CRITERIA FOR SUBDIFFERENTIALS THAT REPRESENT CAPS AND FACES

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The present article is related with the highly peculiar geometry of convex sets in a space of operators. As a rule, the cones of positive operators are devoid of extremal rays (and, therefore, of caps), and, in general, the subdifferentials are not compact in any locally convex topology, but, at the same time, are regenerated by eigensubsets of their Choquet boundaries [1-3]. The nature of the noted effects, restricting the possibility of direct application of the standard methods of the convexity theory, is revealed in Boolean-valued analysis. It turns out that the existing obstacles are apparent in well-known sense: They can be overcome by a suitable choice of the Boolean-valued model, in which we should carry out the investigation. Detailing of this position for problems of study of the inner structure of subdifferentials — of the strictly operator-convex pointwise o-closed and weakly order-bounded sets — has been carried out in [4] (see also [5]). The aim of the present article is to weaken the condition for boundedness in the spirit of the cap theory [6, 7]. The speciality of the approach, to be developed, consists in the introduction of operator caps, which are not so in the classical sense in nontrivial situations (although coincide with them in the scalar case). We give criteria for subdifferentials to be caps and faces of operator sets. In this connection, a significant effect is exposed. Faces (and extreme points), formed by subdifferentials, are "extensional," but the caps are not so. More precisely, not the ordinary caps, but the operator caps, i.e., the subdifferentials that represent descents — representations of scalar caps in a suitable model of set theory, are suited for the study of operator sets.

1. Well-Capped Sets

1.1. Let X be a real vector space, Y be an extended K-space, and U be an operator-convex pointwise o-closed set in the space $\mathscr{L}(X, Y)$ of linear operators from X into Y. (See [8] for the terminology.)

1.2. A subset C of U is called an <u>operator cap of U</u> or, more descriptively, a <u>descended cap</u> of U, if C is a subdifferential, i.e., a weakly order-bounded pointwise o-closed operator-convex set, and, moreover, for arbitrary $x, y \in U$ and multipliers $\alpha, \beta \in [0, 1_Y]$ in the space Y such that $\alpha + \beta = 1_Y$ and $\alpha x + \beta y \in C$ there exists an element b of the basis $B := B(Y)$ of the space Y, for which $bx \in bC$ and $b'y \in b'C$ (here, as usual, $b' := 1_Y - b$ is the complementary projection of b).

1.3. For a set $W \subset \mathscr{L}(X, Y)$ we set $W\uparrow := \{A\uparrow : A \in W\}\uparrow$, where, as always, $\uparrow$ is <u>the symbol of lift</u> in a separable Boolean-valued universe $V^{(B)}$, constructed over B [5]. Thus in particular, $A\uparrow$ is a mapping of the standard name $X^\wedge$ of the space X into the lift $Y\uparrow$ of the considered space Y, which we, according to the Gordon theorem, canonically identify with the <u>descent</u> of the element $\mathscr{R}$ in $V^{(B)}$, representing the field of real numbers. Let us observe that $Y\uparrow = \mathscr{R}$ and $\mathscr{R}\downarrow = Y$.

1.4. <u>Proposition</u>. A subdifferential C is an operator cap of the set U if and only if $C\uparrow$ is a cap of the set $U\uparrow$ in $V^{(B)}$.

$\triangleleft$ Using the rule for the realization of descents and lifts, we carry out the following truth estimates:

$$[C\uparrow \text{ is a cap of } U\uparrow] = [(\forall \alpha \geqslant 0)(\forall \beta \geqslant 0)(\forall x \in U^\uparrow)(\forall y \in U^\uparrow)$$

$$(\alpha + \beta = 1^\wedge \wedge \alpha x + \beta y \in C^\uparrow) \to (x \in C^\uparrow \vee y \in C^\uparrow)] = \bigwedge_{\substack{\alpha \geqslant 0, \beta \geqslant 0 \\ \alpha + \beta = 1_Y}} \bigwedge_{\substack{x, y \in U \\ \alpha x + \beta y \in C}} [x\uparrow \in C^\uparrow] \vee [y\uparrow \in C^\uparrow].$$

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If C is a descended cap, then for multipliers $\alpha, \beta \in [0, 1_Y]$ such that $\alpha + \beta = 1_Y$ and points $x, y \in U$ such that $\alpha x + \beta y \in C$ there exists a projection $b \in B$, for which $x \in bC$ and $y \in b'C$. In other words, $x = bx'$ and $y = b'y'$ for suitable x' and y' in C. It is clear that

$$[x\uparrow \in C^\uparrow] \geq [x\uparrow \in (bC)^\uparrow] \wedge [(bC)^\uparrow = C^\uparrow] = [bC^\uparrow = C^\uparrow] \geq b;$$

$$[y\uparrow \in C^\uparrow] \geq [y\uparrow \in (b'C)^\uparrow] \wedge [(b'C)^\uparrow = C^\uparrow] = [b'C^\uparrow = C^\uparrow] \geq b'.$$

Therefore, by virtue of the above estimate, $[C^\uparrow$ is a cap of $U^\uparrow] = 1_Y$.

