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CONVEX ANALYSIS IN MODULES

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The aim of the present article is to give a complete description of those ordered universal modules over lattice-ordered rings which can have convex analysis. The problem of construction of convex analysis in modules arises from the essential requirements of the classical scalar theory of extremal problems. It is concerned with the fact that the Young transformations of convex operators [1] have the property of module convexity over the ideal center of the ranges of the considered operators. The reflection of this fact serves as the operator convexity of subdifferentials. It is clear that each problem of convex analysis in modules is ultimately a problem of dominated extension of a module homomorphism [2]. There are several theorems of this type (see [3, 4] and the bibliography given there). Let us take special notice of [5], where a complete solution of the corresponding problem for the ring **Z** of integers is given.

In the present article we establish a common defect of the existing Hahn–Banach type theorems for modules. Namely, we establish that there simply does not exist any specific "module" convex analysis. More precisely, to within elementary stipulations, convex analysis exists precisely in the case of the Kantorovich spaces, considered as modules over the algebras of their orthomorphisms. Moreover, in this connection the additive minorants of the module-sublinear operator automatically turn out to be module homomorphisms. The last statement constitutes, in essence, the main result of this article. The idea of its proof is quite clear. Indeed, it is almost obvious that the extreme points of subdifferentials must commute with the multipliers. Besides this, each subgradient is obtained by the "integration" of the extreme points. It remains to observe that the corresponding integrals, i.e., the elements of the subdifferential of a canonical operator, commute with orthomorphisms. Unfortunately, the realization of this idea is made difficult by the fact that K-spaces, considered as modules over the rings of their orthomorphisms, are, as a rule, not injective. This difficulty is overcome, since its essential part occurs only in the "group part" of these modules. However, in this connection, we cannot avoid verification of small, but necessary, facts.

It should be observed that the facts, proved in this article, have a definite qualitative significance for optimization theory, since they put precise meaning in Hurwicz's words [6] "The investigation of the phenomenon of 'indivisibility' has led us beyond the bounds of linear spaces. But. . . in this connection we cannot expect the majority of the important results that hold for linear spaces to remain valid."*

In conclusion, I express deep gratitude to the participants of the seminars of A. D. Aleksandrov and V. L. Markov for valuable discussion of the results of this article.

*Translator's note: This quotation is a retranslation from the Russian.

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1. Sublinear Operators in Modules

Let A be an arbitrary lattice-ordered ring with a positive identity 1_A. Further, let X be an A-module and Y be an ordered A-module. All modules are naturally assumed to be unitary. Let us adjoin a greatest element $+\infty$ to Y, set $Y^{\cdot}=Y\cup\{+\infty\}$, and equip $Y^{\cdot}$ with the natural structure of an A^+-semimodule. Here, as usual, A^+ is the semigroup of the positive elements of A.

An operator $p: X \to Y^{\cdot}$ is said to be A-sublinear with the effective domain $\mathrm{dom}\,(p)=\{x\in X: p(x)<+\infty\}$, if

$$p(\pi_1 x_1+\pi_2 x_2)\leqslant \pi_1 p(x_1)+\pi_2 p(x_2)\ \ (x_1,\ x_2\in X;\ \pi_1,\ \pi_2\in A^+).$$

In the case where dom (p) = X, the operator $p: X\to Y$ (the dot over Y is omitted) is called simply an A-sublinear operator. Let us observe that p(0) = 0 for each A-sublinear operator $p: X \to Y$. Indeed, $p(0) = p(0\cdot 0) \le 0$; in addition, $p(0) = p(0 + 0) \le 2p(0)$. An operator p is said to be A^+-homogeneous, if $p(\pi x) = \pi p(x)$ for all $x\in X$ and $\pi\in A^+$.

Let the symbol $\mathrm{Hom}_A(X, Y)$ denote the set of all the A-sublinear operators $T: X \to Y^{\cdot}$ such that dom (T) is an A-submodule of X and the trace of T on dom (T) is an A-homomorphism, i.e., belongs to $\mathrm{Hom}_A(\mathrm{dom}\,(T), Y^{\cdot})$. The set $\mathrm{Hom}_A(X, Y^{\cdot})$ is also equipped with the natural structure of an A^+-semimodule.

For each sublinear operator $p: X\to Y^{\cdot}$ we define the subdifferential and the subdifferential at a point, respectively, as follows:

$$\partial^A(p)=\{T\in \mathrm{Hom}_A(X,\ Y^{\cdot}): Tx\leqslant p(x)\ \ (x\in X)\},$$
$$\partial_x^A(p)=\{T\in\partial^A(p): Tx=p(x)\,(x\in \mathrm{dom}\,(p))\}.$$

Let us observe that $\partial^A(p) = \partial_0^A(p)$ and, moreover,

$$\partial_x^A(p)=\{T\in \mathrm{Hom}_A(X, Y^{\cdot}): T(\bar{x}-x)\leqslant p(\bar{x})-p(x)\quad (\bar{x}\in X)\}.$$

Since X and Y are, in particular, **Z**-modules, the subdifferentials $\partial^{\mathbf{Z}}(p)$ and $\partial_X^{\mathbf{Z}}(p)$ are defined and are denoted by $\partial(p)$ and $\partial_X(p)$, respectively.

An A-module Y is said to have the property of A-extension if the following asymmetric Hahn–Banach formula holds for set A-sublinear operator $p: X\to Y$ and each A-submodule X_0 of X:

$$\partial^A(p+\delta_Y(X_0))=\partial^A(p)+\partial^A(\delta_Y(X_0)),$$

where, as usual, $\delta_Y(X_0)$ is the indicator operator of X_0, i.e., $\delta_Y(X_0): X\to Y^{\cdot}$ and $\delta_Y(X_0)x = 0$ for $x\in X_0$, and $\delta_Y(X_0)x = +\infty$ in the contrary case. If, besides this, the subdifferential $\partial_X^A(p)$ is nonempty for each $x\in X$, then the module Y is said to admit convex analysis.

<u>Proposition 1.1.</u> If an A-module Y has the property of A-extension and $p: X\to Y$ is an A-sublinear operator, then the following statements are valid:

(1) There exists an operator $T\in\partial^A(p)$ such that Tx = y if and only if $\pi y \le p(\pi x)$ for all $\pi\in A$.

(2) The operator p is A^+-homogeneous if and only if $\partial_X^A(p) \ne \emptyset$ for any $x\in X$.

