ε-SUBDIFFERENTIALS AND ε-OPTIMALITY

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Preliminary Remarks. Let X be a vector space, and $Y\cup\{+\infty\}$ be an ordered vector space Y with an adjunction of a greatest element $+\infty$. We consider a convex operator $F: X\to Y\cup\{+\infty\}$ and a point $\bar{x}$ from the effective set $\operatorname{dom}(F)=\{x\in X: Fx<+\infty\}$ of F. With each positive element $\varepsilon\in Y^+$ we associate the set

$$\partial_{\bar{x},\varepsilon}(F)=\{A\in L(X,Y): Ax-A\bar{x}\leqslant Fx-F\bar{x}+\varepsilon\ (x\in X)\},$$

where, as is usual, L(X, Y) is the space of linear operators acting from X into Y. The set $\partial_{\bar{x},\varepsilon}(F)$ is called the ε-subdifferential of the operator F at the point $\bar{x}$. A point $\bar{x}$ is called ε-optimal for F (or an ε-solution of the unconstrained program Fx → inf) if $0\in\partial_{\bar{x},\varepsilon}(F)$.

The aim of this article is to derive formulas for the calculation of ε-subdifferentials and related conditions for ε-optimality, which are the relevant versions of Lagrange's principle for problems of ε-programming. Obviously, when ε = 0 we recover the classical theory of convex programming. Our exposition is developed on the basis of the theory of the Young transformation of convex operators acting in a Kantorovich space, given in [1, 2]. The results of these papers are used below, as a rule without additional explanation. We emphasize that the idea of ε-programming has its roots in the theory of best approximations, as do the majority of ideas in the theory of extremal problems. In this connection, first of all we must mention the concept of ε-approximation introduced by Buck and explored in detail by Singer [3]. As regards ε-subdifferentials, little is known even for scalar functions. We mention a cycle of papers, mainly connected with the support principle of Phelps–Brondsted–Rockafellar and with the variational principle of Ekeland [4-7]. The first formula for the calculation of the ε-subdifferential of a sum of convex functions was proposed in [8], which serves as a basis for the present article.

In conclusion, I express my sincere thanks to the participants of V. L. Makarov's seminar in the Mathematical Economics Branch of the Institute of Mathematics of the Siberian Branch of the Academy of Sciences of the USSR for many lively and profitable discussions on problems of ε-programming.

Superposition of Convex Operators. Suppose that, as above, Y is an ordered vector space, $F: X\to Y\cup\{+\infty\}$ is a convex operator, and $G: Y\to Z\cup\{+\infty\}$ is an increasing convex operator acting in a K-space Z. If the image F[dom (F)] contains an interior point of the effective set dom (G), then for every $\varepsilon\in Z^+$ we have

$$\partial_{\bar{x},\varepsilon}(G\circ F)=\bigcup_{\substack{\varepsilon_1\geqslant 0,\varepsilon_2\geqslant 0\\ \varepsilon_1+\varepsilon_2=\varepsilon}}\ \bigcup_{B\in\partial_{\bar{x},\varepsilon_1}(G)}\partial_{\bar{x},\varepsilon_2}(B\circ F).$$

Here $\bar{x}\in\operatorname{dom}(F)$ and $F\bar{x}\in\operatorname{dom}(G)$.

For, if $A\in\partial_{\bar{x},\varepsilon_2}(B\circ F)$ for some $B\in\partial_{F\bar{x},\varepsilon_1}(G)$ and ε_1, $\varepsilon_2\geq 0$ such that $\varepsilon_1+\varepsilon_2=\varepsilon$, then

$$Ax-A\bar{x}\leqslant B\circ Fx-B\circ F\bar{x}+\varepsilon_2\leqslant G\circ Fx-G\circ F\bar{x}+\varepsilon_1+\varepsilon_2,$$

i.e., $A\in\partial_{\bar{x},\varepsilon}(G\circ F)$.

Now let $A\in\partial_{\bar{x},\varepsilon}(G\circ F)$. Then, by the definitions, for the Young transformation $(G\circ F)^*$ we have

$$(G\circ F)^*A=\sup_{x\in\operatorname{dom}(G\circ F)}(Ax-G\circ Fx)\leqslant A\bar{x}-G\circ F\bar{x}+\varepsilon.$$

When we apply the exact formula for a change of variables for the Young transformation of a superposition of convex operators, we find an operator $B\in L^+(Y,Z)$ such that

$$\varepsilon\geqslant(B\circ F)^*A+G^*B-A\bar{x}+G\circ F\bar{x}=\sup_{x\in\operatorname{dom}(F)}(Ax-B\circ Fx)+$$

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$$+\sup_{y\in \mathrm{dom}(G)}(By-Gy)-A\bar{x}+G\circ F\bar{x}=\sup_{x\in \mathrm{dom}(F)}\big(Ax-A\bar{x}-$$
$$-\big(B\circ Fx-B\circ F\bar{x}\big)\big)+\sup_{y\in \mathrm{dom}(G)}\big(By-B\circ F\bar{x}-\big(Gy-G\circ F\bar{x}\big)\big).$$

We introduce the following elements:

$$\delta_1=\sup_{y\in \mathrm{dom}(G)}\big(By-B\circ F\bar{x}-\big(Gy-G\circ F\bar{x}\big)\big),$$
$$\delta_2=\sup_{x\in \mathrm{dom}(F)}\big(Ax-A\bar{x}-\big(B\circ Fx-B\circ F\bar{x}\big)\big).$$

Clearly, the following relations hold:

$$\delta_1\geqslant 0,\ \delta_2\geqslant 0,\ \delta_1+\delta_2\leqslant\varepsilon;\ A\in\partial_{\bar{x},\delta_2}(B\circ F),\ \ B\in\partial_{F\bar{x},\delta_1}(G).$$

By setting $\varepsilon_1=\delta_1$ and $\varepsilon_2=\varepsilon-\varepsilon_1$ we obtain the required result.

