Hence, $\hat{t}'_i \in \hat{B}'$ for all i. Since $\hat{t}'_{l,2} \in \hat{B}'c \neq \hat{B}'$ for all l, just as above we have $f'_2 h' \notin \mathbf{Z}(\tilde{B}')$, a contradiction. This proves Lemma 9.

As we remarked earlier, the proof of sufficiency of the theorem is obtained as a result of applying Lemma 8 and repeatedly applying Lemma 9.

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A VARIANT OF NONSTANDARD CONVEX PROGRAMMING

S. S. Kutateladze

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The idea of considering the accuracy of the observance of the optimality criteria in the practical solving of an extremal problem has found a well-known reflection in convex ε-programming [1-5], yielding a tool for the estimation of the approximation of the optimum with respect to a functional. The developed formalism is sufficiently specific and in a certain sense turns out to be artificially complicated in comparison with the usual theory. At the same time, it does not correlate entirely with the existing methods, based on the search of the "practical optimum" with the aid of a "practically accurate" observance of the complementary slackness requirements, corresponding to the classical case ε = 0. As a result one can talk about a definite discrepancy, and even of a gap, between the theoretical and practical views.

The purpose of the present paper is to indicate a possible variant of elimination of the mentioned discontinuity in the light of the concepts of nonstandard analysis [6-8]. In the paper we introduce the concept of an infinitesimally optimal solution, as an admissible point in which the value of the objective function is infinitely close to the ideal one, not necessarily realizing the sense of the program. Thus, the infinitesimal optimum acts as an acceptable pretender to the role of the "practical" optimum since no feasible procedure is capable to distinguish it from the "theoretical" optimum. We give the fundamental formulas of the calculus of infinitesimal subdifferentials, corresponding to the given optimality concept. The obtained rules for external sets coincide in form with their classical analogues in standard convex analysis. Moreover, within the criteria of infinitesimal optimality, there arises actually an approximately fulfilled complementary slackness. All this allows us to consider the described variant of the nonstandard convex programming to be at least admissible.

1. Infinitesimal Subdifferentials

1.1. Let X be a vector space, let Y be an ordered vector space with an adjoined largest element $+\infty$. We consider a convex operator F: X → Y and a point $\bar{x}$ from the effective set dom $F := \{x \in X : Fx < +\infty\}$ of the operator F. For the element $\varepsilon \leqslant 0$ (from the cone of the positive elements Y_+ of the space Y) we define in the accepted manner the ε-subdifferential of F at the point $\bar{x}$, i.e., the set

$$\partial^\varepsilon F(\bar{x}) := \{A \in L(X, Y) : (\forall x \in X) Ax - A\bar{x} \leqslant Fx - F\bar{x} + \varepsilon\},$$

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where L(X, Y) is the space of linear operators acting from X into Y.

1.2. Assume that in Y one has isolated a family E of positive elements, filtered decreasingly. Assuming Y and E standard sets, we define the monad $\mu(E)$ by the relation $\mu(E):=\bigcap\{[0,\varepsilon]: \varepsilon\in{}^{\circ}E\}$. Here, as usual, $^{\circ}E$ is the set of standard elements of E or the standard kernel of E. The elements of $\mu(E)$ are said to be positive infinite small or infinitesimal (relative to E). In the sequel we shall assume without special mention that Y is a K-space while the monad $\mu(E)$ is an exterior cone over $^{\circ}\mathbf{R}$ and in addition $\mu(E)\cap{}^{\circ}Y=0$. (As a rule, in applications, E is the filter of units in Y.) We shall also use the relation of infinite nearness between the elements of Y, i.e.,

$$y_1\approx y_2\leftrightarrow y_1-y_2\in\mu(E)\wedge y_2-y_1\in\mu(E).$$

1.3. We have the equality

$$\bigcap_{\varepsilon\in{}^{\circ}E}\partial^{\varepsilon}F(\bar x)=\bigcup_{\varepsilon\in\mu(E)}\partial^{\varepsilon}F(\bar x).$$

◁ For $A\in L(X,Y)$ we derive successively:

$$A\in\bigcap_{\varepsilon\in{}^{\circ}E}\partial^{\varepsilon}F(\bar x)\leftrightarrow(\forall^{st}\varepsilon\in E)(\forall x\in X)\,Ax-A\bar x\leqslant Fx-F\bar x+\varepsilon\leftrightarrow$$
$$\leftrightarrow(\forall^{st}\varepsilon\in E)\,F^*(A):=\sup_{x\in\operatorname{dom}F}(Ax-Fx)\leqslant A\bar x-F\bar x+\varepsilon\leftrightarrow$$
$$\leftrightarrow(\forall^{st}\varepsilon\in E)\,0\leqslant F^*(A)-(A\bar x-F\bar x)\leqslant\varepsilon\leftrightarrow F^*(A)-(A\bar x-F\bar x)\approx 0\leftrightarrow$$
$$\leftrightarrow(\exists\varepsilon\in Y_+)\,\varepsilon\approx 0\wedge F^*(A)=A\bar x-F\bar x+\varepsilon\leftrightarrow A\in\bigcup_{\varepsilon\in\mu(E)}\partial^{\varepsilon}F(\bar x).\ ▷$$

