INFINITESIMAL TANGENT CONES

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In recent years, an intensive search is carried out in the subdifferential calculus for convenient methods of local one-sided approximation of arbitrary functions and sets. The definition of the subdifferential of a Lipschitz function, given by Clarke [1], serves as the principal starting point. The tangent comes and the subdifferentials corresponding to them, constructed at present, are often defined by hardly visible cumbersome formulas [2-8]. We know that there exists an effective technique, the nonstandard analysis [9-11], for the displacement of formulas — "decrease of quantifiers."

The aim of the present article is to use this technique for the determination of compact criteria of different types of tangents. It turns out that under the assumption of standardness of the parameters of the considered objects, the Bouligand, the Hadamard, and the Clarke cones are defined by an explicit "infinitesimal construction — a direct appeal to infinitely close points and directions. In this connection, in this article we indicate an enumeration of all possible tangents, defined by infinitesimal prefixes. In addition, we have been able to discover the cones, omitted earlier, including those of the Clarke type.

0. Preliminary Information

0.1. In the sequel we will work in a suitable nonstandard model of analysis with a sufficiently strong degree of saturation. For definiteness, we can assume that everything happens within the framework of Hrbacek's outer set theory [11] with the strong idealization axiom. The few deviations from this convention will be explicitly mentioned.

0.2. Let us agree in the sequel to assume the parameters of formal expressions in the following test to be standard objects.

Thus, let X be a real vector space, F be a subset of it, and x' and h' be points of X. In correspondence with the stipulation made by us (in working with the nonstandard set theory), these objects are standard sets. We select two filters in X: $\mathcal{N}_\sigma$ and $\mathcal{N}_\tau$, formed from certain supersets of zero. Further, let $\mu(\sigma)$ and $\mu(\tau)$ be the *monads* of the indicated filters, i.e., intersections of their standard elements [12]. We write $x \approx_\sigma x'$ in place of $x - x' \in \mu(\sigma)$. The expression $h \approx_\tau h'$ is understood analogously. Finally, we denote the *infinitesimality* of a number of α in $\mathbf{R}$ by the symbol $\alpha \approx 0$. It is convenient to set $\mu(\mathbf{R}_+) := \{\alpha \in \mathbf{R} : \alpha \geqslant 0,\ \alpha \approx 0\}$. It is assumed below that the filter $\mathcal{N}_\sigma$ generates the vector topology $\sigma : x \in X \to x + \mathcal{N}_\sigma$ in X.

0.3. A topology τ in a vector space X (over the field $\mathbf{R}$) is said to be *almost vector* if, in the first place, the multiplication by each scalar from $\mathbf{R}$ is continuous and, secondly, addition is continuous with respect to the totality of variables. In this case, the pair (X, τ) (as well as X itself) is called an almost topological vector space.

0.4. Let X be a standard vector space and $\mathcal{N}_\tau$ be the standard filter of supersets of zero. Then there exists a standard almost vector topology τ on X such that $\mathcal{N}_\tau = \tau(0)$ if and only if the monad $\mu(\tau)$ of the filter $\mathcal{N}_\tau$ is an outer vector space over the outer field of standard scalars.

0.5. For a formula φ of the Zermelo–Fraenkel set theory, we set

$$\forall^\cdot x\, \varphi := (\forall x \approx_\sigma x')\, \varphi := \forall x\, (x \in F \wedge x \approx_\sigma x') \to \varphi;$$
$$\forall^\cdot h\, \varphi := (\forall h \approx_\tau h')\, \varphi := \forall h\, (h \in X \wedge h \approx_\tau h') \to \varphi;$$
$$\forall^\cdot \alpha\, \varphi := (\forall \alpha \approx 0)\, \varphi := \forall \alpha\, (\alpha > 0 \wedge \alpha \approx 0) \to \varphi.$$

The quantities $\exists^\cdot x$, $\exists^\cdot h$, and $\exists^\cdot \alpha$ are defined in the natural dual manner, i.e., we assume that

$$\exists^\cdot x\, \varphi := (\exists x \approx_\sigma x')\, \varphi := \exists x\, (x \in F \wedge x \approx_\sigma x') \wedge \varphi;$$

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$$\exists^{\cdot} h\,\varphi := (\exists h \approx_{\tau} h')\,\varphi := \exists h\,(h \in X \wedge h \approx_{\tau} h') \wedge \varphi;$$
$$\exists^{\cdot} \alpha\,\varphi := (\exists \alpha \approx 0)\,\varphi := \exists \alpha\,(\alpha > 0 \wedge \alpha \approx 0) \wedge \varphi.$$

Finally, we agree to write

$$\forall^{st} x\varphi := \forall x\ (x - \text{standard}) \rightarrow \varphi;$$
$$\exists^{st} x\varphi := \exists x\ (x - \text{standard}) \wedge \varphi.$$

1. Bouligand and the Hadamard Cones

1.1. The Bouligand cone Bo(F, x') is defined by the relation

$$\mathrm{Bo}(F, x') := \bigcap_{\substack{U \in \mathcal{N}_{\sigma}(x') \\ \alpha'}} \mathrm{cl}_{\tau} \bigcup_{\substack{x \in F \cap U \\ 0 < \alpha \leqslant \alpha'}} \frac{F - x}{\alpha},$$

where, as usual, $\mathcal{N}_{\sigma}(x') := x' + \mathcal{N}_{\sigma}$. The elements of the Bouligand cone are called *external* (or, less, precisely, outer) *tangents* to the set F at the point x'.