Superposition with an Affine Operator. Let X_1, X be vector spaces, Y a K-space, and $F:X\to Y\cup\{+\infty\}$ a convex operator whose effective set contains an interior point belonging to the image of X_1 under an affine mapping $A_X:x_1\to Ax_1+x$, where $A\in L(X_1,X)$ and $x\in X$. Then, for an element $\bar{x}_1\in X_1$ such that $A_x\bar{x}_1\in\mathrm{dom}(F)$, we have

$$\partial_{\bar{x}_1,\varepsilon}(F\circ A_x)=\partial_{A_x\bar{x}_1,\varepsilon}(F)\circ A.$$

By drawing on the formula established above and using the Archimedean property of Y, we obtain in succession:

$$\partial_{\bar{x}_1,\varepsilon}(F\circ A_x)=\bigcup_{\substack{\varepsilon_1\geqslant 0,\,\varepsilon_2\geqslant 0\\ \varepsilon_1+\varepsilon_2=\varepsilon}}\left\{\partial_{\bar{x}_1,\varepsilon_2}(B\circ A_x):B\in\partial_{A_x\bar{x}_1,\varepsilon_1}(F)\right\}=\bigcup_{\substack{\varepsilon_1\geqslant 0,\,\varepsilon_2\geqslant 0\\ \varepsilon_1+\varepsilon_2=\varepsilon}}\left\{B\circ A:B\in\partial_{A_x\bar{x}_1,\varepsilon_1}(F)\right\}=\partial_{A_x\bar{x}_1,\varepsilon}(F)\circ A,$$

which gives the required result.

Sum of Convex Operators. Let $F_1,\ \ldots,\ F_n:\ X\to Y\cup\{+\infty\}$ be convex operators, where Y is a K-space. If the effective sets of the Hörmander transformations of the operators $F_1,\ldots,F_n$ are in general position, and $\bar{x}\in\mathrm{dom}(F_1)\cap\ldots\cap\mathrm{dom}(F_n)$, then

$$\partial_{\bar{x},\varepsilon}(F_1+\ldots+F_n)=\bigcup_{\substack{\varepsilon_1\geqslant 0,\ldots,\varepsilon_n\geqslant 0\\ \varepsilon_1+\ldots+\varepsilon_n=\varepsilon}}\left(\partial_{\bar{x},\varepsilon_1}(F_1)+\ldots+\partial_{\bar{x},\varepsilon_n}(F_n)\right).$$

It is obvious that the set on the right-hand side of this formula lies in the ε-subdifferential of the sum. If $A\in\partial_{\bar{x},\varepsilon}(F_1+\ldots+F_n)$, then by using Moreau's formula for the Young transformation of a sum of convex operators, and bearing in mind the exactness of this formula, for certain operators $A_1,\ \ldots,\ A_n\in L(X,\ Y)$ such that $A=\sum_{k=1}^{n}A_k$, we obtain

$$\varepsilon-F_1\bar{x}-\ldots-F_n\bar{x}+A\bar{x}\geqslant(F_1+\ldots+F_n)^*A=\sum_{k=1}^{n}F_k^*A_k.$$

Then by defining

$$\delta_k=\sup_{x\in\mathrm{dom}(F_k)}\big(A_kx-A_k\bar{x}-\big(F_kx-F_k\bar{x}\big)\big),$$

we obtain $\delta_1+\ldots+\delta_n\leq\varepsilon$. It is obvious that the elements $\delta_1,\ldots,\delta_n$ are positive. By taking $\varepsilon_k=\delta_k$ for $k=2,\ldots,n$ and $\varepsilon_1=\varepsilon-(\varepsilon_2+\ldots+\varepsilon_n)$ we arrive at the required result.

Maximum of Convex Operators. Let $F_1,\ \ldots,\ F_n:\ X\to Y\cup\{+\infty\}$ be convex operators, the effective sets of the Hörmander transformations of $F_1,\ldots,F_n$ be in general position, and Y be a vector lattice. If Z is a K-space and $A\in L^+(Y,\ Z)$ is a positive linear operator, then for every $\bar{x}\in\mathrm{dom}(F_1\vee\ldots\vee F_n)$ and $\varepsilon\in Z^+$ we have

$$\partial_{\bar{x},\varepsilon}(A\circ(F_1\vee\ldots\vee F_n))=\left\{\sum_{k=1}^{n}\partial_{\bar{x},\varepsilon_k}(A_k\circ F_k):\ A_k\in L^+(Y,Z),\ \sum_{k=1}^{n}A_k=A;\right.$$
$$\left.\varepsilon_k\geqslant 0,\ \sum_{k=1}^{n+1}\varepsilon_k=\varepsilon;\ \sum_{k=1}^{n}A_k\circ F_k\bar{x}\geqslant A\left(F_1\bar{x}\vee\ldots\vee F_n\bar{x}\right)-\varepsilon_{n+1}\right\}.$$