If, in its turn, it is known that $C^\uparrow$ is a cap of $U^\uparrow$ in $V^{(B)}$, then for necessary choice of the parameters α, β, x, and y, on the basis of the above computations, we have $[x\uparrow \in C^\uparrow] \vee [y\uparrow \in C^\uparrow] = 1_Y$. Therefore, $[x\uparrow \in C^\uparrow] \geq b$ and $[y\uparrow \in C^\uparrow] \geq b'$ for suitable $b \in B$. Hence, by virtue of the maximum principle, we deduce the existence of x' and y' in $C^\uparrow\downarrow$ such that $[x\uparrow = x'] \geq b$ and $[y\uparrow = y'] \geq b'$, i.e., $bx\uparrow = bx'$ and $b'y\uparrow = b'y'$. The last equations precisely mean that $x \in bC$ and $y \in b'c$. ▷

1.5. A set is said to be well-capped if it can be expressed as a union of its descended caps. The set $\{x + \alpha(y - x)\colon\ \alpha \in Y_+\}$ in the space $\mathscr{L}(X, Y)$ is called the operator ray from x to y (or a Y-ray).

1.6. Theorem. The following statements are valid:

(1) Each well-capped set is the pointwise o-closure of the strictly operator-convex hull of the set of its extreme points and the extremal operator rays.

(2) The set U is well-capped if and only if the cone H_U, formed by the pointwise o-limits of the extensional nets of elements of the set $\{(\alpha T, \alpha) \in \mathscr{L}(X, Y) \times Y\colon\ \alpha \geq 0,\ T \in U\}$, is well-capped.

◁ (1) Let U be the set under consideration. By virtue of the assumptions made by us, $U^\uparrow$ is a convex subset of the space $\mathscr{L}(X, Y)^\uparrow$ and coincides, as is easily seen, with the space $X^{\wedge\#}$ of the linear forms over $X^\wedge$ in $V^{(B)}$. In addition, $U^\uparrow$ is closed in the multinorm $\{T \to |Tx|\colon\ x \in X^\wedge\}$ in $V^{(B)}$. Using the well-known criteria for subdifferentials [4, 5] and Proposition 1.4, we see that $U^\uparrow$ is well-capped in $V^{(B)}$. Therefore, by the appropriate Choquet theorem [6], $U^\uparrow$ is the convex closure of the set of its extreme points and extremal rays. Using descents, we arrive at the desired statement.

(2) It is clear that the lift $\{(\alpha T, \alpha)\colon\ \alpha \geq 0,\ T \in U\}^\uparrow$ is the conical hull of $U^\uparrow \times 1^\wedge$ in $V^{(B)}$. Hence it is obvious that the set H_U, in which we are interested, is such that $H_U^\uparrow$ is the Hörmander transform of the set $U^\uparrow$ in $V^{(B)}$. Using Proposition 1.4 and a well-known scalar result of [7], we arrive at the desired conclusion. ▷

2. Criteria for Caps

2.1. In connection with Theorem 1.6, it is sufficient to formulate the explicit criteria for caps for the more convenient case of cones of positive operators.

2.2. Proposition. A closed set C is a cap of the cone of positive elements in a topological ordered vector space if and only if for all $c_1 \geq 0$ and $c_2 \geq 0$ such that $c_1 + c_2 \in C$ there exist $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ such that $\alpha_1 + \alpha_2 = 1$, $c_1 \in \alpha_1 C$, $c_2 \in \alpha_2 C$.