(3) The equality $\partial^A(p\circ T) = \partial^A(p)\circ T$ holds for each A-module X_1 and each A-homomorphism $T\in \mathrm{Hom}_A(X_1, X)$.

<u>Proof.</u> Statement (1) is obvious.

(2) If $T\in\partial_x^A(p)$ and $\pi\in A^+$, then $\pi p(x) = \pi Tx = T\pi x \le p(\pi x) \le \pi p(x)$; whence p is A^+-homogeneous. But if it is known beforehand that A is an A-sublinear A^+-homogeneous operator, then

$$\pi p(x)=\pi^+p(x)-\pi^-p(x)=p(\pi^+x)-p(\pi^-x)\leqslant p(\pi^+x-\pi^-x)=p(\pi x).$$

for each $\pi\in A$. Thus, $\partial_X^A(p) \ne \emptyset$ by virtue of (1).

(3) This statement is established in the same manner as the Levine–Rockafeller lemma in [1].

2. Krein – Mil'man Theorem for Groups

Let Y be an ordered Abelian group (a **Z**-module). Let us set $Y_b = Y^+ - Y^+$ and assume that Y_b is an obliterated K-space. Let us recall that the groups obtained from K-spaces by ignoring the multiplications by real numbers are called obliterated K-spaces. The following theorem holds.

Bigard Theorem [5]. An ordered **Z**-module Y has the property of Z-extension if and only if Y_b is an obliterated K-space.

Remark. In the sequel, we will use only the trivial part of the above theorem, which states that K-spaces have the property of **Z**-extension. The Bigard theorem follows in full from the results given below.

Proposition 2.1. Let $p: X \to Y$ be a **Z**-sublinear operator. Then $\partial(np) = n\partial(p)$ for each $n \in \mathbf{N}$.

Proof. The inclusion $n\partial(p) \subset \partial(np)$ is obvious. Let us now assume that $T \in \partial(np)$. We take any operator $T_0 \in \partial(p)$. Then $T - nT_0 \in \partial(n(p - T_0))$. Since $p(x) - T_0x \geq 0$ for all $x \in X$, the image im $(T - nT_0)$ is contained in Y_b. Therefore, the operator $B = (1/n)(T - nT_0)$ has been properly defined. In addition, $B \in \partial(p - T_0)$. We now set $C = B + T_0$. It is clear that $C \in \partial(p)$, and, in addition, $nC = n((1/n)(T - nT_0)) + nT_0 = T$. Finally, $T \in n\partial(p)$.

COROLLARY 2.2. The equality $\sum_{k=1}^{n} \partial(p) = n\partial(p)$ holds for each $n \in \mathbf{N}$.

Proof. It is sufficient to observe that the set on the left-hand side of the relation under consideration is obviously contained in $\partial(np)$.

Proposition 2.3. Let $T_1, T_2 \in \partial(p)$, be such that $nT_1 = nT_2$ for a certain $n \in \mathbf{N}$. Then $T_1 = T_2$.

Proof. Since $T_1 - T_2 \in \partial(p - T_2)$ and $T_2 \in \partial(p)$, it follows that im $(T_1 - T_2) \subset Y_b$. Hence the desired result follows.

Proposition 2.4. Let $p: X \to Y$ be a **Z**-sublinear $\mathbf{Z}^+$-homogeneous operator and let $x \in X$. Then the o-limit

$$p_x(y) = o\text{-}\lim_n (p(nx + y) - p(nx)) = \inf_{n \in N} (p(nx + y) - p(nx))$$

exists for each $y \in X$. In addition, $\partial(p_x) = \partial_x(p)$.

Proof. Setting $z_n = p(nx + y) - p(nx)$, we get $-p(-y) \leq z_m \leq z_n$ for $m \geq n$. Moreover, $\partial(p_x) \subset \partial(p)$ and $p_x(x) = -p_x(-x) = p(x)$.

COROLLARY 2.5. The equality $(np)_x = np_x$ is valid for each $n \in \mathbf{N}$.

Let us also note the following obvious proposition.

Proposition 2.6. Set $h_p(x) = \sup\{Tx : T \in \partial(p)\}$ for each **Z**-sublinear operator $p: X \to Y$. Then h_p is the greatest **Z**-sublinear $\mathbf{Z}^+$-homogeneous operator dominated by p. Moreover, $\partial(h_p) = \partial(p)$.

Let us recall that an operator T from $\partial(p)$ is said to be extreme if the relation $T_1, T_2 \in \partial(p)$ and $T_1 + T_2 = 2T$ imply that $T = T_1 = T_2$. The set of the extreme operators in $\partial(p)$ is denoted by Ch(p). Let us also recall that the symbol $(Y^{\mathscr{A}})_\infty$, where $\mathscr{A}$ is an arbitrary set, denotes the set of all bounded Y-valued functions on $\mathscr{A}$. This set is equipped with the structure of an ordered **Z**-module (of a submodule of the product $Y^{\mathscr{A}}$). The symbol $\varepsilon_{\mathscr{A}}$ denotes the canonical **Z**-sublinear operator

$$\varepsilon_{\mathscr{A}} : (Y^{\mathscr{A}} \to Y, \ \varepsilon_{\mathscr{A}}(f) = \sup\{f(a) : a \in \mathscr{A}\}.$$

If, in addition, $\mathscr{A}$ is a weakly order-bounded set of homomorphisms of X into Y, then the homomorphism $\langle\mathscr{A}\rangle : X \to (Y^{\mathscr{A}})_\infty$ is defined by the relation $\langle\mathscr{A}\rangle x : a \to ax$.

Klein–Mil'man Theorem. The following equality is fulfilled for each **Z**-sublinear operator $p: X \to Y$:

$$\partial(p) = \partial(\varepsilon_{\mathrm{Ch}(p)}) \circ \langle \mathrm{Ch}(p) \rangle.$$

Proof. The proof follows a well-known pattern (cf. [7]); however, it contains certain small nuances.