1.2. The Bouligand cone is the standardization of the $\exists\exists\exists$ -cone, i.e., for each standard element h'

$$h' \in \mathrm{Bo}(F, x') \leftrightarrow \exists^{\cdot} x \exists^{\cdot} \alpha \exists^{\cdot} h\ x + \alpha h \in F.$$

The following equivalences follow from the definition of the Bouligand cone:

$$h' \in \mathrm{Bo}(F, x') \leftrightarrow (\forall U \in \mathcal{N}_{\sigma}(x'))\,(\forall \alpha' \in \mathbf{R})\,(\forall V \in \mathcal{N}_{\tau})\,(\exists x \in F \cap U)$$
$$(\exists 0 < \alpha \leqslant \alpha')\,(\exists h \in h' + V)\ x + \alpha h \in F \leftrightarrow$$
$$\leftrightarrow \forall U \forall \alpha' \forall V \exists x \exists \alpha \exists h\ (x \in F \cap U \wedge h \in h' + V \wedge 0 < \alpha \leqslant \alpha' \wedge x + \alpha h \in F).$$

By virtue of the transfer principle, we deduce that

$$h' \in \mathrm{Bo}(F, x') \leftrightarrow \forall^{st} U \forall^{st} \alpha' \forall^{st} V \exists^{st} x \exists^{st} \alpha$$
$$\exists^{st} h\,(x \in F \cap U \wedge h \in h' + V \wedge 0 < \alpha \leqslant \alpha' \wedge x + \alpha h \in F).$$

Now, using the idealization principle (in the weak form), we get

$$h' \in \mathrm{Bo}(F, x') \rightarrow \exists x \exists \alpha \exists h \forall^{st} U \forall^{st} \alpha' \forall^{st} V\ (x \in F \cap$$
$$\cap\, U \wedge h \in h' + V \wedge 0 < \alpha \leqslant \alpha' \wedge x + \alpha h \in F) \rightarrow (\exists x \approx_{\sigma} x')$$
$$(\exists \alpha \approx 0)\,(\exists h \approx_{\tau} h')\ x + \alpha h \in F \rightarrow \exists^{\cdot} x \exists^{\cdot} \alpha \exists^{\cdot} h\ x + \alpha h \in F.$$

Let, in its turn, the standard element h' occur in the standardization of the $\exists\exists\exists$ -cone. Since the standard elements of a standard filter contain elements of the monad of this filter, we get

$$(\forall^{st} U \in \mathcal{N}_{\sigma}(x'))\,(\forall^{st} \alpha' \in \mathbf{R})\,(\forall^{st} V \in \mathcal{N}_{\tau})$$
$$(\exists x \in F \cap U)\,(\exists 0 < \alpha \leqslant \alpha')\,(\exists h \in h' + V)\ x + \alpha h \in F.$$

By virtue of the transfer principle, we conclude that $h' \in \mathrm{Bo}(F, x')$. ▷

1.3. The above-proved statement can be rewritten in the form

$$\mathrm{Bo}(F, x') = {}^{*}\{h' : \exists^{\cdot} x \exists^{\cdot} \alpha \exists^{\cdot} h\ x + \alpha h \in F\},$$

where * is the symbol of *standardization*. In this connection, we use the transparent notation

$$\exists\exists\exists\,(F, x') := \mathrm{Bo}(F, x').$$

In the sequel, we will use similar kind of notation without specific mention.

1.4. The Hadamard or the hypertangent cone Ha(F, x') is defined by the relation

$$\mathrm{Ha}\,(F, x') := \bigcup_{\substack{U \in \mathcal{N}_{\sigma}(x') \\ \alpha'}} \mathrm{int}_{\tau} \bigcap_{\substack{x \in F \cap U \\ 0 < \alpha \leqslant \alpha'}} \frac{F - x}{\alpha}.$$

The elements of the Hadamard cone are called *internal tangents*.

1.5. The Hadamard cone is the standardization of the $\forall\forall\forall$-cone,

$$\mathrm{Ha}(F, x') = \forall\forall\forall(F, x').$$

In other words, for standard h', F, and x'

$$h' \in \mathrm{Ha}(F, x') \leftrightarrow (x' + \mu(\sigma)) \cap F + \mu(\mathbf{R}_+)(h' + \mu(\tau)) \subset F,$$

where $\mu(\mathbf{R}_+)$ is the outer set of positive infinitesimal numbers.

This statement is obtained by duality arguments from 1.2 if (which is, of course, correct) we forget about the existence of F in $\exists^{\cdot} x$. ▷

2. Clarke Cone

2.1. The *Clark cone* is defined by the relation

$$\mathrm{Cl}(F, x') := \bigcap_{V \in \mathcal{N}_\tau} \bigcup_{\substack{U \in \mathcal{N}_\sigma(x') \\ \alpha'}} \bigcap_{\substack{\in \cap U \\ 0<\alpha\leqslant\alpha'}} \left(\frac{F - x}{\alpha} + V\right).$$

2.2. The following statements are equivalent for standard F, x', and h' (under the conditions of weak idealization):

(1) $h' \in \mathrm{Cl}(F, x')$;

(2) There exist infinitesimal $U \in \mathcal{N}_\sigma(x')$, $V \in \mathcal{N}_\tau$, and $\alpha' > 0$ such that

$$h' \in \bigcap_{\substack{0<\alpha\leqslant\alpha' \\ x \in F \cap U}} \left(\frac{F - x}{\alpha} + V\right);$$

(3) $(\exists U \in \mathcal{N}_\sigma(x'))\exists\alpha'(\forall x \in F \cap U), (\forall 0 < \alpha \leqslant \alpha')(\exists h \approx_\tau h')\, x + \alpha h \in F.$

Using obvious abbreviations, we can write

$$h' \in \mathrm{Cl}(F, x') \leftrightarrow \forall V \exists U \exists \alpha' (\forall x \in F \cap U)(\forall 0 < \alpha \leqslant \alpha')(\exists h \in h' + V),\ x + \alpha h \in F.$$

Using the transfer principle and weak idealization, we successively have

$$h' \in \mathrm{Cl}(F, x') \to \forall^{st} V \exists^{st} U \exists^{st} \alpha' (\forall x \in F \cap U)$$
$$(\forall 0 < \alpha \leqslant \alpha')(\exists h \in h' + V)\ x + \alpha h \in F \to (\forall^{st}\{V_1, \ldots, V_n\}) \exists^{st} U \exists^{st} \alpha' \exists^{st} V (\forall k := 1, \ldots, n) V_k \supset V$$
$$\wedge (\forall x \in F \cap U)(\forall 0 < \alpha \leqslant \alpha')(\exists h \in h' + V)\ x + \alpha h \in F \to$$
$$\to \exists U \exists \alpha' \exists V \forall^{st} V'\ V' \supset V \wedge (\forall x \in F \cap U)$$
$$(\forall 0 < \alpha \leqslant \alpha')(\exists h \in h' + V)\ x + \alpha h \in F.$$

Hence it follows without doubt that (2), and, all the more, (3), holds for certain $V \in \mathcal{N}_\tau$, $V \subset \mu(\tau)$ and $U \in \mathcal{N}_\sigma(x')$, $U \subset \mu(\sigma) + x'$, and infinitesimal α.