◁ → At first, let it be known that C is a cap, $c = c_1 + c_2$, $c_1 \geq 0$, $c_2 \geq 0$, and $c \in C$. Let us suppose that $\lambda c_1 \notin C$ and $\lambda c_2 \notin C$ for any $\lambda > 1$. Then by the definition of a cap, $t := \lambda^{-1}(\lambda c_1) + (1 - \lambda^{-1})\lambda/(\lambda - 1)c_2 \notin C$. At the same time, $t = c$, which contradicts our supposition. Thus, there exists a $\lambda > 1$ such that one of the elements λc_1 and λc_2 belongs to C. For definiteness, we assume that it is λc_1. Let us set $\lambda_0 := \sup\{\lambda > 0\colon \lambda c_1 \in C\}$. Then $\lambda_0 > 1$ and $\lambda c_1 \notin C$ for any $\lambda > \lambda_0$. Since $\lambda^{-1}(\lambda c_1) + (1 - \lambda^{-1})\lambda/(\lambda - 1)c_2 \in C$, it follows that $\lambda/(\lambda - 1)c_2 \in C$ provided $\lambda > \lambda_0$. Since C is closed, we conclude that $\lambda_0/(\lambda_0 - 1)c_2 \in C$. Hence $c_2 \in (\lambda_0 - 1)/\lambda_0 C$ and $c_1 \in 1/\lambda_0 C$.

← Now let $\alpha_1 c_1 + \alpha_2 c_2 \in C$ and none of c_1 and c_2 belong to C for $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_1 + a_2 = 1$ and $c_1, c_2 \geq 0$. If the stated condition is fulfilled, then $\alpha_1 c_1 = \gamma_1 d_1$ and $\alpha_2 c_2 = \gamma_2 d_2$ for certain $\gamma_1 \geq 0$, $\gamma_2 \geq 0$, $\gamma_1 + \gamma_2 = 1$ and $d_1, d_2 \in C$. Since $c_1 = \gamma_1/\alpha_1 d_1$ and $c_2 = \gamma_2/\alpha_2 d_2$, it follows that $\gamma_1/\alpha_1 > 1$ and $\gamma_2/\alpha_2 > 1$. At the same time, the inequality $\gamma_1/\alpha_1 > 1$ ensures that $\gamma_2 = 1 - \gamma_1 < 1 - \alpha_1 = \alpha_2$. We have arrived at a contradiction, i.e., at least one of the points c_1 and c_2 belongs to C. Finally, we conclude that C is a cap. ▷

2.3. Proposition. Let p be a monotone positive sublinear functional on an ordered vector space (X, X_+). The subdifferential ∂p is a cap of the cone $X_+^{\#}$ of positive forms on X if and only if each of the following statements is valid:

(1) $\inf\{p(z): z \geq x_1, z \geq x_2\} = p(x_1) \vee p(x_2)$ for arbitrary $x_1, x_2 \in X$.

(2) The conical segment $\{p < 1\}$ is filtered with respect to growth.

(3) $\bigcap\limits_{\substack{x_1, x_2 \in \{p<1\} \\ \varepsilon>0}} [x_1, \to) \cap [x_2, \to) \cap \{p \leqslant 1+\varepsilon\} \neq \varnothing.$

◁ At first, let it be known that ∂p is a cap in $X_+^{\#}$. From the general rules of the subdifferential calculus, we deduce that

$$\partial((x_1, x_2) \to \inf\{p(z): z \geqslant x_1, z \geqslant x_2\}) = \{(l_1, l_2) \in X^{\#2}: l_1 \geqslant 0, l_2 \geqslant 0, l_1 + l_2 \in \partial p\};$$

$$\partial((x_1, x_2) \to p(x_1) \vee p(x_2)) = \{(\alpha_1 l_1, \alpha_2 l_2): \alpha_1 \geqslant 0, \alpha_2 \geqslant 0, \alpha_1 + \alpha_2 = 1, l_1 \in \partial p, l_2 \in \partial p\}.$$

It remains to take Proposition 2.2 into account and deduce the implications (1) → (2) and (1) → (3).

Now, to complete the proof it is sufficient to verify the implications (2) → (1) and (3) → (1).

Thus, let $t := p(x_1) \vee p(x_2)$ under the conditions (2). Then $(t + \varepsilon)^{-1}x_1 \in \{p < 1\}$ and $(t + \varepsilon)^{-1}x_2 \in \{p < 1\}$ for each $\varepsilon > 0$. By the conditions, $z \geq (t + \varepsilon)^{-1}x_1$, $z \geq (t + \varepsilon)^{-1}x_2$ and $p(z) < 1$ for a certain $z \in X$. Let us set $z_0 := (t + \varepsilon)z$. It is obvious that $p(z_0) = (t + \varepsilon)p(z) < t + \varepsilon$. Hence we deduce that

$$p(x_1) \vee p(x_2) \leqslant \inf\{p(z): z \geqslant x_1, x_2\} \leqslant p(z_0) \leqslant p(x_1) \vee p(x_2) + \varepsilon.$$

Since ε is arbitrary, we conclude that (1) follows from (2). The remaining implication (3) → (1) is verified in the same simple manner. ▷

2.4. Corollary. The following statements are equivalent:

(1) p is the upper envelope of support functions of caps;

(2) p is the upper envelope of discrete functionals;

(3) p is the Minkowski functional of an approximately filtered conical segment, i.e., of an intersection of the filtered (with respect to growth) sets.

