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SUBDIFFERENTIALS IN BOOLEAN-VALUED MODELS OF SET THEORY

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One of the perpetual and key questions of the theory of extremal problems is the following: "How do the optimal values and solutions behave under a change of variables?" Some possible answers to this question have been gathered in the subdifferential calculus of nonsmooth operators. Convex operators and, especially, the "scalar" convex operators, i.e., convex functions, have been studied in greatest detail. The case of convex functions occupies a perfectly exclusive position in connection with the availability of effective geometrical interpretations of local approximations of these functions at interior points of their effective domains. The following are the main connections here:

(1) The subdifferential is a weakly compact convex set.

(2) The elements of the smallest subdifferential, containing a weakly (order) bounded set $\mathscr{A}$, are obtained by successive applications of the operations of taking the convex hull of $\mathscr{A}$ and of passage to the closure.

(3) The extreme points of the smallest subdifferential, generated by a set $\mathscr{A}$, belong to the weak closure of the initial set $\mathscr{A}$.

The determination of the operator versions of the above statements is a well-known problem of local convex analysis [1-3]. There exist a series of particular solutions for special classes of spaces and operators that appeal either to compactness of subdifferential in a suitable operator topology or to a specific geometrical interpretation of separability in concrete function spaces. At the same time, there was no satisfactory general answer, namely, in connection with the fact that for arbitrary spaces and operators, on one hand there is (as a rule!) no compactness in operator topologies and, on the other hand, the "scalar" interpretations of the separability theorems do not furnish adequate characterizations of subdifferentials.

The aim of the present article is to give a solution of the indicated problem. In particular, in the sequel we give an explicit representation of the elements of a subdifferential and its extreme points of special kind by "the dispersion integrals" of the o-extreme points. The method of investigation is the theory of the Boolean-valued models of set theory [4-12], the main connection of which with the K-spaces has been discovered in [13, 14]. A rough plan uses the indicated theory in the following manner. At first, we should choose a Boolean algebra and a model corresponding to it, in which the considered (= "outer") operator is represented by a scalar convex function in the model (= turns into an "inner" convex function). After this, interpreting the inner geometrical meaning of the subdifferential of the function in the outer terminology, we should obtain the desired answer. A directed realization of this plan is possible, but is connected with certain technical difficulties (as the notion of an o-extreme point is "badly" interpreted). In this connection, we undertake a round-about maneuver — the indicated plan is used only for the analysis of the canonical sublinear operator, namely, the operator of taking the supremum. The general case is deduced with regard for the peculiarities of the structure of its subdifferential as well as the fact that each sublinear operator differs from the canonical one by a linear change of variable.

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The proof of the main results uses the deducibility of the Krein–Mil'man and the Mil'man theorems in ZFC (= the Zermelo–Fraenkel set theory with the axiom of choice), and is nonstandard in this sense. It is emphasized that a large part of the facts, given in the sequel, is justified by usual means. The same is true for all that is treated below, if it does not ask for (as this is expected) other large requirements. It can be qualitatively said that it is shown in this article how the central notions of the subdifferential calculus for operators arise under outer deciphering of the corresponding scalar predessors in a suitable model of set theory.

The present article is oriented, first of all, towards specialists in the domain of convex analysis. In this connection, the treatment of the auxiliary logical means is little more detailed than the exposition of necessary facts from the subdifferential calculus. Here let us observe that the results of the present article were announced in [15] and were reported in a seminar of S. L. Sobolev. We express sincere gratitude to the participants of this seminar for valuable discussions.

1. AUXILIARY INFORMATION ABOUT SUBDIFFERENTIALS

1.0. Here we have collected necessary facts from local convex analysis. See [1-3] for precision and details.

1.1. Let X be a real vector space, Y be a K-space, $P:X \to Y$ be a sublinear operator, and $\mathscr{L}(X, Y)$ be the space of the linear operators that act from X into Y. The *subdifferential* $\partial(P)$ of the operator P is defined by the relation

$$\partial(P) := \{A \in \mathscr{L}(X, Y) : (\forall x \in X)\ Ax \leqslant Px\}.$$

1.2. Let $B := \mathscr{B}(Y)$ be the *base* of the K-space Y, i.e., the complete Boolean algebra of the projections onto the components of Y, or, what is the same thing, the algebra of the idempotent elements in $\mathscr{L}(Y) := \mathscr{L}(Y, Y)$, that are majorized by the identity mapping $\mathbf{1} := I_Y$. We take a *partition* of unity in B, i.e., a family $(b_\xi)_{\xi\in\Xi}$ of projections for which $b_\xi \wedge b_\eta = 0$ for $\xi \neq \eta$ and $\sup_{\xi\in\Xi} b_\xi = \mathbf{1}$. If a family $(A_\xi)_{\xi\in\Xi}$ of elements in $\mathscr{L}(X, Y)$ and an operator A from $\mathscr{L}(X, Y)$ are such that $Ax = \sum_{\xi\in\Xi} b_\xi A_\xi x$ for $x \in X$, then A is called the *intermixing* of $(A_\xi)_{\xi\in\Xi}$ with probabilities $(b_\xi)_{\xi\in\Xi}$. The subdifferentials are *strongly cyclic sets*, i.e., they withstand the formation of all possible intermixings of families of their elements. This fact is expressed in words as follows: "A subdifferential coincides with its strongly cyclic hull." Hence it is obvious that each subdifferential is a *strictly operator-convex set* (= coincides with its strictly operator-convex hull), i.e., withstands the formation of the combinations $\sum_{\xi\in\Xi} \alpha_\xi A_\xi$, where $A_\xi \in \partial(P)$ and $\alpha_\xi \in \mathscr{L}(Y)$ are such that $0 \leqslant \alpha_\xi \leqslant \mathbf{1}$ and $\sum_{\xi\in\Xi} \alpha_\xi = \mathbf{1}$. In particular, a subdifferential is an *operator-convex set*, i.e., if $A, B \in \partial(P)$ and $\alpha, \beta \in \mathscr{L}(Y)$, $\alpha, \beta \geqslant 0$, and $\alpha + \beta = \mathbf{1}$, then $\alpha A + \beta B \in \partial(P)$. Besides this, a subdifferential is closed with respect to pointwise r-convergence. Let us recall that a net $(A_\xi)_{\xi\in\Xi}$ in $\mathscr{L}(X, Y)$ is said to be pointwise r-convergent (o-convergent) to $A \in \mathscr{L}(X, Y)$, if for each $x \in X$ the net $(A_\xi x)_{\xi\in\Xi}$ converges with regulator (o-converges) to Ax in Y. It is clear that r-convergence implies o-convergence.

1.3. For a sublinear operator P, the symbol Ch(P) denotes the *set of the extreme points of* $\partial(P)$. The set Ch(P) is strongly cyclic and $\partial(P)$ is regenerated from Ch(P), i.e., it is the smallest subdifferential containing Ch(P). The last fact can be written in the form $\partial(P) = \mathrm{cop}(\mathrm{Ch}(P))$. The symbol $\mathrm{cop}(\mathscr{A})$ denotes the operation of passing to the *supporting hull* of $\mathscr{A}$, i.e., to the smallest (with respect to inclusion) subdifferential containing $\mathscr{A}$.

