where i_1 and i_2 are embedding operators. Then

$$j^{**}: M_{\varphi}^{*} \xrightarrow{i_2^*} \Lambda_{\varphi} \xrightarrow{i_1^*} M_{\varphi^*}.$$

This means that $j^{**}(\Lambda_{\varphi}^{**}) \subset M_{\varphi^*}^{0}$. This proves the theorem.

Remark. It is necessary to emphasize that the spaces, used in the arguments, must be considered as the spaces isometric to the conjugate ones or as their images in the second adjoint space under the canonical mapping.

The function $\varphi_E(t) = \|\chi_{(0,t)}\|_E$ is called the fundamental function of the space E. We know that the fundamental function of an arbitrary symmetric space is increasing and is concave, to within two, on [0, 1]. Conversely, with respect to an arbitrary increasing concave function φ we can construct a symmetric space E such that $\varphi_E = \varphi$ [2].

THEOREM 3. Let $\varphi(t)$ be an increasing concave function on [0, 1]. Then there exists a reflexive symmetric space with the fundamental function φ, if and only if

$$\lim_{t\to 0} \varphi(t) = 0, \tag{2}$$

$$\lim_{t\to 0} \frac{t}{\varphi(t)} = 0. \tag{3}$$

Sufficiency. It follows from Theorem 2 that the embedding $\Lambda_{\varphi} \subset M_{\varphi^*}$ is weakly compact. Then it follows from [1, p. 32] that $(\Lambda_{\varphi}, M_{\varphi^*})_{\theta,q}$ $(0 < \theta < 1$ and $0 < q < \infty)$ is a reflexive symmetric space with the same fundamental function.

Necessity. If (2) is not fulfilled, then $E \simeq L_{\infty}$. If (3) is not fulfilled, then $E \simeq L_1$.

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TOPOLOGICAL CONCEPTS SIMILAR TO CONTINUITY*

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In connection with the development of some areas of applied analysis, and first of all subdifferential calculus, in recent years considerable interest developed in a whole spectrum of topological concepts similar to continuity. Dolecki [1] gave a survey of approaches to the construction and study of the tangents on the basis of constructions of the type of upper and lower limits of sequences of sets. In [2] Penot introduced the concepts of a compact filter and compact net and demonstrated the possibility of using them in applications to sufficient conditions for optimality, to problems on perturbations of programs, etc. The goal of the present paper is to show the effectiveness of the methods of nonstandard analysis in these problems. In the paper we establish a general fact which unifies the languages of the means needed for investigating properties related to continuity. On the basis of this we derive new tests for compactness of filters, subcontinuity, limits of correspondences, and also give simple nonstandard proofs and generalizations of some assertions from [3, 4] connected with them.

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1. Monads and Nets

1.0. In this section the auxiliary information needed concerning the use of nonstandard tools in the theory of filters is collected. Cf. [5, 6] with respect to details of monadology. The approach to subnets used is recounted in [7].

1.1. Let $\mathscr{F}$ be a filter on the set X. Under the usual hypothesis of "standardness of entourage" – under the assumption of standardness all parameters not explicitly described (in the present case $\mathscr{F}$ and X) – a monad $\mu(\mathscr{F})$ is defined by the relation $\mu(\mathscr{F}):=\bigcap{}^{\circ}\mathscr{F}$, where ${}^{\circ}\mathscr{F}$ is the standard kernel of $\mathscr{F}$, i.e., the outer set of standard elements of $\mathscr{F}$. If $\mathscr{B}$ is a basis for the filter $\mathscr{F}$, then we consider that $\mu(\mathscr{B}):=\mu(\mathscr{F})$. It is useful to note that a monad is an inner set if and only if it is standard (and hence the original filter is composed of all supersets of its monad).

