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THE KREIN – MIL'MAN THEOREM AND ITS INVERSE

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In this article we extend the Krein–Mil'man theorem and its inverse to the case of support sets of operators. The question of recovering these sets from extreme points was put forward in the well-known paper of Bonsall, Lindenstrauss, and Phelps [1]. Two approaches to this problem are known. In [2] Oates analyzed the case of operators acting in a K-space of bounded elements; the case of operators with values in K-spaces with a sufficient set of o-continuous functionals was analyzed by Rubinov [3], who discovered a new effect of the process of recovering a subdifferential from weakly extreme points. Both these results appeal in one way or another to compactness; this is not completely satisfactory, since it restricts the class of admissible sets and spaces. As regards general spaces and the inversion of operator forms of the Krein–Mil'man theorem, there are missing results, and the aim of this article is to fill the gaps. It is interesting that this aim can be achieved by the route suggested by Krein and Mil'man in their classical paper [4] (see also [5, 6]). It is proved in particular, that every subdifferential can be recovered not only from the set of its extreme points, but also from a subset (as a rule, proper) consisting of o-extreme points determined in a universal way. It is important to note that the Krein–Mil'man theorem for the subdifferentials of convex operators is connected with the chain-completeness of the range of the operators in question. Whereas, Mil'man's inverse of this theorem, which we establish in a form that is new even in the scalar case, is connected with an essentially stronger property, viz., that the range has the structure of a Kantorovich space.

Finally, we mention that detailed surveys of the topic of extreme points are to be found in [7, 8]. The facts about the subdifferential calculus that we use are presented in detail in [9, 10].

1°. Krein – Mil'man Theorem

Let X be a vector space, Y be some K-space, and $P: X \to Y$ be a sublinear operator. The symbol Ch (P) denotes the set of extreme points of the subdifferential or the support set $\partial(P)$ of the operator P, i.e.,

$$\partial(P) = \{A \in L(X, Y) : Ax \leqslant Px \ (x \in X)\}.$$

We recall that if $\mathfrak{A}$ is a weakly ordered bounded set in the space L(X, Y) of linear operators, then the smallest subdifferential containing $\mathfrak{A}$ is denoted by cop $(\mathfrak{A})$, and it is called the support hull of $\mathfrak{A}$. The symbol $\langle\mathfrak{A}\rangle$ denotes the linear operator acting from X into the K-space $(Y^{\mathfrak{A}})_\infty$ of bounded Y-valued functions on $\mathfrak{A}$:

$$\langle\mathfrak{A}\rangle x : A \to Ax \quad (A \in \mathfrak{A}).$$

The symbol $\varepsilon_{\mathfrak{A}}$ stands for the canonical sublinear operator

$$\varepsilon_{\mathfrak{A}} : (Y^{\mathfrak{A}})_\infty \to Y; \ \varepsilon_{\mathfrak{A}} f = \sup\{f(A) : A \in \mathfrak{A}\}.$$

The support hull of the set $\mathfrak{A}$ has the following representations:

$$\operatorname{cop}(\mathfrak{A}) = \partial(\varepsilon_{\mathfrak{A}}) \circ \langle\mathfrak{A}\rangle = \{\alpha \circ \langle\mathfrak{A}\rangle : \alpha \in L^+((Y^{\mathfrak{A}})_\infty, Y);\ \alpha \circ \Delta_{\mathfrak{A}} = I_Y\},$$

where I_Y denotes the identity mapping of Y into itself, and $\Delta_{\mathfrak{A}} : Y \to (Y^{\mathfrak{A}})_\infty$ is the natural identification of Y with the diagonal of the space $Y^{\mathfrak{A}}$.

Now let Z be another K-space, and let $T \in L^+(Y, Z)$ be a positive linear operator. We denote by $\mathscr{E}(T, P)$ the set of operators $A \in \partial(P)$ such that $T \circ A$ is an extreme point of $\partial(T \circ P)$, i.e.,

$$\mathscr{E}(T, P) = \{A \in \partial(P) :\ T \circ A \in \operatorname{Ch}(T \circ P)\}.$$

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If $\mathfrak{T}$ is a family of positive operators defined on Y, then we set

$$\mathscr{E}(\mathfrak{T}, P) = \bigcap_{T \in \mathfrak{T}} \mathscr{E}(T, P).$$

If, in particular, $\mathfrak{T}$ is the class of o-continuous operators defined on Y, then we write $\mathscr{E}(\mathfrak{T}, P)$ instead of $\mathscr{E}_0(P)$. We stress that the range of the operators being considered is not fixed. The elements of $\mathscr{E}_0(P)$ are called the o-extreme points of ∂(P).

Krein–Mil'man Theorem. Every subdifferential is the support hull of its o-extreme points.