We define the following operators

$$(F_1, \ldots, F_n): X \to Y^n \cup \{+\infty\}, \ (F_1, \ldots, F_n)x = (F_1x, \ldots, F_nx);$$
$$\varkappa: Y^n \to Y, \quad \varkappa(y_1, \ldots, y_n) = y_1 \vee \ldots \vee y_n.$$

Then the following representation follows directly from the definitions

$$A \circ (F_1 \vee \ldots \vee F_n) = A \circ \varkappa \circ (F_1, \ldots, F_n).$$

Consequently, by the formula for calculating the ε-subdifferential of a superposition we have

$$\partial_{\bar{x},\varepsilon}\Big(A \circ (F_1 \vee \ldots \vee F_n)\Big) = \bigcup_{\substack{\varepsilon_1 \geqslant 0,\, \varepsilon_2 \geqslant 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \ \bigcup_{B \in \partial_{(F_1\bar{x}, \ldots, F_n\bar{x}), \varepsilon_1}(A \circ \varkappa)} \partial_{\bar{x},\varepsilon_2}(B \circ (F_1, \ldots, F_n)).$$

Note that $A \circ \varkappa$ is a sublinear operator. Here, for any sublinear operator P it is obvious that

$$\partial_{\bar{x},\varepsilon}(P) = \{C \in \partial(P): C\bar{x} \geqslant P\bar{x} - \varepsilon\},$$

where $\partial(P) = \partial_{0,0}(P)$ is the subdifferential of P.

Since, as is well known,

$$\partial(A \circ \varkappa) = \left\{(y_1, \ldots, y_n) \to \sum_{k=1}^{n} A_k y_k : A_k \in L^+(Y, Z),\ \sum_{k=1}^{n} A_k = A\right\},$$

we obtain the following representation:

$$\partial_{(F_1\bar{x}, \ldots, F_n\bar{x}), \varepsilon_1}(A \circ \varkappa) = \left\{(y_1, \ldots, y_n) \to \sum_{k=1}^{n} A_k y_k :\ A_k \in L^+(Y, Z),\ \sum_{k=1}^{n} A_k = A, \right.$$
$$\left. \sum_{k=1}^{n} A_k \circ F_k\bar{x} \geqslant A(F_1\bar{x} \vee \ldots \vee F_n\bar{x}) - \varepsilon_1\right\}.$$

Now we use the formula for the ε-subdifferential of a sum of convex operators and obtain the required representation.

Superposition with a Regular Convex Operator. Let X be a vector space, Y a K-space, and $\mathfrak{A}$ a weakly order-bounded set in L(X, Y). As is usual, the symbol $\langle\mathfrak{A}\rangle$ stands for the linear operator acting from X into the K-space $(Y^{\mathfrak{A}})_\infty$ of bounded Y-valued functions on $\mathfrak{A}$, in accordance with the rule

$$\langle\mathfrak{A}\rangle x : A \to Ax \quad (A \in \mathfrak{A}).$$

The symbol $\Delta_{\mathfrak{A}}$ denotes the natural identification of Y with the diagonal of the space $Y^{\mathfrak{A}}$, and $\varepsilon_{\mathfrak{A}}$ the canonical sublinear operator

$$\varepsilon_{\mathfrak{A}}: (Y^{\mathfrak{A}})_\infty \to Y, \ \varepsilon_{\mathfrak{A}} f = \sup\{f(A) : A \in \mathfrak{A}\}.$$

Now suppose that $F: X \to Y \cup \{+\infty\}$ is a regular convex operator, i.e., an operator having the representation $F = \varepsilon_{\mathfrak{A}} \circ \langle\mathfrak{A}\rangle_y$ for some $\mathfrak{A} \subset L(X, Y)$ and $y \in (Y^{\mathfrak{A}})_\infty$.

Let $G: Y \to Z \cup \{+\infty\}$ be an increasing convex operator acting in some K-space Z, where the image F[X] contains an interior point of the effective set dom (G). If $\bar{x} \in X$ is such that $F\bar{x} \in \mathrm{dom}(G)$, then for every $\varepsilon \in Z^+$ we have the representation

$$\partial_{\bar{x},\varepsilon}(G \circ F) = \{B \circ \langle\mathfrak{A}\rangle: B \circ \Delta_{\mathfrak{A}} \in \partial_{F\bar{x}, \varepsilon-\delta}(G);\ B \in L^+((Y^{\mathfrak{A}})_\infty, Z);$$
$$0 \leqslant B \circ \Delta_{\mathfrak{A}} F\bar{x} - B \circ \langle\mathfrak{A}\rangle_y \bar{x} \leqslant \delta \leqslant \varepsilon\}.$$

In fact, by using the formula for the subdifferentiation of a superposition with an affine operator, in view of the regularity of F we obtain

$$\partial_{\bar{x},\varepsilon}(G \circ F) = \partial_{\bar{x},\varepsilon}(G \circ \varepsilon_{\mathfrak{A}} \circ \langle\mathfrak{A}\rangle_y) = \partial_{\langle\mathfrak{A}\rangle_y \bar{x}, \varepsilon}(G \circ \varepsilon_{\mathfrak{A}}) \circ \langle\mathfrak{A}\rangle.$$

