure.

Proof. If the support of the measure μ contains an extremal set G of full measure and $\nu > \mu$, then by a theorem of Hardy–Littlewood–Polya [5-7], there exists a weakly measurable map $x \to T_x$, where $T_x > \varepsilon$, such that

$$\nu(f) = \int_Q T_x(f)\, d\mu = \int_G T_x(f)\, d_\mu \quad (f \in C(Q)).$$

Since G is an extremal set, supp $(T_x) \subset \bar{G} \subset$ supp (μ) for all $x \in G$, so that supp $(\nu) \subset$ supp (μ). In other words, $N(\nu) \subset N(\mu)$.

Assume now that the measure μ is extremal. We remark first of all that the point x appears in the boundary $Ch(\mu)$ if and only if

$$\mathrm{co}_H f(x) = \mathrm{co}_{H(\mu)} f(x) \qquad (f \in C(Q)),$$

where co_H, $\mathrm{co}_{H(\mu)}$ are, respectively, the operators of taking the convex hull. Moreover, as we have seen in the preceding section, for the measure μ we have

$$\mu(\mathrm{co}_H f) = \mu(\mathrm{co}_{H(\mu)} f) \qquad (f \in C(Q)).$$

We observe that the operators co_H and $\mathrm{co}_{H(\mu)}$ are uniformly continuous. Thus, if $\{f_n\}$ is a countable everywhere dense subset in C (Q) and

$$G_n = \{x \in Q : \mathrm{co}_H f_n(x) = \mathrm{co}_{H(\mu)} f_n(x)\},$$

then we have

$$\bigcap_{n=1}^{\infty} G_n = \mathrm{Ch}(\mu), \mu\ (G_n) = \mu(Q).$$

Thus $\mu(\mathrm{Ch}(\mu)) = \mu(Q) = \mu(\mathrm{supp}\ (\mu))$. Hence the measure μ is concentrated in the set $\mathrm{Ch}(\mu) \cap \mathrm{supp}\ (\mu)$.

Now let $S \in \mathscr{L}_+(C(Q), R^Q)$ and let S majorize the projection of restriction to $\mathrm{Ch}(\mu) \cap \mathrm{supp}\ (\mu)$ in the Choquet ordering generated by the cone H. Put $\nu_x : f \to Sf(x)$. In order to prove that the projection is extremal we must check that $Sf = 0$ holds for a positive function $f \in C(Q)$ such that $f(x) = 0$ for $x \in \mathrm{Ch}(\mu) \cap \mathrm{supp}\ (\mu)$, i.e., $\nu_x(f) = 0$ for all $x \in Q$. We first take the case $x \in \mathrm{Ch}(\mu) \cap \mathrm{supp}\ (\mu)$. Since $\mu(f)$ is evidently zero, $\nu_x > \varepsilon_x$, and $x \in \mathrm{Ch}(\mu)$, we have $\nu_x(f) = 0$ by the definition of the boundary. If on the other hand $x \notin \mathrm{Ch}(\mu) \cap \mathrm{supp}\ (\mu)$, then $\nu_x > 0$ and hence $\nu_x(f) = 0$, e.g., by Theorem 2.2. Thus, $\mathrm{Ch}(\mu) \cap \mathrm{supp}\ (\mu)$ is an extremal set.

Remark 3.2. The Choquet boundary $\mathrm{Ch}(\mu)$ for an extremal measure μ in fact has two important properties. First, this boundary is itself an extremal set. Indeed, if $x \in \mathrm{Ch}(\mu)$ and $x \notin \mathrm{supp}\ (\mu)$, then we have $\mathrm{co}_{H(\mu)} f(x) = f(x)$ for all $f \in C(Q)$, which means that $x \in \mathrm{Ch}(H)$. Thus, $\mathrm{Ch}(\mu) \cap (Q \setminus \mathrm{supp}\ (\mu)) \subset \mathrm{Ch}(H)$ and hence the part of $\mathrm{Ch}(\mu)$ outside the support of μ is an extremal set. Thus by, e.g., Theorem 2.5, the set $\mathrm{Ch}(\mu)$ is extremal. Second, $\mathrm{Ch}(\mu) \cap \mathrm{supp}\ (\mu)$ is the largest extremal set contained in the support of the measure μ (the existence of such a set in the general case is guaranteed by Theorem 2.5). Indeed, if G is an extremal set, $G \subset \mathrm{supp}\ (\mu)$, the point x lies in G, and $\nu > \varepsilon_x$, then supp $(\nu) \subset \bar{G} \subset$ supp (μ). In other words, $N(\nu) \supset N(\mu)$. The last fact means that $x \in \mathrm{Ch}(\mu)$.

Theorem 3.1 shows that the support of an extreme measure contains an extreme set of full measure. However, one must not assume that the support itself has to be an extremal set. Nevertheless, the situation is different for certain classes of cones which arise fairly frequently.

THEOREM 3.3. If the map $x \to \mathrm{Spr}\ (\varepsilon_x, H)$, where $\mathrm{Spr}\ (\varepsilon_x, H) = \{\nu : \nu > \varepsilon_x\}$, is lower semicontinuous in the extended Hausdorff topology, then a measure is extreme if and only if its support is extreme.

THEOREM 3.4. If the pair (H, C (Q)) has the Lim property, then a measure is superextremal if and only if its support is superextremal.

These theorems are proved by analogous methods; therefore, we give the proof of only one of them.

