CYCLIC MONADS AND THEIR APPLICATION

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UDC 517.11:517.43

In Boolean-valued analysis, a new class of topological spaces has been identified, with the properties of cyclicity or mixing. These objects are descents: mappings from various topological spaces into Boolean-valued models of set theories (see the works of G. Takeuti, E. I. Gordon, A. G. Kusraev, V. A. Lyubetskii, M. Osawa, and others). In the field of nonstandard analysis, A. Robinson developed monadology — a convenient method for studying filters, uniformities, topologies and so on [1, 2]. The aim of this article is to apply this theory to the study of certain cyclic topologies used in this analysis. Below we give criteria for procompact spaces and the contiguous formations associated with them. We give some applications to *o*-convergence in Kantorovich space. In this article we shall use a little-known and heterogeneous technique. Therefore, for convenience we shall accompany certain propositions by additional explanations. This, together with restrictions on the space available, have prevented the author from including a more complete bibliographical and historical background.

0. Preliminary Remarks

0.1. When considering D. Scott and R. Solovei's Boolean-valued models of set theories, we shall use in future the terminology of [3, 4], with no special mention. An explicit presentation of the foundations of Boolean-valued analysis, and suggestions for further reading, can be found in [5-8].

We emphasize that below we shall consider a fixed, complete, Boolean algebra B, and the separable Boolean-valued universe $V^{(B)}$ associated with it. We denote the estimate for the truth of the formula φ in the Zermelo—Frankel theory by the symbol $[\![\varphi]\!]$. For economy of space, we shall call strongly cyclic sets (and the hulls corresponding to them), *cyclic*.

0.2. The use of the technique of nonstandard mathematical analysis, developed by A. Robinson, means the "neoclassical" arrangement, going back to E. Nelson [9]. In other words, sets (in the Zermelo—Frankel theory) are identified with elements of the universe of internal sets, situated in a suitable world of external sets, satisfying the Zermelo axioms. Standard sets form an external class in the universe of internal sets. The Robinson standardization, i.e., the *-transform, and the corresponding world of "classical" sets are not used. Without special mention, *the ordinary proposition for "a standard entourage" will often act*, meaning that parameters in the formal notation of the text, which are not specially mentioned, are assumed to be standard sets. As usual, for a standard filter $\mathcal{F}$ we denote by the symbol $\mu(\mathcal{F})$ the *monad* $\mathcal{F}$, i.e., the external intersection of the standard elements of $\mathcal{F}$. An explicit presentation of the foundations of nonstandard analysis, and bibliographical lists can be obtained from [10, 11].

In future we shall need the concept of the *cyclic hull* mix(A) of an external set A. In the definition of mix(A) we assume that the fixed Boolean algebra B is standard, and the set A is composed of elements of $V^{(B)}$. Thus we say that the element x in $V^{(B)}$ lies in the cyclic hull mix(A) of the external subset A of the universe $V^{(B)}$, if for some interior family $(a_\xi)_{\xi\in\Xi}$ of elements of A, and some interior partition of unity $(a_\xi)_{\xi\in\Xi}$ in the algebra B, the point x is an *intermixing* of $(b_\xi)_{\xi\in\Xi}$ with probabilities $(b_\xi)_{\xi\in\Xi}$, i.e., $b_\xi x = b_\xi a_\xi$ for $\xi\in\Xi$, or equivalently, $x=\sum_{\xi\in\Xi} b_\xi a_\xi$.

0.3. The terminology we use below from topology and the theory of ordered vector space is consistent with that used in [12, 13].

1. Cyclic and Extensional Filters

1.0. In this section we give some information which is necessary and helpful (and largely obvious), about ascents and descents of filters.

Novosibirsk. Translated from Sibirskii Matematicheskii Zhurnal, Vol. 27, No. 1, pp. 100-110, January-February, 1986. Original article submitted January 23, 1985.

 0037-4466/86/2701-0080$12.50

1.1. For nonempty elements $(A_\xi)_{\xi\in\Xi}$ in the universe $V^{(B)}$ and a partition of unity $(b_\xi)_{\xi\in\Xi}$, we have

$$\Big(\sum_{\xi\in\Xi} b_\xi A_\xi\Big)\downarrow = \sum_{\xi\in\Xi} b_\xi A_\xi\downarrow.$$

Proof. Write $A := \sum_{\xi\in\Xi} b_\xi A_\xi$. Clearly, for each $\xi\in\Xi$ we have $[a\in A_\xi]\geqslant[a\in A]\wedge[A=A_\xi]=[A=A_\xi]\geqslant b_\xi$ if only $a\in A\downarrow$. In view of the transfer principle in $V^{(B)}$, we have $[a\in A_\xi]=[(\exists a_\xi\in A_\xi)a=a_\xi]$. Thus, bearing in mind the maximum principle we have $(\exists a_\xi\in A_\xi\downarrow)$ $[a\in A_\xi]=[a=a_\xi]\geqslant b_\xi$. Hence $a=\sum_{\xi\in\Xi} b_\xi a_\xi$.

Suppose that it is now known that $b_\xi a = b_\xi a_\xi$ for some $a_\xi\in A_\xi\downarrow$ and all $\xi\in\Xi$. Then, bearing in mind that $[A=A_\xi]\geqslant b_\xi$ $(\xi\in\Xi)$ by definition of intermixing, we deduce that $[a\in A]\geqslant[a=a_\xi]\wedge[a_\xi\in A_\xi]\wedge[A_\xi=A]\geqslant b_\xi$ for $\xi\in\Xi$, i.e., $[a\in A]\geqslant\bigvee_{\xi\in\Xi} b_\xi=1$ and $a\in A\downarrow$. ▷

1.2. For the cyclic sets A_ξ, where $A_\xi\in\mathscr{P}(V^{(B)})$ for $\xi\in\Xi$, we have

$$\sum_{\xi\in\Xi} b_\xi A_\xi\uparrow = \Big(\sum_{\xi\in\Xi} b_\xi A_\xi\Big)\uparrow.$$

Proof. Bearing in mind that $A_\xi\uparrow\downarrow=A_\xi$ for $\xi\in\Xi$ by the condition, on the basis of 1.1 we deduce that

$$\Big(\sum_{\xi\in\Xi} b_\xi A_\xi\uparrow\Big)\downarrow = \sum_{\xi\in\Xi} b_\xi A_\xi\uparrow\downarrow = \sum_{\xi\in\Xi} b_\xi A_\xi.$$

Hence, recalling that for a nonempty set A inside $V^{(B)}$ we have $A = A\downarrow\uparrow$, we conclude

$$\sum_{\xi\in\Xi} b_\xi A_\xi\uparrow = \Big(\sum_{\xi\in\Xi} b_\xi A_\xi\uparrow\Big)\downarrow\uparrow = \Big(\sum_{\xi\in\Xi} b_\xi A_\xi\Big)\uparrow. \triangleright$$