Let us consider the set P of all the **Z**-sublinear operators p_1 such that $p_1(x) \leq p(x)$ for all $x \in X$ and, moreover, p_1 is extremal for p. The last condition means that if $T_1, T_2 \in \partial(p)$ are such that $T_1 + T_2 \in 2\partial(p_1)$, then $T_1, T_2 \in \partial(p_1)$. It is clear that $p \in P$. Let us order P in the natural manner and consider an arbitrary chain P_0 in P. It is clear that $p(-x) + p_1(x) \geq p_1(x) + p_1(-x) \geq 0$. Thus, the element $p_0(x) = \inf\{p_1(x) : p_1 \in P_0\}$ is defined. Since addition is o-continuous, the operator p_0 is **Z**-sublinear. It is also directly verified that $p_0 \in P$. Thus, by the Zorn lemma, P contains a minimal element q. By virtue of its minimality and Proposition 2.6, $q = h_q$. Therefore, by Proposition 2.4, the operator q_x is defined for each $x \in X$. In addition, if $T_1, T_2 \in \partial(p)$ are such that $T_1 + T_2 \in 2\partial(q_x)$, then $T_1, T_2 \in \partial(q)$ by virtue of extremality of q. By virtue of Proposition 2.4 and Corollary 2.5, we have $T_1x + T_2x = 2q(x)$. Since $T_1x \leq q(x)$ and $T_2x \leq q(x)$, we conclude that $T_1, T_2 \in \partial_x(q) = \partial(q_x)$.

Thus, q_X is extremal for p, i.e., $q = q_X$ for all $x \in X$. The last statement implies that q is a homomorphism, i.e., $q \in \mathrm{Ch}(p)$. Thus, the set Ch(p) is nonempty for each p.

To complete the proof it is sufficient to consider the case where p is a $\mathbf{Z}^+$-homogeneous operator. In this case, as already observed, for each $x \in X$ the operator p_X is extremal for p and, therefore, $\mathrm{Ch}(p_x) \subset \mathrm{Ch}(p)$. Now, using Propositions 1.1 and 2.6, we obtain the desired representation. The theorem is proved.

Proposition 2.7. The equality $\mathrm{Ch}(np) = n\,\mathrm{Ch}(p)$ is fulfilled for each $\mathbf{Z}$-sublinear operator $p: X \to Y$ and each $n \in \mathbf{N}$.

Proof. At first, suppose that $T \in \mathrm{Ch}(np)$. Then, by Proposition 2.1, $T = nS$, where $S \in \partial(p)$. Let us verify that $S \in \mathrm{Ch}(p)$. Indeed, if $2S = S_1 + S_2$, where $S_1, S_2 \in \partial(p)$, then $2T = 2nS = nS_1 + nS_2$. Thus, $nS = nS_1 = nS_2$. By Proposition 2.3, we get $S = S_1 = S_2$, which was required.

Now, if $T \in \mathrm{Ch}(p)$ and $2nT = T_1 + T_2$, where $T_1, T_2 \in \partial(np)$, then, by Proposition 2.1, $T_1 = nS_1$ and $T_2 = nS_2$ for certain $S_1, S_2 \in \partial(p)$. Moreover, $2nT = n(2T) = n(S_1 + S_2)$. Using Proposition 2.3, we have $2T = S_1 + S_2$; whence $T = S_1 = S_2$. Consequently, $T_1 = nS_1 = nT = nS_2 = T_2$. Thus, $nT \in \mathrm{Ch}(np)$. The proposition is proved.

3. Orthomorphisms

Now let Y be a K-space and I_Y be the identity operator in Y. The component generated by I_Y in the K-space of regular operators $L^r(Y)$ is denoted by Orth(Y). The elements of Orth(Y) are called orthomorphisms. The properties of orthomorphisms in K-spaces have been studied in detail in [8]. We isolate the smallest normal subspace Z(Y) of Orth(Y) that contains I_Y. This subspace is called the ideal center of Y. Let us observe that Orth(Y) and Z(Y) are function algebras with respect to the natural ring and order structures. In addition, Z(Y) serves as the foundation of Orth(Y) and Orth(Y) is the centralizer of Z(Y) in the ring $L^r(Y)$. In the sequel we will need the following facts about orthomorphisms.

Proposition 3.1. The following statements are equivalent for each positive operator $T \in L^r(Y)$:

(1) T is an orthomorphism.

(2) $T + I_Y$ is a lattice homomorphism.

(3) $T + I_Y$ has the Magaram property, i.e., it preserves order intervals.

Proof. We will establish that (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1), since the reverse implications are obvious.

(2) $\Rightarrow$ (1). We know [1] that the equality $[0, S] = [0, I_Y]S$ of order intervals in the space of operators is a criterion for S to be a lattice homomorphism. Therefore, since $I_Y \le T + I_Y$, there exists a multiplier γ, $0 \le \gamma \le I_Y$, such that $\gamma T = I_Y - \gamma$. Hence $\gamma(T\,\mathrm{Pr} - \mathrm{Pr}\,T) = 0$ for each projection Pr in Y, since orthomorphisms commute with each other. In particular, for the projection Pr_γ on the kernel ker (γ), which is obviously a component of Y, we get $\gamma T\,\mathrm{Pr}_\gamma = 0$. Besides this, $\gamma T\,\mathrm{Pr}_\gamma = (I_Y - \gamma)\mathrm{Pr}_\gamma = \mathrm{Pr}_\gamma$. Thus, $\ker(\gamma) = \{0\}$. Therefore $T\,\mathrm{Pr} = \mathrm{Pr}\,T$ for each projection Pr. The last statement, as we know [8], implies that T is an orthomorphism.

(3) $\Rightarrow$ (1). We know [8] that the following property is a criterion for an orthomorphism: If $u, v \in Y$ and $u \wedge v = 0$, then $Tu \wedge v = 0$. Thus, let $u \wedge v = 0$. Then $Tu \wedge v \leqslant Tu \leqslant Tu + u = (T + I_Y)u$. Since $T + I_Y$ has the Magaram property, it follows that $Tu \wedge v = Tz + z$ for a certain z in the order interval [0, u]. We have the estimates $v \geqslant Tu \wedge v = Tz + z \geqslant z$ and $u \ge z \ge 0$. Therefore $0 = u \wedge v \geqslant z \wedge z \geqslant 0$. Thus, $z = 0$, and consequently $Tu \wedge v = 0$.

Proposition 3.2. Let A be a subring and sublattice of Orth(Y). For elements $\pi, \gamma \in A^+$ such that $\pi \ge I_Y$, set

$$[\pi^{-1}](\gamma) = \inf\{\delta \in A^+ : \delta\pi \geqslant \gamma\}.$$

Then $[\pi^{-1}]: A \to \mathrm{Orth}(Y)^0$ is an increasing A-sublinear operator and, moreover, $\gamma = [\pi^{-1}](\pi\gamma)$ for all $\gamma \in A^+$.