We set $\bar{y} = \langle\mathfrak{A}\rangle_y \bar{x}$ and consider an operator $B \in \partial_{\bar{y},\varepsilon}(G \circ \varepsilon_{\mathfrak{A}})$. In view of the Archimedean property of Z and the fact that the operators $\varepsilon_{\mathfrak{A}}$ and G are increasing, we obtain that B is a positive operator. Since by the rules for a change of variables in a Young transformation,

$$(G \circ \varepsilon_{\mathfrak{A}})^* B = G^* (B \circ \Delta_{\mathfrak{A}}),$$

consequently we obtain the following estimate:

$$\varepsilon \geqslant (G \circ \varepsilon_{\mathfrak{A}})^* B + G \circ \varepsilon_{\mathfrak{A}} \bar{y} - B\bar{y} = G^* (B \circ \Delta_{\mathfrak{A}}) + G \circ \varepsilon_{\mathfrak{A}} \bar{y} - B\bar{y} =$$
$$= \sup_{y \in \mathrm{dom}(G)} \left(B \circ \Delta_{\mathfrak{A}} y - B \circ \Delta_{\mathfrak{A}} \circ \varepsilon_{\mathfrak{A}} \bar{y} - (Gy - G \circ \varepsilon_{\mathfrak{A}} \bar{y})\right) + B \circ \Delta_{\mathfrak{A}} \circ \varepsilon_{\mathfrak{A}} \bar{y} - B\bar{y}.$$

Thus,

$$B \circ \Delta_{\mathfrak{A}} \in \partial_{\varepsilon_{\mathfrak{A}} \bar{y}, \varepsilon - \left(B \circ \Delta_{\mathfrak{A}} \circ \varepsilon_{\mathfrak{A}} \bar{y} - B\bar{y}\right)} (G).$$

Consequently, the following inclusion holds:

$$\partial_{\bar{y}, \varepsilon} (G \circ \varepsilon_{\mathfrak{A}}) \subset \{B \in L^+ ((Y^{\mathfrak{A}})_\infty, Z) :$$
$$B \circ \Delta_{\mathfrak{A}} \in \partial_{\varepsilon_{\mathfrak{A}} \bar{y}, \varepsilon - \delta} (G); \quad 0 \leqslant B \circ \Delta_{\mathfrak{A}} \circ \varepsilon_{\mathfrak{A}} \bar{y} - B\bar{y} \leqslant \delta \leqslant \varepsilon\}.$$

Now if B lies in the set on the right-hand side of this inclusion, then for any $y \in (Y^{\mathfrak{A}})_\infty$ we have

$$By - B\bar{y} \leqslant B \circ \Delta_{\mathfrak{A}} \circ \varepsilon_{\mathfrak{A}} y - B \circ \Delta_{\mathfrak{A}} \circ \varepsilon_{\mathfrak{A}} \bar{y} + B \circ \Delta_{\mathfrak{A}} \circ \varepsilon_{\mathfrak{A}} \bar{y} - B\bar{y} \leqslant \delta + G \circ \varepsilon_{\mathfrak{A}} y - G \circ \varepsilon_{\mathfrak{A}} \bar{y} + (\varepsilon - \delta).$$

Since from the definition it follows that

$$\varepsilon_{\mathfrak{A}} \bar{y} = \varepsilon_{\mathfrak{A}} \circ \langle \mathfrak{A} \rangle_y \bar{x} = F\bar{x},$$

we obtain finally the representation

$$\partial_{\langle \mathfrak{A} \rangle_y \bar{x}, \varepsilon} (G \circ \varepsilon_{\mathfrak{A}}) = \{B \in L^+ ((Y^{\mathfrak{A}})_\infty, Z):$$
$$B \circ \Delta_{\mathfrak{A}} \in \partial_{F\bar{x}, \varepsilon - \delta} (G); \quad 0 \leqslant B \circ \Delta_{\mathfrak{A}} \circ F\bar{x} - B \circ \langle \mathfrak{A} \rangle_y \bar{x} \leqslant \delta \leqslant \varepsilon\},$$

from which follows the required representation for the ε-subdifferential of a superposition with a regular operator.

ε-Optimality for Regular Programs. We consider the regular convex program

$$Gx \leqslant 0, \quad Fx \to \inf.$$

In other words, we assume that $G, F : X \to Y \cup \{+\infty\}$ are convex operators, which for simplicity, satisfy the condition dom (F) = dom (G) = X. Further, we assume that Y is a K-space, that for every $x \in X$ either $Gx \leq 0$ or $Gx \geq 0$, and that Gx_0 is the identity element in Y for some $x_0 \in X$. We recall also that I_Y denotes the identity mapping of Y onto itself and that the symbol $[0, I_Y]$ denotes the corresponding order interval in the space L(Y, Y) that coincides with the subdifferential $\partial (y \to y^+)$.