If, in its turn, (3) is fulfilled, then by virtue of the definition of the relation $\approx_\tau$ we have

$$\forall^{st} V \exists U \exists \alpha' (\forall x \in F \cap U)(\forall 0 < \alpha \leqslant \alpha')(\exists h \in h' + V)\ x + \alpha h \in F.$$

Therefore, by the transfer principle, $h' \in \mathrm{Cl}(F, x')$. ▷

2.3. The Clarke cone (under the conditions of strong idealization) is the standardization of the $\forall\forall\exists$-cone:

$$\mathrm{Cl}(F, x') = \forall\forall\exists(F, x').$$

In other words,

$$h' \in \mathrm{Cl}(F, x') \leftrightarrow \forall^{\cdot} x \forall^{\cdot} \alpha \exists^{\cdot} h\ x + \alpha h \in F.$$

◁ At first, let $h' \in \mathrm{Cl}(F, x')$. We take arbitrary $x \approx_\sigma x'$ and $\alpha > 0, \alpha \approx 0$. For each standard neighborhood V (an element of the filter $\mathcal{N}_\tau$), by virtue of the transfer principle there **exists** an element h such that $h \in h' + V$ and $x + \alpha h \in F$. Using **strong** idealization, we have

$$\forall^{st} V \exists h (h \in h' + V \wedge x + \alpha h \in V) \to \exists h \forall^{st} V\ h \in h' + V \wedge x + \alpha h \in F \to \exists^{\cdot} h\ x + \alpha h \in F,$$

i.e., $h' \in \forall\forall\exists(F, x')$.

Now, let $h' \in \forall\forall\exists(F, x')$. We take an arbitrary standard neighborhood V in the filter $\mathcal{N}_\tau$ and fix an infinitesimal neighborhood U of x' and an infinitesimal positive number α'. Then, by the condition, for a certain $h \approx_\tau h'$ we have

$$(\forall x \in F \cap U)(\forall 0 < \alpha \leqslant \alpha')\ x + \alpha h \in F.$$

In other words,

$$\forall^{st} V \exists U \exists \alpha' (\forall x \in F \cap U)(\forall 0 < \alpha \leqslant \alpha')(\exists h \in h' + V)\ x + \alpha h \in F.$$

By virtue of the transfer principle, $h' \in \mathrm{Cl}(F, x')$. ▷

2.4. We give an example of application of the nonstandard criterion, found by us, of elements of the Clarke cone for the deduction of its main (and well-known) property. A more general statement will be established below.

2.5. The Clarke cone of an arbitrary set in a topological vector space is convex and closed.

◁ By virtue of the transfer principle, it is sufficient to consider the case in which the parameters — the space, the topology, the set, etc. — are standard. Thus, let $h_0 \in \mathrm{cl}_\tau \mathrm{Cl}(F, x')$. We take a standard neighborhood V in $\mathcal{N}_\tau$, and let the standard elements $V_1, V_2 \in \mathcal{N}_\tau$ be such that $V_1 + V_2 \subset V$. Then there exists a standard element $h' \in \mathrm{Cl}(F, x')$ such that $h' - h_0 \in V_1$. Moreover, for arbitrary $x \approx_\tau x'$ and $\alpha > 0$, $\alpha \approx 0$, we have $h \in h' + V_2$ and $x + \alpha h \in F$ for a certain h. It is clear that $h \in h' + V_2 \subset h_0 + V_1 + V_2 \subset h_0 + V$. Hence $h_0 \in \mathrm{Cl}(F, x')$.

To prove the convexity of the Clarke cone, it is sufficient to observe that $\mu(\tau) + \mu(\mathbf{R}_+)\mu(\tau) \subset \mu(\tau)$ by virtue of the continuity of the mapping $(x, \alpha, h) \to x + \alpha h$. ▷

2.6. Let θ be a vector topology and $\theta \geqslant \tau$. Then

$$\forall\forall\exists(\mathrm{cl}_\theta F, x') \subset \forall\forall\exists(F, x').$$

If, in addition, $\theta \geqslant \sigma$, then

$$\forall\forall\exists(\mathrm{cl}_\theta F, x') = \forall\forall\exists(F, x').$$

◁ Let $h' \in \forall\forall\exists(\mathrm{cl}_\theta F, x')$ be a standard element of the given cone. We take elements $x \in F$ and $\alpha > 0$ such that $x \approx_\sigma x'$ and $\alpha \approx 0$. It is clear that $x \in \mathrm{cl}_\theta F$. Therefore, $x + \alpha h \in \mathrm{cl}_\theta F$ for a certain $h \approx_\tau h'$. We take an infinitesimal neighborhood W in $\mu(\theta)$. The neighborhood αW is also an element of θ(0) and, therefore, $x'' - (x + \alpha h) \in \alpha W$ for a certain $x'' \in F$. Let us set $h'' := (x'' - x)/\alpha$. It is clear that $x + \alpha h'' \in F$ and, moreover, $\alpha h'' \in \alpha h + \alpha W$. Hence $h'' \in h + W \subset h' + \mu(\tau) + W \subset h' + \mu(\tau) + \mu(\theta) \subset h' + \mu(\tau) + \mu(\tau) \subset h' + \mu(\tau)$, i.e., $h'' \approx_\tau h'$. Thus, $h' \in \forall\forall\exists(F, x')$.