1.3. Let $(b_\xi)_{\xi\in\Xi}$ be some partition of unity and let the families of elements $(X_\xi)_{\xi\in\Xi}$, $(Y_\xi)_{\xi\in\Xi}$ be such that $[X_\xi\supset Y_\xi]=1$ $(\xi\in\Xi)$. Then

$$\Big[\sum_{\xi\in\Xi} b_\xi X_\xi \supset \sum_{\xi\in\Xi} b_\xi Y_\xi\Big]=1.$$

Proof. Write $X := \sum_{\xi\in\Xi} b_\xi X_\xi$ and $Y := \sum_{\xi\in\Xi} b_\xi Y_\xi$. Clearly, $[Y\subset X]\geqslant[X=X_\xi]\wedge[X_\xi\supset Y]\geqslant[X=X_\xi]\wedge[X_\xi\supset Y_\xi]\wedge[Y=Y_\xi]\geqslant b_\xi\wedge 1\wedge b_\xi=b_\xi$ for all $\xi\in\Xi$. ▷

1.4. Let X be a nonempty element of $V^{(B)}$. Then

$$[\mathscr{P}_{\rm fin}(X)=\mathscr{P}_{\rm fin}(X\downarrow)^{\uparrow\uparrow}]=1,$$

where, as usual, $\mathscr{P}_{\rm fin}(A)$ is the set of finite subsets of A and $\mathscr{P}_{\rm fin}(X\downarrow)^{\uparrow\uparrow} := \{Y\uparrow : Y\in\mathscr{P}_{\rm fin}(X\downarrow)\}\uparrow$.

Proof. The inclusion $\mathscr{P}_{\rm fin}(X\downarrow)^{\uparrow\uparrow}\subset\mathscr{P}_{\rm fin}(X)$ inside $V^{(B)}$ is obvious (an ascent of a finite set is finite). It remains to carry out the following derivation:

$$[\mathscr{P}_{\rm fin}(X)\subset\mathscr{P}_{\rm fin}(X\downarrow)^{\uparrow\uparrow}]=[(\forall n\in\mathbb{N}^\wedge)(\forall f:n\to X)f(n)\in\mathscr{P}_{\rm fin}(X\downarrow)^{\uparrow\uparrow}]=$$

$$=\bigwedge_{n\in\mathbb{N}}\bigwedge_{[f:n^\wedge\to X]=1}[f(n^\wedge)\in\mathscr{P}_{\rm fin}(X\downarrow)^{\uparrow\uparrow}]=\bigwedge_{n\in\mathbb{N}}\bigwedge_{f:n\to X\downarrow}[f\uparrow(n^\wedge)\in\mathscr{P}_{\rm fin}(X\downarrow)^{\uparrow\uparrow}]=$$

$$=\bigwedge_{n\in\mathbb{N}}\bigwedge_{f:n\to X\downarrow}[f(n)\uparrow\in\mathscr{P}_{\rm fin}(X\downarrow)^{\uparrow\uparrow}]=1. \triangleright$$

1.5. Let $\mathscr{G}$ be a basis of the filter in the set X, where $X\in\mathscr{P}(V^{(B)})$, i.e., X is a subset of $V^{(B)}$. Set

$$\mathscr{G}' := \{F\in\mathscr{P}(X\uparrow)\downarrow : (\exists G\in\mathscr{G})\ [F\supset G\uparrow]=1\};$$
$$\mathscr{G}'' := \{G\uparrow : G\in\mathscr{G}\}.$$

Then $\mathscr{G}'\uparrow$ and $\mathscr{G}''\uparrow$ are bases of the same filter $\mathscr{G}^\uparrow$ in $X\uparrow$ inside $V^{(B)}$.

Proof. We verify that $\mathscr{G}'$ is a basis of a filter in $X\uparrow$ inside $V^{(B)}$. We have

$$[(\forall F_1,F_2\in\mathscr{G}'\uparrow)(\exists F\in\mathscr{G}'\uparrow)F\subset F_1\cap F_2]=\bigwedge_{F_1,F_2\in\mathscr{G}'}[(\exists F\in\mathscr{G}'\uparrow)F\subset F_1\subset F_2].$$

If $F_1, F_2\in\mathscr{G}'$, then there exist $G_1, G_2\in\mathscr{G}$ such that $[F_1\supset G_1\uparrow]=1$ and $[F_2\supset G_2\uparrow]=1$. Take an element $G\in\mathscr{G}$ for which $G\subset G_1\cap G_2$. Then we have $(G_1\cap G_2)\uparrow\in\mathscr{G}'$ and

$$[F_1 \cap F_2 \supset (G_1 \cap G_2)\uparrow] \geqslant [F_1 \supset G_1\uparrow] \wedge [F_2 \supset G_2\uparrow] = 1.$$

Moreover, clearly $\mathcal{G}''\uparrow$ is a basis of a filter in X↑ inside $V^{(B)}$. By construction, $\mathcal{G}' \supset \mathcal{G}''$. Moreover, $\mathcal{G}'\uparrow\downarrow \supset \mathcal{G}''\uparrow\downarrow$ and thus $[\mathcal{G}'\uparrow \supset \mathcal{G}''\uparrow] = 1$. Therefore, a fortiori $[\widetilde{\mathcal{G}'\uparrow} \supset \widetilde{\mathcal{G}''\uparrow}] = 1$, where as usual $\widetilde{\mathcal{B}}$ is the set of supersets of elements of $\mathcal{B}$. In addition,

$$[(\forall F_1 \in \mathcal{G}'\uparrow)(\exists F_2 \in \mathcal{G}''\uparrow) F_1 \supset F_2] = \bigwedge_{F_1 \in \mathcal{G}'} [(\exists F_2 \in \mathcal{G}''\uparrow) F_1 \supset F_2] = 1,$$

since for $G_1 \in \mathcal{G}$ such that $[F_1 \supset G_1\uparrow] = 1$, we have $G_1\uparrow \in \mathcal{G}'\uparrow$. Thus $[\widetilde{\mathcal{G}'\uparrow} \subset \widetilde{\mathcal{G}''\uparrow}] = 1$ by the transfer principle in $V^{(B)}$.

1.6. The filter $\mathcal{G}^\uparrow$ inside $V^{(B)}$ constructed in 1.5 is called an *ascent of* $\mathcal{G}$.

1.7. Let $\mathcal{G}$ be a basis of a filter in X↓ for nonempty X in $V^{(B)}$. Moreover, let mix$(\mathcal{G})$ be the set of intermixings of nonempty families of elements of $\mathcal{G}$. Then if $\mathcal{G}$ consists of cyclic sets, then mix$(\mathcal{G})$ is a basis of a filter in X↓ and mix$(\mathcal{G}) \supset \mathcal{G}$. Moreover, we have the equation $\mathcal{G}^\uparrow = \mathrm{mix}(\mathcal{G})^\uparrow$.