An admissible point $\bar{x}$ is ε-optimal in a regular program if and only if the following system of conditions is consistent

$$\bar{\alpha}, \bar{\beta} \in [0, I_Y], \ \bar{\alpha} + \bar{\beta} = I_Y, \ \mathrm{Ker}(\bar{\alpha}) = \{0\};$$

$$\varepsilon_1 \geqslant 0, \ \varepsilon_2 \geqslant 0, \ 0 \leqslant \varepsilon_1 + \varepsilon_2 \leqslant \bar{\alpha}\varepsilon + \bar{\beta} \circ G\bar{x};$$
$$0 \in \partial_{\bar{x}, \varepsilon_1} (\bar{\alpha} \circ F) + \partial_{\bar{x}, \varepsilon_2} (\bar{\beta} \circ G).$$

If this system is consistent, then for any admissible element x we have

$$\bar{\alpha} \circ F\bar{x} \leqslant \bar{\alpha} \circ Fx + \bar{\beta} \circ Gx - \bar{\beta} \circ G\bar{x} + \varepsilon_1 + \varepsilon_2 \leqslant \bar{\alpha}(Fx + \varepsilon).$$

By using the properties of a multiplier $\bar{\alpha} \in [0, I_Y]$, we obtain from the last relation that $\bar{x}$ is ε-optimal.

Suppose that it is known in turn that an admissible point $\bar{x}$ is an ε-solution of the program in question. It is clear that in this case the objective of the program is bounded from below, and consequently there is defined an element $y \in Y$, which is the value of the program, i.e., $y = \inf\{Fx : x \in X, Gx \leqslant 0\}$. Obviously,

$$0 \leqslant \inf_{x \in X} ((Fx - y) \vee Gx) \leqslant \inf_{x : Gx \leqslant 0} (Fx - y) \leqslant 0.$$

Hence it follows, e.g., by the vector minimax theorem, that the following system of conditions is consistent for certain multipliers $\bar{\alpha}, \bar{\beta} \in [0, I_Y]$:

$$\bar{\alpha} + \bar{\beta} = I_Y;$$

$$0 = \inf_{x \in X} (\bar{\alpha}(Fx - y) + \bar{\beta} \circ Gx).$$

By considering a point x_0 at which Gx_0 is the identity element, we obtain in a standard way from the condition $\bar{\alpha}(Fx_0 - y) + \bar{\beta} \circ Gx_0 \geq 0$ that Ker $(\bar{\alpha}) = \{0\}$. In addition, since $\bar{x}$ is ε-optimal, we have $y \geq F\bar{x} - \varepsilon$, from which it follows that for any $x \in X$

$$-\bar{\alpha}\varepsilon \leqslant \bar{\alpha} \circ Fx - \bar{\alpha} \circ F\bar{x} + \bar{\beta} \circ Gx.$$

Thus, the element $\bar{\alpha}\varepsilon + \bar{\beta} \circ G\bar{x}$ is positive, and

$$0 \in \partial_{\bar{x}, \bar{\alpha}\varepsilon + \bar{\beta} \circ G\bar{x}} (\bar{\alpha} \circ F + \bar{\beta} \circ G).$$

We obtain the required result by using the formula for the ε-subdifferential of a sum of convex operators.

ε-Optimality for Programs Regular in the Sense of Slater. We consider the program

$$Ax = A\bar{x}, \quad Gx \leqslant 0, \quad Fx \to \inf,$$

where X, X_1 are vector spaces, $A \in L(X, X_1)$ is a linear operator, and $G : X \to Z \cup \{+\infty\}$ and $F : X \to Y \cup \{+\infty\}$ are convex operators. For simplicity we assume that G and F are defined everywhere, i.e., dom (G) = dom (F) = X.

We assume that the convex program in question is regular in the sense of Slater, i.e., Z is an Archimedean ordered vector space, Y is a K-space of bounded elements, and the element Gx_0 is an interior point of the cone Z^+ for some admissible element x_0.

An admissible point $\bar{x}$ is ε-optimal in a program regular in the sense of Slater, if and only if the following system of conditions is consistent:

$$\bar{\gamma} \in L^+(Z, Y), \quad \bar{\mu} \in L(X_1, Y);$$
$$\varepsilon_1 \geqslant 0, \ \varepsilon_2 \geqslant 0, \quad \varepsilon_1 + \varepsilon_2 \leqslant \bar{\gamma} \circ G\bar{x} + \varepsilon;$$
$$0 \in \partial_{\bar{x}, \varepsilon_1}(F) + \partial_{\bar{x}, \varepsilon_2}(\bar{\gamma} \circ G) + \bar{\mu} \circ A.$$

If this system is consistent, then for any admissible point x we have

$$F\bar{x} \leqslant Fx + \bar{\gamma} \circ Gx - \bar{\gamma} \circ G\bar{x} + \varepsilon_1 + \varepsilon_2 - \bar{\mu}(Ax) + \bar{\mu}(A\bar{x}) \leqslant Fx + \bar{\gamma} \circ Gx + \varepsilon \leqslant Fx + \varepsilon,$$

so that $\bar{x}$ is an ε-optimal point.