Now, let $\theta \geqslant \sigma$ and $h' \in \forall\forall\exists(F, x')$. We take an infinitesimal positive α and some element $x \in \mathrm{cl}_\theta F$ such that $x \approx_\sigma x'$. We select $x'' \in F$ such that $x - x'' \in \alpha W$, where $W \subset \mu(\theta)$ is an infinitesimal symmetric neighborhood of zero in θ. Since $\theta \geqslant \sigma$, it follows that $\mu(\theta) \subset \mu(\sigma)$, i.e., $x - x'' \in \mu(\theta) \subset \mu(\sigma)$. In other words, $x \approx_\sigma x' \approx_\sigma x''$. By definition (the element h', as usual, is assumed to be standard!) $x'' + \alpha h \in F$ for a certain $h \approx_\tau h'$. Let us set $h'' := (x'' - x)/\alpha + h$. It is clear that, in addition,

$$h'' \in h + W \subset h + \mu(\theta) \subset h' + \mu(\theta) + \mu(\tau) \subset h' + \mu(\tau) + \mu(\tau) \subset h' + \mu(\tau),$$

i.e., $h'' \approx_\tau h'$. Moreover,

$$x + \alpha h'' = x + (x'' - x) + \alpha h = x'' + \alpha h \in F \subset \mathrm{cl}_\theta F.$$

Finally, $h' \in \forall\forall\exists(\mathrm{cl}_\theta F, x')$. ▷

3. Cones of Infinitesimal Type

3.1. The above nonstandard criteria of the Bouligand, the Hadamard, and the Clarke cones show that these cones are taken from a list of eight possible cones with infinitesimal prefix $Q'x Q'\alpha Q'h$ (here Q is either ∀ or ∃). It is clear that for complete description of all these cones it is sufficient to give the characterizations of the ∀∃∃-cone and the ∀∃∀-cone.

3.2. The following representation is valid:

$$\forall\exists\exists(F, x') = \bigcap_{\substack{\alpha' \\ V \in \mathcal{N}_\tau}} \bigcup_{U \in \mathcal{N}_\sigma(x')} \bigcap_{x \in F \cap U} \left(V + \bigcup_{0 < \alpha \leqslant \alpha'} \frac{F - x}{\alpha}\right).$$

◁ To prove this statement, we should, at first, understand that the desired equation is the abbreviated expression for the following statement: For standard h', F, and x'

$$\forall^{\cdot} x \exists^{\cdot} \alpha \exists^{\cdot} h \ x+\alpha h \in F \leftrightarrow$$
$$(\forall V \in \mathcal{N}_\tau) \forall \alpha' (\exists U \in \mathcal{N}_\sigma(x')) (\forall x \in F \cap U)$$
$$(\exists 0 < \alpha \leqslant \alpha') (\exists h \in h'+V) \quad x+\alpha h \in F.$$

Therefore, for $h' \in \forall\exists\exists(F, x')$ and standard $V \in \mathcal{N}_\tau$ and $\alpha > 0$ we can take an internal subset of the monad $\mu(\mathcal{N}_\sigma(x'))$ as the desired neighborhood U. In its turn, successive application of the transfer principle and the strong idealization principle gives

$$\forall^{st} V \forall^{st} \alpha' (\forall x \approx_\sigma x') (\exists 0 < \alpha \leqslant \alpha') (\exists h \in h' + V) \quad x + \alpha h \in F$$
$$\rightarrow (\forall x \approx_\sigma x') (\forall^{st} \{V_1, \ldots, V_n\}) (\forall^{st} \{\alpha'_1, \ldots \alpha'_n\})$$
$$\exists h \exists \alpha (\forall k := 1, \ldots, n)\, 0 < \alpha \leqslant \alpha'_k \wedge h \in h' + V_k \wedge x + \alpha h \in F$$
$$\rightarrow (\forall x \approx_\sigma x') \exists h \exists \alpha \forall^{st} V h \in h' + V \wedge \forall^{st} \alpha' 0 < \alpha \leqslant \alpha' \wedge x + \alpha h \in F$$

$$\rightarrow \forall^{\cdot} x \exists^{\cdot} h (\exists \alpha \approx 0) \quad x + \alpha h \in F \rightarrow h' \in {}^*\{h' : \forall^{\cdot} x \exists^{\cdot} \alpha \exists^{\cdot} h \ x + \alpha h \in F\} \rightarrow h' \in \forall\exists\exists (F, x').$$

By the same token, the proof is complete. ▷

3.3. The following estimates are valid:

$$(\exists U \in \mathcal{N}_\sigma(x')) (\exists V \in \mathcal{N}_\tau) (\exists \alpha : F \cap U \rightarrow (0, 1] \wedge$$
$$\wedge \lim_{\mathcal{N}_\sigma(x')} \alpha(x') = 0) (\forall x \in F \cap U) (\forall h \in h' + V) \quad x + \alpha(x) h$$
$$\in F \rightarrow h' \in \forall\exists\forall (F, x') \rightarrow \forall \alpha' (\exists U \in \mathcal{N}_\sigma(x'))$$
$$(\exists V \in \mathcal{N}_\tau) (\exists \alpha : F \cap U \rightarrow (0, \alpha']) \quad (\forall x \in F \cap U)$$
$$(\forall h \in h' + V) \quad x + \alpha(x) h \in F.$$

◁ Let us consider standard sets F, x', and h' and assume that the premise of the statement is fulfilled. Using the transfer principle and considering the criterion of continuity of a standard function, we deduce that $x+\alpha(x)h \in F$ for $x \approx_\sigma x'$ and $h \approx_\tau h'$, i.e., h' belongs to the standardization of the outer set $\{h' : \forall^{\cdot} x \exists^{\cdot} \alpha \forall^{\cdot} h \ x+\alpha h \in F\}$. Thus, the first implication is proved.