◁ (1) → (2) is ensured by the fact that the extreme points of a cap of the cone of the positive forms are discrete functionals.

(2) → (3), if $p(x) = \sup\{p_\xi(x): \xi \in \Xi\}$ for $x \in X$, where $p_\xi(x) = T_\xi(x)^+$ and T_ξ is a suitable discrete functional for each ξ. It is clear that p is the Minkowski functional of the intersection $S := \bigcap\limits_{\xi \in \Xi} \{p_\xi < 1\}$ and, therefore, S is approximately directed by virtue of Proposition 2.3.

(3) → (1) follows from the general properties of the Minkowski functional and 2.3. ▷

2.5. Now let X be an ordered vector space and $\mathscr{L}_+(X, Y)$ be the cone of the positive linear operators that act from X into Y.

2.6. Theorem. The following statements are equivalent for each increasing positive sublinear operator $P: X \to Y$:

(1) The subdifferential ∂P is a descended cap of the cone $\mathscr{L}_+(X, Y)$.

(2) For all $x_1, x_2 \in X$

$$\inf\{Pz: z \geqslant x_1, z \geqslant x_2\} = Px_1 \vee Px_2.$$

(3) If $A_1, A_2 \in \mathscr{L}_+(X, Y)$ are such that $A_1 + A_2 \in \partial P$, then there exist multipliers $\alpha_1, \alpha_2 \in [0, 1_Y]$, for which $\alpha_1 + \alpha_2 = 1_Y$ and, moreover, $A_1 \in \alpha_1 \partial P$ and $A_2 \in \alpha_2 \partial P$.

(4) For arbitrary $x_1, x_2 \in X$ such that $Px_1 \leq 1_Y$ and $Px_2 \leq 1_Y$ and arbitrary $\varepsilon > 0$ there exist a partition of unity $(b_\xi)_{\xi \in \Xi}$ and a family $(z_\xi)_{\xi \in \Xi}$ of elements of X that satisfy the relations

$$z_\xi \geqslant x_1, \ z_\xi \geqslant x_2, \ b_\xi P z_\xi \leqslant (1+\varepsilon) b_\xi \quad (\xi \in \Xi).$$

(5) The lift $\partial P^{\uparrow}$ is a cap of the cone of the positive forms on the standard name $X^{\wedge}$ of the space X in the Boolean-valued universe $V^{(B)}$, constructed over the basis B of the considered K-space Y.

(6) The set $\{P\uparrow < 1\}$ is filtered with respect to growth in $V^{(B)}$.

◁ By virtue of 1.6, we have (1) ↔ (5) since $\partial P\uparrow = \partial P^{\uparrow}$ in $V^{(B)}$. The equivalences (1) ↔ (2) ↔ (6) are ensured by 2.3 and the transfer principle of the Boolean-valued analysis and the equivalence (1) ↔ (4) is ensured by Proposition 2.3 since

$$[\partial P\uparrow \text{ is a cap}] =$$
$$= [(\forall x_1, x_2 \in X^{\wedge})(\forall \varepsilon > 0)(\exists z \in X^{\wedge})\, z \geq x_1 \wedge z \geq x_2 \wedge P\uparrow z \leq 1^{\wedge} + \varepsilon] =$$
$$= \bigwedge_{x_1, x_2 \in X} [(\exists z \in X^{\wedge})\, z \geq x_1^{\wedge} \wedge z \geq x_2^{\wedge} \wedge Pz \leq 1_Y + \varepsilon].$$

It remains to use the estimation rule and the Gordon theorem [5]. Finally, the equivalence (2) ↔ (3) is ensured, e.g., by the rules of subdifferentiation. ▷

2.6. Corollary. The extreme points of an operator cap of the cone of positive operators are discrete operators.

2.7. Corollary. An increasing positive sublinear operator P is the pointwise supremum of a set of discrete operators if and only if the lift $P\uparrow$ is the Minkowski functional of an approximately filtered conical segment in a Boolean-valued universe.