<u>Proof.</u> Let $U, V \in \mathrm{mix}(\mathcal{G})$. This means that we have sets Ξ and H, partitions of unity $(b_\xi)_{\xi\in\Xi}$, $(c_\eta)_{\eta\in H}$ and families $(U_\xi)_{\xi\in\Xi}$, $(V_\eta)_{\eta\in H}$ of elements of $\mathcal{G}$, for which $b_\xi U = b_\xi U_\xi$ $(\xi \in \Xi)$ and $c_\eta V = c_\eta V_\eta$ $(\eta \in H)$. Let $W_{(\xi,\eta)} \subset U_\xi \cap V_\eta$ be some element in the basis $\mathcal{G}$. Set $d_{(\xi,\eta)} := b_\xi \wedge c_\eta$. Clearly, $(d_{(\xi,\eta)})_{(\xi,\eta)\in\Xi\times H}$ is a partition of unity. Consider $W := \Sigma_{(\xi,\eta)\in\Xi\times H} d_{(\xi,\eta)} W_{(\xi,\eta)}$, i.e., the set of corresponding intermixings of elements $W_{(\xi,\eta)}$. Clearly, $d_{(\xi,\eta)} U = b_\xi c_\eta U = c_\eta b_\xi U_\xi \supset d_{(\xi,\eta)} W_{(\xi,\eta)}$ and analogously $d_{(\xi,\eta)} V \supset d_{(\xi,\eta)} W_{(\xi,\eta)}$. By the same token, $W \subset U \cap V$ and $W \in \mathrm{mix}(\mathcal{G})$.

Since $\mathcal{G}$ consists of cyclic sets, then bearing in mind 1.2 and 1.3, it is clear that $\mathrm{mix}(\mathcal{G})' = \mathrm{mix}(\mathcal{G}')$, which completes the proof.

1.8. For the filter $\mathcal{F}$ in X inside $V^{(B)}$, set $\mathcal{F}^\downarrow := \widetilde{\{F\downarrow : F \in \mathcal{F}\downarrow\}}$. We call the filter $\mathcal{F}^\downarrow$ in X↓ the *descent* of $\mathcal{F}$. A basis of the filter $\mathcal{G}$ in X↓ is called *extensional*, if $\mathcal{G}^{\uparrow\downarrow} = \widetilde{\mathcal{G}}$. A basis of the filter $\mathcal{G}$ in X↓ is called *cyclic*, if $\widetilde{\mathcal{G}}$ has a basis of cyclic sets. (We note that in [14], extensional filters were called cyclic).

1.9. The filter $\mathcal{F}$ is extensional if and only if $\mathcal{F}$ is cyclic and $\mathcal{F} = \widetilde{\mathrm{mix}(\mathcal{F})}$.

<u>Proof.</u> This follows from 1.2, 1.3, and 1.7.

1.10. For extensional filters $\mathcal{F}$ and $\mathcal{G}$ in X↓, we have $\mathcal{F} \supset \mathcal{G} \leftrightarrow [\mathcal{F}^\uparrow \supset \mathcal{G}^\uparrow] = 1$.

<u>Proof.</u> If $\mathcal{F} \supset \mathcal{G}$, then $\mathcal{F}' \supset \mathcal{G}'$ and moreover $[\mathcal{F}^\uparrow \supset \mathcal{G}^\uparrow] = 1$. Hence $\mathcal{F}^\uparrow\downarrow \supset \mathcal{G}^\uparrow\downarrow$, i.e., $\mathcal{F}^{\uparrow\downarrow} \supset \mathcal{G}^{\uparrow\downarrow}$. It remains to refer to 1.8.

1.11. The maximal elements in the set of extensional filters are called *proultrafilters*.

1.12. Maximal elements in the set of cyclic filters are proultrafilters.

<u>Proof.</u> If $\mathcal{A}$ is a proultrafilter and $\mathcal{F}$ is a cyclic filter majorizing it, then $\mathcal{A} \subset \mathcal{F} \subset \mathrm{mix}(\mathcal{F})$. Hence $\mathcal{A} = \mathcal{F}$. Conversely, let $\mathcal{A}$ be a maximal cyclic filter. Then $\mathcal{A} = \mathrm{mix}(\mathcal{A})$ and thus $\mathcal{A}$ is a proultrafilter.

1.13. The proultrafilters in X↓ are precisely the descents of ultrafilters in X.

<u>Proof.</u> This is a direct corollary of 1.8.

1.14. We have the following statements:

(1) if f:X → Y inside $V^{(B)}$ and $[\mathcal{F}$ is a filter in X] = 1, then

$$f(\mathcal{F})^\downarrow = f\downarrow(\mathcal{F}^\downarrow);$$

(2) for an extensional mapping f:X↓ → Y↓ and a filter $\mathcal{F}$ in X↓, we have

$$f(\mathcal{F})^\uparrow = f\uparrow(\mathcal{F}^\uparrow);$$

(3) the image of an extensional filter under an extensional mapping is extensional;

(4) the image of proultrafilter under an extensional mapping is a proultrafilter.

<u>Proof.</u> (1) Using the definition and property of the descent f↓ of the mapping f, we have

$$G \in f(\mathcal{F})^\downarrow \leftrightarrow (\exists U \in f(\mathcal{F})\downarrow) G \supset U\downarrow \leftrightarrow (\exists F \in \mathcal{F}\downarrow) G \supset f(F)\downarrow \leftrightarrow$$
$$\leftrightarrow (\exists F \in \mathcal{F}\downarrow) G \supset f\downarrow(F\downarrow) \leftrightarrow (\exists F \in \mathcal{F}^\downarrow) G \supset f\downarrow(F) \leftrightarrow G \in f\downarrow(\mathcal{F}^\downarrow).$$

(2) Using the properties of the ascent $f\uparrow$, we have the estimates

$$[G \in f\uparrow(\mathscr{F}^{\uparrow})] = [(\exists U \in f\uparrow(\mathscr{F}^{\uparrow}))G \supset U] = [(\exists F \in \mathscr{F}^{\uparrow})G \supset f\uparrow(F)] =$$
$$= \bigvee_{F \in \mathscr{F}} [G \supset f\uparrow(F\uparrow)] = \bigvee_{F \in \mathscr{F}} [G \supset f(F)\uparrow] =$$
$$= \bigvee_{U \in f(\mathscr{F})'} [G \supset U] = [(\exists U \in f(\mathscr{F})'\uparrow)G \supset U] = [(\exists U \in f(\mathscr{F})^{\uparrow})G \supset U] = [G \in f(\mathscr{F})^{\uparrow}].$$

(3) Applying (2) and (1) successively, we have

$$f(\mathscr{F})^{\uparrow\downarrow} = f\uparrow(\mathscr{F}^{\uparrow})^{\downarrow} = f\uparrow\downarrow(\mathscr{F}^{\uparrow\downarrow}) = f(\mathscr{F}^{\uparrow\downarrow}).$$

This equation ensures the required statement.