Now suppose that $\bar{x}$ is an ε-solution of the program in question. By Lagrange's principle, the value of the problem under consideration is the value of the unconstrained program for a suitable Lagrangian. In other words, for certain Lagrange multipliers $\bar{\gamma} \in L^+(Z, Y)$ and $\bar{\mu} \in L(X_1, Y)$ we have

$$-\varepsilon + F\bar{x} \leqslant \inf_{x \in X} (Fx + \bar{\gamma} \circ Gx + \bar{\mu}(Ax - A\bar{x})).$$

Thus, $\varepsilon + \bar{\gamma} \circ G\bar{x}$ is positive, and in addition

$$-\bar{\mu} \circ A \in \partial_{\bar{x}, \varepsilon + \bar{\gamma} \circ G\bar{x}} (F + \bar{\gamma} \circ G).$$

The required result follows from using the formula for the ε-subdifferential of a sum.

As a corollary of this condition we prove that under our assumptions on the space and the operator G, the Lebesgue set $U = \{x \in X : Gx \leqslant 0\}$ satisfies

$$\partial_{\bar{x}, \varepsilon}(\delta_Y(U)) = \bigcup_{\substack{\gamma \in L^+(Z, Y) \\ 0 \leqslant \delta \leqslant \gamma \circ G\bar{x} + \varepsilon}} \partial_{\bar{x}, \delta}(\gamma \circ G).$$

Here, as usual, $\delta_Y(U)$ is the indicator operator of the set U, and the point $\bar{x}$ lies in U.

In fact, it is sufficient to note that an operator A lies in the left-hand side of the last relation if and only if $\bar{x}$ is an ε-solution of the following program, which is regular in the sense of Slater:

$$Gx \leqslant 0, \quad -Ax \to \inf.$$

ε-Optimality in the Sense of Pareto. Again we consider a program regular in the sense of Slater in the above form (although here we assume that Y satisfies only the same conditions as on Z). An admissible point x^0 is called ε-optimal in the sense of Pareto (relative to the strong identity $\mathbf{1}_Y$ in Y), if $Fx = Fx^0 - \varepsilon \mathbf{1}_Y$ for every admissible point x such that $Fx - Fx^0 \le -\varepsilon 1_Y$. Here ε is a positive real number.

If an admissible point x^0 is ε-optimal in the sense of Pareto in a program regular in the sense of Slater, and if $0 \le \varepsilon < 1$, then for certain linear functionals α^0, β^0, and γ^0 on the spaces Y, Z, and X_1, respectively, the following system of conditions is consistent

$$\alpha^0 > 0,\ \beta^0 \geqslant 0;\ \varepsilon_1 \geqslant 0,\ \varepsilon_2 \geqslant 0;$$
$$\varepsilon_1 + \varepsilon_2 \leqslant \varepsilon + \beta^0 \circ Gx^0;$$
$$0 \in \partial_{x^0,\varepsilon_1}(\alpha^0 \circ F) + \partial_{x^0,\varepsilon_2}(\beta^0 \circ G) + \gamma^0 \circ A.$$

If these conditions hold for some admissible point x^0 and if $\alpha^0(1_Y) = 1$, then x^0 is an ε-optimal point in the sense of Pareto.

Suppose that x^0 is ε-optimal in the sense of Pareto. Then, it is easily verified (see [1]) that

$$-\varepsilon \leqslant \inf_{x \in X} \varepsilon_{1_Y} \circ \Phi x,$$

where we have used the following notation:

$$\Phi x = (F_U x - F_U x^0) \vee \varepsilon_{1_Z} \otimes 1_Y \circ Gx;$$
$$1_Z = -Gx_0;\ U = \{x \in X:\ Ax = A\bar{x}\};\ F_U = F + \delta_Y(U);$$
$$\varepsilon_{1_Z} : z \to \inf\{t \in R : z \leqslant t1_Z\},\quad \varepsilon_{1_Z} : Z \to R;$$
$$\varepsilon_{1_Y} : y \to \inf\{t \in R : y \leqslant t1_Y\},\quad \varepsilon_{1_Y} : Y \to R;$$
$$\varepsilon_{1_Z} \otimes 1_Y : z \to \varepsilon_{1_Z}(z)\, 1_Y,\quad \varepsilon_{1_Z} \otimes 1_Y : Z \to Y.$$

By applying the rule for a change of variables in Young transformations, we find that for certain linear functionals α_1, α^0 on Y the following conditions hold:

$$\alpha^0 \geqslant 0,\ \alpha_1 \geqslant 0;\ (\alpha^0 + \alpha_1)(1_Y) = 1;$$
$$-\varepsilon \leqslant \inf_{x \in X} (\alpha^0(F_U x - F_U x^0) + \alpha_1 \circ G_0 x).$$

Here we have used the notation $G_0 = \varepsilon_{1_Z} \otimes 1_Y \circ G$.