To prove the second implication, we take a standard number α' and select infinitesimal neighborhoods U in x' + $\mu(\sigma)$ and V in $\mu(\tau)$. By the condition, $(\forall x \in F \cap U) (\exists \alpha \in (0, \alpha']) (\forall h \in h' + V)\ x+\alpha h \in F$. In the "interior world" (in the universe of interior sets) we can apply the axiom of choice and try to find the function $\alpha : F \cap U \rightarrow (0, \alpha']$ with the desired property. It remains to refer to the transfer principle. ▷

3.4. Besides the above-indicated eight infinitesimal cones of classical type, there are nine more pairs of cones that contain the Hadamard cone and are contained in the Bouligand cone. It is clear that these cones are generated by changing the order of the quantifiers. Five new pairs are constructed in composite manner by the type of the $\forall\exists\forall$ -cone. The remaining pairs are generated by permutations and dualizations of the Clarke cone and the $\forall\exists\exists$ -cone. For example, in the natural notation, we have

$$\forall \alpha \forall h \exists x (F, x') = \bigcap_{U \in \mathcal{N}_\sigma(x')} \bigcup_{\alpha'} \mathrm{int}_\tau \bigcap_{0<\alpha<\alpha'} \bigcup_{x \in F \cap U} \frac{F - x}{\alpha};$$

$$\exists h \exists x \forall \alpha (F, x') = \bigcup_{\alpha'} \bigcap_{U \in \mathcal{N}_\sigma(x')} \mathrm{cl}_\tau \bigcup_{x \in F \cap U} \bigcap_{0<\alpha<\alpha'} \frac{F - x}{\alpha};$$

$$\exists h \forall x \forall \alpha (F, x') = \bigcup_{\substack{U \in \mathcal{N}_\sigma(x') \\ \alpha'}} \mathrm{cl}_\tau \bigcap_{\substack{0<\alpha<\alpha' \\ x \in F \cap U}} \frac{F - x}{\alpha}.$$

The last cone is the Clarke cone and is convex in the case where $\mu(\sigma) + \mu(\mathbf{R}_+)\mu(\tau) \subset \mu(\sigma)$.

4. Generalized Clarke Cone

4.1. From the qualitative point of view, the approximating infinitesimal cones are the results of the examination of a set under a microscope. Thus, it is not completely natural to speak about the exact occurrence of an element in a set. In this connection, we fix one more (standard) filter $\mathcal{N}_\theta$ and the corresponding outer relation $\approx_\theta$.

4.2. Let us consider a set F and a point x'. Let us set

$$\mathrm{Cl}^{\approx}(F, x') := \bigcap_{\substack{V\in\mathcal{N}_\tau \\ W\in\mathcal{N}_\theta}} \bigcup_{\substack{U\in\mathcal{N}_\sigma(x') \\ \alpha'}} \bigcap_{\substack{x\in F\cap U \\ 0<\alpha\leqslant\alpha'}} \left(\frac{F+W-x}{\alpha}+V\right).$$

The set $\mathrm{Cl}^{\approx}(F, x')$ is called the *generalized Clarke cone*. It is clear that this cone is wider than the Clarke cone and coincide with it if $\mathcal{N}_\theta$ is the filter of all supersets of zero. It is useful to emphasize that if the filter $\mathcal{N}_\theta$ is not sufficiently fine, i.e., contains Mahlo sets, then the cone $\mathrm{Cl}^{\approx}(F,x')$ is too wide-coincides with the whole space.

4.3. For standard F and x', the set $\mathrm{Cl}^{\approx}(F, x')$ is the standardization of the $\forall\forall\exists\exists$ -cone, i.e., a standard element h' occurs in $\mathrm{Cl}^{\approx}(F, x')$ if and only if

$$\forall^{\cdot} x \forall^{\cdot} \alpha \exists^{\cdot} h (\exists x''\in F)\ x'' \approx_\theta x+\alpha h.$$

◁ The proof of this statement is analogous to that given in 2.3. ▷

4.4. In the sequel, we will assume that the mapping (x, t, y, z) → x + ty + z, acting from the space X × **R** × X × X with the topology $\sigma \times \tau_{\mathbf{R}} \times \tau \times \theta$ into the space X with the topology σ, is continuous, i.e., $\mu(\sigma)+\mu(\mathbf{R}_+)\mu(\tau)+\mu(\theta)\subset\mu(\sigma)$ under the assumption of standardness of parameters.

4.5. The generalized Clarke cone is τ-closed and convex.

◁ The proof of this statement is analogous to that of 2.5. ▷

4.6. The Rockafellar formula

$$\forall\forall\forall\exists(F, x')+\mathrm{Cl}^{\approx}(F, x')\subset\forall\forall\forall\exists(F, x')$$

is valid, where the convex cone $\forall\forall\forall\exists(F, x')$ is defined by the relation

$$\forall\forall\forall\exists(F, x') := \bigcap_{W\in\mathcal{N}_\theta} \bigcup_{\substack{U\in\mathcal{N}_\sigma(x') \\ \alpha'}} \mathrm{int}_\tau \bigcap_{\substack{x\in F\cap U \\ 0<\alpha\leqslant\alpha'}} \frac{F+W-x}{\alpha},$$

i.e., represents the standardization

$$\forall\forall\forall\exists(F, x') = {}^*\{h' : \forall^{\cdot} x\forall^{\cdot}\alpha\forall^{\cdot} h(\exists x''\in F)\ x+\alpha h\approx_\theta x''\}$$

(for standard parameters).

◁ Only the verification of the Rockafellar formula is nontrivial. In this connection, by virtue of the transfer principle, we can restrict ourselves to the case of standard parameters.

Thus, let k' occur in $\forall\forall\forall\exists(F, x')$ and $h'\in\forall\forall\exists\exists(F, x')$, where k' and h' are standard elements. Then

$$(x'+\mu(\sigma))\cap F+\mu(\mathbf{R}_+)(k'+\mu(\tau))\subset F+\mu(\theta);$$
$$(\forall x\approx_\sigma x')(\forall\alpha\approx 0)(\exists h\approx_\tau h')(\exists x''\in F)\ x+\alpha h\approx_\theta x''.$$

Hence, for α > 0 and $\alpha\approx 0$ we deduce that

$$x'+\alpha(h'+k'+\mu(\tau)) = x'+\alpha h+\alpha(k'+(h'-h)+\mu(\tau))\subset$$
$$\subset x''+\mu(\theta)+\alpha(k'+\mu(\tau)+\mu(\tau))\subset x''+\alpha(k'+\mu(\tau))+\mu(\theta)$$

for a certain $x''\in F$. Besides this,

$$x''\in x'+\alpha h+\mu(\theta)\subset x'+\alpha h'+\alpha\mu(\tau)+\mu(\tau)\subset x'+\mu(\sigma)+\alpha\mu(\tau)+\mu(\theta)\subset x'+\mu(\sigma).$$