1.4. The external set, occurring in both sides of the equality of 1.3, is called an infinitesimal subdifferential of F at the point $\bar x$ and is denoted by $DF(\bar x)$. The elements of $DF(\bar x)$ are called infinitesimal subgradients of F at the point $\bar x$. No special indications are made on the set E since the probability of a misunderstanding is insignificant.

1.5. Assume that the assumption of the "standard entourage" holds, i.e., the parameters X, F, $\bar x$ are standard sets. Then the standardization of the infinitesimal subdifferential F at the point x coincides with the (zero) subdifferential of F at $\bar x$, i.e.,

$$^*DF(\bar x)=\partial F(\bar x).$$

◁ For a standard $A\in{}^{\circ}L(X,Y)$, by virtue of the transfer principle we have

$$A\in{}^*DF(\bar x)\leftrightarrow A\in DF(\bar x)\leftrightarrow(\forall^{st}\varepsilon\in E)(\forall x\in X)Ax-A\bar x\leqslant Fx-F\bar x+\varepsilon\leftrightarrow$$
$$\leftrightarrow(\forall\varepsilon\in E)(\forall x\in X)Ax-A\bar x\leqslant Fx-F\bar x+\varepsilon\leftrightarrow A\in\partial F(\bar x),$$

since inf E = 0 on the basis of the relation $\mu(E)\cap{}^{\circ}Y=0$. ▷

2. Fundamental Formulas for the Computation of Infinitesimal Subdifferentials

2.1. Let Z be a standard K-space and let G: Y → Z be an increasing convex operator. If the sets X × epi G and epi F × Z are in general position, then

$$D(GF)(\bar x)=\bigcup_{B\in DG(F\bar x)}D(BF)(\bar x).$$

If, in addition, the parameters (with the possible exception of the point $\bar x$) are standard, then for the standard kernels we have the representation

$$^{\circ}D(GF)(\bar x)=\bigcup_{B\in{}^{\circ}DG(F\bar x)}{}^{\circ}D(BF)(\bar x).$$

◁ We mention that, by assumption, the monad $\mu(E)$ is a normal exterior subsemigroup in Z, i.e.,

$$\varepsilon\in\mu(E)\to[0,\varepsilon]\subset\mu(E);$$
$$\mu(E)+\mu(E)\subset\mu(E).$$

Taking this into account and making use of both 1.3 and the rule for the computation of ε-subdifferentials, we obtain successively

$$D(GF)(\bar x)=\bigcup_{\varepsilon\in\mu(E)}\partial^{\varepsilon}(GF)(\bar x)=\bigcup_{\varepsilon\in\mu(E)}\ \bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon\\ \varepsilon_1\geqslant0,\varepsilon_2\geqslant0}}\ \bigcup_{B\in\partial^{\varepsilon_1}G(F\bar x)}\partial^{\varepsilon_2}(BF)(\bar x)=$$

$$= \bigcup_{\substack{\varepsilon_1\geqslant 0,\varepsilon_2\geqslant 0\\ \varepsilon_1\approx 0,\varepsilon_2\approx 0}} \bigcup_{B\in\partial^{\varepsilon_1}G(F\bar{x})} \partial^{\varepsilon_2}(BF)(\bar{x}) = \bigcup_{\varepsilon_1\geqslant 0,\varepsilon_1\approx 0} \bigcup_{B\in\partial^{\varepsilon_1}G(F\bar{x})} \bigcup_{\varepsilon_2\geqslant 0,\varepsilon_2\approx 0} \partial^{\varepsilon_2}(BF)(\bar{x}) = \bigcup_{\varepsilon_1\geqslant 0,\varepsilon_1\approx 0} \bigcup_{B\in\partial^{\varepsilon_1}G(F\bar{x})} D(BF)(\bar{x}).$$

Assume now that the assumption of the standard entourage holds and $A \in {}^{\circ}D(GF)(\bar{x})$. Then, for some infinitely small ε we have

$$(GF)^*(A) = \sup_{x\in \operatorname{dom}(GF)} (Ax - GFx) \leqslant A\bar{x} - GF\bar{x} + \varepsilon.$$