1.4. Let $T \in \mathscr{L}_+(Y, Z)$, where Z is another K-space, and let $A \in \partial(P)$, where $P:X \to Y$. The operator A is said to be a *T-extreme* operator if $TA \in \mathrm{Ch}(TP)$. The set of all the *o-extreme points* of P, i.e., the set of the points that are T-extreme points for each o-continuous operator T, is denoted by $\mathscr{E}_0(P)$. The equality $\partial(P) = \mathrm{cop}(\mathscr{E}_0(P))$ holds. At the same time, $\mathscr{E}_0(P)$ is not necessarily even a cyclic set, i.e., may not withstand intermixing of finite families of its elements.

1.5. Let $\mathscr{A}$ be a nonempty set and $l_\infty(\mathscr{A}, Y)$ be the space constituted from order-bounded Y-valued functions on $\mathscr{A}$. This space is equipped with the natural structure of a K-space (and even of a module over the ring of orthomorphisms of Y). The symbol $\varepsilon_{\mathscr{A}}$ denotes the *canonical sublinear operator*

$$\varepsilon_{\mathscr{A}}(f) := \sup f(\mathscr{A}) \quad (f \in l_\infty(\mathscr{A}, Y)).$$

If $\mathscr{A}\subset\mathscr{L}(X, Y)$, then we set $\langle\mathscr{A}\rangle x : A\to Ax$. By the same token, an operator $\langle\mathscr{A}\rangle\in\mathscr{L}(X, Y^{\mathscr{A}})$ is defined. The set $\mathscr{A}$ is said to be *weakly order-bounded* if $\langle\mathscr{A}\rangle$ acts in $l_\infty(\mathscr{A}, Y)$. Let us observe that the following equality holds for such an $\mathscr{A}$:

$$\operatorname{cop}(\mathscr{A})=\partial(\varepsilon_{\mathscr{A}})\circ\langle\mathscr{A}\rangle;$$
$$\partial(\varepsilon_{\mathscr{A}})=\{\alpha\in\mathscr{L}_+(l_\infty(\mathscr{A}, Y), Y):\alpha\circ\Delta_{\mathscr{A}}=1\}.$$

Here $\Delta_{\mathscr{A}}$ is the diagonal embedding: $\Delta_{\mathscr{A}}y:=(y)_{A\in\mathscr{A}}$ for $y\in Y$; it transforms Y into a set of constants in the space $l_\infty(\mathscr{A}, Y)$.

It is important that each sublinear operator P differs from the canonical operator by only a linear change of variable. More precisely,

$$P=\varepsilon_{\partial(P)}\circ\langle\partial(P)\rangle;\quad \partial(P)=\partial(\varepsilon_{\mathscr{E}(P)})\circ\langle\mathscr{E}_0(P)\rangle$$

(= the Krein–Mil'man theorem for operators).

1.6. *The lattice homomorphisms that preserve constants [i.e., lie in $\partial(\varepsilon_{\mathscr{A}})$] are the extreme points of $\partial(\varepsilon_{\mathscr{A}})$. Moreover, an operator $S\in\partial(\varepsilon_{\mathscr{A}})$ is T-extreme if and only if*

$$T|Sf|=TS|f|\quad(f\in l_\infty(\mathscr{A}, Y)).$$

◁ Indeed, the following equation serves as a criterion for T-extremality of S (see [6]):

$$Tg_+=\inf_{u\in l_\infty(\mathscr{A},Y)}T\left((\varepsilon_{\mathscr{A}}u-Su)\vee(\varepsilon_{\mathscr{A}}(u-f)-S(u-f)+g)\right)$$

for arbitrary $f, g\in l_\infty(\mathscr{A}, Y)$. In other words, T-extremality is equivalent to the following equations:

$$0=\inf_u T\left((\varepsilon_{\mathscr{A}}(u+f)-S(u+f))\vee(\varepsilon_{\mathscr{A}}(u-f)-S(u-f))\right)=$$
$$=\inf_u T\left(\varepsilon_{\mathscr{A}}((u+f-\Delta_{\mathscr{A}}Su-\Delta_{\mathscr{A}}Sf)\vee(u-f-\Delta_{\mathscr{A}}Su+\Delta_{\mathscr{A}}Sf)\right)=\inf_u T\left(\varepsilon_{\mathscr{A}}(u-\Delta_{\mathscr{A}}Su)+|f-\Delta_{\mathscr{A}}Sf|\right).$$

By virtue of the theorem on vector minimax, we can write

$$(\text{S is a T-extreme operator})\leftrightarrow(\forall R\in\partial(T\varepsilon_{\mathscr{A}}))\,0\geqslant\inf_u R\left(u-\Delta_{\mathscr{A}}Su+|f-\Delta_{\mathscr{A}}Sf|\right)=$$
$$=R|f-\Delta_{\mathscr{A}}Sf|+\inf_u(Ru-R\Delta_{\mathscr{A}}Su)=R|f-\Delta_{\mathscr{A}}Sf|+\inf_u(Ru-TSu)\leftrightarrow$$
$$\leftrightarrow(R=TS\to R|f-\Delta_{\mathscr{A}}Sf|=0)\leftrightarrow TS|f-\Delta_{\mathscr{A}}Sf|=0.$$

It remains to observe that

$$0=TS|f-\Delta_{\mathscr{A}}Sf|\geqslant T(S(|f|-\Delta_{\mathscr{A}}|Sf|))=TS|f|-T|Sf|\geqslant 0\to$$
$$\to TS|f|=T|Sf|\to TS|f-\Delta_{\mathscr{A}}Sf|=T|Sf-S\Delta_{\mathscr{A}}Sf|=0.\quad\triangle$$

1.7. *The following equality holds for the canonical operator $\varepsilon_{\mathscr{A}}$:*

$$\mathscr{E}_0(\varepsilon_{\mathscr{A}})=\operatorname{Ch}(\varepsilon_{\mathscr{A}}).$$

In addition,

$$P=\varepsilon_{\mathscr{A}}\circ\langle\mathscr{A}\rangle\to\operatorname{Ch}(P)\subset\mathscr{E}_0(\varepsilon_{\mathscr{A}})\circ\langle\mathscr{A}\rangle$$

for each sublinear operator P (= the Mil'man theorem for operators).

Let us observe here that the δ-*functions* $\varepsilon_A : f\to f(A)$ for $f\in l_\infty(\mathscr{A}, Y)$ and $A\in\mathscr{A}$ lie clearly in $\operatorname{Ch}(\varepsilon_{\mathscr{A}})$. The intermixings of the family $(\varepsilon_A)_{A\in\mathscr{A}}$ are called *pure states* on $\mathscr{A}$. It is obvious that the pure states are the o-extreme points of the canonical operator.

2. AUXILIARY INFORMATION ABOUT BOOLEAN-VALUED MODELS

2.0. Here we have gathered necessary facts about the structure of the Boolean-valued models and the laws of working with them. See [4, 5, 10-12] for details.

2.1. Let B be a complete Boolean algebra. For each ordinal α we set

$$V_\alpha^{(B)} := \{x : (\exists \beta \in \alpha)\, x : \mathrm{dom}(x) \to B \wedge \mathrm{dom}(x) \subset V_\beta^{(B)}\}.$$

After this recursive definition, we introduce the *Boolean-valued universe* $V^{(B)}$ or, as it is also called, the class of B-sets by the relation

$$V^{(B)} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(B)},$$

where On is the class of all ordinals.