Proof. First of all, let us observe that $\pi(\delta_1 \wedge \delta_2) = \pi\delta_1 \wedge \pi\delta_2 \geqslant \gamma$ for $\pi\delta_1 \ge \gamma$ and $\pi\delta_2 \ge \gamma$. Hence $[\pi^{-1}](\gamma) \leqslant \gamma$ and $\pi[\pi^{-1}](\gamma) \geqslant \gamma$. If $\gamma_2 \ge \gamma_1$, then $\pi([\pi^{-1}](\gamma_2) \wedge \gamma_1) \geqslant \gamma_2 \wedge \pi\gamma_1 \geqslant \gamma_2 \wedge \gamma_1 = \gamma_1$. Therefore $[\pi^{-1}](\gamma_2) \wedge \gamma_1 \geqslant [\pi^{-1}](\gamma_1)$. Thus, the operator $[\pi^{-1}]$ is increasing.

Let us now observe that, by what we have already proved, $\pi([\pi^{-1}](\gamma_1) + [\pi^{-1}](\gamma_2)) \geqslant \gamma_1 + \gamma_2$ for $\gamma_1, \gamma_2 \in A^+$. Consequently, $[\pi^{-1}](\gamma_1 + \gamma_2) \leqslant [\pi^{-1}](\gamma_1) + [\pi^{-1}](\gamma_2)$. Moreover, if $\mu, \gamma \in A^+$, then $\pi\mu[\pi^{-1}](\gamma) = \mu\pi[\pi^{-1}](\gamma) \geqslant \mu\gamma$, i.e., $[\pi^{-1}](\mu\gamma) \leqslant \mu[\pi^{-1}](\gamma)$. In other words, the operator $[\pi^{-1}]$ is A-sublinear.

To complete the proof, let us observe that $[\pi^{-1}](\pi\gamma) \leqslant \gamma$ and $\pi[\pi^{-1}](\pi\gamma) \geqslant \pi\gamma$. Hence $\pi\gamma = \pi[\pi^{-1}](\pi\gamma)$. Now, since $\ker(\pi) = \{0\}$ by virtue of the condition $\pi \geq I_Y$, it follows that $\gamma = [\pi^{-1}](\pi\gamma)$, which was required to be proved.

4. Construction of the Subdifferential of a Module-Sublinear Operator

In this section it is assumed that the ordered A-module Y is such that $Y_b = Y^+ - Y^+$ is an (obliterated) K-space and that the ring A is a subring and sublattice of the ring of orthomorphisms $\mathrm{Orth}(Y_b)$ and acts naturally in Y_b.

Proposition 4.1. Let $Y = Y_b$ and $\mathscr{A}$ be an arbitrary set. If the group $(Y^{\mathscr{A}})_\infty$ is equipped with the natural structure of a Z(Y)-module, then

$$\partial(\varepsilon_{\mathscr{A}}) \subset \mathrm{Hom}_{Z(Y)}((Y^{\mathscr{A}})_\infty, Y).$$

Proof. Let Pr be an arbitrary projection in Y and $\alpha \in \partial(\varepsilon_{\mathscr{A}})$. Then $-\mathrm{Pr}\varepsilon_{\mathscr{A}}(-y) \leqslant \alpha\mathrm{Pr}y \leqslant \varepsilon_{\mathscr{A}}(\mathrm{Pr}y) = \mathrm{Pr}\varepsilon_{\mathscr{A}}(y)$ for all $y \in (Y^{\mathscr{A}})_\infty$. Thus, $\mathrm{Pr}^d \alpha \mathrm{Pr} = 0$ for the complementary projection $\mathrm{Pr}^d = I_Y - \mathrm{Pr}$. Therefore $\alpha\mathrm{Pr} = \mathrm{Pr}\alpha\mathrm{Pr}$. In spite of this $\mathrm{Pr}\alpha\mathrm{Pr}^d = 0$. Finally, $\alpha\mathrm{Pr} = \mathrm{Pr}\alpha\mathrm{Pr} + \mathrm{Pr}\alpha\mathrm{Pr}^d = \mathrm{Pr}\alpha$. It follows from the last relation that the operator α commutes with finite-valued elements of the form $t_1\mathrm{Pr}_1 + \ldots + t_m\mathrm{Pr}_m$, where $t_k \in \mathbf{R}^+$, and Pr_k are projections in Y. Since for each $\pi \in Z(Y)$ and each $n \in \mathrm{N}$ there exist finite-valued elements α_n and β_n such that $0 \leqslant \pi - \alpha_n \leqslant (1/n)I_Y$ and $0 \leqslant \beta_n - \pi \leqslant (1/n)I_Y$, it follows from the relations $\alpha_n\alpha \leqslant \alpha\pi \leqslant \beta_n\alpha$ that α is a Z(Y)-homomorphism.

Proposition 4.2. Let p be an A-sublinear operator. Then $\partial^{A\cap Z(Y_b)}(p) \subset \partial^A(p)$.

Proof. Let us take $\pi \in A^+$ and set $\alpha_n = \pi \wedge nI_{Y_b}$ for each $n \in \mathrm{N}$. Consider $T \in \partial^{A\cap Z(Y_b)}(p)$ and a point x from the domain of the operator p. Then

$$(\pi - \alpha_n)p(x) \geqslant p((\pi - \alpha_n)x) \geqslant T(\pi - \alpha_n)x = T\pi x - \alpha_n Tx.$$

Thus, $\pi p(x) - T\pi x \geq \alpha_n(p(x) - Tx)$. Since $p(x) - Tx \in Y_b$, it follows from the last inequality that $\pi p(x) - T\pi x \geq \pi p(x) - \pi Tx$. Since x is arbitrary, we get $T\pi = \pi T$, i.e., $T \in \partial^A(p)$. The proposition is proved.

THEOREM 4.3. $\partial(p) = \partial^A(p)$ for each A-sublinear operator $p: X \to Y$.

Proof. First of all, let us establish that if $T \in \mathrm{Ch}(p)$, then $T \in \partial^{A\cap Z(Y_b)}(p)$. We take $\pi \in A^+ \cap Z(Y_b)$. Let us observe that $\pi \leq n1_A$ for a certain $n \in \mathrm{N}$, since $1_A = I_{Y_b}$. Since 1_A acts in X as well as in Y as the corresponding identity operator, we get

$$nT = n1_A T = \pi T + (n1_A - \pi)T; \quad nT = T\pi + T(n1_A - \pi);$$
$$2nT = (\pi T + T(n1_A - \pi)) + (T\pi + (n1_A - \pi)T).$$

By virtue of the obvious relations

$$\pi T + T(n1_A - \pi) \in \partial(np), \quad T\pi + (n1_A - \pi)T \in \partial(np)$$

and Proposition 2.7, by which $nT \in \mathrm{Ch}(np)$, we get $nT = \pi T + T(n1_A - \pi)$. Thus, $T\pi = \pi T$.