Proof. We consider the set $\mathfrak{F}$ of all subdifferential operators $P' : X \to Y$ such that $\partial(P') \subset \partial(P)$, and in addition, that $\partial(T \circ P')$ is an extreme subset of $\partial(T \circ P)$ for any o-continuous operator T defined on Y. Clearly, $P \in \mathfrak{F}$. We order $\mathfrak{F}$ in a natural way, setting $P_1 \geqslant P_2 \Leftrightarrow \partial(P_1) \supset \partial(P_2)$; it can be verified that $\mathfrak{F}$ is inductive.

Consider an arbitrary chain $\mathfrak{C}$ from $\mathfrak{F}$. First note that the family $\{P'x : P' \in \mathfrak{C}\}$ is a descending filtration for any $x \in X$. Here, $P(-x) + P'x \geq P'x + P'(-x) \geq 0$, so that there is defined an element

$$P_0x = \inf\{P'x : P' \in \mathfrak{C}\}.$$

Obviously, the operator $P_0 : X \to Y$ is sublinear. We verify that $P_0 \in \mathfrak{F}$. For this we take an o-continuous operator T, numbers $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 + \alpha_2 = 1$, and operators $A_1, A_2 \in \partial(T \circ P)$ satisfying the condition $\alpha_1 A_1 + \alpha_2 A_2 \in \partial(T \circ P_0)$. Since the set $\partial(T \circ P')$ is extremal and contains $\partial(T \circ P_0)$ for any $P' \in \mathfrak{C}$, we have $A_1 \in \partial(T \circ P')$ and $A_2 \in \partial(T \circ P')$. Because T is o-continuous, we obtain

$$A_1x \leqslant \inf\{T \circ P'x : P' \in \mathfrak{C}\} = T \inf\{P'x : P' \in \mathfrak{C}\} = T \circ P_0x.$$

Hence, $A_1 \in \partial(T \circ P_0)$. Similarly, $A_2 \in \partial(T \circ P_0)$.

By Zorn's lemma, $\mathfrak{F}$ contains a minimal element Q. We denote by Q_x the directional derivative $Q'(x)$ of Q at $x \in X$. Clearly, $Q_x \leq Q$. In addition, for any o-continuous operator T we have

$$\partial(T \circ Q_x) = \partial(T \circ Q'(x)) = \partial((T \circ Q)'(x)) = \partial_x(T \circ Q) = \{B \in \partial(T \circ Q) : Bx = T \circ Qx\}.$$

Suppose that $\alpha_1 A_1 + \alpha_2 A_2 \in \partial(T \circ Q_x)$ for some $A_1, A_2 \in \partial(T \circ P)$ and for $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 + \alpha_2 = 1$. Then $A_1, A_2 \in \partial(T \circ Q)$ because the set $\partial(T \circ Q)$ is extremal and $\partial(T \circ Q_x) \subset \partial(T \circ Q)$. Moreover, in view of what we have proved already, we have $\alpha_1 A_1 x + \alpha_2 A_2 x = T \circ Qx$. Hence,

$$0 \geqslant \alpha_1(A_1x - T \circ Qx) + \alpha_2(A_2x - T \circ Qx) = 0.$$

Thus, $A_1, A_2 \in \partial_x(T \circ Q) = \partial(T \circ Q_x)$. Finally, we find that Q_x lies in $\mathfrak{F}$. Because Q is a minimal element, it follows that $Q_x = Q$. Since $x \in X$ is an arbitrary element, we find that Q is a linear operator. Hence, $Q \in \mathscr{E}_0(P)$.

Thus, we have proved that any subdifferential contains o-extreme points. Therefore, to complete the proof it is enough to remark that $\mathscr{E}_0(P) \supset \mathscr{E}_0(P'(x))$ for every $x \in X$. This holds because, as we have noted, $\partial(T \circ Q'(x))$ is an extremal subset of $\partial(T \circ Q)$ for any o-continuous operator T and an arbitrary sublinear operator Q. Thus, the following estimates hold for any $x \in X$:

$$Px \geqslant \sup\{Ax : A \in \mathscr{E}_0(P)\} \geqslant \sup\{Ax : A \in \mathscr{E}_0(P'(x))\} = Px.$$

The theorem is proved completely.

Remark. It is clear from the proof that the Krein–Mil'man theorem is valid under weaker conditions. Namely, it is enough to require that the space Y and the ranges of the o-continuous operators in question only have the property of chain completeness, and are not necessarily K-spaces.

Here it is appropriate to mention the following. Since $\mathrm{Ch}(P) = \mathscr{E}(I_Y, P)$, the Krein–Mil'man theorem in the above form means, in particular, that a subdifferential can be recovered from the set of its extreme points.

2°. Properties of Extreme Points

In this section we give the conditions for extreme points that we need later. In addition, we describe the o-extreme points of the subdifferential of a canonical operator.