(4) If $f: X\downarrow \to Y\downarrow$ is an extensional mapping and $\mathscr{F}$ is a proultrafilter, then $\mathscr{F}^{\uparrow}$ is an ultrafilter in X inside $V^{(B)}$. Therefore, $f\uparrow(\mathscr{F}^{\uparrow})$ is an ultrafilter in Y inside $V^{(B)}$. By the same token, $f\uparrow(\mathscr{F}^{\uparrow})^{\downarrow}$ is a proultrafilter. It remains to note that $f\uparrow(\mathscr{F}^{\uparrow})^{\downarrow} = f(\mathscr{F}^{\uparrow\downarrow}) = f(\mathscr{F})$ in view of (3).

2. Cyclic Monads, Essential and Proideal Points

2.0. In this section we give a test for a filter to be cyclic, and we introduce some concepts connected with this which will be useful in future.

2.1. The monad $\mu(\mathscr{F})$ of the filter $\mathscr{F}$ is called *cyclic*, if it coincides with its cyclic hull $\operatorname{mix}(\mu(\mathscr{F}))$.

2.2. Nonstandard Criterion for a Filter to Be Cyclic. A standard filter is cyclic if and only if its monad is cyclic.

Proof. Let $\mathscr{F}$ be a standard filter. Suppose that it is cyclic. Take an internal set Ξ and an internal partition of unity $(b_\xi)_{\xi\in\Xi}$ and a family of points $(x_\xi)_{\xi\in\Xi}$ of the monad $\mu(\mathscr{F})$. By the condition, $\mathscr{F}$ has a basis $\mathscr{G}$ of cyclic sets, and therefore $\mu(\mathscr{F}) = \bigcap\{G : G \in {}^\circ\mathscr{G}\}$, where as usual ${}^\circ\mathscr{G}$ is the set of standard elements of $\mathscr{G}$. If x is the intermixing $(x_\xi)_{\xi\in\Xi}$ with probabilities $(b_\xi)_{\xi\in\Xi}$, then x lies in each standard G in $\mathscr{G}$ (since $x_\xi \in G$ for $\xi \in \Xi$). Moreover, $\mu(\mathscr{F}) \supset \operatorname{mix}(\mu(\mathscr{F})) \supset \mu(\mathscr{F})$.

If it is known already that the monad $\mu(\mathscr{F})$ is a cyclic external set, then taking an infinitely small element $F \in \mathscr{F}$ [i.e., such that $F \subset \mu(\mathscr{F})$], we see that $F_0 := \operatorname{mix}(F) \subset \operatorname{mix} \times (\mu(\mathscr{F})) \subset \mu(\mathscr{F})$. Thus the internal set F_0 is infinitely small and lies in $\mathscr{F}$. Thus $(\forall^{st} F \in \mathscr{F}) \times (\exists F_0 \in \mathscr{F})(F_0 = \operatorname{mix}(F_0) \wedge F \supset F_0)$. By the Leibniz rule, we deduce that $\mathscr{F}$ has a cyclic basis.

2.3. THEOREM. For a standard filter $\mathscr{F}$ in $X\downarrow$, set

$$\mathscr{F}\uparrow\downarrow := \widetilde{\{F\uparrow\downarrow : F \in \mathscr{F}\}}.$$

Then $\operatorname{mix}(\mu(\mathscr{F})) = \mu(\mathscr{F}\uparrow\downarrow)$ and $\mathscr{F}\uparrow\downarrow$ is the greatest cyclic filter which is cruder than $\mathscr{F}$.

Proof. Clearly, $\mathscr{F}\uparrow\downarrow \subset \mathscr{F}$ and thus by 2.2 $\mu(\mathscr{F}\uparrow\downarrow) \supset \mu(\mathscr{F})$ and $\mu(\mathscr{F}\uparrow\downarrow) \supset \operatorname{mix}(\mu(\mathscr{F}))$. Now let $x \in \mu(\mathscr{F}\uparrow\downarrow)$. By the definition of a monad and the properties of intermixing, we have

$$(\forall^{st} F \in \mathscr{F})(\exists (b_\xi)_{\xi\in\Xi})(\exists (x_\xi)_{\xi\in\Xi})(\forall \xi \in \Xi) x_\xi \in F \wedge b_\xi x = b_\xi x_\xi.$$

Clearly, by the same token we have

$$(\forall^{st\,fin} \mathscr{F}_0 \subset \mathscr{F})(\exists (b_\xi)_{\xi\in\Xi})(\exists (x_\xi)_{\xi\in\Xi})(\forall F_0 \in \mathscr{F}_0)\cdot$$
$$\cdot(\forall \xi \in \Xi)(x_\xi \in F \wedge b_\xi x_\xi = b_\xi x).$$

Applying the principle of idealization in its strong form, we have

$$(\exists (b_\xi)_{\xi\in\Xi})(\exists (x_\xi)_{\xi\in\Xi})(\forall^{st} F \in \mathscr{F})(\forall \xi \in \Xi)(x_\xi \in F \wedge b_\xi x_\xi = b_\xi x).$$

The latter means that there exist elements $(x_\xi)_{\xi\in\Xi}$ in the monad $\mu(\mathscr{F})$, such that $x = \sum_{\xi\in\Xi} b_\xi x_\xi$, i.e., $x \in \operatorname{mix}(\mu(\mathscr{F}))$. Finally, we obtain the equation $\mu(\mathscr{F}\uparrow\downarrow) = \operatorname{mix}(\mu(\mathscr{F}))$.

Now let $\mathscr{G}$ be a cyclic filter, where $\mathscr{G} \subset \mathscr{F}$. Moreover, $\operatorname{mix}(\mu(\mathscr{G})) = \mu(\mathscr{G}) \supset \operatorname{mix}(\mu(\mathscr{F})) = \mu(\mathscr{F}\uparrow\downarrow)$. Thus $\mathscr{G} \subset \mathscr{F}\uparrow\downarrow$.

2.4. Let x be an internal point in $X\downarrow$. We define a standard filter (x) in $X\downarrow$ by the relation

$$(x) := {}^*\{U \subset X\!\downarrow : x \in U\},$$

where * is the *symbol of standardization*. Thus (x) consists precisely of those standard subsets of X↓ which contain x. The element x is called an *essential point of* X↓ [written $x \in e(X)$], if $(x)\!\uparrow\downarrow$ is a proultrafilter in X↓.

2.5. Each point x in a monad of a standard proultrafilter $\mathcal{F}$ is essential. Moreover, we have the equations

$$\mathcal{F} = (x)^{\uparrow\downarrow} = (x)\!\uparrow\downarrow = {}^*\{U\!\uparrow\downarrow : x \in U \wedge U \subset X\!\downarrow\}.$$

Proof. Since (see [2]) the monad $\mu(\mathcal{F})$, by the condition, affects the monad of the ultrafilter (x), then $(x) \supset \mathcal{F}$. Therefore $(x)\!\uparrow\downarrow \supset \mathcal{F}\!\uparrow\downarrow = \mathcal{F}$. On the basis of 1.12, we deduce that $\mathcal{F} = (x)\!\uparrow\downarrow$. In view of 1.5, we have the equation $(x)\!\uparrow\downarrow\uparrow = (x)^{\uparrow}$. Thus by 1.13 x is an essential point. Finally, $(x)^{\uparrow\downarrow} = \mathcal{F}^{\uparrow\downarrow} = \mathcal{F} = (x)\!\uparrow\downarrow$.