Proof of Theorem 3.4. First assume we know that supp (μ) is a superextremal set, and let ν be a maximal measure majorizing μ in the Choquet ordering. By the Hardy–Littlewood–Polya theorem, we have

$$\nu(f) = \int_Q T_x(f)\, d\mu = \int_{\mathrm{supp}(\mu)} T_x(f)\, d\mu \quad (f \in C(Q))$$

for some family $x \to T_x$, where $T_x > \varepsilon_x$. As is well known, the measure T_x is maximal for almost all x. Thus if $\mu(f) = 0$ for a positive function f, the function $x \to T_x(f)$ vanishes almost everywhere. Hence $\nu(f) = 0$.

Now assume that we know the measure μ is superextremal. Take a point $x \in \operatorname{supp}(\mu)$ and a maximal measure ν such that $\nu \succ \varepsilon_x$. Since the pair (H, C(Q)) has the Lim property, there exists an extended continuous additive balayage Ψ such that $\Psi(\varepsilon_x) = \nu$. Furthermore, it is obvious that

$$\Psi(\mu)(f) = \int_Q \Psi(\varepsilon_x)(f)\, d\mu \quad (f \in C(Q)).$$

Since $\Psi(\mu)$ is a maximal measure and $\Psi(\mu) \succ \mu$, if $f \in N(\mu)$, then $f \in N(\Psi(\mu))$. Thus, $\Psi(\varepsilon_x)(|f|) = 0$ for all $x \in \operatorname{supp}(\mu)$. In particular, $\nu(|f|) = 0$, which means that supp (μ) is superextremal. The theorem is proved.

LITERATURE CITED

1. E. Alfsen, Compact Convex Sets and Boundary Integrals, Springer-Verlag, New York (1971).
2. L. V. Kantorovich, B. Z. Vulikh, and A. G. Pinsker, Functional Analysis in Partially Ordered Spaces [in Russian], Gostekhizdat, Moscow–Leningrad (1950).
3. B. Z. Vulikh, Introduction to the Theory of Partially Ordered Spaces, Gordon (1967).
4. P. Mayer, Probability and Potential [Russian translation], Mir, Moscow (1973).
5. C. C. Kutateladze, "Choquet boundaries in K-spaces," Usp. Mat. Nauk, 30, No. 4, 107-146 (1975).
6. G. Vincent-Smith, "A Choquet boundary theory for measures taking values in a Stone algebra," J. London Math. Soc., 44, No. 3, 553-558 (1969).
7. V. L. Levin, "Convex integral functionals and lifting theory," Usp. Mat. Nauk, 30, No. 2, 115-178 (1975).

A PROBLEM IN THE THEORY OF PHYSICAL STRUCTURES

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In his research on the foundations of physics, Kulakov [1, 2] has proposed a mathematical model of the structure of a physical law. Under the name of a binary physical structure of rank (m, n), this model has been considered for the simplest cases m = n = 2 and m = n + 1 = 3 in [3-5] by the method of parametrization. In this paper we present a more perfect formulation of the axioms of a physical structure and we use the functional method described by us in [6]. Due to the length of the analysis, a complete proof is carried out for $m \geq n = 2$ (Sec. 2).

§1. Mathematical Formulation of Theory of Physical Structures

For greater clarity, it is possible to describe the concept of a physical structure as follows. Let us consider two sets $\mathfrak{M}$ and $\mathfrak{N}$, and a function $a: \langle i, \alpha\rangle \in \mathfrak{M}\times\mathfrak{N} \to a_{i\alpha} \in R$. Next, let $m \geq n \geq 2$ be integers. We shall say that a triple $\langle \mathfrak{M}, \mathfrak{N}, a\rangle$ forms a binary physical structure of rank (m, n) if for any elements $i_1, i_2, \ldots, i_m \in \mathfrak{M}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathfrak{N}$ we have the equation $\Phi(a_{i_1\alpha_1}, a_{i_1\alpha_2}, \ldots, a_{i_m\alpha_n}) = 0$. On the sets $\mathfrak{M}$ and $\mathfrak{N}$ and the functions a and Φ we shall impose certain sufficient natural restrictions without which our problem is apparently meaningless from a mathematical point of view. Let us present exact formulations.

Let us consider two sets $\mathfrak{M}$ and $\mathfrak{N}$, and a function $a: \langle i, \alpha\rangle \to a_{i\alpha} \in R$. For convenience we shall write (i, α) instead of $a_{i\alpha}$.

Definition 1. Two elements $i, j \in \mathfrak{M}$ (or $\alpha, \beta \in \mathfrak{N}$) are said to be equivalent if for any $\gamma \in \mathfrak{N}$ (or $k \in \mathfrak{M}$) we have the equation $(i, \gamma) = (j, \gamma)$ [or $(k, \alpha) = (k, \beta)$].

It is assumed that the sets $\mathfrak{M}$ and $\mathfrak{N}$ are factored according to the above equivalence relation.

Definition 2. An element $\tilde{i} \in \mathfrak{M}$ (or $\tilde{\alpha} \in \mathfrak{N}$) is called a quasizero element with respect to $\mathfrak{N}$ (or $\mathfrak{M}$) if the set of values of the function $a: \{\tilde{i}\}\times\mathfrak{N} \to R$ (or $a: \mathfrak{M}\times\{\tilde{\alpha}\} \to R$) is finite.

Let us denote by $\tilde{\mathfrak{M}}$ and $\tilde{\mathfrak{N}}$ the sets of nonzero elements. We shall assume that they are finite and that $\mathfrak{M} \setminus \tilde{\mathfrak{M}}, \mathfrak{N} \setminus \tilde{\mathfrak{N}}$ are not empty.

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