THEOREM 1. For an operator A from ∂(P) the following assertions are equivalent:

(1) A is contained in $\mathscr{E}(T, P)$.

(2) If operators $T_1, T_2; A_1, A_2$ satisfy

$$T_1, T_2 \in L^+(Y, Z);\ A_1, A_2 \in L(X, Z);$$

$$T_1+T_2=T;\ A_1\in\partial(T_1\circ P);\ A_2\in\partial(T_2\circ P);\ T\circ A=A_1+A_2,$$

then $T_1\circ A = A_1$ and $T_2\circ A = A_2$.

(3) For the operator $\mathscr{A}:(x, y)\to y-Ax$, defined on the space $X\times Y$, ordered by the cone $\{(x, y)\in X\times Y: y\geq Px\}$, the equality of ordered intervals holds: $[0, T]\circ\mathscr{A}=[0, T\circ\mathscr{A}]$.

(4) For any $x\in X$ and $y\in Y$ we have

$$Ty^+=\inf_{u\in X}\{T((Pu-Au)\vee(P(u-x)-A(u-x)+y))\}.$$

Proof. We prove the theorem by the scheme (1) ⇒ (2) ⇒ (3) ⇒ (1) and (3) ⇔ (4)., because the equivalence of the first three assertions is not needed under the additional assumptions about the structure of the ordering in the spaces Y and Z.

(1) ⇒ (2). If the operators T_1, T_2; A_1, A_2 satisfy the conditions (2), then

$$2T\circ A=(T_1\circ A+A_2)+(A_1+T_2\circ A);$$
$$T_1\circ A+A_2\in\partial(T\circ P);\ A_1+\partial(T_2\circ A)\in\partial(T\circ P).$$

Thus, by (1) we have $T\circ A = T_1\circ A + A_2 = A_1 + T_2\circ A$. Hence it follows that $T_1\circ A = A_1$ and $T_2\circ A = A_2$.

(2) ⇒ (3). First of all let us note that the operator $(x, y)\to Uy - Vx$, where $U\in L(Y, Z)$, and $V\in L(X, Z)$, is positive on the space $X\times Y$ with the indicated ordering, if and only if $U\in L^+(Y, Z)$ and $V\in\partial(U\circ P)$. Hence it follows that the operator $\mathscr{A}$ is positive because $A\in\partial(P)$ and so $[0, T]\circ A\subset[0, T\circ\mathscr{A}]$. If A satisfies (2) and the operator $\mathscr{B}$: $(x, y)\to Uy-Vx$ is positive and majorized by the operator $T\circ\mathscr{A}$, then U, $T-U\in L^+(Y, Z)$. Here, $V\in\partial(U\circ P)$ and $T\circ A-V\in\partial((T-U)\circ P)$. By applying (2) to the operators $T_1 = U$, $T_2 = T - U$, $A_1 = V$, and $A_2 = T\circ A - V$, we obtain $V = A_1 = T_1\circ A = U\circ A$, so that $\mathscr{B}=U\circ\mathscr{A}$, where $U\in[0, T]$.

(3) ⇒ (1). Suppose that A satisfies (3) and that $T\circ A = \alpha_1A_1 + \alpha_2A_2$, where A_1, $A_2\in\partial(T\circ P)$ and α_1, $\alpha_2>0$ with $\alpha_1+\alpha_2=1$. We consider the operator $\mathscr{B}:(x, y)\to\alpha_1T_y-\alpha_1A_1x$. Then $\mathscr{B}\in[0, T\circ\mathscr{A}]$ because

$$(T\circ\mathscr{A}-\mathscr{B})(\cdot, 0)=-\alpha_2A_2;\ (T\circ\mathscr{A}-\mathscr{B})(0, \cdot)=\alpha_2T.$$

Thus, $T_1\circ\mathscr{A}=\mathscr{B}$ for some $T_1\in[0, T]$; in other words, $T_1=\alpha_1T$ and $T_1\circ A = \alpha_1A_1$. The latter means that $A_1 = T\circ A$.

(3) ⇔ (4). Note that for every $(x, y)\in X\times Y$ we have

$$(x, y\vee Px),\ (0, y\vee Px-y)\in\{(x, y)\in X\times Y: y\geqslant Px\}.$$

Hence, the following set is nonempty:

$$U_{(x, y)}=\{(u, v)\in X\times Y: v\geqslant Pu;\ v-y\geqslant P(u-x)\}.$$

Thus, the following operator is properly defined:

$$P_1:(x, y)\to\inf\{T\circ\mathscr{A}(u, v):(u, v)\in U_{(x, y)}\}.$$

Clearly, the operator $P_1: X\times Y\to Z$ is sublinear, and $\partial(P_1)=[0, T\circ\mathscr{A}]$. It is known that a sublinear operator $P_2=(y\to Ty^+)\circ\mathscr{A}$ satisfies $\partial(P_2)=[0, T]\circ\mathscr{A}$. Thus, by the Minkowski duality indicated in (3), the equality of ordered intervals holds if and only if $P_1 = P_2$. This means that for all $x\in X$ and $y\in Y$,