2.6. The image of an essential point under an extensional mapping is an essential point in the image.

Proof. Let x be an essential point of X↓ and let f:X↓ → Y↓ be an extensional mapping. There exists a proultrafilter $\mathcal{F}$ such that $x \in \mu(\mathcal{F})$. Clearly, $f(x) \in f(\mu(\mathcal{F})) = \mu(f(\mathcal{F}))$. In fact, bearing in mind strong idealization

$$y \in \mu(f(\mathcal{F})) \leftrightarrow (\forall^{st} F \in \mathcal{F})\, y \in f(F) \leftrightarrow$$
$$\leftrightarrow (\forall^{st\,fin} \mathcal{F}_0 \subset \mathcal{F})\,(\exists x)\,(\forall F \in \mathcal{F}_0)\, x \in F \wedge y = f(x) \leftrightarrow$$
$$\leftrightarrow (\exists x)\,(\forall^{st} F \in \mathcal{F})\, x \in F \wedge y = f(x) \leftrightarrow (\exists x \in \mu(\mathcal{F}))\, y = f(x) \leftrightarrow$$
$$\leftrightarrow y \in f(\mu(\mathcal{F})).$$

It remains to refer to 1.14.

2.7. Let E be some standard set and let X be a standard element of $V^{(B)}$. Consider the product $X^{E^\wedge}$ inside $V^{(B)}$, where $E^\wedge$ is the standard name of E in $V^{(B)}$. If x is an essential point of $X^{E^\wedge}\!\downarrow$, then for any standard $e \in E$, the point $x\!\downarrow(e)$ is essential in X↓.

Proof. As $x \in X^{E^\wedge}\!\downarrow$, then $[x : E^\wedge \to X] = 1$, i.e., $x\!\downarrow : E \to X\!\downarrow$ and for any $e \in E$ we have $[x\!\downarrow(e) = x(e^\wedge)] = 1$, by definition of the descent x↓.

Consider the mapping taking the element $x \in X^{E^\wedge}\!\downarrow$ into the point $x(e^\wedge)$ in X↓ for a fixed standard $e \in E$. Clerly, for $x_1, x_2 \in X^{E^\wedge}\!\downarrow$ we have

$$[x_1 = x_2] = [(\forall e \in E^\wedge)\, x_1(e) = x_2(e)] = \bigwedge_{e \in E} [x_1(e^\wedge) = x_2(e^\wedge)] \leqslant [x_1(e^\wedge) = x_2(e^\wedge)],$$

i.e., the standard mapping introduced above is extensional. On the basis of 2.8, we conclude that $x(e^\wedge)$ is an essential point in X↓. It remains to recall that $x\!\downarrow(e) = x(e^\wedge)$ by definition of descent.

2.8. Let $\mathcal{F}$ be a cyclic filter in X↓ and let ${}^e\mu(\mathcal{F}) := \mu(\mathcal{F}) \cap e(X)$ be the set of essential points of its monad. Then

$${}^e\mu(\mathcal{F}) = {}^e\mu(\mathcal{F}^{\uparrow\downarrow}).$$

Proof. Let $x \in {}^e\mu(\mathcal{F})$. Thus x lies in the monad of some proultrafilter $\mathcal{G}$. Hence $\mu(\mathcal{G}) \cap \mu(\mathcal{F}) \neq \varnothing$ and therefore $\mathcal{G} \supset \mathcal{F}$. Bearing in mind 1.10, $\mathcal{G}^{\uparrow\downarrow} \supset \mathcal{F}^{\uparrow\downarrow}$ and $x \in \mu(\mathcal{G}) \subset \mu(\mathcal{F}^{\uparrow\downarrow})$. If it is now known that $x \in {}^e\mu(\mathcal{F}\!\uparrow\downarrow)$, then there exists an ultrafilter $\mathcal{G}$ in X inside $V^{(B)}$ such that $x \in \mu(\mathcal{G}^{\downarrow})$ and $\mathcal{G} \supset \mathcal{F}^{\uparrow}$. Since $\mathcal{F} = \mathcal{F}\!\uparrow\downarrow \subset \mathcal{F}^{\uparrow\downarrow} \subset \mathcal{G}^{\downarrow}$ in view of 1.7, then $\mu(\mathcal{F}) \supset \mu(\mathcal{G}^{\downarrow})$. Hence $x \in {}^e\mu(\mathcal{F})$.

2.9. Let A be a subset of the descent X↓ we are considering. The set $(X \backslash A\!\uparrow)\!\downarrow$ is called the *procomplement* or the *cyclic complement* of A and is denoted by A^c. The point $x \in X\!\downarrow$ is called *proideal*, if x lies in the procomplement of each finite standard subset of X↓. We denote the set of all proideal points of X↓ by p(X).

2.10. If the set X↓ has no proideal points, then X is a finite set inside $V^{(B)}$.

Proof. By the idealization principle, there exists a finite standard set Y in X↓ such that $T^c = \emptyset$. Thus $[X \backslash Y\!\uparrow = \varnothing\!\uparrow] = 1$, i.e., X = Y↑.

2.11. If X is an infinite set inside $V^{(B)}$, then the proideal points of X↓ form a cyclic monad. The ascent of the cyclic filter with monad p(X) is the filter of the complements of the finite subsets of X inside $V^{(B)}$.

Proof. The procomplements of the finite subsets of $X\downarrow$ form a basis of a filter. In fact, since $(Y\cup Z)\uparrow\downarrow \supset Y\uparrow\downarrow \cup Z\uparrow\downarrow$, then $(Y\cup Z)\uparrow \supset Y\uparrow \cup Z\uparrow$ and $[X\setminus(Y\cup Z)\uparrow \subset X\setminus(Y\uparrow\cup Z\uparrow)]=1$. Thus $(Y\cup Z)^c \subset (X\setminus Y\uparrow)\downarrow \cap (X\setminus Z\uparrow)\downarrow = Y^c \cap Z^c$. Hence, on the basis of 2.2 p(X) is a cyclic monad. Denote by ${}_p\mathcal{F}$ the filter with the monad p(X), i.e., the filter of procomplements of the finite sets of $X\downarrow$. Moreover, let ${}_{cf}\mathcal{F}(X)$ be the filter of the complements of the finite sets in X inside $V^{(B)}$ (a cofinite filter in X). Bearing in mind 1.4, we have

$$[Y\in_{cf}\mathcal{F}(X)] = [(\exists Z\in\mathcal{P}_{\text{fin}}(X))\, Y\supset X\setminus Z] =$$
$$= \bigvee_{A\in\mathcal{P}_{\text{fin}}(X\downarrow)} [Y\supset X\setminus A\uparrow] = \bigvee_{A\in\mathcal{P}_{\text{fin}}(X\downarrow)} [Y\supset A^c\uparrow] = \bigvee_{Z\in{}_p\mathcal{F}} [Y\supset Z\uparrow] = [Y\in {}_p\mathcal{F}^\uparrow].$$

Thus ${}_{cf}\mathcal{F}(X) = {}_p\mathcal{F}^\uparrow$.