Finally,

$$x'+\alpha(h'+k'+\mu(\tau))\subset(x'+\mu(\sigma))\cap F+\alpha(k+\mu(\tau)+\mu(\theta))\subset F+\mu(\theta)+\mu(\theta)\subset F+\mu(\theta),$$

which was desired to be proved. ▷

5. Directional Epiderivatives

5.1. Let $\overline{\mathbf{R}}$ be the standard extended number line. For *a finite number* $t\in{}^{\approx}\mathbf{R} := \{t\in\mathbf{R} : (\exists^{\mathrm{st}} n\in\mathbf{N})\ |t|\leqslant n\}$, we denote, as usual, the *standard part of* t by st(t). Thus, $(\forall s,\ t\in{}^{\approx}\mathbf{R})$ $t\approx s \leftrightarrow \mathrm{st}(t)=\mathrm{st}(s)$. Let us also set

$$\mu(-\infty) := \{s\in\overline{\mathbf{R}} :\ s\leqslant{}^{\approx}\mathbf{R}\};$$

$$\mu(+\infty) := \{s \in \overline{\mathbf{R}} : s \geqslant {}^*\mathbf{R}\}$$

which are monads of infinities and suppose that

$$t \in \mu(-\infty) \leftrightarrow \mathrm{st}(t) = -\infty;$$
$$t \in \mu(+\infty) \leftrightarrow \mathrm{st}(t) = +\infty.$$

5.2. For a standard t from **R** and an arbitrary s from $\overline{\mathbf{R}}$

$$\mathrm{st}(s) \leqslant t \leftrightarrow (\forall^{\mathrm{st}} \varepsilon > 0)\ s < t + \varepsilon.$$

◁ ← If s < t + ε, then st(s) ⩽ st(t + ε) = st(t) + st(ε) = t + ε. Therefore, $((\forall \varepsilon > 0)$ st(s) ⩽ t + ε) → st(s) ⩽ t.

→ Let st(s) = t. Since $s \in \mu(s)$, we have s < t + ε for each standard ε > 0. But if st(s) < t, then the monad of the point st(s) lies on the left of t. Hence s is on the left of t and, therefore, a fortriori, s < t + ε. ▷

5.3. Let f:X → $\overline{\mathbf{R}}$ be a standard function defined on a standard X and $\mathscr{F}$ be a standard filter in X. For each standard $t \in \mathbf{R}$

$$\sup_{V \in \mathscr{F}} \inf f(V) \leqslant t \leftrightarrow (\exists v \in \mu(\mathscr{F}))\ \mathrm{st}(f(v)) \leqslant t;$$

$$\inf_{V \in \mathscr{F}} \sup f(V) \leqslant t \leftrightarrow (\forall v \in \mu(\mathscr{F}))\ \mathrm{st}(f(v)) \leqslant t.$$

◁ At first, let us verify the first equivalence. Applying successively the transfer principle and the idealization principle, we deduce that

$$\sup_{V \in \mathscr{F}} \inf f(V) \leqslant t \to (\forall V \in \mathscr{F})\ \inf f(V) \leqslant t \to$$
$$\to (\forall V \in \mathscr{F})(\forall \varepsilon > 0)\ \inf f(V) < t + \varepsilon \to \forall \varepsilon \forall V (\exists v \in V)\ f(v) < t + \varepsilon \to$$
$$\to \forall^{\mathrm{st}} \varepsilon \forall^{\mathrm{st}} V \exists v\ v \in V \wedge f(v) < t + \varepsilon \to \exists v \forall^{\mathrm{st}} \varepsilon \forall^{\mathrm{st}} V\ v \in V \wedge f(v) < t + \varepsilon \to$$
$$\to (\exists v \in \mu(\mathscr{F}))(\forall^{\mathrm{st}} \varepsilon > 0)\ f(v) < t + \varepsilon \to (\exists v \in \mu(\mathscr{F}))\ \mathrm{st}(f(v)) \leqslant t$$

(here we have considered 5.2). Let us now observe that $v \in \mu(\mathscr{F}) \subset V$ for each standard element V of the filter $\mathscr{F}$. Therefore, inf f(V) ⩽ t [since inf f(V) ⩽ f(v) < t + ε for each ε > 0]. Hence, by virtue of the transfer principle, inf f(V) ⩽ t for each interior V from $\mathscr{F}$, which was desired.

By virtue of what we have already proved and since −f and t are standard, we deduce that

$$\inf_{V \in \mathscr{F}} \sup f(V) \geqslant t \leftrightarrow -\inf_{V \in \mathscr{F}} \sup f(V) \leqslant -t \leftrightarrow$$

$$\leftrightarrow \sup_{V \in \mathscr{F}} \inf(-f)(V) \leqslant -t \leftrightarrow (\exists v \in \mu(\mathscr{F}))\ \mathrm{st}(-f(v)) \leqslant -t \leftrightarrow (\exists v \in \mu(\mathscr{F}))\ \mathrm{st}(f(v)) \geqslant t.$$

Thus, we get

$$\inf_{V \in \mathscr{F}} \sup f(V) < t \leftrightarrow \neg\left(\inf_{V \in \mathscr{F}} \sup f(V) \geqslant t\right) \leftrightarrow \neg((\exists v \in \mu(\mathscr{F}))\ \mathrm{st}(f(v)) \geqslant t) \leftrightarrow (\forall v \in \mu(\mathscr{F}))\ \mathrm{st}(f(v)) < t.$$