2.2. Let φ be an arbitrary formula of the theory ZFC. Interpreting the connectors and the quantifiers in the Boolean algebra B in the natural manner, we introduce the valuation $[\varphi]$ of the formula φ, considering the character of its construction from the atomic formulas $x \in y$ and x = y and defining the valuations of the latter formulas for $x, y \in V^{(B)}$ by the recursion scheme:

$$[x \in y] := \bigvee_{z \in \mathrm{dom}(y)} y(z) \wedge [z = x];$$

$$[x = y] := \bigwedge_{z \in \mathrm{dom}(x)} x(z) \Rightarrow [z \in y] \wedge \bigwedge_{z \in \mathrm{dom}(y)} y(z) \Rightarrow [z \in x].$$

2.3. The universe $V^{(B)}$ with the indicated law of valuation of formulas is a model of set theory in the sense that $[\varphi] = 1$ or, as it is said, φ *is true inside* $V^{(B)}$ for each theorem φ of the theory ZFC. The last fact is called the *transfer principle*.

Let us observe here the following general convention. If x is an element of $V^{(B)}$ and $\varphi(\cdot)$ is a formula of ZFC, then the phrase "x satisfies φ inside $V^{(B)}$" or, in short, "$\varphi(x)$ inside $V^{(B)}$" means that $[\varphi(x)] = 1$.

2.4. For each element $x \in V^{(B)}$ and arbitrary $b \in B$ we define a function

$$bx : z \to bx(z) \quad (z \in \mathrm{dom}(x)).$$

(This expression means that $b\emptyset := \emptyset$ for $b \in B$.)

The following statements are valid for B-valued sets x and y and each element $b \in B$ *:*

$$[x \in by] = b[x \in y];$$
$$[bx = by] = b \Rightarrow [x = y];$$
$$[x = bx] = [b'x = \varnothing] = b' \Rightarrow [x = \varnothing].$$

(Here b' is the complement of b.)

2.5. There exists a natural equivalence relation in the class $V^{(B)}$: $x \sim y := [x = y] = 1$. Selecting one representation (of smallest rank) in each equivalence class (this is the so-called Frege—Russel—Scott device), we arrive at the *separable universe* $\overline{V}^{(B)}$, in which

$$x = y \leftrightarrow [x = y] = 1.$$

It is easily seen that the valuations of the formulas do not change under passage to equivalent elements. In this connection we will effect the identification $V^{(B)} := \overline{V}^{(B)}$ in the sequel without mention. We emphasize that the element bx is correctly defined in $V^{(B)}$ for $x \in V^{(B)}$ and $b \in B$, since $[x_1 = x_2] = 1 \to [bx_1 = bx_2] = b \Rightarrow [x_1 = x_2] = 1$ by virtue of 2.4. In this connection, we often use the expression $\mathbf{0} = \emptyset$, keeping in mind, in particular, that $\mathbf{0}\emptyset = \emptyset = 0x$ for $x \in V^{(B)}$.

2.6. The following *intermixing principle* is valid in $V^{(B)}$.

Let $(b_\xi)_{\xi \in \Xi}$ be a partition of unity in B. *For each family* $(x_\xi)_{\xi \in \Xi}$ *of elements of the universe* $V^{(B)}$ *there exists a unique intermixing* $(x_\xi)_{\xi \in \Xi}$ *with probabilities* $(b_\xi)_{\xi \in \Xi}$, *i.e., an element x of the separable universe (denoted by* $\sum_{\xi \in \Xi} b_\xi x_\xi$ *such that*

$$[x = x_\xi] \geqslant b_\xi \quad (\xi \in \Xi).$$

In addition,

$$x = \sum_{\xi \in \Xi} b_\xi x_\xi \leftrightarrow (\forall \xi \in \Xi)\, b_\xi x = b_\xi x_\xi.$$

In particular, bx is the intermixing of x and **0** with probabilities b and b', respectively.

It is useful to emphasize that the use of the term "intermixing" in different situations (cf. 1.2) not only does not lead to unpleasant conflicts (as would appear at a first glance), but, indeed, as becomes obvious from the sequel, is justified (since we are concerned with unique, to within inessential stipulations, objects). Similar situation is not unique — the **remark made** by us, e.g., can be applied to the term "cyclic," etc.

2.7. The following *maximum principle* is valid in $V^{(B)}$.

For each formula φ *of the theory ZFC there exists a B-valued set* x *such that*

$$[\exists x \varphi(x)] = [\varphi(x)].$$

2.8. For each element x of *the Von Neumann universe* V **defined** by the recursion scheme

$$V_\alpha := \{x : (\exists \beta \in \alpha) \quad x \in \mathscr{P}(V_\beta)\};$$
$$V := \bigcup_{\alpha \in \mathrm{On}} V_\alpha,$$

i.e., for each set x, we set

$$\varnothing^\wedge := \varnothing; \quad \mathrm{dom}\,(x^\wedge) := \{y^\wedge : y \in x\}, \quad \mathrm{im}\,(x^\wedge) := \{1\}.$$

(it is more **precise to speak** about the selected representative of the equivalence class of $x^\wedge$ in $V^{(B)}$). The element $x^\wedge$ of $V^{(B)}$ is called the *standard name of* x. By the same token, a *canonical embedding* of V into $V^{(B)}$ arises. In this connection, the following equality is fulfilled for $x, x_1, \ldots, x_n \in V$ and $y \in V^{(B)}$:

$$[y \in x^\wedge] = \bigvee_{z \in x} [y = z^\wedge];$$
$$\varphi(x_1, \ldots, x_n) \leftrightarrow [\varphi(x_1^\wedge, \ldots, x_n^\wedge)] = 1$$

for each bounded formula φ of the theory ZFC. (A formula is said to be *bounded* if connected variables occur in it under the signs of bounded quantifiers, i.e., quantifiers extended to some sets.)

3. ELEMENTARY LOWERINGS AND LIFTINGS

3.0. Here we gether the basic facts about the representation of elementary objects in Boolean-valued models (see [10-12]).

3.1. Let φ be a formula of **ZFC** and fix a set y of elements of the Boolean-valued universe. Further, let $A_\varphi := A_{\varphi(\cdot,\, y)} := \{x : \varphi(x,\, y)\}$ be the class of the sets that can be defined by means of y. The lowering $A_\varphi{\downarrow}$ of the class A_φ is defined by the relation

$$A_\varphi{\downarrow} := \{t : t \in V^{(B)} \wedge [\varphi(t,\, y)] = 1\}.$$

If $t \in A_\varphi{\downarrow}$, then t is said to *satisfy* $\varphi(\cdot, y)$ *inside* $V^{(B)}$.

The lowering of each class is *strongly cyclic*, i.e., withstands all possible intermixings of its elements. In this connection, two nonempty classes inside $V^{(B)}$ coincide if and only if they have the same elements inside $V^{(B)}$.

3.2. *For each element* x *of* $V^{(B)}$, *its lowering* $x{\downarrow}$ *is given by the rule*

$$x{\downarrow} := \{t : t \in V^{(B)} \wedge [t \in x] = 1\},$$

i.e., $x{\downarrow} = A_{\cdot \in x}{\downarrow}$. The class $x{\downarrow}$ is a set. In addition, $x{\downarrow} \subset \mathrm{scyc}\,(\mathrm{dom}\,(x))$, where scyc is the symbol for passage to the *strongly cyclic hull*. It is useful to emphasize that the following relation holds for each nonempty (inside $V^{(B)}$) set x:

$$(\exists z \in x{\downarrow})[(\exists z \in x)\varphi(z)] = [\varphi(z)].$$

3.3. *Let* F *be a correspondence from* X *into* Y *inside* $V^{(B)}$. *Then there exists a unique correspondence* $F{\downarrow}$ *from* $X{\downarrow}$ *into* $Y{\downarrow}$ *such that*

$$F{\downarrow}(A{\downarrow}) = F(A){\downarrow}$$

for each (nonempty) subset A *of* X *inside* $V^{(B)}$.