3. Transforms of Precompact and Compact Spaces

3.0. In this section, we apply cyclic monads to obtain the necessary descriptions of descents – transformations of topological spaces into Boolean-valued models of set theories. The idea of the following results is closely connected with the classical works of Robinson [1] and Luxemburg [2]. Below, for simplicity we shall always consider an internal (in the sense of $V^{(B)}$) nonempty uniform space $(X, \mathcal{U})$. The ordinary assumption of "standard entourage" also acts in this section, i.e., in particular when we are using nonstandard methods, B, X, $\mathcal{U}$ and so on will be assumed to be standard sets. Since this is understood, we shall write $x \approx y$ instead of $(x, y)\in\mu(\mathcal{U}^\downarrow)$.

3.1. The uniform space $(X\downarrow, \mathcal{U}^\downarrow)$ is called *precompact*, if $(X, \mathcal{U})$ is compact inside $V^{(B)}$. An analogous meaning is attached to the term *"procomplete boundedness,"* and so on. We shall sometimes use the terminology "cyclic compactness."

3.2. Nonstandard Criteria for Precompactness. For a standard space X, the following statements are equivalent:

(1) $X\downarrow$ is a procompact space;

(2) each essential point of $X\downarrow$ is nearly-standard;

(3) each essential point of $X\downarrow$ is nearly-standard.

Proof. (1) → (2). Let x be an essential point of $X\downarrow$. Then x lies in the monad of the proultrafilter $(x)^{\uparrow\downarrow}$. Thus inside $V^{(B)}$ it is true that there exists an element $y \in X$ such that $(x)^\uparrow$ converges to y. In view of the maximum principle and Leibnitz' rule (in the internal world), we may conclude that there exists a standard element $y\in X\downarrow$ such that $(x)^{\uparrow\downarrow}\supset \mathcal{U}^\downarrow(y)$. Hence it follows that $\mu((x)^{\uparrow\downarrow})\subset\mathcal{U}^\downarrow(y)$, and thus $x \approx y$. In other words, x is nearly standard point.

(2) → (3) is obvious.

(3) → (1). We need to verify that an ultrafilter in X inside $V^{(B)}$ has a point of tangency. Without loss of generality, we may assume that $\mathcal{F}$ is not a principal ultrafilter. Therefore $\mathcal{F}$ is finer than the filter of complements of finite sets inside $V^{(B)}$. Referring to 2.6, we see that $\mu(\mathcal{F}^\downarrow)\subset p(X)$. If $x\in\mu(\mathcal{F}^\downarrow)$, then on the basis of 2.8 $\mathcal{F}=(x)^\uparrow$, and moreover x is an essential point. By the condition, such a point is nearly standard, i.e., there exists standard $y\in X\downarrow$, such that $\mathcal{U}^\downarrow(y)\cap\mu(\mathcal{F}^\downarrow)\neq\varnothing$. By the same token, y is a tangency point of $\mathcal{F}$ inside $V^{(B)}$.

3.3. From Theorem 3.2, it is easy to see the difference between a Boolean-valued criterion for procompactness, and the usual one: "a compact space is a space with nearly standard points." The existence of a colossal number of procompact and noncompact spaces ensures a variety of examples of nonstandard and nonideal points. We also note here that the simultaneous application of 3.2 and 2.7 enables us to give a nonstandard proof for the natural analog of Tikhonov's theorem for a product of procompact spaces – "the descent of Tikhonov's theorem into $V^{(B)}$."

3.4. Nonstandard Criterion for Proprecompactness. A standard space is the descent of a totally bounded uniform space, if and only if each essential point of it is pre-nearly standard.

Proof. → Let x be an essential point of $X\downarrow$. Then $(x)^\uparrow$ is an ultrafilter inside $V^{(B)}$, and thus $(x)^\uparrow$ is a Cauchy filter in X, in view of the total boundedness of X in $V^{(B)}$. The descent of a Cauchy filter is a Cauchy filter in a descent. Thus x is an element of the monad of a Cauchy filter, i.e., x is a pre-nearly standard.

← Take an ultrafilter $\mathcal{F}$ in X inside $V^{(B)}$. We need to show that $\mathcal{F}$ is a Cauchy filter in $V^{(B)}$. Take a point x in the monad of the descent $\mathcal{F}\downarrow$. Then x is essential, and therefore pre-nearly standard. Thus the microhalo of x, i.e., the set $\mathcal{U}\downarrow(x)$, is a monad of a Cauchy filter. By the same token, $\mathcal{F}\downarrow$ is a Cauchy filter.

4. Ordered Convergence

4.0. In this section, we apply cyclic monads to the description of o-convergence in a K-space Y. To save space, we shall only consider filters containing ordered intervals (or, equivalently, filters with bounded monads). In addition, in accordance with our stated aims we shall assume that the K-space Y is extended. On the basis of Gordon's theorem [15], the space Y can be canonically realized as the descent $\mathcal{R}\downarrow$ of the element $\mathcal{R}$, which is the field of real numbers **R** in the Boolean-valued universe $V^{(B)}$, constructed over the *basis* B of the space Y. We shall use the symbol E to denote the ordered *filter of unity* in Y, i.e., $E:=\{\varepsilon\in Y_+ : [\varepsilon=0]=0\}$. The notation $x\approx y$ expresses the *infinite closeness* of the elements x and $y\in Y$, generated by the descent of the ordinary topology of $\mathcal{R}$ in $V^{(B)}$, i.e., $x\approx y\leftrightarrow(\forall^{st}\varepsilon\in E)|x-y|<\varepsilon$. Here, and in future, we shall assume that $a<b$ for a, $b\in Y$ if $[a<b]=1$, i.e., $a>b\leftrightarrow a-b\in E$. Thus we have a deviation from the conventions of the theory of ordered vector spaces. Clearly, this is because of the need to observe the principles of introducing notation for descents and ascents.

4.1. Let ${}^{\approx}Y$ be the *nearly standard part* of Y, i.e., the microhalo of the external set of standard elements of Y. For $y\in{}^{\approx}Y$, we denote by the symbol $^\circ y$ [or st(y)] the *standard part* of y, i.e., the natural standard element which is infinitely close to y.