$$T(y-Ax)^+=\inf_{u\in X, v\in Y}\{Tv-T\circ Au: v\geqslant Pu\vee(y+P(u-x))\}=$$
$$=\inf_{u\in X}\{T(Pu\vee(y+P(u-x))-T\circ Au\}=\inf_{u\in X}\{T((Pu\vee(y+P(u-$$
$$-x))-Au)\}=\inf_{u\in X}\{T((Pu-Au)\vee(P(u-x)-A(u-x)+y-Ax))\}.$$

Since $x\in X$ and $y\in Y$ are arbitrary, the inequality we have derived is equivalent to (4). The theorem is proved completely.

A number of useful properties of extreme points can be obtained as corollaries of Theorem 1. We only state these results because their proof is a straightforward verification.

Proposition 1. If T is an o-continuous operator and $A\in\mathscr{E}(T, P)$, then the projection Pr_T onto the component of essential positivity of T is such that $\mathrm{Pr}_T\circ A\in\mathrm{Ch}(\mathrm{Pr}_T\circ P)$.

Proposition 2. If $\mathfrak{T}$ is the set of o-continuous lattice homomorphisms defined on Y, then $\mathscr{E}(\mathfrak{T}, P)=\mathrm{Ch}(P)$.

Proposition 3. If the cone $\{(x, y) \in X \times Y : y \geqslant Px\}$ is minihedral, then $\mathscr{E}(T, P) \supset \mathrm{Ch}(P)$ for every operator T. Here, $\mathscr{E}_0(P) = \mathrm{Ch}(P)$.

Proposition 4. If $\mathfrak{T}$ is the ordered interval [0, T], then $\mathscr{E}(\mathfrak{T}, P) = \mathscr{E}(T, P)$.

Proposition 5. Let Γ be a set of o-continuous lattice homomorphisms of Z into a K-space W such that Γ is bounded from above and is filtered ascendingly. For any $\mathfrak{T} \subset L^+(Y, Z)$,

$$\mathscr{E}(\sup \Gamma \circ \mathfrak{T}, P) = \mathscr{E}(\Gamma \circ \mathfrak{T}, P).$$

COROLLARY. If Z = W and Γ is an arbitrary family of projections, then $\mathscr{E}(\sup \Gamma \circ \mathfrak{T}, P) = \mathscr{E}(\Gamma \circ \mathfrak{T}, P)$.

Remark. It follows from the above, in particular, that to find o-extreme points it is sufficient to consider only "large" operators with "small" ranges. Thus, if Y is a space of type (L) and $\mathfrak{T}$ is the family of all operators on Y with values in regularly ordered spaces, then $\mathscr{E}(\mathfrak{T}, P) = \mathscr{E}(\mathbf{1}, P)$, where **1** is a strong unit in the space of continuous functionals on Y.

THEOREM 2. The following assertions are equivalent:

(1) An operator A is in Ch (P).

(2) For any operators $A_1, \ A_2 \in \partial(P)$ and multipliers $\alpha_1, \alpha_2 \in [0, I_Y]$ such that $\alpha_1 + \alpha_2 = I_Y$ and $\alpha_1 \circ A_1 + \alpha_2 \circ A_2 = A$, we can find a projection Pr into Y with $\mathrm{Pr} \circ A = \mathrm{Pr} \circ A_1$ and $\mathrm{Pr}^d \circ A = \mathrm{Pr}^d \circ A_2$, where $\mathrm{Pr}^d = I_Y - \mathrm{Pr}$.

(3) If operators $A_1, \ldots, A_n \in \partial(P)$ and multipliers $\alpha_1, \ldots, \alpha_n \in [0, I_Y]$ satisfy

$$\sum_{k=1}^{n} \alpha_k = I_Y, \quad \sum_{k=1}^{n} \alpha_k \circ A_k = A,$$

then $\alpha_k \circ A = \alpha_k \circ A_k$ for every k = 1, . . . , n.