1.2. Let $\Gamma\subset X\times Y$ be a correspondence from X to Y and the filter $\mathscr{F}$ hit $\operatorname{dom}\Gamma$, i.e., $(\forall F\in\mathscr{F})F\cap\operatorname{dom}\Gamma\neq\varnothing$ or, what is the same, $\mu(\mathscr{F})\cap\operatorname{dom}\Gamma\neq\varnothing$. We consider the image $\Gamma(\mathscr{F})$ of the filter $\mathscr{F}$, defined by the relation

$$\Gamma(\mathscr{F}):=\{G\subset Y\colon\ (\exists F\in\mathscr{F})G\supset\Gamma(F)\}.$$

The monad of the image is the image of the monad, i.e.,

$$\mu(\Gamma(\mathscr{F}))=\Gamma(\mu(\mathscr{F}))$$

(we use the strong form of the principle of idealization here).

1.3. Let Ξ be a direction, i.e., a nonempty directed set. According to the principle of idealization there are in Ξ inner elements which majorize ${}^{\circ}\Xi$. They are called remote or infinitely large in Ξ. We consider a standard basis for the filter of tails $\mathscr{B}:=\{\sigma(\xi)\colon\ \xi\in\Xi\}$, where σ is the order in Ξ. It is clear that the monad of the filter of tails is composed of remote elements of the direction considered. We use the notation ${}^{a}\Xi:=\mu(\mathscr{B})$ and $\xi\approx+\infty\leftrightarrow\xi\in{}^{a}\Xi$.

1.4. Let Ξ, H be two directions and $\xi:=\xi(\cdot)\colon \mathrm{H}\to\Xi$ be a map. The following assertions are equivalent:

(1) $\xi({}^{a}\mathrm{H})\subset{}^{a}\Xi$;

(2) $(\forall\xi\in\Xi)(\exists\eta\in\mathrm{H})(\forall\eta'\geqslant\eta)\quad \xi(\eta')\geqslant\xi.$

◅ In fact, (1) means that the filter of tails Ξ is coarser than the image of the filter of tails of H, i.e., that in each tail of the direction Ξ there lies the image of some tail of H. The last assertion is the content of (2). ▻

1.5. If the equivalent conditions 1.4(1) and 1.4(2) hold, one says that H is a subdirection of Ξ [with respect to $\xi(\cdot)$].

1.6. Let X be some set and $x:=x(\cdot)\colon\Xi\to X$ be some net of elements of X [we also write $(x_\xi)_{\xi\in\Xi}$ or simply (x_ξ)]. In addition, let $(y_\eta)_{\eta\in\mathrm{H}}$ be another net of elements of X. One says that (y_η) is a Moore subnet of (x_ξ), if H is a subdirection of Ξ with respect to a $\xi(\cdot)$, such that $y_\eta=x_{\xi(\eta)}$ for all $\eta\in\mathrm{H}$, i.e., $y = x\circ\xi$. We stress that by virtue of 1.2 one has $y({}^{a}\mathrm{H})\subset x({}^{a}\Xi)$.

1.7. The last property of Moore subnets cited is taken in [7] as the basis of a freer definition of subnet, which immediately attracts attention to direct connection with filters, namely: the net $(y_\eta)_{\eta\in\mathrm{H}}$ of elements of X is called a subnet (or a subnet in the extended sense of the word) of the net $(x_\xi)_{\xi\in\Xi}$ of elements of X, if

$$(\forall\xi\in\Xi)(\exists\eta\in\mathrm{H})(\forall\eta'\geqslant\eta)(\exists\xi'\geqslant\xi)\quad x(\xi')=y(\eta'),$$

i.e., when each tail of the net x contains some tail of y. In the language of monads it is clear that $y({}^{a}\mathrm{H})\subset x({}^{a}\Xi)$, or in intuitive notation

$$(\forall\eta\approx+\infty)(\exists\xi\approx+\infty)\quad y_\eta=x_\xi.$$

Here, aiming at picturesqueness one often writes: $(x_\eta)_{\eta\in\mathrm{H}}$ is a subnet of the net $(x_\xi)_{\xi\in\Xi}$ (which can lead to misunderstandings). It is useful to stress that in general subnets are not necessarily Moore subnets. We also note that two nets in one set are called equivalent, if each of them is a subnet of the other, i.e., if their monads coincide [7].