3. Criteria for Faces

3.1. We pass to the characteristics of the subdifferentials that are faces. In this section, X and Y may be assumed to be modules over a single lattice-ordered ring $\mathbf{A}$ with identity [3, 9]. The sublinear operator P is assumed to be $\mathbf{A}_+$-homogeneous. We begin with the analysis of a generalized notion of cap. To this end, we assume X to be an ordered module and P to be an increasing positive operator. Further, let Z be also an ordered $\mathbf{A}$-module that admits convex analysis and T be a positive module homomorphism from Y into Z.

3.2. The following statements are equivalent:

(1) For all $x_1, x_2 \in X$

$$\inf\{TPz: \ z \geq x_1,\ z \geq x_2\} = T(Px_1 \vee Px_2).$$

(2) For aribtrary $A_1, A_2 \in \mathscr{L}_+(X, Z)$ such that $A_1 + A_2 \in \partial(TP)$, there exist homomorphisms T_1 and T_2 in $\mathscr{L}_+(Y, Z)$ such that

$$T_1 + T_2 = T, \quad A_1 \in \partial(T_1P), \quad A_2 \in \partial(T_2P).$$

◁ Let

$$Q_1(x_1, x_2) := \inf\{TPz: \ z \geq x_1, x_2\}; \quad Q_2(x_1, x_2) := T(Px_1 \vee Px_2).$$

It is clear that Q_1 and Q_2 are sublinear operators and $Q_1 \geq Q_2$. Thus, the equality in (1), in which we are interested, can be rewritten as the inclusion $\partial Q_1 \subset \partial Q_2$. It remains to observe that for the homomorphism $(A_1, A_2):(x_1, x_2) \to A_1x_1 + A_2x_2$, where $A_i \in \mathscr{L}_+(X, Z)$ for $i := 1, 2$,

$$(A_1, A_2) \in \partial Q_1 \leftrightarrow A_1 \geq 0,\ A_2 \geq 0,\ A_1 + A_2 \in \partial(TP);$$
$$(A_1, A_2) \in \partial Q_2 \leftrightarrow (\exists T_1 \geq 0, T_2 \geq 0)\, T_1 + T_2 = T,$$
$$A_1 \in \partial(T_1P), \quad A_2 \in \partial(T_2P).$$

Indeed, the last equivalences are verified by direct computation [8]. ▷

3.3. The operator P (and its subdifferential ∂P), satisfying the equivalent conditions, formulated in 3.2, is called a T-cap of the semimodule $\mathscr{L}_+(X, Y)$.

3.4. The following statements are valid:

(1) Each T-cap is an S-cap for $S \in [0, T]$.

(2) Let P be a T-cap and $TA \in \mathrm{Ch}(TP)$ for $A \in \partial P$ (i.e., A is a T-extreme point of ∂P). Then $[0, TA] = [0, T]A$.

◁ (1) For the sake of symmetry, let us set $T' := S$ and $T'' := T - S'$. We have

$$0 \leq \inf\{T'Pz\colon z \geq x_1, x_2\} - T'(Px_1 \vee Px_2) + \inf\{T''Pz\colon z \geq x_1, x_2\} - \\ - T''(Px_1 \vee Px_2) \leq \inf\{(T' + T'')Pz\colon z \geq x_1, x_2\} - T(Px_1 \vee Px_2) = 0.$$

(2) Let $0 \leq S \leq TA$. On the basis of 3.2, there exist $T_1 \geq 0$ and $T_2 \geq 0$ such that $T_1 + T_2 = T$, $S \in \partial(T_1P)$, and $TA - S \in \partial(T_2P)$. Thus, $2TA = (S + T_2A) + ((TA - S) + T_1A)$, i.e., $T_1A = S$ and $S \in [0, T]A$, since $S + T_2A \in \partial(TP)$ and $(TA - S) + T_1A \in \partial(TP)$. ▷

3.5. Theorem. The following statements are equivalent:

(1) The subdifferential $\partial(TQ)$ is a face of the subdifferential $\partial(TP)$.

(2) For arbitrary homomorphisms $T_1, T_2 \in \mathscr{L}(X, Y)$ and $A_1, A_2 \in \mathscr{L}(X, Z)$ such that

$$T_1 \geq 0, \quad T_2 \geq 0, \quad T_1 + T_2 = T;$$
$$A_1 \in \partial(T_1P), \quad A_2 \in \partial(T_2P), \quad A_1 + A_2 \in \partial(TQ),$$

the relations $A_1 \in \partial(T_1Q)$ and $A_2 \in \partial(T_2Q)$ are valid.