By the formula of the change of variable in the Young transformation and taking into account the transfer principle, there exists a standard operator $B \in {}^{\circ}L(Y, Z)$ such that B is positive, i.e., $B \in {}^{\circ}L_+(Y, Z)$ and, in addition,

$$(GF)^*(A) = (BF)^*(A) + G^*(B).$$

From here it follows that

$$\varepsilon \geqslant \sup_{x\in \operatorname{dom} F} (Ax - BFx) + \sup_{y\in \operatorname{dom} G} (By - Gy) - A\bar{x} + GF\bar{x} =$$

$$= \sup_{x\in \operatorname{dom} F} (Ax - A\bar{x} - (BFx - BF\bar{x})) + \sup_{y\in \operatorname{dom} G} (By - BF\bar{x} - (Gy - GF\bar{x})).$$

We set

$$\varepsilon_1 := \sup_{y\in \operatorname{dom} G} (By - BF\bar{x} - (Gy - GF\bar{x})),$$

$$\varepsilon_2 := \sup_{x\in \operatorname{dom} F} (Ax - A\bar{x} - (BFx - BF\bar{x})).$$

It is clear that $A \in \partial^{\varepsilon_2}(BF)(\bar{x})$, i.e., $A \in {}^{\circ}D(BF)(\bar{x})$, and $B \in \partial^{\varepsilon_1}G(F\bar{x})$, i.e., $B \in {}^{\circ}DG(F\bar{x})$, since $\varepsilon_1 \approx 0$ and $\varepsilon_2 \approx 0$. ▷

2.2. Let $F_1,\dots,F_n$: $X \to Y^{\cdot}$ be convex operators, n being a standard natural number. If $F_1,\dots,F_n$ are in general position, then for the points $\bar{x} \in \operatorname{dom} F_1 \cap \dots \cap \operatorname{dom} F_n$ we have

$$D(F_1 + \dots + F_n)(\bar{x}) = DF_1(\bar{x}) + \dots + DF_n(\bar{x}).$$

◁ The proof consists in the application of 1.3 and of the rule of ε-subdifferentiation of a sum, taking into account that the sum of a standard number of infinitely small terms is infinitely small. ▷

2.3. Let $F_1,\dots,F_n$: $X \to Y^{\cdot}$ be convex operators, n being a standard number. We assume that $F_1,\dots,F_n$ are in general position, Y is a vector lattice and $\bar{x} \in \operatorname{dom} F_1 \vee \dots \vee F_n$. If Z is a standard K-space and $A \in L_+(Y, Z)$ is a positive linear operator, then the element $B \in L(X, Z)$ is the infinitesimal subgradient of the operator $A(F_1 \vee \dots \vee F_n)$ at the point $\bar{x}$ if and only if the following system of conditions is consistent:

$$A = \sum_{k=1}^{n} A_k; \quad A_k \in L_+(Y, Z) \quad (k := 1, \dots, n);$$

$$\sum_{k=1}^{n} A_k\bar{x} \approx A(F_1\bar{x} \vee \dots \vee F_n\bar{x}); \quad B \in \sum_{k=1}^{n} D(A_kF_k)(\bar{x}).$$

◁ We define the operators

$$(F_1, \dots, F_n): X \to (Y^n)^{\cdot}; \ (F_1, \dots, F_n)x := (F_1x, \dots, F_nx);$$

$$\varkappa: Y^n \to Y; \quad \varkappa(y_1, \dots, y_n) := y_1 \vee \dots \vee y_n.$$

Then we have the representation

$$A(F_1 \vee \dots \vee F_n) = A\varkappa(F_1, \dots, F_n).$$

From here, taking into account 2.1 and recalling that $A\varkappa$ is a sublinear operator, we derive what we intended to prove. ▷

2.4. Let X be a vector space, let Y be some K-space and let $\mathscr{A}$ be a weak order bounded set in L(X, Y). We consider a regular convex operator $F := \varepsilon_{\mathscr{A}}\langle\mathscr{A}\rangle_y$, where, as usually, $\varepsilon_{\mathscr{A}}$ is the canonical sublinear operator

$$\varepsilon_{\mathscr{A}}: l_\infty(\mathscr{A}, Y) \to Y, \ \varepsilon_{\mathscr{A}}f := \sup f(\mathscr{A}) \quad (f \in l_\infty(\mathscr{A}, Y))$$

and the affine operator $\langle\mathscr{A}\rangle_y$ for $y \in l_\infty(\mathscr{A}, y)$ acts according to the rule $\langle\mathscr{A}\rangle_y x := \langle\mathscr{A}\rangle x + y$; $\langle\mathscr{A}\rangle x$: $A \in \mathscr{A} \to Ax$.