1.8. If $\mathscr{F}$ is a filter on X and (x_ξ) is a net of elements of X, then one says that the net considered is subordinate to $\mathscr{F}$ under the condition that $\xi\approx+\infty\to x_\xi\in\mu(\mathscr{F})$. In other words, the net (x_ξ) is subordinate to $\mathscr{F}$, if the filter of its tails is finer than $\mathscr{F}$. Here one takes the liberty of writing $x_\xi \downarrow \mathcal{F}$, having in mind the analogy with the topological notations for convergence. We also note here that if $\mathcal{F}$ is an ultrafilter, F coincides with the filter of tails of any net (x_ξ) subordinate to it. In particular, such a net (x_ξ) is itself an ultrafilter [7].

1.9. THEOREM. Let $\varphi = \varphi(x, y, z)$ be a formula of Zermelo–Frankel theory, containing no free parameters other than x, y, and z, while z is a standard set. In addition let $\mathcal{F}$ be a filter on X, and $\mathcal{G}$ be a filter on Y. The following assertions are equivalent:

(1) $(\forall G \in \mathcal{G})(\exists F \in \mathcal{F})(\forall x \in F)(\exists y \in G)\quad \varphi(x, y, z)$;

(2) $(\forall x \in \mu(\mathcal{F}))(\exists y \in \mu(\mathcal{G}))\quad \varphi(x, y, z)$;

(3) for any net $(x_\xi)_{\xi \in \Xi}$ of elements of X, subordinate to $\mathcal{F}$, one can find a net $(y_\eta)_{\eta \in H}$ of elements of Y, subordinate to $\mathcal{G}$, and a strict subnet $(x_{\xi(\eta)})_{\eta \in H}$ of the net $(x_\xi)_{\xi \in \Xi}$ such that for all $\eta \in H$ one will have $\varphi(x_{\xi(\eta)}, y_\eta, z)$, i.e., symbolically,

$$(\forall x_\xi \downarrow \mathcal{F})(\exists y_\eta \downarrow \mathcal{G})\quad \varphi(x_{\xi(\eta)}, y_\eta, z);$$

(4) for any net $(x_\xi)_{\xi \in \Xi}$ of elements of X, subordinate to $\mathcal{F}$, one can find a net $(y_\eta)_{\eta \in H}$ of elements of Y, subordinate to $\mathcal{G}$, and a subnet $(x_\eta)_{\eta \in H}$ of the net $(x_\xi)_{\xi \in \Xi}$ such that for all $\eta \in H$, one will have $\varphi(x_\eta, y_\eta, z)$, i.e., symbolically,

$$(\forall x_\xi \downarrow \mathcal{F})(\exists y_\eta \downarrow \mathcal{G})\quad \varphi(x_\eta, y_\eta, z);$$

(5) for any ultranet $(x_\xi)_{\xi \in \Xi}$ of elements of X, subordinate to $\mathcal{F}$, one can find an ultranet $(y_\eta)_{\eta \in H}$, subordinate to $\mathcal{G}$, and an ultranet $(x_\eta)_{\eta \in H}$, equivalent to $(x_\xi)_{\xi \in \Xi}$, such that $\varphi(x_\eta, y_\eta, z)$ for all $\eta \in H$.

$\lhd$ (1) → (2). Let $x \in \mu(\mathcal{F})$. By the transfer principle, for each standard G there is a standard F such that $(\forall x \in F)(\exists y \in G)\, \varphi(x, y, z)$. Hence, for $x \in \mu(\mathcal{F})$ one will have $(\forall G \in {}^\circ\mathcal{G})(\exists y \in G)\, \varphi(x, y, z)$. Drawing on the idealization principle, we deduce that $(\exists y)(\forall G \in {}^\circ\mathcal{G})\ y \in G \wedge \varphi(x, y, z)$. Thus, $y \in \mu(\mathcal{G})$ and $\varphi(x, y, z)$.