(3) The operator $(x, y) \to y + Q(-x)$, acting from the module $X \times Y$ with the semimodule $\operatorname{epi}P := \{(x, y) \in X \times Y\colon y \geq Px\}$ into the module Y, is a T-cap.

(4) For all $x_1, x_2 \in X$

$$\inf_{z \in X} T((P(x_1 - z) + Qz - Qx_1) \vee (P(x_2 - z) + Qz - Qx_2)) = 0.$$

◁ (1) → (2). Let the homomorphisms T_1, T_2, A_1 and A_2 be selected according to the conditions (2). Let us consider an element S of the subdifferential ∂Q. The following relations are obviously valid:

$$A_1 + T_2S \in \partial(TP), \quad A_2 + T_1S \in \partial(TP);$$
$$(A_1 + T_2S) + (A_2 + T_1S) = (A_1 + A_2) + TS \in 2\partial Q.$$

Therefore, on the basis of (1), the homomorphism $A_1 + T_2S$ belongs to $\partial(TQ)$, i.e., $A_1x + T_2Sx \leq TQx$ for all $x \in X$. Hence

$$A_1x + T_2Qx = \sup\{A_1x + T_2Sx\colon S \in \partial Q\} \leq TQx$$

for each $x \in X$. Consequently, $A_1 \in \partial(T_1Q)$. It is analogously established that $A_2 \in \partial(T_2Q)$ (since $A_2 + T_2S \in \partial(TQ)$ for each $S \in \partial Q$).

(2) → (3). Let us set $\mathscr{P}(x, y) := y + Q(-x)$ and take the homomorphisms $\mathscr{A}_1, \mathscr{A}_2 \in \mathscr{L}_+(X \times Y, Z)$ such that $\mathscr{A}_1 + \mathscr{A}_2 \in \partial(T\mathscr{P})$. We set $T_iy := \mathscr{A}_i(0, y)$ for $i := 1, 2$ and $y \in Y$. It is clear that $T_1 \geq 0$ and $T_2 \geq 0$, since $0 \times Y_+ \in \operatorname{epi}P$. Moreover, $(T_1 + T_2)y = \mathscr{A}_1(0, y) + \mathscr{A}_2(0, y) \leq T(y + Q(0)) = Ty$ for all $y \in Y$. Therefore, $T_1 + T_2 = T$. It remains to verify that $\mathscr{A}_1 \in \partial(T_1\mathscr{P})$ and $\mathscr{A}_2 \in \partial(T_2\mathscr{P})$. Let us set $A_ix := \mathscr{A}_i(-x, 0)$ for $x \in X$. Then we get

$$\mathscr{A}_i(x, Px) = T_iPx + \mathscr{A}_i(x, 0) = T_iPx - \mathscr{A}_i(-x, 0) = T_iPx - A_ix \geq 0.$$

Therefore, $A_i \in \partial(T_iP)$ for $i := 1, 2$. Moreover, $(A_1 + A_2)x = (\mathscr{A}_1 + \mathscr{A}_2)(0, -x) \leq T\mathscr{P}(-x, 0) = TQx$. By virtue of (2), we conclude that $A_i \in \partial(T_iQ)$. Thus,

$$\mathscr{A}_i(x, y) = T_iy - A_ix \leq T_iy + T_iQ(-x) = T_i\mathscr{P}(x, y)$$

for all $(x, y) \in X \times Y$. By the same token, $\mathscr{P}$ is a T-cap.

(3) → (4). Considering the definition of a T-cap, for arbitrary $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ we get

$$T((y_1 + Q(-x_1)) \vee (y_2 + Q(-x_2))) = \\ = \inf_{x,y}\{T(y + Q(-x))\colon y - y_1 \geq P(x - x_1), y - y_2 \geq P(x - x_2)\}.$$

Hence, since T is positive, we get

$$T((y_1 + Qx_1) \vee (y_2 + Qx_2)) = \\ = \inf_{z \in X} T((y_1 + P(z + x_1) + Q(-z)) \vee (y_2 + P(z + x_2) + Q(-z))) = \\ = \inf_{z \in X} T((y_1 + P(x_1 - z) + Qz) \vee (y_2 + P(x_2 - z) + Qz)).$$

Setting $y_1 := Qx_2$ and $y_2 := Qx_1$, we arrive at (4).