2.5. If G: Y → Z is an increasing convex operator, acting in the standard K-space Z and if in the image F(X) one has an algebraically interior point dom G, while the element $\bar{x}$ from X is such that $F\bar{x} \in \operatorname{dom} G$, then we have the representation (see [2])

$$D(GF)(\bar{x}) = \{B\langle\mathscr{A}\rangle : B\Delta_{\mathscr{A}} \in DG(F\bar{x});\quad B \geqslant 0;\quad B\Delta_{\mathscr{A}}F\bar{x} \approx B\langle\mathscr{A}\rangle_y\bar{x}\}.$$

◁ If $C \in D(GF)(\bar{x})$, then by 1.3 we have $C \in \partial^{\varepsilon}(GF)(\bar{x})$ for some $\varepsilon \approx 0$. It remains to use the corresponding rule for ε-subdifferentiation. If, however, $B \geqslant 0$, $B\Delta_{\mathscr{A}} \in DG(F\bar{x})$ and $B\Delta_{\mathscr{A}} \times F\bar{x} \approx B\langle\mathscr{A}\rangle_y\bar{x}$, then for some $\varepsilon \approx 0$ we shall have, naturally, $B\Delta_{\mathscr{A}} \in \partial^{\varepsilon}G(F\bar{x})$. In addition, we set $\delta := B\Delta_{\mathscr{A}}F\bar{x} - B\langle\mathscr{A}\rangle_y\bar{x}$. Then $\delta \geqslant 0$ and $\delta \approx 0$ by assumption. Thus, $B\langle\mathscr{A}\rangle \in \partial^{\varepsilon+\delta}(GF)(\bar{x})$. It remains to note that $\delta + \varepsilon \approx 0$. ▷

2.6. Assume that, under the conditions of 2.5, the mapping G is a sublinear Maharam operator. Then

$$D(GF)(\bar{x}) = \bigcup_{T \in DG(F\bar{x})} \ \bigcup_{\delta \geqslant 0,\, T\delta \approx 0} T\partial^{\delta}F(\bar{x}).$$

◁ By virtue of 2.1 one can assume that G: = T. If for each $x \in X$ we have $Cx - C\bar{x} \leqslant Fx - F\bar{x} + \delta$ and $T\delta \approx 0$, then undoubtedly $TC \in \partial^{T\delta}(TF)(\bar{x}) \subset D(TF)(\bar{x})$. In order to conclude the proof, we select $B \in D(TF)(\bar{x})$. According to 1.3 we have an infinitely small ε such that $B \in \partial^{\varepsilon}(TF)(\bar{x})$. Making use of the corresponding rule for ε-subdifferentiation from [9], we find $\delta \geqslant 0$ and $C \in \partial^{\delta}F(\bar{x})$ such that $T\delta \leqslant \varepsilon$ and $B = TC$. This is what we intended to prove. ▷

2.7. Let Ξ be some set and let $(F_{\xi})_{\xi \in \Xi}$ be a uniformly regular family of convex operators. We have the representations

$$D\Big(o\text{-}\sum_{\xi \in \Xi} F_{\xi}\Big)(\bar{x}) = \bigcup_{\substack{\delta \in l_1(\Xi, Y)\\ \delta \geqslant 0,\, \delta \approx 0}} o\text{-}\sum_{\xi \in \Xi} \partial^{\delta(\xi)}F_{\xi}(\bar{x});$$

$$D\Big(\sup_{\xi \in \Xi} F_{\xi}\Big)(\bar{x}) = \bigcup \Big\{o\text{-}\sum_{\xi \in \Xi} \alpha_{\xi}\partial^{\delta(\xi)}F_{\xi}(\bar{x}):\quad 0 \leqslant \alpha_{\xi} \leqslant 1_Y,$$

$$o\text{-}\sum_{\xi \in \Xi} \alpha_{\xi} = 1_Y;\quad o\text{-}\sum_{\xi \in \Xi} \alpha_{\xi}F_{\xi}(\bar{x}) \approx \sup_{\xi \in \Xi} F_{\xi}(\bar{x});\ o\text{-}\sum_{\xi \in \Xi} \alpha_{\xi}\delta(\xi) \approx 0\Big\}.$$

◁ The proof follows directly from 2.6 and by taking into account the results of [9]. ▷

2.8. It is useful to note that formulas 2.2-2.7 admit refinements similar to those occurring in Subsec. 2.1 in the case of the standard entourage (in which it is possible that the point $\bar{x}$ is not included). We also point out that according to the given model, one introduces the full spectrum of all possible formulas of subdifferential calculus (convolutions, Lebesgue sets, etc.).

3. Infinitesimal Subdifferentials at Generalized Points

3.1. Assume that, as above, F: X → Y is a convex operator, acting in a standard K-space Y and let $\mathscr{X} := \mathscr{X}(\cdot)$ be a generalized point in dom F, i.e., a net of elements of dom F. We say that an operator $A \in L(X, Y)$ is an infinitesimal subgradient of F at the generalized point $\mathscr{X}$ if for some infinitely small positive ε we have

$$F^*(A) \leqslant \liminf(A\mathscr{X} - F\mathscr{X}) + \varepsilon.$$

Thus, under the assumption of the standard character of the entourage, the infinitesimal subgradient is a usual support operator at the generalized point [10, 11]. We shall denote by the symbol $DF(\mathscr{X})$ the collection of all infinitesimal subgradients F in $\mathscr{X}$. For obvious reasons, this set is called the infinitesimal subdifferential of F in $\mathscr{X}$.