Finally, on the basis of what we have proved, we conclude that

$$\inf_{V \in \mathscr{F}} \sup f(V) \leqslant t \leftrightarrow (\forall \varepsilon > 0) \inf_{V \in \mathscr{F}} \sup f(V) < t + \varepsilon \leftrightarrow (\forall^{\mathrm{st}} \varepsilon > 0)(\forall v \in \mu(\mathscr{F}))\ \mathrm{st}(f(v)) < t + \varepsilon \leftrightarrow$$

$$\leftrightarrow (\forall v \in \mu(\mathscr{F}))(\forall^{\mathrm{st}} \varepsilon > 0)\ \mathrm{st}(f(v)) < t + \varepsilon \leftrightarrow (\forall v \in \mu(\mathscr{F}))\ \mathrm{st}(f(v)) \leqslant t,$$

since the number st(f(v)) is standard. ▷

5.4. Let X and Y be standard sets, f:X × Y → $\overline{\mathbf{R}}$ be a standard function, and $\mathscr{F}$ and $\mathscr{G}$ be standard filters in X and Y, respectively. Then for each standard real number t

$$\sup_{V \in \mathscr{F}} \inf_{U \in \mathscr{G}} \sup_{u \in U} \inf_{v \in V} f(u, v) \leqslant t \leftrightarrow (\forall u \in \mu(\mathscr{G}))(\exists v \in \mu(\mathscr{F}))\ \mathrm{st}(f(u, v)) \leqslant t.$$

◁ Let us set $F_V(u) := \inf\{f(u, v) : v \in V\}$. Let us observe that F_V is a standard function, provided V is a standard set. Applying the transfer principle, Proposition 5.3, and (strong) idealization, we successively deduce that

$$\sup_{V\in\mathcal{F}}\inf_{U\in\mathcal{G}}\sup_{u\in U}\inf_{v\in V} f(u,v)\leqslant t \leftrightarrow (\forall V\in\mathcal{F})\ \inf_{U\in\mathcal{G}}\sup_{u\in U} F_V(u)\leqslant t \leftrightarrow$$

$$\leftrightarrow (\forall^{st} V\in\mathcal{F})\ \inf_{U\in\mathcal{G}}\sup F_V(U)\leqslant t \leftrightarrow (\forall^{st} V\in\mathcal{F})(\forall u\in\mu(\mathcal{G}))\quad \mathrm{st}(F_V(u))\leqslant t\leftrightarrow$$

$$\leftrightarrow (\forall u\in\mu(\mathcal{G}))(\forall^{st} V\in\mathcal{F})(\forall^{st}\varepsilon>0)\quad \inf_{v\in V} f(u,v)<t+\varepsilon\rightarrow$$

$$\rightarrow (\forall u\in\mu(\mathcal{G}))(\forall^{st}\varepsilon>0)(\forall^{st} V\in\mathcal{F})(\exists v\in V)\quad f(u,v)<t+\varepsilon\rightarrow$$

$$\rightarrow (\forall u\in\mu(\mathcal{G}))(\exists v\in\mu(\mathcal{F}))(\forall^{st}\varepsilon>0)\ f(u,v)<t+\varepsilon\rightarrow$$

$$\rightarrow (\forall u\in\mu(\mathcal{G}))(\exists v\in\mu(\mathcal{F}))\ \mathrm{st}(f(u,v))\leqslant t.$$

From the last relation, for an interior element $U\subset\mu(\mathcal{G})$ of the filter $\mathcal{F}$ and a standard element V of the filter $\mathcal{G}$ we deduce

$$\sup_{u\in U}\inf_{v\in V} f(u,v)\leqslant t\rightarrow \inf_{U\in\mathcal{G}}\sup_{u\in U}\inf_{v\in V} f(u,v)\leqslant t$$

$$\rightarrow (\forall^{st} V\in\mathcal{F})\ \inf_{U\in\mathcal{G}}\sup_{u\in U}\inf_{v\in V} f(u,v)\leqslant t\rightarrow (\forall V\in\mathcal{F})\ \inf_{U\in\mathcal{G}}\sup_{u\in U}\inf_{v\in V} f(u,v)\leqslant t$$

by virtue of the transfer principle. ▷

5.5. The above criteria enable us to give nonstandard criteria of directional derivatives and epiderivatives corresponding to infinitesimal tangent cones. Let us dwell, for illustration, on the Rockafellar epiderivative for a continuous function [4, 7].

5.6. Let $f:X\to\bar{\mathbb{R}}$ be a standard function that is continuous at a standard point x' of its effective domain dom f. Then the following statements are equivalent for each standard number t' and direction h':

(1) $(h',t')\in \mathrm{Cl}(\mathrm{epi} f, (x', f(x')))$;

(2) $f^{\uparrow}(x')h'\leqslant t'$, where $f^{\uparrow}(x')$ is the Rockafellar derivative.

(3) $(\forall x\approx_\sigma x')\ \ x\in \mathrm{dom} f\rightarrow \forall^{\cdot}\alpha\exists^{\cdot}h\ \ \mathrm{st}\left(\frac{f(x+\alpha h)-f(x)}{\alpha}\right)\leqslant t'$;

(4) The following estimate is valid:

$$\sup_{V\in\mathcal{N}_\tau}\ \inf_{\substack{U\in\mathcal{N}_\sigma(x')\\ \alpha'}}\ \sup_{\substack{x\in\mathrm{dom} f\cap U\\ 0<\alpha\leqslant\alpha'}}\ \inf_{h\in h'+V}\frac{f(x+\alpha h)-f(x)}{\alpha}\leqslant t'.$$

◁ By definition, the epigraph of the Rockafellar derivative $f^{\uparrow}(x')$ is the Clarke cone of the epigraph epi f of the function f at the point (x', f(x')), which ensures that (1) ↔ (2). Considering the construction of monads in a product, continuity of f at the point x', i.e., $x\in\mathrm{dom} f\wedge x\approx_\sigma x'\rightarrow f(x)\approx_\tau f(x')$, and the nonstandard criterion 2.3 of elements of the Clarke cone, and setting $F:=\mathrm{dom} f$, we deduce that

$$(h',t')\in\mathrm{Cl}(\mathrm{epi} f, (x', f(x')))\leftrightarrow \forall^{\cdot}x\forall^{\cdot}\alpha\exists^{\cdot}h(\exists t\approx t')\ \ (x+\alpha h, f(x)+\alpha t)\in\mathrm{epi} f$$

$$\leftrightarrow \forall^{\cdot}x\forall^{\cdot}\alpha\exists^{\cdot}h(\exists t\approx t')\quad t\geqslant s:=\frac{f(x+\alpha h)-f(x)}{\alpha}\rightarrow \forall^{\cdot}x\forall^{\cdot}\alpha\exists^{\cdot}h(\exists t\approx t')\quad \mathrm{st}(t)\geqslant\mathrm{st}(s)$$

$$\rightarrow \forall^{\cdot}x\forall^{\cdot}\alpha\exists^{\cdot}h\quad t'\geqslant\mathrm{st}(s)\rightarrow \forall^{\cdot}x\forall^{\cdot}\alpha\exists^{\cdot}h(\exists t\approx t')\quad t\geqslant s.$$