4.2. We have the following propositions:

(1) for $x, y\in{}^{\approx}Y$ and $\alpha\in{}^{\approx}\mathbf{R}$, we have

$$^\circ(x+y)={}^\circ x+{}^\circ y;\ {}^\circ(x\vee y)={}^\circ x\vee{}^\circ y;$$
$$^\circ(\alpha x)={}^\circ\alpha{}^\circ x;\ x\leqslant y\rightarrow{}^\circ x\leqslant{}^\circ y;$$

(2) for $z\in{}^\circ Y$ and $y\in{}^{\approx}Y$, we have

$$^\circ y\leqslant z\leftrightarrow(\forall^{st}\varepsilon>0)y<z+\varepsilon\leftrightarrow(\forall^{st}\varepsilon>0)y\leqslant z+\varepsilon;$$

(3) an essential point of Y has a standard part if and only if it lies in some standard interval.

Proof. (1) Operations in Y are descents of operations in $\mathcal{R}$, where they are obviously continuous.

(2) In view of the above, we have

$$^\circ y\leqslant z\rightarrow(\forall^{st}\varepsilon>0)|y-{}^\circ y|<\varepsilon\wedge{}^\circ y\leqslant z\rightarrow(\forall^{st}\varepsilon>0)y<z+\varepsilon\rightarrow$$
$$\rightarrow(\forall^{st}\varepsilon>0)y\leqslant z+\varepsilon\rightarrow(\forall^{st}\varepsilon>0){}^\circ y\leqslant{}^\circ(z+\varepsilon)\rightarrow(\forall^{st}\varepsilon>0){}^\circ y\leqslant z+\varepsilon\rightarrow{}^\circ y\leqslant z.$$

(3) If y is nearly standard, then in view of (2) it is obvious that $y\in[{}^\circ y-1^\wedge, {}^\circ y+1^\wedge]$. Conversely, if the essential point y lies in the standard interval $[a, b]$, then in view of the procompactness of $[a, b]$, y is nearly standard by 3.2.

4.3. For each filter $\mathcal{F}$ we have the relations

$$\inf_{F\in\mathcal{F}}\sup F=\inf_{F\in\mathcal{F}^{\uparrow\downarrow}}\sup F=\inf_{F\in\mathcal{F}^{\uparrow}}\sup F=\inf_{F\in\mathcal{F}^{\uparrow\downarrow}}\sup F;$$
$$\sup_{F\in\mathcal{F}}\inf F=\sup_{F\in\mathcal{F}^{\Uparrow}}\inf F=\sup_{F\in\mathcal{F}^{\uparrow}}\inf F=\sup_{F\in\mathcal{F}^{\Uparrow}}\inf F.$$

Proof. The required equations are obvious. In fact (see [16, 17]) we have

$$\inf_{F\in\mathcal{F}^{\uparrow}}\sup F=\inf\{\sup F : F\in\mathcal{F}^{\uparrow}\}=\inf\{\sup F\uparrow : F\in\mathcal{F}\}\uparrow,$$

since we have the following relations for estimates of truth:

$$[z\in\{\sup F : F\in\mathcal{F}^{\uparrow}\}]=[(\exists F\in\mathcal{F}^{\uparrow})z=\sup F]=\bigvee_{F\in\mathcal{F}}[z=\sup F\uparrow]=$$
$$=[(\exists y\in\{\sup F\uparrow : F\in\mathcal{F}\}\uparrow)z=y]=[z\in\{\sup F\uparrow : F\in\mathcal{F}\}\uparrow].$$

It remains to recall that $\sup F=\sup F\uparrow=\sup F\uparrow\downarrow$, and apply 2.3.

4.4. THEOREM. For a standard filter $\mathcal{F}$ in Y and standard $z \in$ Y, we have the following propositions:

(1) $\inf_{F\in\mathcal{F}} \sup F\leqslant z \leftrightarrow (\forall y\in {}^{\cdot}\mu(\mathcal{F}\Updownarrow))^{\circ}y\leqslant z \leftrightarrow (\forall y\in{}^{e}\mu(\mathcal{F}\Updownarrow))^{\circ}y\leqslant z$

(2) $\sup_{F\in\mathcal{F}} \inf F\geqslant z \leftrightarrow (\forall y\in {}^{\cdot}\mu(\mathcal{F}\Updownarrow))^{\circ}y\geqslant z \leftrightarrow (\forall y\in{}^{e}\mu(\mathcal{F}\Updownarrow))^{\circ}y\geqslant z;$

(3) $\inf_{F\in\mathcal{F}} \sup F\geqslant z \leftrightarrow (\exists y\in {}^{\cdot}\mu(\mathcal{F}\Updownarrow))^{\circ}y\geqslant z \leftrightarrow (\exists y\in{}^{e}\mu(\mathcal{F}\Updownarrow))^{\circ}y\geqslant z;$

(4) $\sup_{F\in\mathcal{F}} \inf F\leqslant z \leftrightarrow (\exists y\in {}^{\cdot}\mu(\mathcal{F}\Updownarrow))^{\circ}y\leqslant z \leftrightarrow (\exists y\in{}^{e}\mu(\mathcal{F}\Updownarrow))^{\circ}y\leqslant z;$

(5) $\mathcal{F}^{\circ}\to z \leftrightarrow (\forall y\in{}^{e}\mu(\mathcal{F}\Updownarrow))y\approx z \leftrightarrow ((\forall y\in\mu(\mathcal{F}^{\Updownarrow}))\,y\approx z.$

Here ${}^{\cdot}\mu(\mathcal{F}\uparrow\downarrow):=\mu(\mathcal{F}\uparrow\downarrow)\cap{}^{\approx}Y$, and as usual, ${}^{e}\mu(\mathcal{F}\uparrow\downarrow)$ is the set of essential points of the monad $\mu(\mathcal{F}\uparrow\downarrow)$, i.e., ${}^{e}\mu(\mathcal{F}\uparrow\downarrow)=\mu(\mathcal{F}\uparrow\downarrow)\cap{}^{e}(\mathcal{R})$.

Proof. It is sufficient to verify (1) and (3).

(1) Suppose that $\inf_{F\in\mathcal{F}}\sup F\leqslant z$, and bearing in mind the inclusion ${}^{e}\mu(\mathcal{F}\uparrow\downarrow)\subset{}^{\cdot}\mu(\mathcal{F}\uparrow\downarrow)$, consider $y\in{}^{\cdot}\mu(\mathcal{F}\uparrow\downarrow)$. If we take an arbitrary $y'\in\mu(\mathcal{F})$, then for any standard bounded $F\in\mathcal{F}$ we have $y'\leqslant\sup F$. On the basis of 2.3, for $y'\in\mu(\mathcal{F}\uparrow\downarrow)$ we also have $y'\leqslant\sup F$. Thus for all standard F in $\mathcal{F}$ we have ${}^{\circ}y\leqslant\sup F$. By Leibnitz' rule, this relation holds for any $F\in\mathcal{F}$. Thus ${}^{\circ}y\leqslant\inf_{F\in\mathcal{F}}\sup F\leqslant z$.