Proof. (1) ⇒ (2). If $\alpha_1, \alpha_2; A_1, A_2$ satisfy the conditions in (2), then since $\mathrm{Ch}(P) = \mathscr{E}(T_Y, P)$ and by Theorem 1, we have $\alpha_1 \circ A_1 = \alpha_2 \circ A$. Let Pr denote the projection onto a component of essential positivity of α_1. It is clear from the properties of multipliers that $\mathrm{Pr} \circ A_1 = \mathrm{Pr} \circ A$. In addition, by noting that $\alpha_1 \circ \mathrm{Pr}^d = 0$, we have

$$\mathrm{Pr}^d \circ A = \mathrm{Pr}^d \circ (\alpha_1 \circ A_1 + \alpha_2 \circ A_2) = \alpha_1 \circ \mathrm{Pr}^d \circ A_1 + \alpha_2 \circ \mathrm{Pr}^d \circ A_2 = \alpha_2 \circ \mathrm{Pr}^d \circ A_2 = (I_Y - \alpha_1) \circ \mathrm{Pr}^d \circ A_2 = \mathrm{Pr}^d \circ A_2.$$

(2) ⇒ (3). To be specific we verify that under the conditions (3), $\alpha_n \circ A = \alpha_n \circ A_n$. For this we first note that

$$\sum_{k=1}^{n-1} \alpha_k \circ A_k \in \partial\left(\sum_{k=1}^{n-1} \alpha_k \circ (x \to A_1 x \vee \ldots \vee A_{n-1} x)\right).$$

By using the rules of the calculus of subdifferentials we obtain the compatibility of the following system of conditions:

$$\sum_{k=1}^{n-1} \alpha_k \circ A_k = \left(\sum_{k=1}^{n-1} \alpha_k\right) \circ \sum_{s=1}^{n-1} \beta_s \circ A_s;$$

$$\sum_{s=1}^{n-1} \beta_s = I_Y; \quad \beta_1, \ldots, \beta_{n-1} \in [0, I_Y].$$

Since the operator $A' = \sum_{s=1}^{n-1} \beta_s \circ A_s$ is in ∂(P) and, in addition, $\sum_{k=1}^{n-1} \alpha_k = I_Y - \alpha_n$, from the equality $A = (I_Y - \alpha_n) \circ A' + \alpha_n \circ A_n$ we obtain $\mathrm{Pr} \circ A_n = \mathrm{Pr} \circ A$ for some projection Pr into Y and $\mathrm{Pr}^d \circ A' = \mathrm{Pr}^d \circ A$. Since multipliers commute with one another we have

$$\mathrm{Pr}^d \circ A = \mathrm{Pr}^d \circ (I_Y - \alpha_n) \circ A' + \mathrm{Pr}^d \circ \alpha_n \circ A_n =$$
$$= (I_Y - \alpha_n) \circ \mathrm{Pr}^d \circ A' + \alpha_n \circ \mathrm{Pr}^d \circ A_n = (I_Y - \alpha_n) \circ \mathrm{Pr}^d \circ A + \alpha_n \circ \mathrm{Pr}^d \circ A_n.$$

Thus, $\alpha_n \circ \mathrm{Pr}^d \circ A = \alpha_n \circ \mathrm{Pr}^d \circ A_n$. Finally, we obtain

$$\alpha_n \circ A = \mathrm{Pr}^d \circ \alpha_n \circ A + \mathrm{Pr} \circ \alpha_n \circ A_n = \alpha_n \circ \mathrm{Pr}^d \circ A_n + \mathrm{Pr} \circ \alpha_n \circ A_n = \alpha_n \circ A_n.$$

(3) ⇒ (1). This follows from Theorem 1.

COROLLARY 1. If $A_1, A_2 \in \mathrm{Ch}(P)$ and Pr is an arbitrary projection into Y, then $\mathrm{Pr} \circ A_1 + \mathrm{Pr}^d \circ A_2 \in \mathrm{Ch}$ (P).

Remark. The last corollary gives us a reason for calling a set $\mathfrak{A}$ in L(X, Y) cyclic if $\Pr\circ A_1+\Pr^d\circ A_2\in\mathfrak{A}$ for any $A_1, A_2\in\mathfrak{A}$ and any projection Pr into Y. The smallest cyclic set containing a given set $\mathfrak{A}$ is called the cyclic hull of $\mathfrak{A}$, and is denoted by $c(\mathfrak{A})$. It can be verified that

$$c(\mathfrak{A})=\left\{\sum_{k=1}^{n}\Pr_k\circ A_k : A_k\in\mathfrak{A};\quad \sum_{k=1}^{n}\Pr_k=I_Y;\quad k=1,2,\ldots\right\}.$$

Thus, we obtain

COROLLARY 2. $\mathrm{Ch}(P)=c(\mathrm{Ch}(P))$.

Remark. It is not difficult to verify that for every multiplier $\alpha\in[0, I_Y]$ and an arbitrary $\varepsilon > 0$ there exists an operator $\alpha_\varepsilon\in \mathrm{co}(c(\{0, I_Y\}))$ such that $0 \le \alpha-\alpha_\varepsilon\le \varepsilon I_Y$. Hence, it is clear that for K-spaces with a sufficient number of o-continuous functionals, Mil'man's inversion of the Krein–Mil'man theorem holds up to a transition to a cyclic hull. In particular, for locally convex K-spaces with the property (A), the closure in the weak operator topology of the cyclic hull of the set of o-extreme points contains all the extreme points of the subdifferential in question. A more general result will be established in the next section.