We derive two basic rules of subdifferentiation at a generalized point; these present interest due to the fact that exact formulas for the corresponding ε-subdifferentials are not known.

3.2. Let $F_1, \ldots, F_n$ be a standard collection of convex operators in the general position and assume that the generalized point $\mathscr{X}$ lies in $\operatorname{dom} F_1 \cap \ldots \cap \operatorname{dom} F_n$. Then

$$D(F_1 + \ldots + F_n)(\mathscr{X}) = DF_1(\mathscr{X}) + \ldots + DF_n(\mathscr{X}).$$

◁ Let $A_k \in DF_k(\mathscr{X})$ for k: = 1,...,n, i.e.,

$$F_k^*(A_k) \leqslant \liminf(A_k\mathscr{X} - F_k\mathscr{X}) + \varepsilon_k$$

for appropriate infinitely small $\varepsilon_1, \ldots, \varepsilon_n$. Moreover,

$$(F_1 + \ldots + F_n)^* (A_1 + \ldots + A_n) \leqslant \sum_{k=1}^{n} F_k^* (A_k) \leqslant$$

$$\leqslant \sum_{k=1}^{n} (\liminf (A_k\mathscr{X} - F_k\mathscr{X}) + \varepsilon_k) \leqslant \liminf \sum_{k=1}^{n} (A_k\mathscr{X} - F_k\mathscr{X}) + \sum_{k=1}^{n} \varepsilon_k$$

by virtue of the usual properties of the Young transformation and of the lower limits. It remains to note that $\varepsilon_1 + \ldots + \varepsilon_n \approx 0$ and to derive the validity of the inclusion $\supset$ for the sets considered in our equality.

For the verification of the opposite inclusion, reducing it to the case $n = 2$, we take $A \in D(F_1+F_2)(\mathscr{X})$. Then for some $\varepsilon \approx 0$ and A_1, A_2 such that $A_1 + A_2 = A$, we have

$$(F_1 + F_2)^* (A) = F_1^* (A_1) + F_2^* (A_2),$$
$$F_1^* (A_1) + F_2^* (A_2) - \liminf (A\mathscr{X} - (F_1 + F_2) (\mathscr{X})) \leqslant \varepsilon.$$

We define

$$\delta_1 := F_1^* (A_1) - \liminf (A_1\mathscr{X} - F_1\mathscr{X}),$$
$$\delta_2 := F_2^* (A_2) - \liminf (A_2\mathscr{X} - F_2\mathscr{X}).$$

It is clear that for $k := 1, 2$ we have

$$0 \leqslant \sup_{x \in \operatorname{dom} F_k} (A_k x - F_k x) - \limsup (A_k\mathscr{X} - F_k\mathscr{X}) \leqslant \delta_k.$$

Thus, it remains to verify the infinite smallness of δ_1 and δ_2. We have

$$\delta_1 + \delta_2 \leqslant \varepsilon + \liminf (A\mathscr{X} - (F_1 + F_2) (\mathscr{X})) - \sum_{k=1}^{2} \liminf (A_k\mathscr{X} - F_k\mathscr{X}) \leqslant$$
$$\leqslant (\varepsilon + \limsup (A_1\mathscr{X} - F_1\mathscr{X}) - \liminf (A_1\mathscr{X} - F_1\mathscr{X})) \wedge$$
$$\wedge (\varepsilon + \limsup (A_2\mathscr{X} - F_2\mathscr{X}) - \liminf (A_2\mathscr{X} - F_2\mathscr{X})) \leqslant$$

$$\leqslant (\varepsilon + F_1^* (A_1) - \liminf (A_1\mathscr{X} - F_1\mathscr{X})) \wedge (\varepsilon + F_2^* (A_2) - \liminf (A_2\mathscr{X} - F_2\mathscr{X})) = \varepsilon + \delta_1 \wedge \delta_2.$$

From here $0 \leqslant \delta_1 \vee \delta_2 \leqslant \varepsilon$, which concludes the proof. ▷