Let us now consider the operator $p_1 = p - T$, where $T \in \mathrm{Ch}(p)$. It is clear that $\mathrm{im}(p - T) \subset Y_b$. By virtue of what we have already proved, $\mathrm{Ch}(p_1) \subset \partial^{A\cap Z(Y_b)}(p_1)$. Moreover, by the Krein–Mil'man theorem and Proposition 4.1, we have

$$\partial(p_1) = \partial(\varepsilon_{\mathrm{Ch}(p_1)}) \circ \langle \mathrm{Ch}(p_1) \rangle,$$
$$\partial(\varepsilon_{\mathrm{Ch}(p_1)}) \subset \mathrm{Hom}_{A\cap Z(Y_b)}(((Y_b)^{\mathrm{Ch}(p_1)})_\infty, Y_b).$$

Hence it follows immediately that $\partial(p_1) = \partial^{A\cap Z(Y_b)}(p)_1$. Now, if $S \in \partial(p)$, then $S - T \in \partial(p_1)$, and therefore the operator S−T is an $A \cap Z(Y_b)$-homomorphism. The operator T is of the same type, i.e., $S \in \partial^{A\cap Z(Y_b)}(p)$. A reference to Proposition 4.2 completes the proof.

COROLLARY 4.4. Each ordered A-module Y has the property of A-extension.

COROLLARY 4.5. Let $p_1, p_2 : X \to Y^{\cdot}$ be A-sublinear operators. If the effective domains dom (p_1) and dom (p_2) are strongly situated in the common position, i.e.,

$$X_0 = \mathrm{dom}\,(p_1) \cap X_0 - \mathrm{dom}\,(p_2) \cap X_0$$

for each A-module X_0 containing $\mathrm{dom}\,(p_1) \cap \mathrm{dom}\,(p_2)$, then the symmetric Hahn–Banach formula holds:

$$\partial_x^A (p_1 + p_2) = \partial_x^A (p_1) + \partial_x^A (p_2)$$

for each $x \in \mathrm{dom}\,(p_1) \cap \mathrm{dom}\,(p_2)$.

Proof. If $T_1 \in \partial^A{}_x(p_1)$ and $T_2 \in \partial^A{}_x(p_2)$, then $\mathrm{dom}\,(T_1) \supset \mathrm{dom}\,(p_1)$, $\mathrm{dom}\,(T_2) \supset \mathrm{dom}\,(p_2)$, and $T_1x = p_1(x)$, $T_2x = p_2(x)$. Therefore $\mathrm{dom}\,(T_1 + T_2) \supset \mathrm{dom}\,(p_1) \cap \mathrm{dom}\,(p_2) = \mathrm{dom}\,(p_1 + p_2)$ and $(T_1 + T_2)x = (p_1 + p_2)(x)$. Consequently, $T_1 + T_2 \in \partial^A{}_x(p_1 + p_2)$. Thus, to complete the proof we should verify that $\partial^A(p_1 + p_2) \subset \partial^A(p_1) + \partial^A(p_2)$.

Let $T \in \partial^A(p_1 + p_2)$ and set $X_0 = \mathrm{dom}\,(T)$. It is clear that $X_0 \supset \mathrm{dom}\,(p_1) \cap \mathrm{dom}\,(p_2)$. By virtue of the conditions satisfied by the effective domains of the considered operators, we have

$$X_0 \times X_0 = \mathrm{dom}\,(p_1) \cap X_0 \times \mathrm{dom}\,(p_2) \cap X_0 - \mathrm{im}\,(\Delta),$$

where $\Delta x_0 = (x_0, x_0)$ for $x_0 \in X_0$. Indeed, by the condition, for $x_0 \in X_0$, we have $(x_0, 0) = (x_1, x_2) - (x_2, x_2)$, where $x_0 = x_1 - x_2$; $x_1, x_2 \in X_0$; $x_1 \in \mathrm{dom}\,(p_1)$ and $x_2 \in \mathrm{dom}\,(p_2)$. An analogous representation holds also for the element $(0, x_0)$. Thus, for arbitrary $x, y \in X_0$ there exist elements $h, g \in X_0$ such that

$$x + h \in \mathrm{dom}\,(p_1), \quad y + h \in \mathrm{dom}\,(p_2);$$
$$-x + g \in \mathrm{dom}\,(p_1), \quad -y + g \in \mathrm{dom}\,(p_2).$$

In addition, $p_1(x + h) + p_2(y + h) - Th \geq -p_1(-x + g) + p_1(h + g) - p_2(-y + g) + p_2(h + g) - Th \geq -p_1(-x + g) - p_2(-y + g) + Tg$. Thus, the element

$$p(x, y) = \inf \{p_1(x + h) + p_2(y + h) - Th : h \in X_0,\ x + h \in \mathrm{dom}\,(p_1) \cap X_0,\ y + h \in \mathrm{dom}\,(p_2) \cap X_0\}$$

is defined. The resulting operator $p : X_0 \times X_0 \to Y$ is certainly **Z**-sublinear. Besides this, $\pi \inf U = \inf \pi(U)$ for each $\pi \in A^+$ and each nonempty subset U of Y^+. Hence, we successively get

$$\pi p(x, y) = \inf \{\pi p_1(x + h) + \pi p_2(y + h) - \pi Th : h \in X_0,\ x + h \in \mathrm{dom}\,(p_1),$$
$$y + h \in \mathrm{dom}\,(p_2)\} \geqslant \inf \{p_1(\pi x + \pi h) + p_2(\pi y + \pi h) - T\pi h : h \in X_0,$$
$$x + h \in \mathrm{dom}\,(p_1),\ y + h \in \mathrm{dom}\,(p_2)\} \geqslant \inf \{p_1(\pi x + \pi h) + p_2(\pi y + \pi h) -$$
$$- T\pi h : \pi h \in X_0,\ \pi x + \pi h \in \mathrm{dom}\,(p_1),\ \pi y + \pi h \in \mathrm{dom}\,(p_2)\} \geqslant$$
$$\geqslant \inf \{p_1(\pi x + h) + p_2(\pi y + h) - Th : h \in X_0,\ \pi x + h \in \mathrm{dom}\,(p_1),$$
$$\pi y + h \in \mathrm{dom}\,(p_2)\} = p(\pi x, \pi y).$$