THEOREM 3. The set of extreme points of the subdifferential of a canonical operator $\varepsilon_\mathfrak{A}$ is the same as the set of lattice homomorphisms of the space $(Y^\mathfrak{A})_\infty$ into Y that lie in $\partial(\varepsilon_\mathfrak{A})$; here $\mathrm{Ch}(\varepsilon_\mathfrak{A})=\mathscr{E}_0(\varepsilon_\mathfrak{A})$.

Proof. We prove the first part of the assertion.

First let $\mu\in\mathrm{Ch}(\varepsilon_\mathfrak{A})$ and $\mu'\in[0, \mu]$. Clearly, the operator $\alpha=\mu'\circ\Delta_\mathfrak{A}$ satisfies $\alpha\in[0, I_Y]$. Here $\mu'\in\partial(\alpha\circ\varepsilon_\mathfrak{A})$. Thus, by the rules of subdifferentiation we have $\mu'=\alpha\circ\mu_1$ for some $\mu_1\in\partial(\varepsilon_\mathfrak{A})$. Similarly, $\mu-\mu'=(I_Y-\alpha)\circ\mu_2$ for some $\mu_2\in\partial(\varepsilon_\mathfrak{A})$. Thus, the representation $\mu=\alpha\circ\mu_1+(I_Y-\alpha)\circ\mu_2$ holds. Hence, by Theorem 2, $\alpha\circ\mu=\alpha\circ\mu_1=\mu'$. Thus, $[0, \mu]=[0, I_Y]\circ\mu$, i.e., μ is a lattice homomorphism.

Now suppose that $\mu\in\partial(\varepsilon_\mathfrak{A})$ and that μ is a lattice homomorphism. If $\mu=\alpha_1\mu_1+\alpha_2\mu_2$, where $\mu_1, \mu_2\in\partial(\varepsilon_\mathfrak{A})$ and $\alpha_1, \alpha_2 > 0$ with $\alpha_1+\alpha_2=1$, then $\alpha_1\mu_1=\alpha\circ\mu$ for some $\alpha\in[0, I_Y]$. Because $\mu_1\circ\Delta_\mathfrak{A}=\mu\circ\Delta_\mathfrak{A}=I_Y$, we have $\alpha=\alpha_1 I_Y$, so that $\mu_1=\mu$.

To prove the second part of the theorem it is sufficient to establish that the cone $\{(f, y)\in(Y^\mathfrak{A})_\infty\times Y : y\geqslant \varepsilon_\mathfrak{A}^+ f\}$ is minihedral. Here, $\varepsilon_\mathfrak{A}^+ f=(\varepsilon_\mathfrak{A} f)^+$. In fact, by Proposition 3 we have $\mathscr{E}_0(\varepsilon_\mathfrak{A}^+)=\mathrm{Ch}(\varepsilon_\mathfrak{A}^+)$. In addition, if $\alpha_1\mu_1+\alpha_2\mu_2\in\partial(\varepsilon_\mathfrak{A})$ and $\mu_1, \mu_2\in\partial(\varepsilon_\mathfrak{A}^+)$, then in view of the relation

$$\partial(\varepsilon_\mathfrak{A}^+)=[0, I_Y]\circ\partial(\varepsilon_\mathfrak{A}),$$

which follows from a rule for subdifferentiation, we obtain $\mu_1\circ\Delta_\mathfrak{A}, \mu_2\circ\Delta_\mathfrak{A}\in[0, I_Y]$, and $\alpha_1\mu_1\circ\Delta_\mathfrak{A}+\alpha_2\mu_2\circ\Delta_\mathfrak{A}=I_Y$. Hence it follows that $\mu_1\circ\Delta_\mathfrak{A}=\mu_2\circ\Delta_\mathfrak{A}=I_Y$. Hence, $\mu_1, \mu_2\in\partial(\varepsilon_\mathfrak{A})$. Thus, $\partial(\varepsilon_\mathfrak{A})$ is an extremal subset in $\partial(\varepsilon_\mathfrak{A}^+)$. Consequently, $\mathrm{Ch}(\varepsilon_\mathfrak{A})\subset\mathrm{Ch}(\varepsilon_\mathfrak{A}^+)=\mathscr{E}_0(\varepsilon_\mathfrak{A}^+)$. Therefore, if $\mu\in\mathrm{Ch}(\varepsilon_\mathfrak{A})$, then $T\circ\mu\in\mathrm{Ch}(T\circ\varepsilon_\mathfrak{A}^+)$ for any operator T, so that $T\circ\mu\in\mathrm{Ch}(T\circ\varepsilon_\mathfrak{A})$ because $\partial(T\circ\varepsilon_\mathfrak{A})\subset\partial(T\circ\varepsilon_\mathfrak{A}^+)$. Thus, $\mathrm{Ch}(\varepsilon_\mathfrak{A})\subset\mathscr{E}_0(\varepsilon_\mathfrak{A})$. The reverse inclusion is obvious, and the theorem is proved.