To prove the last implication, let us observe that it is obvious for st(s) = −∞. But if s is a finite number, then we set $t:=t'+|\mathrm{st}(s)-s|$. It is clear that $t'\approx t$ and $t-s\geqslant t'-\mathrm{st}(s)+s-s\geqslant 0$. Thus, by virtue of 5.4,

$$f^{\uparrow}(x')h'\leqslant t'\leftrightarrow \forall^{\cdot}x\forall^{\cdot}\alpha\exists^{\cdot}h\ \ \mathrm{st}\left(\frac{f(x+\alpha h)-f(x)}{\alpha}\right)\leqslant t'\leftrightarrow \sup_{V\in\mathcal{N}_\tau}\ \inf_{\substack{U\in\mathcal{N}_\sigma(x')\\ \alpha'}}\ \sup_{\substack{x\in U\cap\mathrm{dom} f\\ 0<\alpha\leqslant\alpha'}}\ \inf_{h\in h'+V}\frac{f(x+\alpha h)-f(x)}{\alpha}\leqslant t',$$

which completes the proof. ▷

LITERATURE CITED

1. F. Clarke, "Generalized gradients and applications," Trans. Am. Math. Soc., 205, 247-262 (1975).
2. E. Sachs, "Differentiability in optimality theory," Optimization, 9, No. 4, 497-513 (1978).

3. J.-B. Hiriart-Urruty, "Tangent cones, generalized gradients and mathematical programming in Banach spaces," Math. Oper. Res., 4, No. 1, 79-97 (1979).
4. R. T. Rockafellar, "Generalized directional derivatives and subgradients of nonconvex functions," Can. J. Math., 32, No. 2, 257-280 (1980).
5. A. D. Ioffe, "Nonsmooth analysis: differential calculus of nondifferentiable mappings," Trans. Am. Math. Soc., 266, No. 1, 1-56 (1981).
6. A. G. Kusraev and S. S. Kutateladze, "Local convex analysis," in: Current Problems in Mathematics [in Russian], R. V. Gamkrelidze (ed.), Vol. 19, Vsesoyz. Inst. Nauchn. i Tekhn. Informatsii, Moscow (1982), pp. 156-207.
7. Sz. Dolecki, "Tangency and differentiation: some applications of convergence theory," Ann. Mat. Pure Appl., 80, No. 4, 223-255 (1982).
8. J. P. Penot, "Compact filters, nets, and relations," J. Math. Anal. Appl., 93, No. 2, 400-417 (1983).
9. E. Nelson, "Internal set theory," Bull. Am. Math. Soc., 83, No. 6, 1165-1198 (1977).
10. A. Robinson, Nonstandard Analysis, North-Holland, Amsterdam—London (1970).
11. K. Hrbacek, "Axiomatic foundations for nonstandard analysis," Fund. Math., 98, No. 1, 1-24 (1978).
12. W. A. J. Luxemburg, "A general theory of monads," in: Applications of Model Theory to Algebra, Analysis, and Probability, Holt, Rinehart, and Winston, New York (1966), pp. 18-86.

SMOOTHNESS OF CONVEX SURFACES AND GENERALIZED SOLUTIONS OF MONGE—AMPÈRE EQUATION ON THE BASIS OF DIFFERENTIAL PROPERTIES OF QUASICONFORMAL MAPS

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In this article we consider questions of the connection between the order of smoothness of a two-dimensional convex surface in three-dimensional Euclidean space E^3 and the order of smoothness of its intrinsic metric.

For convex surfaces with C^n-smooth metric ($n \geqslant 2$) and positive Gaussian curvature Pogorelov [1, p. 118] proved that the surface itself belongs to the class $C^{n-1,\alpha}$ for any $0 < \alpha < 1$ (if the metric of the surface is analytic, then the surface is analytic). Sabitov [2] proved Pogorelov's theorem in Hölder classes of smoothness of the metric; it can be stated in the following form (see also [3]): *if the curvature of a convex surface in a* C^2*-smooth metric is positive and belongs to the class* $C^{n-2,\alpha}$ ($n \geqslant 2$, $0 < \alpha < 1$), *then the surface itself is* $C^{n,\alpha}$*-smooth.* Convex surfaces of bounded positive curvature were considered by the authors in [4]. In this article we assume that the curvature is positive and has an (n, α)-approximative differential at a fixed point (for the definition, see Sec. 1.1).

We state the main results of the paper.

THEOREM 1. Let $F: z = F(x, y)$, $x^2 + y^2 \leqslant r^2$ be a convex surface whose upper and lower curvatures $K^0(x, y)$, $K_0(x, y)$ ([4]) admit the estimate $0 < m \leqslant K_0(x, y) \leqslant K^0(x, y) \leqslant M$, $x^2 + y^2 \leqslant r^2$. If there is a polynomial $P_n(x, y)$, $n = 0, 1, 2, \ldots$, of degree not greater than n such that for $x^2 + y^2 \leqslant r^2$

$$|K_0(x, y) - P_n(x, y)| \leqslant C(x^2 + y^2)^{\frac{n+\alpha}{2}},$$

$$|K^0(x, y) - P_n(x, y)| \leqslant C(x^2 + y^2)^{\frac{n+\alpha}{2}}, \quad 0 < \alpha < 1,$$

then there is a polynomial $Q_{n+2}(x, y)$ of degree no greater than n + 2 such that

$$|F(x, y) - Q_{n+2}(x, y)| \leqslant C'(x^2 + y^2)^{\frac{n+2+\alpha}{2}},$$

where the constant C' depends on C, m, M, r, n, α, and $\max\limits_{x^2+y^2 \leqslant r^2} |F(x, y) - F(0, 0)|$.

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