(4) → (1). Let A_1 and A_2 be elements of $\partial(TP)$ such that $A_1 + A_2 \in 2\partial(TQ)$. For x_1, $x_2 \in X$ and arbitrary $z \in X$ we have

$$A_1x_1 + A_2x_2 = A_1(x_1 - z) + A_2(x_2 - z) + (A_1 + A_2)z \leqslant$$
$$\leqslant TP(x_1 - z) + TP(x_2 - z) + TQz_1 - TQx_1 + TQz - TQx_2 + TQx_1 + TQx_2.$$

Taking infimum with respect to z, we deduce that

$$A_1x_1 + A_2x_2 \leqslant \inf_{z \in X} (T(P(x_1 - z) + Qz - Qx_1) + T(P(x_2 - z) + Qz - Qx_2)) + TQx_1 + TQx_2 \leqslant$$
$$\leqslant 2 \inf_{z \in X} T((P(x_1 - z) + Qz - Qx_1) \vee (P(x_2 - z) + Qz - Qx_1)) + TQx_1 + TQx_2.$$

Taking (4) into account, we conclude that $A_1 \in \partial(TQ)$ and $A_2 \in \partial(TQ)$. ▷

3.6. <u>Remark</u>. If Z is a K-space, then the equivalent statements of Theorem 5 are equivalent to the statement that the lift $\partial(TQ)^{\uparrow}$ is a face of the lift $\partial(TP)^{\uparrow}$ in the Boolean-valued universe constructed over the basis Z.

3.7. <u>Remark</u>. The part (1) ↔ (4) of Theorem 3.5 is a generalization of the well-known Buck–Phelps–Jolley criterion for extreme points [10]. As an application of it, we give a criterion for a face in the language of representing measures.

3.8. Let $\mathscr{A}$ be a weakly order-bounded set in $\mathscr{L}(X, Y)$ and $Px := \sup\{Ax: A \in \mathscr{A}\}$. Further, let $Q: X \to Y$ be a sublinear operator such that $Q \leq P$. As usual, let $\ell_\infty(\mathscr{A}, Y)$ denote the set of the bounded Y-valued functions on $\mathscr{A}$, equipped with the natural structure of an ordered module. Let us consider <u>the canonical sublinear operator</u>

$$\varepsilon_{\mathscr{A}}: l_\infty(\mathscr{A}, Y) \to Y, \quad \varepsilon_{\mathscr{A}}f := \sup f(\mathscr{A}) \quad (f \in l_\infty(\mathscr{A}, Y))$$

and the mappings connected with it

$$\langle \mathscr{A} \rangle \in \mathscr{L}(X, l_\infty(\mathscr{A}, Y)), \quad \langle \mathscr{A} \rangle x: A \to Ax \quad (A \in \mathscr{A});$$
$$\Delta_{\mathscr{A}}: Y \to Y^{\mathscr{A}}, \quad \Delta_{\mathscr{A}}y: A \to y \quad (A \in \mathscr{A}).$$

We know that the following representation is valid for each increasing sublinear operator $R: Y \to Z$:

$$\partial(R\varepsilon_{\mathscr{A}}) = \{\beta \in \mathscr{L}_+(l_\infty(\mathscr{A}, Y), Z): \beta\Delta_{\mathscr{A}} \in \partial R\}.$$

Let us also observe that $P = \varepsilon_{\mathscr{A}}\langle \mathscr{A} \rangle$ by definition.

3.9. <u>Theorem</u>. The set $\partial(TQ)$ is a face of $\partial(TP)$ is and only if for each $\beta \in \mathscr{L}_+(\ell_\infty(\mathscr{A}, Y), Z)$ such that $\beta\Delta_{\mathscr{A}} = T$ and $\beta\langle \mathscr{A} \rangle \in \partial(TQ)$,

$$\beta((\Delta_{\mathscr{A}}Qx_1 - \langle \mathscr{A} \rangle x_1) \wedge (\Delta_{\mathscr{A}}Qx_2 - \langle \mathscr{A} \rangle x_2)) \geqslant 0$$

for all $x_1, x_2 \in X$, or, equivalently,

$$\beta(\langle \mathscr{A} \rangle x - \Delta_{\mathscr{A}}Qx)_+ = 0$$

for each $x \in X$.