Thus, we only need to establish that the cone

$$\{(f, y)\in(Y^\mathfrak{A})_\infty\times Y : y\geqslant\varepsilon_\mathfrak{A}^+ f\}=\{(f, y)\in(Y^\mathfrak{A})_\infty\times Y : y\geqslant 0,\quad f\leqslant\Delta_\mathfrak{A} y\}$$

is minihedral. For this we show that for any two elements (f, y) and (g, z) from $(Y^\mathfrak{A})_\infty\times Y$,

$$(f, y)\wedge(g, z)=\left((f-\Delta_\mathfrak{A}(y-z)^+)\wedge(g-\Delta_\mathfrak{A}(z-y)^+), y\wedge z\right).$$

We denote by h the first projection of the right-hand side of the last relation, and note that $(h, y\wedge z)$ is a lower bound of (f, y). In fact,

$$f-h=f+(\Delta_\mathfrak{A}(y-z)^+-f)\wedge(\Delta_\mathfrak{A}(z-y)^+-g)=\Delta_\mathfrak{A}(y-z)^+\wedge(\Delta_\mathfrak{A}(z-y)^++f-g)\leqslant\Delta_\mathfrak{A}(y-z)^+,$$

i.e., $\varepsilon_\mathfrak{A}^+(f-h)\leqslant(y-z)^+=y-y\wedge z$. Similarly, it can be verified that (g, z) is majorized by $(h, y\wedge z)$.

Now let (h', p) be an arbitrary lower bound of (f, y) and (g, z). Then $\varepsilon_\mathfrak{A}^+(f-h')\leqslant y-p$, and $\varepsilon_\mathfrak{A}^+(g-h')\leqslant z-p$. Thus,

$$y-p\geqslant 0;\quad z-p\geqslant 0;\quad f-h'\leqslant\Delta_\mathfrak{A}(y-p);\quad g-h'\leqslant\Delta_\mathfrak{A}(z-p).$$

Consequently, $p \leqslant y \wedge z$ and

$$h - h' \leqslant (f - \Delta_{\mathfrak{A}}(y-z)^+) \vee (g - \Delta_{\mathfrak{A}}(z-y)^+) - (f - \Delta_{\mathfrak{A}}(y-p)) \vee \vee (g - \Delta_{\mathfrak{A}}(z-p)) = (f - \Delta_{\mathfrak{A}}(y - y \wedge z)) \vee (g - \Delta_{\mathfrak{A}}(z - y \wedge z)) - - (f - \Delta_{\mathfrak{A}} y) \vee (g - \Delta_{\mathfrak{A}} z) + \Delta_{\mathfrak{A}} p = \Delta_{\mathfrak{A}}(y \wedge z - p).$$

i.e., (h', p) is a lower bound of $(h, y \wedge z)$.

The theorem is proved completely.

3°. Inverse of the Krein – Mil'man Theorem

We need the following refinement of a theorem of Kantorovich about the extension of positive operators.

LEMMA. Let X be an ordered vector space, and let X_0 be a majorizing subspace in X, where the cone $X_0 \cap X^+$ is reproducing in X_0. Next, let $T_0 : X_0 \to Y$ be a positive operator into some K-space Y that is such that $[0, T_0] = [0, I_Y] \circ T_0$. Then T_0 has a positive extension $T : X \to Y$ such that $[0, T] = [0, I_Y] \circ T$.

The lemma is proved by analogy with the classical form; we mention only the relevant changes.

If $X = \{x_0 + tx_1 : x_0 \in X_0,\ t \in R\}$ for some $x_1 \in X \setminus X_0$, then we set $Tx_1 = \inf\{T_0 x_0 : x_0 \in X_0;\ x_0 \geqslant x_1\}$ and $T|X_0 = T_0$. Clearly, T is a positive operator. If $T' \in [0, T]$, then $T'x_0 = \alpha \circ Tx_0$ for all $x_0 \in X_0$ and for some multiplier $\alpha \in [0, I_Y]$. Since

$$T'x_1 \leqslant \inf\{T'x_0 : x_0 \in X_0;\ x_0 \geqslant x_1\} = \alpha \circ Tx_1;$$
$$(T - T')x_1 \leqslant \inf\{(T - T')x_0 : x_0 \in X_0;\ x_0 \geqslant x_1\} = (I_Y - \alpha) \circ Tx_1,$$

we have $T' = \alpha \circ T$.