In other words, the operator p is A-sublinear. By Corollary 4.4, a certain $T_0 \in \mathrm{Hom}_A(X_0 \times X_0, Y)$ belongs to $T_0 \in \partial^A(p)$. We set $T_1x = T_0(x, 0)$ and $T_2x = (0, x)$ for $x \in X_0$ and $T_1x = T_2x = +\infty$ for $x \in X \setminus X_0$. It is clear that dom × $(T_1) = \mathrm{dom}\,(T_2) = X_0$. Moreover, for $h \in X_0$ we have

$$T_1h \leqslant p(h, 0) \leqslant p_1(h + 0) + p_2(0 + 0) - T0 = p_1(h),$$
$$T_2h \leqslant p(0, h) \leqslant p_1(0 + 0) + p_2(h + 0) - T0 = p_2(h).$$

Thus, $T_1 \in \partial^A(p_1)$ and $T_2 \in \partial^A(p_2)$, and, moreover, $T = T_1 + T_2$. The corollary is proved.

5. Modules That Admit Convex Analysis

In this section we give fundamental results on the characterization of modules that admit convex analysis.

THEOREM 5.1. If an ordered A-module Y has the property of A-extension, then Y_b is an obliterated K-space.

Proof. At first, we establish that bounded sets in Y have supremums. For this we should show that each family $[a_\xi, b_\xi]$ $(\xi \in \Xi)$ of pairwise intersecting order intervals, i.e., of intervals $[a_\xi, b_\xi]$ such that $a_\xi \leq b_\eta$ for all $\xi, \eta \in \Xi$, has a common point.

Let us consider an A-module X that is the direct sum of Ξ copies of the ring A. Further, let X_0 be the A-submodule of X defined as follows:

$$X_0 = \left\{\pi = \pi(\cdot) \in X : \sum_{\xi \in \Xi} \pi(\xi) = 0\right\}.$$

Let us consider the operator $p:X \to Y$ defined by the relation

$$p(\pi) = \sum_{\xi \in \Xi} \pi(\xi)^+ b_\xi - \pi(\xi)^- a_\xi = \sum_{\xi \in \Xi} \pi(\xi) a_\xi + \pi(\xi)^+ (b_\xi - a_\xi).$$

It is clear that the operator p is A-sublinear. Moreover, by virtue of the definition, for each element $\pi \in X_0$ we have

$$0 = \sum_{\xi \in \Xi} \pi(\xi) = \sum_{\xi \in \Xi} \pi^+(\xi) - \pi^-(\xi) = \sum_{\eta \in \Xi} \pi(\eta)^+ - \sum_{\xi \in \Xi} \pi(\xi)^-.$$

Applying the lemma about the double decomposition of the positive elements, we find a family $\pi_{\xi\eta}$ $(\xi, \eta \in \Xi)$ of positive elements of A such that

$$\pi(\eta)^+ = \sum_{\xi \in \Xi} \pi_{\xi\eta},\ \pi(\xi)^- = \sum_{\eta \in \Xi} \pi_{\xi\eta}\ (\xi, \eta \in \Xi).$$

Then for each $\pi \in X_0$ we have

$$p(\pi) = \sum_{\eta \in \Xi} \pi(\eta)^+ b_\eta - \sum_{\xi \in \Xi} \pi(\xi)^- a_\xi = \sum_{\xi,\eta} \pi_{\xi\eta}(b_\eta - a_\xi) \geqslant 0.$$

Therefore, there exists an operator $T \in \partial^A(p)$ such that $T\pi = 0$ for $\pi \in X_0$. We take any index $\xi \in \Xi$ and set $\pi_\xi(\xi) = 1_A$ and $\pi_\xi(\eta) = 0$ for $\xi \neq \eta$. Then, since $\pi_\xi - \pi_\eta \in X_0$ for all ξ and η, it follows that $T\pi_\xi = T\pi_\zeta$ for all $\xi \in \Xi$ and a fixed $\zeta \in \Xi$. In other words, $-p(-\pi_\xi) \leq T\pi_\zeta \leq p(\pi_\eta)$ for arbitrary $\xi, \eta \in \Xi$. It remains to observe that $p(\pi_\eta) = b_\eta$ and $p(-\pi_\xi) = -a_\xi$.

To prove that the conditionally complete group Y_b is an obliterated K-space, it is sufficient to restore the operation of multiplication by $1/2$ in Y_b.

Let us consider a $y \in Y^+$ and set $p(y) = \inf\{z \in Y^+ : 2z \geqslant y\}$. Since the set on the right-hand side of the last equation is filtered by descent, the relations $p(y) \leq y$ and $2p(y) \geq y$ are fulfilled by virtue of the o-continuity of addition. Hence $2(\pi_1 p(y_1) + \pi_2 p(y_2)) \geq \pi_1 y_1 + \pi_2 y_2$ for arbitrary $\pi_1, \pi_2 \in A^+$ and arbitrary $y_1, y_2 \in Y^+$. Consequently, $p(\pi_1 y_1 + \pi_2 y_2) \leq \pi_1 p(y_1) + \pi_2 p(y_2)$. Moreover, $p: Y^+ \to Y$ is an increasing operator. Indeed, if $y_2 \geq y_1$, then $2(p(y_2) \wedge y_1) = 2p(y_2) \wedge 2y_1 \geqslant y_2 \wedge 2y_1 \geqslant y_2 \wedge y_1 = y_1$ and therefore $p(y_2) \geqslant p(y_2) \wedge y_1 \geqslant p(y_1)$. In addition, let us observe that $p(2y) = y$ for each $y \in Y^+$. Indeed, $p(2y) \leq y$ and $2p(2y) \geq 2y$. Therefore $2p(2y) = 2y$; whence $(y - p(2y)) = -(y - p(2y))$.

Let us now consider the operator $q: Y_b \to Y$ defined by the relation $q(y) = p(y^+)$. By virtue of what we have already established, q is an increasing A-sublinear operator. Therefore $\partial^A(q) \neq \emptyset$. Now, for $y \in Y_b$ we set

$$[1/2]y = \sup\{Ty : T \in \partial^A(q)\}.$$

We take $y \in Y^+$. Then $\pi y = \pi^+ y - \pi^- y = p(2\pi^+ y) - p(\pi^- y) = q(2\pi^+ y) - q(2\pi^- y) \leq q(2\pi^+ y - 2\pi^- y) = q(2\pi y) = q(\pi(2y))$ for each $\pi \in A$. Therefore, by virtue of Proposition 1.1, there exists an operator $T \in \partial^A(q)$ such that $T2y = y \leq q(2y) \leq p(2y) = y$. Therefore $2q(y) = q(2y) = y$, since q is a $\mathbf{Z}^+$-homogeneous operator. Thus, $[1/2](2y) = y$ for all $y \in Y^+$. Hence it follows immediately that the operator $[1/2]$ is an increasing A-homomorphism. It is clear that this operator is the desired one. The theorem is proved.