(2) → (3). Let $(x_\xi)_{\xi \in \Xi}$ be a standard net in X, subordinate to $\mathcal{F}$. For each standard G from $\mathcal{G}$ and $\xi \in \Xi$ we set

$$A_{(G,\xi)} := \{\xi' \geqslant \xi\colon (\forall \xi'' \geqslant \xi')(\exists y \in G)\quad \varphi(x_{\xi''}, y, z)\}.$$

On the basis of 1.1 and 1.2 we see that ${}^a\Xi \subset A_{(G,\xi)}$. Considering that $A_{(G,\xi)}$ is an inner set, by Cauchy's principle we conclude that ${}^\circ A_{(G,\xi)} \neq \varnothing$. Thus, on the direction $H := \mathcal{G} \times \Xi$ (with the natural ordering) there are given standard maps $\xi\colon H \to \Xi$ and $y\colon H \to Y$ such that $\xi(\eta) \in A_{(G,\xi)}$ and $y_\eta \in G$ for $G \in \mathcal{G}$ and $\xi \in \Xi$, for which $\eta = (G, \xi)$. It is clear that $\xi(\eta) \approx +\infty$ and $y_\eta \in \mu(\mathcal{G})$ for $\eta \approx +\infty$.

(3) → (4). Obvious.

(4) → (1). If (1) does not hold, then by hypothesis

$$(\exists G \in \mathcal{G})(\forall F \in \mathcal{F})(\exists x \in F)(\forall y \in G)\quad \neg\varphi(x, y, z).$$

For $F \in \mathcal{F}$ we take $x_F \in F$ so that $\neg\varphi(x, y, z)$ for all $y \in G$. We note that the net $(x_F)_{F \in \mathcal{F}}$ of elements of X obtained, just like the set G, can be considered standard on the basis of the transfer principle. There is no doubt that $x_F \downarrow \mathcal{F}$ and hence, by virtue of (3) one can find a direction H and a subnet $(x_\eta)_{\eta \in H}$ of the net $(x_F)_{F \in \mathcal{F}}$ such that for some net $(y_\eta)_{\eta \in H}$ one will have $\varphi(x_\eta, y_\eta, z)$ for each $\eta \in H$. By the definition of 1.7, x_η for each infinitely large η coincides with x_F for some remote F, i.e., $x_\eta \in \mu(\mathcal{F})$. By hypothesis, $y_\eta \in \mu(\mathcal{G})$ and all the more $y_\eta \in G$. It turns out here that $\varphi(x_\eta, y_\eta, z)$ and $\neg\varphi(x_\eta, y_\eta, z)$, which is impossible. The contradiction found indicates that the assumption made was false. Thus, (1) holds [provided (4) holds].

(1) ↔ (5). To prove the equivalence required it suffices to note that it becomes obvious if $\mathcal{F}$ and $\mathcal{G}$ are ultrafilters (cf. 1.8). It remains to recall that each monad is a union of monads of ultrafilters. $\rhd$

1.10. In applications it happens to be convenient to consider the concrete version of 1.9 corresponding to cases in which one of the filters is discrete. Thus, using the natural notation, we deduce that

$$(\exists x \in \mu(\mathcal{F}))\quad \varphi(x, y) \leftrightarrow (\exists x_\xi \downarrow \mathcal{F})\quad \varphi(x_\xi, y);$$
$$(\forall x \in \mu(\mathcal{F}))\quad \varphi(x, y) \leftrightarrow (\forall x_\xi \downarrow \mathcal{F})(\exists x_\eta \downarrow \mathcal{F})\quad \varphi(x_\eta, y).$$

2. Limits of Correspondences

2.0. In this section we collect auxiliary, generally familiar information about limits of families of sets used in what follows. Cf. [1, 8] with respect to the history of the question.