◁ Using Theorem 3.5, we deduce the following criterion for a face:

$$0 = \inf_{z \in X} T((\varepsilon_{\mathscr{A}}\langle \mathscr{A} \rangle(x_1 - z) + Qz - Qx_1) \vee (\varepsilon_{\mathscr{A}}\langle \mathscr{A} \rangle(x_2 - z) + Qz - Qx_2)) =$$
$$= \inf_{z \in X} T\varepsilon_{\mathscr{A}}((\Delta_{\mathscr{A}}Qz - \langle \mathscr{A} \rangle z + \langle \mathscr{A} \rangle x_1 - \Delta_{\mathscr{A}}Qx_1) \vee$$
$$\vee (\Delta_{\mathscr{A}}Qz - \langle \mathscr{A} \rangle z + \langle \mathscr{A} \rangle x_2 - \Delta_{\mathscr{A}}Qx_2)) = \inf_{z \in X} T\varepsilon_{\mathscr{A}}((\Delta_{\mathscr{A}}Qz - \langle \mathscr{A} \rangle z) + (\langle \mathscr{A} \rangle x_1 - \Delta_{\mathscr{A}}Qx_1) \vee (\langle \mathscr{A} \rangle x_2 - \Delta_{\mathscr{A}}Qx_2)).$$

If $\beta \geq 0$, $\beta\Delta_{\mathscr{A}} = T$ and $\beta\langle \mathscr{A} \rangle \in \partial(TQ)$, then

$$0 \geqslant \inf_{z \in X}(\beta\Delta_{\mathscr{A}}Qz - \beta\langle \mathscr{A} \rangle z) + \beta((\langle \mathscr{A} \rangle x_1 - \Delta_{\mathscr{A}}Qx_1) \vee (\langle \mathscr{A} \rangle x_2 - \Delta_{\mathscr{A}}Qx_2)) =$$
$$= \beta((\langle \mathscr{A} \rangle x_1 - \Delta_{\mathscr{A}}Qx_1) \vee (\langle \mathscr{A} \rangle x_2 - \Delta_{\mathscr{A}}Qx_2)).$$

By the same token, the necessity of the inequalities, to be proved, is established. To verify their sufficiency, we use the rules of change of variables in the Young transformations [8]. By the theorem on vector minimax, there exists an operator β in $\partial(T\varepsilon_{\mathscr{A}})$ such that the infimum z_0 (which is a positive element of Z), in which we are interested, can be expressed in the form

$$z_0 = \inf_{z \in X}(\beta\Delta_{\mathscr{A}}Qz - \beta\langle\mathscr{A}\rangle z) + \beta\left((\langle\mathscr{A}\rangle x_1 - \Delta_{\mathscr{A}}Qx_1) \vee (\langle\mathscr{A}\rangle x_2 - \Delta_{\mathscr{A}}Qx_2)\right).$$

Hence the set $U := \{\beta\Delta_{\mathscr{A}}Qz - \beta\langle\mathscr{A}\rangle z : z \in X\}$ is bounded below and, therefore, by virtue of the positive homogeneity of Q and $\langle\mathscr{A}\rangle$ we have $\inf U = 0$. The last equation means that $TQz - \beta\langle\mathscr{A}\rangle z \geqslant 0$ for $z \in X$, i.e., $\beta\langle\mathscr{A}\rangle \in \partial(TQ)$. Therefore, by the condition, $z_0 \leq 0$, which ensures the equality $z_0 = 0$.

The inequalities, being investigated, for $x_1 := x$ and $x_2 := 0$ lead to the desired equation. In its turn, the last equation is equivalent to the implication

$$(\forall\beta' \geqslant 0) \quad \beta' \leqslant \beta \rightarrow \beta'(\langle\mathscr{A}\rangle x - \Delta_{\mathscr{A}}Qx) \leqslant 0$$

for each $x \in X$. It remains to observe that

$$\beta\left((\Delta_{\mathscr{A}}Qx_1 - \langle\mathscr{A}\rangle x_1) \wedge (\Delta_{\mathscr{A}}Qx_2 - \langle\mathscr{A}\rangle x_2)\right) = \beta_1(\Delta_{\mathscr{A}}Qx_1 - \langle\mathscr{A}\rangle x_1) + \beta_2(\Delta_{\mathscr{A}}Qx_2 - \langle\mathscr{A}\rangle x_2)$$

for a suitable choice of positive β_1 and β_2 that constitute β, i.e., $\beta_1 + \beta_2 = \beta$. This observation completes the proof. ▷

3.9. In conclusion, we indicate that certain statements of this article were announced in [11, 12]. Let us observe that the formulations of the statements 1.6(2), 2.7, and 3.9 are given in [11] with errors.

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