If $X = \bigcup_t X_t$, where (X_t) is a family of subspaces containing X_0 and filtered ascendingly, and if the positive operators $T_t : X_t \to Y$ have the properties: $[0, T_t] = [0, I_Y] \circ T_t$ and T_s is the trace of T_t on X_s, provided that $X_t \supset X_s$, then by considering the natural extension T of T_0 to X, for $x \in X_t \subset X$ we obtain

$$(Tx)^+ = (T_t x)^+ = \inf\{T_t x_t : x_t \in X_t;\ x_t \geqslant 0;\ x_t \geqslant x\} \geqslant \inf\{Tx' : x' \in X;\ x' \geqslant 0;\ x' \geqslant x\} \geqslant (Tx)^+.$$

By the Minkowski duality the latter means that $[0, T] = [0, I_Y] \circ T$.

COROLLARY 1. Let X be a vector space, X_0 be a subspace of X, and $P : X \to Y$ be a sublinear operator acting into a K-space Y. Let $\mathrm{Ch}_{X_0}(P)$ denote the collection of operators from L(X, Y) for which the traces on X_0 are extreme points of the subdifferential of the restriction $P : X_0 \to Y$, then

$$\mathrm{Ch}_{X_0}(P) \subset \mathrm{Ch}(P) + \partial(\delta_Y(X_0)),$$

where $\delta_Y(X_0)$ is the indicator of X_0.

The proof consists of the converse of Theorem 1 followed by an application of the lemma.

Remark. The refinement of the Hahn–Banach theorem contained in Corollary 1 was in all probability first noticed in scalar form by Bonsall [11], who suggested a new proof of the Krein–Mil'man theorem (see also [12, 13]).

COROLLARY 2. Let X_1, X be vector spaces, let Y be a K-space, $P : X \to Y$ be a sublinear operator, and $A \in L(X_1, X)$. Then $\mathrm{Ch}(P \circ A) \subset \mathrm{Ch}(P) \circ A$.

Proof. If $C \in \mathrm{Ch}(P \circ A)$, then by a lemma of Levin–Rockafellar, $C = B \circ A$ for some $B \in \partial(P)$. Obviously, $B \in \mathrm{Ch}_{A[X_1]}(P)$. The result is obtained by applying Corollary 1.

Thus, the following inverse of the Krein–Mil'man theorem holds.

Mil'man's Theorem. If a subdifferential $\partial(P)$ is the support hull of a set $\mathfrak{A}$, then $\mathrm{Ch}(P) \subset \mathrm{Ch}(\varepsilon_{\mathfrak{A}}) \circ \langle \mathfrak{A} \rangle$.

COROLLARY. $\mathrm{Ch}(P) \subset \mathrm{Ch}(\varepsilon_{\mathscr{E}_0(P)}) \circ \langle \mathscr{E}_0(P) \rangle$.

Remark. It must be stressed that in applying Mil'man's theorem we must take Theorem 3 into account.

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SUBLINEAR OPERATORS DEFINED ON A CONE IN A FINITE-DIMENSIONAL SPACE

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In this paper we study sublinear operators defined on a cone in a finite-dimensional Banach space and with values in Lindenstrauss spaces, in particular, spaces of continuous functions on compact spaces. Such operators arise, for example, in the investigation of models in economic dynamics [1-3]. We prove that such sublinear operators can be approximated by linear operators, and this permits us to approximate complex models in economic dynamics by simpler ones whose behavior is described by linear operators.

Sublinear operators defined on a cone are investigated by methods of continuous sections (selections), just as continuous sublinear operators defined on the whole space were investigated in [4, 5].

Our recourse to new methods of investigation has to do with the fact that the classical Hahn–Banach–Kantorovich extension principle for linear operators does not carry over to the operators under consideration for two fundamental reasons. First, the domain of these operators is a cone, and, second, the range spaces of the operators are, generally speaking, not K-spaces (boundedly complete vector lattices). On the other hand, we note advantages of the method of sections, which enables us, by considering sections that are continuous in various topologies, to approximate sublinear operators not only by continuous linear operators, but also by compact operators. Moreover, this method makes essential use of the theory of sublinear functionals, which is quite well developed (see, e.g., [6-8]).

Relatively little is known about sublinear operators, especially about those defined on a cone. The basic work is exhausted by the list of publications cited above on point-set mappings and their applications in models of economic dynamics. In addition to this, two publications (as far as we know) deal with problems involving the approximation of sublinear operators defined on a cone. In [9] operators with values in spaces with the property of chain completeness were studied from the algebraic point of view, and in the author's Candidate Dissertation section methods were used to investigate some classes of operators with values in spaces of continuous functions on compact spaces. These results were published partly in [10, pp. 59-65].

In the present paper the main emphasis is on sublinear operators defined on a cone in a finite-dimensional Banach space and with values in Lindenstrauss spaces.

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