It is easily seen that $F{\downarrow}$ is defined by the rule

$$\langle x, y\rangle \in F{\downarrow} \leftrightarrow [\langle x, y\rangle \in F] = 1.$$

3.4. Let $x \in \mathscr{P}(V^{(B)})$, i.e., x is a set formed from B-valued sets. Let us set $\varnothing{\uparrow} := \varnothing$ and

$$\mathrm{dom}\,(x{\uparrow}) := x,\ \mathrm{im}\,(x{\uparrow}) := \{1\}$$

for $x \neq \varnothing$. The element $x{\uparrow}$ (of the separable universe $V^{(B)}$, i.e., the selected representative of the corresponding class) is called the *lifting* of x.

It is clear that $x{\uparrow}{\downarrow} = \mathrm{scyc}(x)$ and $x{\downarrow}{\uparrow} = x$ for each nonempty (inside $V^{(B)}$) set x. Let us observe here that if $x \in V$ and $\hat{x}$ is its *standard domain*, i.e.,

$$\hat{x} := \{z^{\wedge} : z \in V \wedge z \in x\},$$

then $\hat{x}{\uparrow} = x^{\wedge}$.

3.5. *Let $X, Y \in \mathscr{P}(V^{(B)})$ and F be a correspondence from X into Y. Then there exists a unique correspondence $F{\uparrow}$ from $X{\uparrow}$ into $Y{\uparrow}$ inside $V^{(B)}$ such that*

$$F{\uparrow}(A{\uparrow}) = F(A){\uparrow}$$

for each subset A of X if and only if F is extensional. The latter property means that

$$y_1 \in F(x_1) \to [x_1 = x_2] \leqslant \bigvee_{y_2 \in F(x_2)} [y_1 = y_2].$$

It is easily seen that $F{\uparrow}$ is the lifting of the set

$$\mathrm{dom}\,(F{\uparrow}) := \{\langle x, y\rangle^{B} : \langle x, y\rangle \in F\},$$

where $\langle x, y\rangle^{B}$ is the unique element of $V^{(B)}$ that corresponds by the maximum principle to the formula

$$\exists z \forall u\, (u \in z \leftrightarrow u = \{x\} \vee u = \{x, y\}).$$

It is clear that this can be easily shown by direct construction of this element.

We give here a criterion for intermixing of functions inside $V^{(B)}$, which will be needed in the sequel.

Let Ξ be a set and $(f_\xi)_{\xi\in\Xi}$ be a family of elements of $V^{(B)}$ that are functions from a nonempty set X into Y inside $V^{(B)}$ and suppose that $(b_\xi)_{\xi\in\Xi}$ is a partition of unity. Then the intermixing $f := \sum_{\xi\in\Xi} b_\xi f_\xi$ is a function from X into Y inside $V^{(B)}$ such that

$$\left[(\forall x \in X)\, f(x) = \sum_{\xi\in\Xi} b_\xi f_\xi(x)\right] = 1.$$

◁ For $x \in X{\downarrow}$ we set $g(x) := \sum_{\xi\in\Xi} b_\xi f_\xi(x)$. It is clear that $g(x) \in Y{\downarrow}$ and, moreover, $[g(x) = f_\xi(x)] \geqslant b_\xi$ for all $\xi \in \Xi$. Let us establish that the mapping $g: X{\downarrow} \to Y{\downarrow}$ is extensional, considering that $f_\xi{\downarrow}$ is extensional. Indeed, for $x_1, x_2 \in X{\downarrow}$ we have

$$[x_1 = x_2] = \Big(\bigvee_{\xi\in\Xi} b_\xi\Big) \wedge [x_1 = x_2] = \bigvee_{\xi\in\Xi} (b_\xi \wedge [x_1 = x_2]) \leqslant$$
$$\leqslant \bigvee_{\xi\in\Xi} [f_\xi(x_1) = g(x_1)] \wedge [x_1 = x_2] \wedge [f_\xi(x_2) = g(x_2)] \leqslant$$

$$\leqslant \bigvee_{\xi\in\Xi} [f_\xi(x_1) = g(x_1)] \wedge [f_\xi(x_1) = f_\xi(x_2)] \wedge [f_\xi(x_2) = g(x_2)] \leqslant \bigvee_{\xi\in\Xi} [g(x_1) = g(x_2)] = [g(x_1) = g(x_2)].$$

Thus, the lifting $g{\uparrow}$ of the mapping g exists. Let us establish that $g{\uparrow} = f$. To this end, let us observe that by virtue of the transfer principle we have

$$[g{\uparrow} = f_\xi] = [(\forall x \in X)\, g{\uparrow}(x) = f_\xi(x)] = \bigwedge_{x\in X{\downarrow}} [g{\uparrow}(x) = f_\xi(x)] = \bigwedge_{x\in X{\downarrow}} [g(x) = f_\xi(x)] \geqslant b_\xi$$

for $\xi \in \Xi$. It remains to refer to the uniqueness of the intermixing. ▷

3.6. Let x be a set and $f: x \to Y{\downarrow}$, where $Y \in V^{(B)}$. Since $Y = Y{\downarrow}{\uparrow}$, we can consider f to be a mapping of $\hat{x}$ into $\mathrm{dom}\,(Y)$. It is clear that f is extensional and, therefore, it makes sense to speak about the element $f{\uparrow}$ of $V^{(B)}$. Let us observe that $[f{\uparrow}: x^{\wedge} \to Y] = 1$, by virtue of what we have remarked earlier, and, in addition, for each $g \in V^{(B)}$ such that $[g{:}x^\wedge \to Y] = \mathbf{1}$, there exists a unique mapping $f{:}x \to Y{\downarrow}$ such that $g = f{\uparrow}$. Of course, the lowering $g{\downarrow}$ of the mapping g (transferred from $\hat{x}$ to x) is such a mapping.

4. LOWERINGS AND LIFTINGS CONNECTED WITH THE SUBDIFFERENTIAL OF THE CANONICAL OPERATOR

4.0. In this section we give necessary facts about the transfer of the **notions** connected with the representation of the canonical operator in a suitable Boolean-valued model.

4.1. By virtue of the maximum principle, there exists an object $\mathscr{R}$ in $V^{(B)}$ such that

$$[\mathscr{R} \cdot \text{ is the K-space of real numbers } \iota] = \mathbf{1}.$$

Here it is implied that $\mathscr{R}$ is the supporting set of the space of real numbers inside $V^{(B)}$. Let us observe here that $\mathbf{R}^\wedge$ (= the standard name of the field R of real numbers), being an Archimedeally ordered field inside $V^{(B)}$, is a dense subfield of $\mathscr{R}$ inside $V^{(B)}$ (up to isomorphism).