3.3. Let Z be a standard K-space and let $G: Y \to Z$ be an increasing convex operator. If the sets $X \times \operatorname{epi} G$ and $\operatorname{epi} F \times Z$ are in general position, then for a generalized point $\mathscr{X}$ in dom (GF) we have

$$D(GF)(\mathscr{X}) = \bigcup_{B \in DG(F\mathscr{X})} D(BF)(\mathscr{X}).$$

◁If it is known that

$$(BF)^* (A) \leqslant \liminf (A\mathscr{X} - BF\mathscr{X}) + \varepsilon_1;$$
$$G^* (B) \leqslant \liminf (BF\mathscr{X} - GF\mathscr{X}) + \varepsilon_2$$

for some infinitely small ε_1 and ε_2, then

$$(GF)^* (A) \leqslant (BF)^* (A) + G^* (B) \leqslant$$

$$\leqslant \liminf (A\mathscr{X} - BF\mathscr{X}) + \varepsilon_1 + \liminf (BF\mathscr{X} - GF\mathscr{X}) + \varepsilon_2 \leqslant \liminf (A\mathscr{X} - GF\mathscr{X}) + \varepsilon_1 + \varepsilon_2.$$

Consequently, $A \in D(GF)(\mathscr{X})$ and the right-hand side of the analyzed formula symbolizes the set occurring in its left-hand side.

In order to conclude the proof we take $A \in D(GF)(\mathscr{X})$. Then there exist an infinitely small ε and an operator B such that

$$(GF)^* (A) = (BF)^* (A) + G^* (B) \leqslant \liminf (A\mathscr{X} - GF\mathscr{X}) + \varepsilon.$$

We set

$$\delta_1 := (BF)^* (A) - \liminf (A\mathscr{X} - BF\mathscr{X}),$$
$$\delta_2 := G^* (B) - \liminf (BF\mathscr{X} - GF\mathscr{X}).$$

Taking into account the properties of the upper and lower limits, we derive, firstly,

$$\delta_1 \geqslant (BF)^* (A) - \limsup (A\mathscr{X} - BF\mathscr{X}) \geqslant 0,$$
$$\delta_2 \geqslant G^* (B) - \limsup (BF\mathscr{X} - GF\mathscr{X}) \geqslant 0$$

and, secondly,

$$\delta_1+\delta_2\leqslant \liminf(A\mathscr{X}-GF\mathscr{X})+\varepsilon-\liminf(A\mathscr{X}-BF\mathscr{X})-\liminf(BF\mathscr{X}-GF\mathscr{X})\leqslant$$
$$\leqslant(\limsup(A\mathscr{X}-BF\mathscr{X})-\liminf(A\mathscr{X}-BF\mathscr{X})+\varepsilon)\wedge$$
$$\wedge(\limsup(BF\mathscr{X}-GF\mathscr{X})-\liminf(BF\mathscr{X}-GF\mathscr{X})+\varepsilon)\leqslant\delta_1\wedge\delta_2+\varepsilon,$$

since we have the obvious inequalities

$$\limsup(BF\mathscr{X}-GF\mathscr{X})\leqslant G^*(B);$$
$$\limsup(A\mathscr{X}-BF\mathscr{X})\leqslant(BF)^*(A).$$

Thus, $0\leqslant\delta_1\vee\delta_2\leqslant\varepsilon$ and $\delta_1\approx 0$, $\delta_2\approx 0$. This means that $B\in DG(F\mathscr{X})$ and $A\in D(BF)(\mathscr{X})$. ▷

4. Tests for Infinitesimal Solutions

4.1. A point $\bar{x}\in\operatorname{dom}F$ is said to be an infinitesimal solution of the unconditional program Fx → inf, where F: X → Y, if $0\in DF(\bar{x})$, i.e., if $\bar{x}$ is admissible and $F\bar{x}\approx\inf\{Fx\colon x\in X\}$. In a natural manner one defines an infinitesimal solution of an arbitrary program.

4.2. In a standard unconditional program Fx → inf one has an infinitesimal solution if and only if, firstly, the image F(X) is bounded from below and, secondly, there exists a standard generalized solution $(x_\varepsilon)_{\varepsilon\in E}$ of the considered program, i.e., $x_\varepsilon\in\operatorname{dom}F$ and $y\leqslant Fx\leqslant y+\varepsilon$ for all $\varepsilon\in E$, where $y:=\inf F(X)$ is the value of the program.

◁ By virtue of the idealization and transfer principles, with the aid of 1.3 we derive that

$$(\exists\bar{x}\in X)0\in DF(\bar{x})\leftrightarrow(\exists x\in X)(\forall^{st}\varepsilon\in E)0\in\partial^\varepsilon F(x)\leftrightarrow(\forall^{st\,fin}E_0\subset E)(\exists x\in X)(\forall\varepsilon\in E_0)0\in\partial^\varepsilon F(x)\leftrightarrow$$
$$\leftrightarrow(\forall^{st}\varepsilon\in E)(\exists x_\varepsilon\in X)0\in\partial^\varepsilon F(x_\varepsilon)\leftrightarrow(\forall\varepsilon\in E)(\exists x_\varepsilon\in X)(\forall x\in X)Fx\geqslant Fx_\varepsilon-\varepsilon.\ \triangleright$$

4.3. We consider a regular convex program

$$Gx\leqslant 0,\quad Fx\to\inf.$$

Thus, G, F: X → $Y^\bullet$ (for the sake of simplicity, dom F = dom G = X), for each $x\in X$ either $Gx\geqslant 0$ or $Gx\geqslant 0$, in addition, for some $x_0\in X$ the element $-Gx_0$ is the identity in Y.