Suppose that it is now known that for each $y\in{}^{e}\mu(\mathcal{F}\uparrow\downarrow)$ we have ${}^{\circ}y\leqslant z$ (the latter, of course, is satisfied if $(\forall y\in{}^{\cdot}\mu(\mathcal{F}\uparrow\downarrow))\,{}^{\circ}y\leqslant z$). Consider an arbitrary standard proultrafilter, containing $\mathcal{F}^{\uparrow\downarrow}$. Choose some infinitely small element U of this proultrafilter $\mathcal{G}$. By the condition, $(\forall u\in U)^{\circ}u\leqslant z$, since in view of 2.8 $\mu(\mathcal{G})\subset{}^{e}\mu(\mathcal{F}^{\uparrow\downarrow})={}^{e}\mu(\mathcal{F}\uparrow\downarrow)$. On the basis of 4.3, for each standard $\varepsilon\in$E we have $(\forall u\in U)u<z+\varepsilon$. In other words, $(\forall^{st}\varepsilon\in E)(\exists U\in\mathcal{G})(\forall u\in U)$ $u<z+\varepsilon$. Bearing in mind the standardness of z and the other parameters, using Leibnitz' rule we obtain

$$(\forall\varepsilon\in E)(\exists U\in\mathcal{G})\sup U\leqslant z+\varepsilon\to\bigwedge_{\varepsilon\in E}[(\exists U\in\mathcal{G}^{\uparrow})\sup U\leqslant z+\varepsilon]=1\to$$

$$\to[(\forall\varepsilon>0)(\exists U\in\mathcal{G}^{\uparrow})\sup U\leqslant z+\varepsilon]=1\to\Big[(\forall\varepsilon>0)\inf_{U\in\mathcal{G}^{\uparrow}}\sup U\leqslant z+\varepsilon\Big]=1\to$$

$$\to\Big[\inf_{U\in\mathcal{G}^{\uparrow}}\sup U\leqslant z\Big]=1\to\inf_{U\in\mathcal{G}}\sup U\leqslant z.$$

Once again applying Leibnitz' rule, we see that for any proultrafilter $\mathcal{G}$, containing $\mathcal{F}^{\uparrow\downarrow}$, we have $\inf_{U\in\mathcal{G}}\sup U\leqslant z$. In the field of real numbers **R**, for the filter $\mathcal{A}$ and the number t, it is clear that

$$\inf_{A\in\mathcal{A}}\sup A\leqslant t\leftrightarrow(\forall\mathcal{G}\supset\mathcal{A})\Big(\mathcal{G}\text{ is an ultrafilter}\to\inf_{G\in\mathcal{G}}\sup G\leqslant t\Big)$$

(the simple nonstandard proof is easily obtained, for example from [18]). Thus by the transfer principle of Boolean analysis

$$\Big[\inf_{F\in\mathcal{F}^{\uparrow}}\sup F\leqslant z\Big]=\bigwedge_{\substack{\mathcal{G}\supset\mathcal{F}^{\Updownarrow}\\ \mathcal{G}\text{ is a pro-ultrafilter}}}\Big[\inf_{G\in\mathcal{G}^{\uparrow}}\sup G\leqslant z\Big]=1.$$

It remains to refer to 4.3.

(3) First, in the wider set ${}^{\cdot}\mu(\mathcal{F}\uparrow\downarrow)$, let there exist an element y such that ${}^{\circ}y\geqslant z$. For any standard $F\in\mathcal{F}$ we have $y\in F\uparrow\downarrow$. Thus for $\varepsilon\in{}^{\circ}$E we have $y>z-\varepsilon$ and $\sup F=\sup F\uparrow\downarrow>z-\varepsilon$. By Leibnitz' rule we see that $(\forall^{st}F\in\mathcal{F})(\forall^{st}\varepsilon>0)\sup F\geqslant z$, i.e., $(\forall F\in\mathcal{F})\sup F\geqslant z$ and $\inf_{F\in\mathcal{F}}$ sup F $\geqslant$ z.

In order to prove the relations which we have not yet verified, we first note that in view of the properties of the upper limit in **R**, the transfer principle of Boolean-valued analysis and Proposition 4.3, we have

$$\Big[(\exists\mathcal{G})(\mathcal{G}\text{ is an ultrafilter in }\mathcal{R}\wedge\mathcal{G}\supset\mathcal{F}^{\uparrow})\wedge\inf_{G\in\mathcal{G}}\sup G\geqslant z\Big]=1.$$

On the basis of the maximum principle, there exists a proultrafilter $\mathcal{G}$ such that $\mathcal{G}\supset\mathcal{F}^{\uparrow\downarrow}$ and $\inf_{G\in\mathcal{G}}\sup G\geqslant z$. Using the transfer principle and idealization, bearing in mind 4.3, we then obtain successively

$$
\begin{gathered}
(\forall^{st} G \in \mathcal{G}) \sup G \geqslant z \leftrightarrow (\forall^{st} G \in \mathcal{G})[\sup G{\uparrow} \geqslant z] = 1 \leftrightarrow \\
\leftrightarrow (\forall^{st} G \in \mathcal{G})[(\forall \varepsilon > 0)(\exists g \in G{\uparrow}) g > z - \varepsilon] = 1 \leftrightarrow (\forall^{st} G \in \mathcal{G})(\forall \varepsilon > 0)(\exists g \in G{\uparrow}{\downarrow}) g > z - \varepsilon \leftrightarrow \\
\leftrightarrow (\forall^{st} G \in \mathcal{G})(\forall^{st} \varepsilon > 0)(\exists g \in G{\uparrow}{\downarrow}) g > z - \varepsilon \leftrightarrow (\forall^{st\,fin} \mathcal{G}_0 \subset \mathcal{G})(\forall^{st\,fin} \mathrm{E}_0 \subset \mathrm{E})(\exists g) \\
(\forall G \in \mathcal{G}_0)(\forall \varepsilon \in \mathrm{E}_0)(g \in G{\uparrow}{\downarrow} \wedge g > z - \varepsilon) \leftrightarrow (\exists g)(\forall^{st} G \in \mathcal{G})(\forall^{st} \varepsilon > 0)(g \in G{\uparrow}{\downarrow} \wedge g > z - \varepsilon) \leftrightarrow \\
\leftrightarrow (\exists g \in \mu(\mathcal{G}^{\uparrow\downarrow}))\, {}^{\circ}g \geqslant z \leftrightarrow (\exists g \in \mu(\mathcal{G}))\, {}^{\circ}g \geqslant z.
\end{gathered}
$$

It remains to refer to 2.8 and to note that $\mu(\mathcal{G}) \subset {}^{e}\mu(\mathcal{F}^{\uparrow\downarrow}) = {}^{e}\mu(\mathcal{F}{\uparrow}{\downarrow}) \subset \mu(\mathcal{F}{\uparrow}{\downarrow})$.

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