Let us realize a lowering of structures from $\mathscr{R}$ into $\mathscr{R}{\downarrow}$ by the following general rules (cf. 3.3):

$$x + y = z \leftrightarrow [x + y = z] = \mathbf{1};$$
$$xy = z \leftrightarrow [xy = z] = \mathbf{1};$$
$$x \leqslant y \leftrightarrow [x \leqslant y] = \mathbf{1};$$
$$\lambda x = y \leftrightarrow [\lambda^\wedge x = y] = \mathbf{1} \quad (x, y, z \in \mathscr{R}{\downarrow},\ \lambda \in \mathbf{R}).$$

Gordon Theorem [13]. The set $\mathscr{R}{\downarrow}$ with the lowered structures is an extended K-space with the base $\mathscr{B}(\mathscr{R}{\downarrow})$, isomorphic to B. Such an isomorphism is realized by identifying B with the lowering of the field $\{0^\wedge, 1^\wedge\}$, i.e., by the mapping $\iota: B \to \mathscr{B}(\mathscr{R}{\downarrow})$ defined by the rule

$$[\iota(b) = 1^\wedge] = b; \quad [\iota(b) = 0^\wedge] = b' \quad (0; 1 \in \mathbf{R}).$$

In addition, for all $x, y \in \mathscr{R}$

$$[\iota(b)x = \iota(b)y] = b \Rightarrow [x = y];$$
$$b\iota(b)x = bx, \quad b'\iota(b)x = 0.$$

In particular, the following equivalences are valid:

$$\iota(b)x = \iota(b)y \leftrightarrow [x = y] \geqslant b;$$
$$\iota(b)x \geqslant \iota(b)y \leftrightarrow [x \geqslant y] \geqslant b.$$

4.2. Let $\mathscr{A}$ be a nonempty set. By virtue of the maximum principle, there exists an object $l_\infty(\mathscr{A}^\wedge, \mathscr{R})$ such that

$[l_\infty(\mathscr{A}^\wedge, \mathscr{R})$ is the K-space of the bounded functions with domain $\mathscr{A}^\wedge$ and the range in $\mathscr{R}] = \mathbf{1}$.

Let us consider the **lowering**

$$l_\infty(\mathscr{A}^\wedge, \mathscr{R}){\downarrow} := \{t \in V^{(B)}: [t \in l_\infty(\mathscr{A}^\wedge, \mathscr{R})] = \mathbf{1}\}.$$

Let us realize the lowering of the algebraic operations and the order relation from $l_\infty(\mathscr{A}^\wedge, \mathscr{R})$ into $l_\infty(\mathscr{A}^\wedge, \mathscr{R}){\downarrow}$. It is obvious that, by the same token, $l_\infty(\mathscr{A}^\wedge, \mathscr{R}){\downarrow}$ turns into a K-space and, all the more, into an (extended) module over $\mathscr{R}{\downarrow}$ (see [8, 9]).

The mapping "lifting," associating with a bounded $\mathscr{R}{\downarrow}$-valued function on $\mathscr{A}$ its lifting (a bounded $\mathscr{R}$-valued function on $\mathscr{A}^\wedge$ inside $V^{(B)}$), ***realizes an*** *algebraic and order isomorphism of $l_\infty(\mathscr{A}, \mathscr{R}{\downarrow})$ and $l_\infty(\mathscr{A}^\wedge, \mathscr{R}){\downarrow}$.*

◁ This statement is almost obvious if we glance at the canvas of construction. For completeness, we explain certain steps.

Thus, let $f \in l_\infty(\mathscr{A}, \mathscr{R}{\downarrow})$. Then, as remarked in 3.6, $[f{\uparrow}: \mathscr{A}^\wedge \to \mathscr{R}] = \mathbf{1}$. In addition, $[f(A) = f{\uparrow}(A^\wedge)] = \mathbf{1}$ for $A \in \mathscr{A}$. It is clear from the definition of order in $\mathscr{R}{\downarrow}$ that $f{\uparrow}(\mathscr{A}^\wedge)$ is bounded inside $V^{(B)}$ and, therefore, $f{\uparrow} \in l_\infty(\mathscr{A}^\wedge, \mathscr{R}){\downarrow}$. We are interested in the operator

Up: $f \to f\uparrow$ from $l_\infty(\mathscr{A}, \mathscr{R}\downarrow)$ into $l_\infty(\mathscr{A}^\wedge, \mathscr{R})\downarrow$. Let $g \in l_\infty(\mathscr{A}, \mathscr{R})\downarrow$. Then $[g: \mathscr{A}^\wedge \to \mathscr{R} \wedge (\exists t \in \mathscr{R})\, t \geqslant |g(\mathscr{A}^\wedge)|] = 1$. It is clear that $g = \mathrm{Up}(g\downarrow)$, i.e., Up is an epimorphism. The other assertions about Up are verified in equally simple manner. ▷

The meaning of the above statement, in particular, is that $l_\infty(\mathscr{A}^\wedge, \mathscr{R})\downarrow$ can be considered, on one hand, as yet another realization of the space $l_\infty(\mathscr{A}, \mathscr{R}\downarrow)$ and, on the other hand, as $\mathrm{dom}(l_\infty(\mathscr{A}^\wedge, \mathscr{R}))$.

4.3. Let us consider an object $l_\infty(\mathscr{A}^\wedge, \mathscr{R})^\#$ in $V^{(B)}$ such that

$[l_\infty(\mathscr{A}^\wedge, \mathscr{R})^\#$ is the adjoint space of $l_\infty(\mathscr{A}^\wedge, \mathscr{R})$] = 1.

The lowering $l_\infty(\mathscr{A}^\wedge, \mathscr{R})^\#\downarrow$ is provided by the lowered structures. In particular, it is clear that $l_\infty(\mathscr{A}^\wedge, \mathscr{R})^\#\downarrow$ is an $\mathscr{R}\downarrow$-module.

Let $\mu \in l_\infty(\mathscr{A}^\wedge, \mathscr{R})^\#\downarrow$, i.e.,

$$[\mu \text{ is an } \mathscr{R} \text{ homomorphism of } l_\infty(\mathscr{A}^\wedge, \mathscr{R}) \text{ into } \mathscr{R}] = 1.$$

Further, let $\mu\downarrow: l_\infty(\mathscr{A}^\wedge, \mathscr{R})\downarrow \to \mathscr{R}\downarrow$ be the lowering of μ. For $f \in l_\infty(\mathscr{A}, \mathscr{R}\downarrow)$, let us set

$$\mu_\downarrow(f) := \mu\downarrow(f\uparrow).$$

The mapping "lowering" $\mu \to \mu_\downarrow$ realizes an isomorphism of the $\mathscr{R}\downarrow$-modules of $l_\infty(\mathscr{A}^\wedge, \mathscr{R})^\#\downarrow$ and the space of $\mathscr{R}\downarrow$-homomorphisms $\mathrm{Hom}_{\mathscr{R}\downarrow}(l_\infty(\mathscr{A}, \mathscr{R}\downarrow), \mathscr{R}\downarrow)$.

◁ The only statement that is not completely obvious is that each $\mathscr{R}\downarrow$-module homomorphism $T: l_\infty(\mathscr{A}, \mathscr{R}\downarrow) \to \mathscr{R}\downarrow$ (and, indeed, each $\mathscr{R}\downarrow_+$-homogeneous mapping) represents the lowering of a suitable mapping inside $V^{(B)}$. For the verification of this statement, we set

$$t(f) := T(f\downarrow) \quad (f \in l_\infty(\mathscr{A}^\wedge, \mathscr{R})\downarrow).$$

It should be verified that t is an extensional mapping since it is clear that t is an $\mathscr{R}\downarrow$-homomorphism of $l_\infty(\mathscr{A}^\wedge, \mathscr{R})\downarrow$ into $\mathscr{R}\downarrow$.