By considering a point x_0 occurring in the definition of a program regular in the sense of Slater we have

$$-\varepsilon \leqslant \alpha^0(Fx_0 - Fx^0) - \alpha_1(1_Y) = \alpha^0(1_Y) - 1 + \alpha^0(Fx_0 - Fx^0).$$

Thus, $\alpha^0 \ne 0$ since $0 \le \varepsilon < 1$. In addition, $-\varepsilon \le \alpha_1 \circ G_0 x^0$ and

$$0 \in \partial_{x^0,\varepsilon + \alpha_1 \circ G_0 x^0}(\alpha^0 \circ F_U + \alpha_1 \circ G_0) = \partial_{x^0,\varepsilon + \alpha_1 \circ G_0 x^0}(\alpha^0 \circ F + \alpha_1 \circ G_0 + \delta_R(U)).$$

Hence, by taking into account the way in which the set U is defined, we have

$$0 \in \partial_{x^0,\varepsilon + \alpha_1 \circ G_0 x^0}(\alpha^0 \circ F + \alpha_1 \circ G_0) + \gamma^0 \circ A$$

for some linear functional γ^0. Hence, by the formula for the ε-subdifferential of a sum, we can find elements $\varepsilon_1 \ge 0$ and $\delta \ge 0$ such that

$$\varepsilon_1 + \delta = \varepsilon + \alpha_1 \circ G_0 x^0;$$
$$0 \in \partial_{x^0,\varepsilon_1}(\alpha^0 \circ F) + \partial_{x^0,\delta}(\alpha_1 \circ G_0) + \gamma^0 \circ A.$$

Now we note that

$$\partial_{x^0,\delta}(\alpha_1 \circ G_0) = \partial_{x^0,\delta}(\alpha_1(1_Y)\varepsilon_{1_Z} \circ G).$$

By using the rule for the ε-subdifferentiation of a superposition, we find positive elements ε_2 and ε_3 such that $\varepsilon_2 + \varepsilon_3 = \delta$, and in addition,

$$\beta^0 \circ Gx^0 \geqslant \alpha_1(1_Y)\varepsilon_{1_Z} \circ Gx^0 - \varepsilon_3;$$
$$0 \in \partial_{x^0,\varepsilon_1}(\alpha^0 \circ F) + \partial_{x^0,\varepsilon_2}(\beta^0 \circ G) + \gamma^0 \circ A$$

for some positive functional β^0 on Z. Moreover, the following estimate holds:

$$\varepsilon_1 + \varepsilon_2 = \varepsilon_1 + \delta - \varepsilon_3 = \varepsilon + \alpha_1 \circ G^0 x_0 - \varepsilon_3 = \varepsilon + \alpha_1 (1_Y) \varepsilon_{1_Z} \circ Gx^0 - \varepsilon_3 \leqslant \beta^0 \circ Gx^0 + \varepsilon,$$

which we had to establish.

If the conditions of the second part of the condition in question hold, then it is easily verified that $-\varepsilon \leq \alpha^0(Fx - Fx^0)$ for any admissible element x. Hence, if $Fx - Fx^0 \leq -\varepsilon 1_Y$, then in view of the relation $\alpha^0(1_Y) = 1$ we obtain $\alpha^0(Fx^0 - Fx) = \varepsilon$. In other words $\alpha^0(Fx^0 - Fx - \varepsilon 1_Y) = 0$. This means that $Fx = Fx^0 - \varepsilon 1_Y$, which we had to prove.

Generalized ε-Solutions. Let Y be a K-space and $F: X \to Y \cup \{+\infty\}$ a convex operator. We consider a subset U in X. For convenience we assume that U is contained in the effective set dom (F). A subset $\bar{U}$ of U is called a generalized ε-solution of the program $x \in U$, $Fx \to \inf$, if $\inf F[U] \geq \inf F[\bar{U}] - \varepsilon$.

We consider the space $X^{\bar{U}}$ and an operator

$$\bar{F}: X^{\bar{U}} \to Y^{\bar{U}} \cup \{+\infty\}, \quad \bar{F}\mathfrak{x}: \bar{x} \to F\mathfrak{x}(\bar{x}).$$

We set $\bar{\mathfrak{x}}: \bar{x} \to \bar{x}$ and assume that $\bar{F}\mathfrak{x} \in (Y^{\bar{U}})_\infty$ for any element $\mathfrak{x} \in (\mathrm{dom}\,(F))^{\bar{U}}$, and in addition, that $\bar{\mathfrak{x}}$ is an interior point of $(\mathrm{dom}\,(F))^{\bar{U}}$.

A set $\bar{U}$ is a generalized ε-solution of the program $x \in U$, $Fx \to \inf$ if and only if the following system of conditions is consistent

$$\bar{\alpha} \in L^+((Y^{\bar{U}})_\infty, Y), \quad \alpha \circ \Delta_{\bar{U}} = I_Y; \quad \bar{\alpha} \circ \bar{F}\bar{\mathfrak{x}} = \inf_{\bar{x} \in \bar{U}} F\bar{x};$$
$$\varepsilon_1 \geqslant 0, \ \varepsilon_2 \geqslant 0, \ \varepsilon_1 + \varepsilon_2 = \varepsilon;$$
$$0 \in \partial_{\bar{\mathfrak{x}}, \varepsilon_1}(\bar{\alpha} \circ \bar{F}) + \partial_{\bar{\mathfrak{x}}, \varepsilon_2}(\delta_Y(U^{\bar{U}})).$$

As is well known, if $\bar{\alpha} \in \partial_{-\bar{F}\bar{\mathfrak{x}}}(\varepsilon_{\bar{U}})$, then

$$\bar{\alpha} \in L^+((Y^{\bar{U}})_\infty, Y), \quad \bar{\alpha} \circ \bar{F}\bar{\mathfrak{x}} = \inf_{\bar{x} \in \bar{U}} F\bar{x},$$

and for any admissible $\mathfrak{x}$ we have

$$\bar{\alpha} \circ \bar{F}\mathfrak{x} \geqslant \inf_{\bar{x} \in \bar{U}} \bar{F}\mathfrak{x}(\bar{x}) \geqslant \inf_{x \in U} Fx \geqslant \inf_{\bar{x} \in \bar{U}} F\bar{x} - \varepsilon = \bar{\alpha} \circ \bar{F}\bar{\mathfrak{x}} - \varepsilon.$$