4.4. In the case of a standard entourage, an admissible internal point $\bar{x}$ is an infinitesimal solution of the considered regular program if and only if the following system of conditions is consistent:

$$\alpha,\beta\in{}^\circ[0,1_Y],\ \alpha+\beta=1_Y,\ \ker\alpha=0;$$
$$\beta G\bar{x}\approx 0,\ 0\in D(\alpha F)(\bar{x})+D(\beta G)(\bar{x}).$$

◁ ← Under the consistency of the considered system, for an admissible x and some infinitely small ε_1 and ε_2 we have

$$\alpha F\bar{x}\leqslant\alpha Fx+\beta Gx-\beta G\bar{x}+\varepsilon_1+\varepsilon_2\leqslant\alpha Fx+\varepsilon$$

for each standard $\varepsilon\in{}^\circ E$. In particular, $\alpha(F\bar{x}-Fx)\leqslant\alpha\varepsilon$ for $\varepsilon\in{}^\circ E$ since α is a standard mapping. By virtue of the condition ker α = 0 and of the general properties of the multipliers, we see that $\bar{x}$ is an infinitesimal solution.

→ Let $y:=\inf\{Fx\colon x\in X,\ Gx\leqslant 0\}$ be the value of the considered program. By assumption and by the transfer principle, y is a standard element. Thus, making use again of the transfer principle, by the theorem on the vector minimax we find standard multipliers $\alpha,\beta\in{}^\circ[0,1_Y]$ such that

$$\alpha+\beta=1_Y;$$
$$0=\inf_{x\in X}(\alpha(Fx-y)+\beta Gx).$$

By a usual argument [2] one shows that ker α = 0. In addition, since $\bar{x}$ is an infinitesimally optimal solution, for some infinitely small ε we have $F\bar{x}-y=\varepsilon$. Consequently, for any $x\in X$ we have the estimates

$$-\alpha\varepsilon\leqslant\alpha Fx-\alpha F\bar{x}+\beta Gx.$$

In particular, $0\geqslant\beta G\bar{x}\geqslant-\alpha\varepsilon\geqslant-\varepsilon$, i.e., $\beta G\bar{x}\approx 0$ and

$$0 \in \partial^{\alpha\varepsilon+\beta G\bar{x}}(\alpha F + \beta G)(\bar{x}) \subset D(\alpha F + \beta G)(\bar{x}),$$

since $\alpha\varepsilon + \beta G\bar{x} \approx 0$. ▷

4.5. We consider a Slater regular program

$$Ax = A\bar{x}; \quad Gx \leqslant 0, \quad Fx \to \inf,$$

i.e., firstly, $A \in L(X, \mathfrak{X})$ is a linear operator with values in some vector space X; the mappings G: X → Z and F: X → Y$^{\cdot}$ are convex operators (for econvenience, dom F = dom G = X), and, secondly, Z is an Archimedean ordered vector space, Y is the standard K-space of the bounded elements and, thirdly, for some admissible point x_0 the element $-Gx_0$ is a strong identity in Z.

4.6. An admissible point $\bar{x}$ is an infinitesimal solution of a Slater regular program if and only if the following system of conditions is consistent:

$$\gamma \in L_+(Z, Y), \quad \mu \in L(\mathfrak{X}, Y); \quad \gamma G\bar{x} \approx 0;$$
$$0 \in DF(\bar{x}) + D(\gamma G)(\bar{x}) + \mu A.$$

◁ ← Under the consistency of the considered system, for each admissible point x and some infinitely small ε_1 and ε_2 we have

$$F\bar{x} \leqslant Fx + \varepsilon_1 + \gamma Gx - \gamma G\bar{x} + \varepsilon_2 - \mu(Ax) + \mu(A\bar{x}) \leqslant Fx + \varepsilon_1 + \varepsilon_2 + \gamma G\bar{x} \leqslant Fx + \varepsilon$$

for any standard $\varepsilon \in {}^{\circ}E$.

→ If $\bar{x}$ is an infinitesimal solution, then it is also an ε-solution for an appropriate infinitely small ε. It remains to use the corresponding ε-optimality criterion. ▷

4.7. An admissible point $\bar{x}$ is said to be Pareto optimal if $\bar{x}$ is Pareto ε-optimal for any infinitely small ε (relative to the strong identity $\mathbf{1}_Y$ in Y).