We carry out the proof of the extensionality of t (without appealing to its additivity). First of all, for an element $b \in B$ and the element $\iota(b)$ of $V^{(B)}$, representing the intermixing of $1^\wedge$ and $0^\wedge$ with probabilities b and b', respectively (see 4.1), we have $\iota(b) \in \mathscr{R}\downarrow$. In addition, for the functions f and g from $\mathscr{A}^\wedge$ into $\mathscr{R}$ inside $V^{(B)}$, we successively deduce that

$$[f = g] \geqslant b \leftrightarrow [(\forall A \in \mathscr{A}^\wedge)\, f(A) = g(A)] \geqslant b \leftrightarrow \bigwedge_{A \in \mathscr{A}} [f(A^\wedge) = g(A^\wedge)] \geqslant b \leftrightarrow \bigwedge_{A \in \mathscr{A}} [f\downarrow(A) = g\downarrow(A)] \geqslant b \leftrightarrow$$

$$\leftrightarrow (\forall A \in \mathscr{A})\, \iota(b) f\downarrow(A) = \iota(b) g\downarrow(A) \leftrightarrow \iota(b) f\downarrow = \iota(b) g\downarrow.$$

Hence, with regard for the positive homogeneity of T, for $f, g \in l_\infty(\mathscr{A}^\wedge, \mathscr{R})\downarrow$ we get

$$[f = g] \geqslant b \leftrightarrow \iota(b) f\downarrow = \iota(b) g\downarrow \to T(\iota(b) f\downarrow) = T(\iota(b) g\downarrow) \to \iota(b) T(f\downarrow) = \iota(b) T(g\downarrow) \leftrightarrow [T(f\downarrow) = T(g\downarrow)] \geqslant b$$

by virtue of the Gordon theorem. ▷

Let us denote the inverse mapping of the lowering $\mu \to \mu_\downarrow$ as $t \to t^\uparrow$, where $t \in \mathrm{Hom}_{\mathscr{R}\downarrow}(l_\infty(\mathscr{A}, \mathscr{R}\downarrow), \mathscr{R}\downarrow)$. Therefore, in detailed expression,

$$t^\uparrow(f) := t(f\downarrow) \quad (f \in l_\infty(\mathscr{A}^\wedge, \mathscr{R})\downarrow).$$

4.4. Let $\varepsilon_{\mathscr{A}^\wedge}$ be the canonical operator on $\mathscr{A}^\wedge$ inside $V^{(B)}$, i.e., the object of $V^{(B)}$ such that

$$[\varepsilon_{\mathscr{A}^\wedge}: l_\infty(\mathscr{A}^\wedge, \mathscr{R}) \to \mathscr{R} \wedge (\forall f \in l_\infty(\mathscr{A}^\wedge, \mathscr{R}))\, \varepsilon_{\mathscr{A}^\wedge}(f) = \sup f(\mathscr{A}^\wedge)] = 1.$$

An obvious computation shows that

$$[\varepsilon_{\mathscr{A}^\wedge}(f\uparrow) = \varepsilon_{\mathscr{A}}(f)] = 1$$

for each element $f \in l_\infty(\mathscr{A}, \mathscr{R}\downarrow)$.

Let $\partial\left(\varepsilon_{\mathscr{A}^\wedge}\right)$ *be the subdifferential of the functional* $\varepsilon_{\mathscr{A}^\wedge}$ *inside* $V^{(B)}$ *and* $\mathrm{Ch}\left(\varepsilon_{\mathscr{A}^\wedge}\right)$ *be the set of the extreme points of* $\partial\left(\varepsilon_{\mathscr{A}^\wedge}\right)$ *inside* $V^{(B)}$. *Then for each* $t \in \mathrm{Hom}_{\mathscr{R}\downarrow}(l_\infty(\mathscr{A}, \mathscr{R}\downarrow), \mathscr{R}\downarrow)$ *and each* $\mu \in l_\infty(\mathscr{A}^\wedge, \mathscr{R})^{\#}\downarrow$

$$t^\uparrow \in \partial\left(\varepsilon_{\mathscr{A}^\wedge}\right)\downarrow \leftrightarrow t \in \partial\left(\varepsilon_{\mathscr{A}}\right);$$
$$t^\uparrow \in \mathrm{Ch}\left(\varepsilon_{\mathscr{A}^\wedge}\right)\downarrow \leftrightarrow t \in \mathrm{Ch}\left(\varepsilon_{\mathscr{A}}\right);$$
$$\mu_\downarrow \in \partial\left(\varepsilon_{\mathscr{A}}\right) \leftrightarrow \mu \in \partial\left(\varepsilon_{\mathscr{A}^\wedge}\right)\downarrow;$$
$$\mu_\downarrow \in \mathrm{Ch}\left(\varepsilon_{\mathscr{A}}\right) \leftrightarrow \mu \in \mathrm{Ch}\left(\varepsilon_{\mathscr{A}^\wedge}\right)\downarrow.$$

◁ Using successively what we have established earlier, we deduce that

$$t^\uparrow \in \partial\left(\varepsilon_{\mathscr{A}^\wedge}\right)\downarrow \leftrightarrow \left[t^\uparrow \in \partial\left(\varepsilon_{\mathscr{A}^\wedge}\right)\right] = 1 \leftrightarrow \left[(\forall f \in l_\infty(\mathscr{A}^\wedge, \mathscr{R}))\, t^\uparrow(f) \leqslant \varepsilon_{\mathscr{A}^\wedge}(f)\right] = 1 \leftrightarrow$$

$$\leftrightarrow \bigwedge_{f \in l_\infty(\mathscr{A}^\wedge, \mathscr{R})\downarrow} \left[t^\uparrow(f) \leqslant \varepsilon_{\mathscr{A}^\wedge}(f)\right] = 1 \leftrightarrow \bigwedge_{f \in l_\infty(\mathscr{A}^\wedge, \mathscr{R})\downarrow} \left[t(f\downarrow) \leqslant \varepsilon_{\mathscr{A}^\wedge}(f\downarrow\uparrow)\right] = 1 \leftrightarrow$$

$$\leftrightarrow \bigwedge_{f \in l_\infty(\mathscr{A}^\wedge, \mathscr{R})\downarrow} \left[t(f\downarrow) \leqslant \varepsilon_{\mathscr{A}}(f\downarrow)\right] = 1 \leftrightarrow \bigwedge_{g \in l_\infty(\mathscr{A}, \mathscr{R}\downarrow)} \left[t(g) \leqslant \varepsilon_{\mathscr{A}}(g)\right] = 1 \leftrightarrow$$

$$\leftrightarrow (\forall g \in l_\infty(\mathscr{A}, \mathscr{R}\downarrow))\left[t(g) \leqslant \varepsilon_{\mathscr{A}}(g)\right] = 1 \leftrightarrow (\forall g \in l_\infty(\mathscr{A}, \mathscr{R}\downarrow)) \quad t(g) \leqslant \varepsilon_{\mathscr{A}}(g) \leftrightarrow t \in \partial\left(\varepsilon_{\mathscr{A}}\right).$$