Thus, $\bar{\mathfrak{x}}$ is an ε-solution in the program

$$\mathfrak{x} \in U^{\bar{U}}, \ \bar{\alpha} \circ \bar{F}\mathfrak{x} \to \inf.$$

Conversely, if $\bar{\mathfrak{x}}$ is an ε-solution of this problem, then for any $x \in U$ we have

$$-\varepsilon + \inf_{\bar{x} \in \bar{U}} F\bar{x} = -\varepsilon + \bar{\alpha} \circ \bar{F}\bar{\mathfrak{x}} \leqslant \bar{\alpha} \circ \bar{F}(\bar{x} \to x) = Fx.$$

Thus, the set $\bar{U}$ is a generalized ε-solution in the program $x \in U$, $Fx \to \inf$ if and only if

$$0 \in \partial_{\bar{\mathfrak{x}}, \varepsilon}(\bar{\alpha} \circ \bar{F} + \delta_Y(U^{\bar{U}})).$$

The required result is then obtained by applying the formula for the ε-subdifferential of a sum.

Concluding Remarks. Now we dwell on two additional aspects of ε-subdifferentiation that are useful in investigating concrete programming problems.

Structure of an ε-Subdifferential. Let $F: X \to Y \cup \{+\infty\}$ be a convex operator, where Y is a K-space. We assume that $\bar{x}$ is an interior point of dom (F). For each $h \in X$ we define the ε-derivative of F at $\bar{x}$ in the direction h by

$$F^\varepsilon(\bar{x})\,h = \inf_{\alpha > 0} \frac{F(\bar{x} + \alpha h) - F(\bar{x}) + \varepsilon}{\alpha}.$$

Clearly, the operator $F^\varepsilon(\bar{x}): X \to Y$ is sublinear, and

$$\partial_{\bar{x}, \varepsilon}(F) = \partial(F^\varepsilon(\bar{x})).$$

Thus, an ε-subdifferential coincides with the usual subdifferential of some everywhere defined sublinear operator. Hence it follows, in particular, that the set of extreme points of an ε-subdifferential (with respect to an interior point) is nonempty, and in addition, an ε-subdifferential can be recovered from the set of its extreme points [9]. Hence it easily follows that in ε-programming problems, the finite-dimensional calculation can be carried out in precisely the same way as in the usual problems.

Calculation of ε-Derivatives. It is clear that a point $\bar{x} \in \operatorname{dom}(F)$ is ε-optimal for F if and only if the ε-derivative of F at $\bar{x}$ is positive. It is a fact that in programming problems it is often convenient to use directional derivatives rather than subdifferentials. Here it must be emphasized that the calculation of ε-derivatives in an explicit form is significantly more complicated than that of ε-subdifferentials. This is connected with the fact that the concept of an ε-derivative for ε > 0 has a nonlocal character, in contrast to the concept of an 0-derivative, that is, the usual directional derivative. In fact, it is convenient to calculate these derivatives only for functions with known Young transformations. Whereas the formulas for the calculation of ε-subdifferentials given above can, of course, be turned into formulas for ε-derivatives. We mention some of them, assuming for simplicity that all the operators in question are everywhere defined:

$$(F_1 + F_2)^{\varepsilon}(\bar{x}) = \sup_{\substack{\varepsilon_1 \geqslant 0,\, \varepsilon_2 \geqslant 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \left(F_1^{\varepsilon_1}(\bar{x}) + F_2^{\varepsilon_2}(\bar{x})\right);$$

$$(G \circ F)^{\varepsilon}(\bar{x}) = \sup_{\substack{\varepsilon_1 \geqslant 0,\, \varepsilon_2 \geqslant 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \sup_{B \in \partial_{F\bar{x}, \varepsilon_1}(G)} (B \circ F)^{\varepsilon_2}(\bar{x});$$

$$(A \circ F_1 \vee F_2)^{\varepsilon}(\bar{x}) =$$

$$= \sup_{\substack{\varepsilon_1 \geqslant 0,\, \varepsilon_2 \geqslant 0,\, \varepsilon_3 \geqslant 0 \\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon}} \sup_{\substack{A_1 \geqslant 0,\, A_2 \geqslant 0,\, A_1 + A_2 = A; \\ A_1 \circ F_1 \bar{x} + A_2 \circ F_2 \bar{x} \geqslant A(F_1 \bar{x} \vee F_2 \bar{x}) - \varepsilon_3}} \left((A_1 \circ F_1)^{\varepsilon_1}(\bar{x}) + (A_2 \circ F_2)^{\varepsilon_2}(\bar{x})\right);$$

$$(F \circ A_x)^{\varepsilon}(\bar{x}_1) = F^{\varepsilon}(A_x \bar{x}_1) \circ A.$$

Clearly, the validity of these formulas is ensured by the results derived above and by Minkowskii duality.

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