Remark. The scheme of proof of the conditional completeness of Y is, in essence, suggested by A. D. Ioffe in the framework of the theory of fans, developed by him. Hence the divisibility of the group Y_b can be deduced with the help of a result of [5]. Here we observe that Theorem 5.1 clearly contains the well-known Bonnice–Silverman–Tu theorem [2].

THEOREM 5.2. Let A be a d-ring, i.e., $(\pi_1\pi_2)^+ = \pi_1^+\pi_2$ and $(\pi_2\pi_1)^+ = \pi_2\pi_1^+$ for arbitrary $\pi_1 \in A$ and $\pi_2 \in A^+$.

An ordered A-module Y has the property of A-extension if and only if Y_b is an obliterated K-space and the natural linear representation of A into Y_b is a lattice and ring homomorphism onto a sublattice and subring of the ring of orthomorphisms $\mathrm{Orth}(Y_b)$. In addition, $\partial^A(p) = \partial(p)$ for each A-sublinear operator $p: X \to Y$.

Proof. At first, suppose that Y has the property of A-extension. Then, by Theorem 5.1, Y_b is an (obliterated) K-space. Let us consider the natural linear representation φ of the ring A into the space Y_b defined by the relation $\varphi(\pi)y = \pi y$, where $y \in Y_b$ and $\pi \in A$. First of all, we will establish that φ is a lattice homomorphism. To this end, for $y \in Y^+$ we define an operator $p: A \to Y$ by the relation $p(\pi) = \pi^+ y$. This operator is an increasing A-sublinear operator. Therefore if $T \in \partial^A(p)$, then $0 \leq T1_A \leq y$. Thus, $T\pi = \pi T1_A = \pi y_1$, where $y_1 = T1_A$ and $y_1 \in [0, y]$. If, in its turn, the element $y_1 \in [0, y]$ is fixed and we set $T\pi = \pi y_1$ for $\pi \in A$, then we obtain an element of $\partial^A(p)$. Since the operator p is A^+-homogeneous, we get the following relations from Proposition 1.1:

$$\varphi(\pi^+)y = \pi^+ y = p(\pi) = \sup\{T\pi : T \in \partial^A(p)\} = \sup \pi[0, y] = \varphi(\pi)^+ y.$$

Let us now verify that $\mathrm{im}(\varphi) \subset \mathrm{Orth}(Y_b)$. To this end, we fix elements $\pi \in A^+$ and $z, y \in Y^+$ such that $0 \le z \le \pi y$. Then $\pi_1 z \leqslant \pi_1^+ z \leqslant \pi_1^+ \pi y = (\pi_1\pi)^+ y = p(\pi_1\pi)$ for each $\pi_1 \in A$. Therefore, by Proposition 1.1, there exists an operator $T \in \partial_A(p)$ such that $T\pi = z$. Thus, $z = \pi T 1_A$, where $T1_A \in [0, y]$. Therefore, the operator $\varphi(\pi)$ has the Magaram property. Since π is arbitrary, we conclude from Proposition 3.1 that $\varphi(\pi)$ is an orthomorphism.

To complete the proof, it is sufficient to establish that if φ is a lattice homomorphism of A into the K-space $\mathrm{Orth}(Y_b)$, then $\partial(p) = \partial^A(p)$ for each A-sublinear operator $p: X \to Y$. At first, let us consider the case $Y = Y_b$. We take a $T \in \partial(p)$ and a point $x \in X$. Let us consider the operator $t\pi = T\pi x$, where $\pi \in A$. Since $t\pi \le p(\pi x) \le \pi^+ p(x) + \pi^- p(-x)$, it follows that $\ker(t) \supset \ker(\varphi)$, and therefore the operator t admits a lowering $\bar{t}$ on the lattice-ordered ring $\bar{A} = A / \ker(\varphi)$. Let us equip Y with the associated structure of an exact module over $\bar{A}$. In this connection, $\bar{A}$ may be considered as a subring and sublattice of Orth(Y). In addition, let us observe that $p(\pi_1 x) = p(\pi_2 x)$ for all $\pi_1, \pi_2 \in \bar{\pi}$, $\bar{\pi} \in \bar{A}$, since $p(\pi_1 x) - p(\pi_2 x) \le p((\pi_1 - \pi_2)x) \le (\pi_1 - \pi_2)^+ p(x) + (\pi_1 - \pi_2)^- p(-x)$. Thus the operator $\bar{p}: \bar{A} \to Y$ that acts by the rule $\bar{p}(\bar{\pi}) = p(\pi x)$ for $\pi \in \bar{\pi}$ is properly defined. It is clear that the operator $\bar{p}$ is $\bar{A}$-sublinear. Here $\bar{t} \in \partial(\bar{p})$. By virtue of Theorem 4.3, we have $\partial(\bar{p}) = \partial^{\bar{A}}(\bar{p})$, i.e., $\bar{t}\bar{\pi} = \bar{\pi}\bar{t}1_{\bar{A}}$ for $\bar{\pi} \in \bar{A}$. Hence $T\pi x = \pi T x$, i.e., $T \in \partial^A(p)$.

Let us now consider the general case and again take a $T \in \partial(p)$ and a point $x \in X$. Let us observe that for each $\pi \in A$ we have

$$p(\pi x) - \pi T x = p(\pi^+ x - \pi^- x) - (\pi^+ - \pi^-)Tx \leqslant \pi^+(p(x) - Tx) + \pi^-(p(-x) - T(-x)).$$

Thus, the relation $q(\pi) = p(\pi x) - \pi T x$ defines an operator that acts from A into Y_b. It is clear that this operator in A-sublinear, and therefore, by what we have already proved, $\partial(q) = \partial^A(q)$. The operator $s\pi = T\pi x - \pi T x$ obviously belongs to $\partial(q)$, and therefore $s\pi = \pi s 1_A = \pi(Tx - Tx) = 0$. This means that $T \in \partial^A(p)$. The theorem is completely proved.