In the proof of the second equivalence, we have found it convenient to use the fact that the extreme points of the subdifferential of the canonical operator are lattice homomorphisms that lie in this subdifferential (see 1.6). Hence we have

$$t^\uparrow \in \mathrm{Ch}\left(\varepsilon_{\mathscr{A}^\wedge}\right) \leftrightarrow \left[t^\uparrow \in \mathrm{Ch}\left(\varepsilon_{\mathscr{A}^\wedge}\right)\right] = 1 \leftrightarrow$$
$$\leftrightarrow \left[t^\uparrow \in \partial\left(\varepsilon_{\mathscr{A}^\wedge}\right)\right] \wedge \left[(\forall f \in l_\infty(\mathscr{A}^\wedge, \mathscr{R}))\, t^\uparrow(|f|) = |t^\uparrow(f)|\right] = 1 \leftrightarrow$$
$$\leftrightarrow t \in \partial\left(\varepsilon_{\mathscr{A}}\right) \wedge \bigwedge_{f \in l_\infty(\mathscr{A}^\wedge, \mathscr{R})\downarrow} \left[t^\uparrow(|f|) = |t^\uparrow(t)|\right] = 1 \leftrightarrow$$
$$\leftrightarrow t \in \partial\left(\varepsilon_{\mathscr{A}}\right) \wedge (\forall f \in l_\infty(\mathscr{A}, \mathscr{R}\downarrow))\, t(|f|) = |t(f)| \leftrightarrow t \in \mathrm{Ch}\left(\varepsilon_{\mathscr{A}}\right).$$

The remaining two equivalences are other expressions of what we have already established. ▷

4.5. *The pure states on* $\mathscr{A}$ *are precisely the* δ*-functions at the points of the standard name* $\mathscr{A}^\wedge$ *inside* $V^{(B)}$. *In other words, the following equivalence holds for* $t \in \mathrm{Hom}_{\mathscr{R}\downarrow}(l_\infty(\mathscr{A}, \mathscr{R}\downarrow), \mathscr{R}\downarrow)$ *:*

$$(t \text{ is a pure state on } \mathscr{A}) \leftrightarrow \left[(\exists A \in \mathscr{A}^\wedge)\, t^\uparrow = \varepsilon_A\right] = 1.$$

◁ It is clear that

$$1 = \left[(\exists A \in \mathscr{A}^\wedge)\, t^\uparrow = \varepsilon_A\right] = \bigwedge_{A \in \mathscr{A}} \left[t^\uparrow = \varepsilon_{A^\wedge}\right].$$

As is obvious, the last equality holds if and only if there exist a partition of unity $(b_\xi)_{\xi\in\Xi}$ and a family $(A_\xi)_{\xi\in\Xi}$ of points of $\mathscr{A}$ such that $t^\uparrow$ is an intermixing of $\left(\varepsilon_{A_\xi^\wedge}\right)_{\xi\in\Xi}$ with probabilities $(b_\xi)_{\xi\in\Xi}$.

Further, using (3.6) and the Gordon theorem, we deduce that

$$t^\uparrow = \sum_{\xi\in\Xi} b_\xi \varepsilon_{A_\xi^\wedge} \leftrightarrow \left[(\forall f \in l_\infty(\mathscr{A}^\wedge, \mathscr{R}))\, t^\uparrow(f) = \sum_{\xi\in\Xi} b_\xi \varepsilon_{A_\xi^\wedge}(f)\right] = 1 \leftrightarrow$$

$$\leftrightarrow (\forall f \in l_\infty(\mathscr{A}, \mathscr{R}\downarrow))\, t^\uparrow(f\uparrow) = \sum_{\xi\in\Xi} b_\xi f\uparrow(A_\xi^\wedge) \leftrightarrow (\forall f \in l_\infty(\mathscr{A}, \mathscr{R}\downarrow))\, t(f) = \sum_{\xi\in\Xi} b_\xi f(A_\xi) \leftrightarrow$$

$$\leftrightarrow (\forall f \in l_\infty(\mathscr{A}, \mathscr{R}\downarrow))(\forall \xi \in \Xi)\, b_\xi t(f) = b_\xi f(A_\xi) \leftrightarrow (\forall f \in l_\infty(\mathscr{A}, \mathscr{R}\downarrow))(\forall \xi \in \Xi)\left[\iota(b_\xi)\, t(f) = \iota(b_\xi)\, \varepsilon_{A_\xi}(f)\right] \geqslant b_\xi \leftrightarrow$$

$$\leftrightarrow (\forall f \in l_\infty(\mathscr{A}, \mathscr{R}\downarrow))\, t(f) = \sum_{\xi\in\Xi} \iota(b_\xi)\, \varepsilon_{A_\xi}(f) \leftrightarrow t = \sum_{\xi\in\Xi} \iota(b_\xi)\, \varepsilon_{A_\xi}.$$

The established equivalence makes the desired statement obvious. ▷

5. STRUCTURE OF THE SUBDIFFERENTIAL OF THE CANONICAL OPERATOR

5.0. In this section we describe the structure of the extreme points and elements of the subdifferential of the canonical operator. The method of obtaining the desired descriptions consists in the interpretation in the outer terminology of the Krein–Mil'man and the Mil'man theorems, formulated for functionals in necessary Boolean-valued model.

5.1. *Each extreme point of the subdifferential of the canonical operator is the pointwise r-limit of a net of pure states.*

◁ Let us consider an extreme point of the subdifferential of the canonical operator that acts from $l_\infty(\mathscr{A}, Y_0)$ into Y_0 for a certain K-space Y_0. By virtue of the Krein–Mil'man theorem for operators, we can assume that the considered extreme point is the restriction to $l_\infty(\mathscr{A}, Y_0)$ of an extreme point t of the subdifferential of the canonical operator $\varepsilon_{\mathscr{A}}$ that acts from $l_\infty(\mathscr{A}, Y)$ into Y, where Y is a maximal extension of Y_0 [16]. Thus, to prove the desired statement in full measure, it is sufficient (and clearly necessary) to analyze only the case of the extended K-space Y. In addition, choosing a Boolean-valued universe $V^{(B)}$, where $B := \mathscr{B}(Y)$ is the base of the considered space Y (coinciding with the base of the initial space Y_0), we see that $Y = \mathscr{R}\downarrow$, where $\mathscr{R}$ is the object that plays the role of **R** in $V^{(B)}$.

First of all, let us observe that, as established in [8], if $t \in \mathscr{L}(l_\infty(\mathscr{A}, \mathscr{R}\downarrow), \mathscr{R}\downarrow)$ and $t \in \partial(\varepsilon_{\mathscr{A}})$, then t is automatically a module homomorphism, i.e., $t \in \mathrm{Hom}_{\mathscr{R}\downarrow}(l_\infty(\mathscr{A}, \mathscr{R}\downarrow), \mathscr{R}\downarrow)$. Working in $V^{(B)}$, now on the basis of 4.4 we see that $t^\uparrow \in \mathrm{Ch}\left(\varepsilon_{\mathscr{A}^\wedge}\right)\downarrow$. Further, on the basis of the classical Mil'man theorem, the δ-functions are complete in the weak topology in the set of extreme points of the subdifferential of the (scalar) canonical operator. By virtue of the transfer principle, we conclude that

$$(\forall k := 1, \ldots, m)\left[(\exists A \in \mathscr{A}^\wedge)\,|t^\uparrow(f_k\uparrow) - f_k\uparrow(A)| \leqslant \frac{1}{n^\wedge}\right] = 1$$

for all $f_1, \ldots, f_m \in l_\infty(\mathscr{A}, \mathscr{R}\downarrow)$ and $n := 1, 2, \ldots$. Using 4.5 and setting $\gamma := \langle\{f_1, \ldots, f_m\}, n\rangle$, we find a pure state t_γ such that

$$|t_\gamma(f_k) - t(f_k)| \leqslant \frac{1}{n} 1^\wedge \quad (k := 1, \ldots, m).$$

Equipping the index set $\{\gamma\}$ with natural order and turning it by the same token into a direction, we see that the net, so obtained, of pure states (t_γ) r-converges to t. ▷

5.2. *Each element of the subdifferential of the canonical operator is the pointwise r-limit of a net of the elements of the strictly operator-convex hull of the set of δ-functions.*

◁ Reasoning as in 5.1, we reduce the whole thing to the case of the canonical operator that acts in the lowering $\mathscr{R}\downarrow$.