4.8. Assume that the point $\bar{x}$ is infinitesimally Pareto optimal in a Slater regular program. Then for some linear functionals α, β, γ on the spaces Y, Z, and $\mathfrak{X}$, respectively, the following system of conditions is consistent:

$$\alpha > 0, \quad \beta \geqslant 0, \quad \beta G\bar{x} \approx 0;$$
$$0 \in D(\alpha F)(\bar{x}) + D(\beta G)(\bar{x}) + \gamma A.$$

If, in turn, the given relations hold for some admissible point x, while $\alpha(\mathbf{1}_Y) = 1$ and $\ker\alpha \cap Y_+ = 0$, then $\bar{x}$ is an infinitesimal, Pareto optimal solution of the considered program.

◁ The first part of the statement follows from the usual rule for Pareto ε-optimality and by taking into account the previously mentioned properties of infinitely small quantities. If, however, the assumption of the second part of the proposition holds, then, making use of the definition, for any admissible $x \in X$ we derive

$$0 \leqslant \alpha(Fx - F\bar{x}) + \beta Gx - \beta G\bar{x} + \varepsilon_1 + \varepsilon_2 \leqslant \alpha(Fx - F\bar{x}) + \varepsilon_1 + \varepsilon_2 - \beta G\bar{x}$$

under appropriate infinitely small ε_1, ε_2. We set $\varepsilon := \varepsilon_1 + \varepsilon_2 - \beta G\bar{x}$. Clearly, ε ≈ 0 and, in addition, ε ⩾ 0. If now for an admissible x we have $Fx - F\bar{x} \leqslant -\varepsilon\mathbf{1}_Y$, then we have the equality $\alpha(F\bar{x} - Fx) = \varepsilon$. In other words, $\alpha(Fx - F\bar{x} - \varepsilon\mathbf{1}_Y) = 0$ and $Fx - F\bar{x} = \varepsilon\mathbf{1}_Y$. The latter means exactly that $\bar{x}$ is a Pareto ε-optimal solution. ▷

4.9. By the described model one can obtain the rules of the infinitesimal solutions also in other fundamental forms of the problems of convex programming; for example, one can derive nonstandard analogues of theorems on the characterization in a natural manner of the determined infinitesimally optimal trajectories in finite-step terminal dynamic problems (see [4]).

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SOLVABILITY OF THE CAUCHY PROBLEM FOR AN EVOLUTION EQUATION IN A BANACH SPACE WITH A NONDENSELY DEFINED OPERATOR COEFFICIENT GENERATING A SEMIGROUP WITH A SINGULARITY

Yu. T. Sil'chenko and P. E. Sobolevskii UDC 517.937

1. This work is devoted to the Cauchy problem for the first-order equation

$$v'(t)+A(t)v(t)=f(t) \quad (0<t\leqslant 1),\ v(0)=v_0 \tag{1}$$

in a Banach space E. As is known [1, 2], various boundary-value problems for equations of parabolic type may be reduced to this problem. A solution of problem (1) is, by definition, a function v(t) continuous on [0, 1], such that v'(t), A(t)v(t) are continuous on (0, 1] and relations (1) are satisfied.

Here A(t) is defined for every $t\in[0,1]$ and it is a linear operator with a constant domain D. We assume that it has a bounded inverse $A^{-1}(t)$ and that f(t) is a given function continuous for t > 0.

It is usually assumed that the domain D of A(t) is a set dense in E and that A(t) generates, for every t, a strongly continuous or analytic semigroup. Nevertheless, in some problems nondensely defined operators and semigroups with a singularity at zero appear.

Let us assume that for every fixed $t\in[0,1]$ there exists an operator-valued function $T_t(\tau)=\exp\{-\tau A(t)\}$ with the properties:

1) $T_t(\tau)$ is a bounded linear operator from E into D ($\tau > 0$);

2) $T_t(\tau)T_t(s)=T_t(\tau+s)$ $(\tau, s>0)$;

3) $\lim\limits_{\tau\to+0} T_t(\tau)v=v$ for $v\in D$;

4) $T_t(\tau)$ is differentiable with respect to τ ($\tau > 0$) and $\frac{d}{d\tau}T_t(\tau)=-A(t)T_t(\tau)$;

5) $T_t(\tau)$ commutes with $A(\tau)$;

6) we have the estimates

$$\|T_t(\tau)\|\leqslant c\tau^{-\alpha}\exp(-\rho\tau),\quad \|T_t'(\tau)\|\leqslant c\tau^{-\beta}\exp(-\rho\tau) \tag{2}$$

for some $\rho>0$, $\alpha\geqslant 0$, $\beta\geqslant 1$.

In contrast with the strongly continuous semigroups, these properties do not imply that D is dense in E.

We shall call $T_t(\tau)$ a semigroup generated by A(t).

We are going to give examples of operators generating semigroups with the indicated properties.

A. Let

$$l(y)=y^{(n)}+p_1(x)y^{(n-1)}+\ldots+p_n(x)y \tag{3}$$

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