Remark. The condition imposed earlier by us on the ring A can be altered; however, it is impossible to completely get rid of this kind of condition if we desire to preserve the A^+-homogeneity of a $\mathbf{Z}^+$-homogeneous A-sublinear operator. Let us here observe that Theorem 5.2 shows that the property of extension necessarily holds in a strengthened form, i.e., a group homomorphism defined on a subgroup and dominated by a module-sublinear operator admits an extension to a module homomorphism.

We will need the following definition in the sequel. A subring A of the ring of orthomorphisms is said to be almost rational if for each $n \in \mathbf{N}$ there exists a decreasing net of multipliers (π_ξ) from A such that $(1/n)y = o\text{-}\lim_\xi \pi_\xi y = \inf_\xi \pi_\xi y$ for each $y \in Y^+$.

Proposition 5.3. The ring A is almost rational if and only if each A-sublinear operator is A^+-homogeneous.

Proof. At first, we assume that A-sublinear operators are A^+-homogeneous. We take $y \in Y^+$ and, using Proposition 3.2, consider the A-sublinear operator $\gamma \to [\pi^{-1}](\gamma^+)y$, where $\gamma \in A$. By assumption, this operator is A^+-homogeneous, i.e., $y = [\pi^{-1}](\pi 1_A)y = \pi[\pi^{-1}](1_A)y$. Since y is arbitrary, it follows that $[\pi^{-1}](1_A) = \pi^{-1}$. As π let us consider the operator $n1_A$. Then, by virtue of the definition of the operator $[\pi^{-1}]$, we get

$$[n1_A]^{-1}(1_A) = \inf\{\delta \in A^+ : n\delta \geqslant 1_A\},$$

whence the ring A is almost rational.

Now, suppose that the ring A is almost rational. Let us consider an A-sublinear operator $p: X \to Y$. First of all, we observe that for each $\pi \in A$ such that $0 \le \pi \le 1_A$ we have $p(\pi x) = \pi p(x)$ for all $x \in X$ even without the supposition that A is almost rational. Indeed, $p(x) = p(\pi x + (1_A - \pi)x) \le \pi p(x) + (1_A - \pi)p(x) = p(x)$. Thus, by virtue of Proposition 4.2, to establish the A^+-homogeneity of p it is sufficient to verify that p is a $\mathbf{Z}^+$-homogeneous operator. To verify the $\mathbf{Z}^+$-homogeneity of p, we take an $n \in \mathbf{N}$ and choose a family of multipliers (π_ξ) such that $\pi_\xi \in A$, and, in addition, $0 \le \pi_\xi \le 1_A$ and $\pi_\xi \downarrow (1/n)1_A$. Let us set $\omega_\xi = (1_A - (n-1)\pi_\xi)^+$. It is clear that $\omega_\xi \in A^+$. Moreover, $1_A - (n-1)\pi_\xi \le 1_A - ((n-1)/n)1_A = (1/n)1_A$. Thus, $\omega_\xi \le (1/n)1_A$ and, in addition, $\omega_\xi \uparrow (1/n)1_A$. We take an element $x \in X$. Then $np(x) - p(nx) \in Y^+$, and therefore

$$0 \leqslant \omega_\xi(np(x) - p(nx)) = n\omega_\xi p(x) - p(n\omega_\xi x) = 0.$$

Taking limit, we verify that p is a $\mathbf{Z}^+$-homogeneous operator.

THEOREM 5.4. An ordered A-module Y admits convex analysis if and only if Y_b is an obliterated K-space and the natural linear representation of A into Y_b is a ring and lattice homomorphism onto an almost rational ring of orthomorphisms.

Proof. The operators $\pi \to \pi^{+}y$ and $z \to z^{+}$, where $\pi \in A$, $y \in Y^{+}$, and $z \in Y_b$, are obviously A-sublinear. Therefore, if the A-module Y admits convex analysis, then these operators are A^{+}-homogeneous by virtue of Proposition 1.1. By virtue of Proposition 3.1, this means that the natural linear representation of A into Y_b is a ring and lattice homomorphism onto a subring and sublattice of Orth(Y_b). By virtue of Proposition 5.3, this subring is almost rational. To complete the proof, it is sufficient to realize the necessary factorizations, as has been done in the proof of Theorem 5.2, and to refer to this theorem and Proposition 5.3.

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CONSTRUCTION OF THE m-JUMP

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The notion of the m-jump has appeared in Yu. L. Ershov's works [1, 2] in connection with the hierarchy of m-degrees, introduced by him, and turns out to be very natural and useful. In the present article we prove theorems, from which it follows, in particular, that the segments of m-degrees and pm-degrees formed by a set and by the m-jump of this set are, in general, not isomorphic (as semilattices) for various sets. Indeed, for sets A and B of natural numbers, let $\bar{A}$ be the complement of A, $A \oplus B \leftrightharpoons \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$, and a segment of the m-degrees be $[a, b] \leftrightharpoons \{u \mid u \text{ is an m-degree and } a \le u \le b\}$, where the m-degrees a and b (the endpoints of the segments) are fixed. Any such segment is a countable distributive upper semilattice with 0 and 1 [2, 3]. Consequently, an arbitrary segment of m-degrees contains a countable number of initial segments (with a greatest element, i.e., these are also countable distributive upper semilattices with 0 and 1). But there exists a continuum of nonisomorphic semilattices of this kind (an example of a suitable family of semilattices is given at the end of this article); on the other hand, each such semilattice is realized in the form of an initial segment in a certain segment of m-degrees that is formed by a certain set and its jump (Theorem 1). Consequently, for each fixed segment of m-degrees formed by a set and its jump there exists a continuum of segments of m-degrees of this type that are not isomorphic to it.

See, e.g., [4, 5] for the main definitions. Let Φ be a one-place universal p. r. function [5]. For an arbitrary set A, its pm-cylindrification A^{pm} is $\Phi^{-1}(A)$ and the m-jump $mjA \leftrightharpoons (A \oplus \bar{A})^{pm}$. We always have $A <_m mjA$ and $A \oplus \bar{A} <_m mjA$. See [1, 2] for these and many other properties of pm and mj.

Each denumerable distributive upper semilattice L with 0 and 1 is the direct limit of a certain sequence

$$D_0 \xrightarrow{\chi_0} D_1 \xrightarrow{\chi_1} \ldots$$

of finite distributive lattices with the embeddings that preserve unions, 0, and 1; the converse is also true [2]. In the sequel, such a sequence for L is fixed for convenience of construction.

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