Thus, let X be the strictly operator-convex hull of the set of δ-functions and $t \in \partial(\varepsilon_{\mathscr{A}})$. It is clear that X consists of the $\mathscr{R}\downarrow$-homomorphisms and that the element t is also an $\mathscr{R}\downarrow$-homomorphism. By the same token, $\mathfrak{X} := \{s^\uparrow : s \in X\}$ is a strongly cyclic set of elements of $V^{(B)}$, where $B := \mathscr{B}(\mathscr{R}\downarrow)$, and, in addition, $[\alpha\mathfrak{X} + \beta\mathfrak{X} \subset \mathfrak{X}] = 1$ for $\alpha, \beta \in \mathscr{R}\downarrow$ provided $[\alpha, \beta \geqslant 0^\wedge \wedge \alpha + \beta = 1^\wedge] = 1$. Here we have used the fact that $l_\infty(\mathscr{A}^\wedge, \mathscr{R})^\#\downarrow$ is an $\mathscr{R}\downarrow$-module. Finally, using 4.4, we see that $\mathfrak{X}\uparrow$ is a convex subset of $\partial\left(\varepsilon_{\mathscr{A}^\wedge}\right)$ inside $V^{(B)}$. Indeed, we have

$$[(\forall\alpha, \beta \in \mathscr{R})(\alpha \geqslant 0^\wedge \wedge \beta \geqslant 0^\wedge \wedge \alpha + \beta = 1^\wedge) \to (\alpha\mathfrak{X}\uparrow + \beta\mathfrak{X}\uparrow \subset \mathfrak{X}\uparrow)] =$$

$$= \bigwedge_{\substack{\alpha,\beta \in \mathscr{R}\downarrow \\ \alpha\geqslant 0, \beta\geqslant 0, \alpha+\beta=1}} \; \bigwedge_{p^\uparrow, q^\uparrow \in \mathfrak{X}} [\alpha p^\uparrow + \beta q^\uparrow \in \mathfrak{X}\uparrow] = \bigwedge_{\substack{\alpha,\beta \in \mathscr{R}\downarrow \\ \alpha\geqslant 0, \beta\geqslant 0, \alpha+\beta=1}} \; \bigwedge_{p,q \in X} [(\alpha p)^\uparrow + (\beta q)^\uparrow \in \mathfrak{X}\uparrow] = 1.$$

Therefore, by virtue of the classical Krein–Mil'man theorem, $\mathfrak{X}\uparrow$ is dense in the weak topology in $\partial\left(\varepsilon_{\mathscr{A}^\wedge}\right)$ inside $V^{(B)}$. Since $t^\uparrow \in \partial\left(\varepsilon_{\mathscr{A}^\wedge}\right)\downarrow$, we find the desired net of elements of X that converges with regulator to t (see 5.1). ▷

6. STRUCTURE OF THE SUBDIFFERENTIALS OF ARBITRARY OPERATORS

6.0. In this section, we give the basic results about the structure of the subdifferentials of sublinear operators that act in a K-space.

6.1. *Each extreme point of the subdifferential is the pointwise r-limit of a net of elements of the strongly cyclic hull of the set of o-extreme points.*

◁ Let $P:X \to Y$ be the operator under consideration and $T \in \mathrm{Ch}(P)$. By virtue of 1.7, we have $T = t_0 \circ \langle \mathscr{E}_0(P) \rangle$ for a certain $t \in \mathrm{Ch}\left(\varepsilon_{\mathscr{E}_0(P)}\right)$. Let (t_γ) be a net of pure states that pointwise r-converges to t (its existence is ensured by 5.1). It is clear that $(t_\gamma \circ \langle \mathscr{E}_0(P) \rangle)$ is the desired net. ▷

6.2. *The extreme points of the smallest subdifferential that contains a given weakly order-bounded set $\mathscr{A}$ are the pointwise r-limits of suitable nets of intermixings of elements of $\mathscr{A}$.*

◁ It is obvious that the set of the extreme points of $\mathrm{cop}(\mathscr{A})$ is contained in $\mathrm{Ch}(\varepsilon_{\mathscr{A}}) \circ \langle \mathscr{A} \rangle$. It remains to refer to 5.2. ▷

6.3. *A weakly order-bounded set is a subdifferential if and only if it is operator-convex and pointwise o-closed.*

◁ It is clear that an operator-convex and pointwise o-closed weakly order-bounded set $\mathscr{A}$ is trivially strictly operator-convex. Since r-convergence implies o-convergence, we deduce from 5.2 that

$$\mathscr{A} \subset \partial(\varepsilon_{\mathscr{A}}) \circ \langle \mathscr{A} \rangle \subset \mathscr{A}$$

(the left-hand inclusion is valid without any stipulations). Thus, $\mathscr{A}$ is a subdifferential. The remaining unproved part of the statement is obvious. ▷

6.4. *A weakly order-bounded set is a subdifferential if and only if it is cyclic, convex, and pointwise r-closed.*

◁ It is easily seen that cyclicity, in combination with convexity and r-closedness, ensures strict operator-convexity and pointwise o-closedness. A reference to 6.3 completes the proof. ▷

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CERTAIN PROBLEMS OF ANALYTIC CONTINUATION FROM INTERIOR SETS

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The problems of analytic continuation of the regeneration of analytic functions from their values on certain sets are connected with numerous applications, in particular, with the problems of processing and interpretation of the readings of physical devices.

In the present article we will consider problems of analytic continuation from interior sets, i.e., from sets contained in the interior of the domain of regularity of analytic functions. We restrict ourselves to the consideration of analytic functions of two real variables in bounded domains.

1. Let D be a bounded domain of the (x, y)-plane and f(x, y) be an analytic function in the domain* D.

Let M be a subset of D.

Definition 1. The set M is called a *uniqueness set* for the domain D if each function f(x, y) that is analytic in D is uniquely determined by its values on M.

The following definition is an equivalent definition of the uniqueness set.

Definition 1'. The set M is called a *uniqueness set* for the domain D if for each analytic function f(x, y) in D the equality

$$f(x, y) = 0 \quad \forall (x, y) \in M$$

implies that

$$f(x, y) = 0 \quad \forall (x, y) \in D.$$

In the present article, we will assume that the considered sets are closed.

It is easily seen that the property of being a uniqueness set depends on the domain D. The set M may be a uniqueness set for the domain D and may not be a uniqueness set for a subdomain D_1 of D.

Let D_h be a rectangle:

$$D_h = \{(x, y) : h < x < a, \ |y| < b, \ a > 0, \ b > 1\},$$

and M be a segment of a curve Γ:

$$\Gamma = \left\{(x, y) \colon y = \sin \frac{1}{x}, \quad x > 0\right\}$$

or a set that has a limit point on the curve Γ:

$$(x_0, y_0) \in \Gamma \cap D_h.$$

It is easy to show that in the case

$$h < 0$$

the indicated set M is a uniqueness set, and in the case

*A function f(x, y) is said to be *analytic in a domain* D if it can be continued as an analytic function of two complex variables into a complex neighborhood of D.

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