2.1. Let $\Gamma \subset X \times Y$ be an inner correspondence from the standard set X to the standard set Y. Let us assume that in X there is singled out a filter $\mathcal{F}$, and in Y a topology τ. We set (cf. [9])

$$\forall\forall(\Gamma) := {}^*\{y' : (\forall x \in \mu(\mathcal{F}) \cap \operatorname{dom}\Gamma)(\forall y \approx y')\quad (x, y) \in \Gamma\};$$
$$\exists\forall(\Gamma) := {}^*\{y' : (\exists x \in \mu(\mathcal{F}) \cap \operatorname{dom}\Gamma)(\forall y \approx y')\quad (x, y) \in \Gamma\};$$
$$\forall\exists(\Gamma) := {}^*\{y' : (\forall x \in \mu(\mathcal{F}) \cap \operatorname{dom}\Gamma)(\exists y \approx y')\quad (x, y) \in \Gamma\};$$
$$\exists\exists(\Gamma) := {}^*\{y' : (\exists x \in \mu(\mathcal{F}) \cap \operatorname{dom}\Gamma)(\exists y \approx y')\quad (x, y) \in \Gamma\},$$

where, as usual, $*$ is the symbol of <u>standardization,</u> and the notation $y \approx y'$ means that $y \in \mu(\tau(y'))$. The set $Q_1Q_2(\Gamma)$ is called the Q_1Q_2-<u>limit</u> of Γ (here Q is one of the quantifiers $\forall$ or $\exists$).

2.2. In topology one usually restricts oneself to the case when Γ is a standard correspondence defined on some element of the filter $\mathcal{F}$. Here one studies the $\exists\exists$-limit and $\forall\exists$-limit. The first is called the <u>upper limit,</u> and the second the <u>lower limit of Γ along $\mathcal{F}$.</u>

If one considers the net $(x_\xi)_{\xi\in\Xi}$ in the domain of definition of Γ, then keeping in mind the filter of tails of a net, one sets

$$\mathrm{Li}_{\xi\in\Xi}\Gamma := \liminf_{\xi\in\Xi}\Gamma(x_\xi) := \forall\exists(\Gamma);$$
$$\mathrm{Ls}_{\xi\in\Xi}\Gamma := \limsup_{\xi\in\Xi}\Gamma(x_\xi) := \exists\exists(\Gamma).$$

In such cases one most often speaks of <u>Kuratowsky limits.</u>

2.3. For a standard correspondence Γ one has the representations

$$\exists\exists(\Gamma) = \bigcap_{F\in\mathcal{F}} \operatorname{cl} \bigcup_{x\in F} \Gamma(x);$$
$$\forall\exists(\Gamma) = \bigcap_{F\in\ddot{\mathcal{F}}} \operatorname{cl} \bigcup_{x\in F} \Gamma(x),$$

where $\ddot{\mathcal{F}}$ is the so-called <u>grill</u> of $\mathcal{F}$, i.e., the family composed of all subsets of X defining the monad $\mu(\mathcal{F})$. In other words,

$$\ddot{\mathcal{F}} = {}^*\{F' \subset X : F' \cap \mu(\mathcal{F}) \neq \varnothing\} = \{F' \subset X : (\forall F \subset \mathcal{F})\quad F \cap F' \neq \varnothing\}.$$

We note in this connection the relations

$$\exists\forall(\Gamma) = \bigcup_{F\in\ddot{\mathcal{F}}} \operatorname{int} \bigcap_{x\in F} \Gamma(x);$$
$$\forall\forall(\Gamma) = \bigcup_{F\in\mathcal{F}} \operatorname{int} \bigcap_{x\in F} \Gamma(x).$$

2.3. A description of limits in the language of nets follows instantly from the theorem of 1.9.

2.4. The element y lies in the $\forall\exists$-limit of Γ if and only if for each net $(x_\xi)_{\xi\in\Xi}$ of elements of $\operatorname{dom}\Gamma$, subordinate to $\mathcal{F}$, one can find a subnet $(x_\eta)_{\eta\in H}$ of the net $(x_\xi)_{\xi\in\Xi}$ and a net $(y_\eta)_{\eta\in H}$, converging to y, such that $(x_\eta, y_\eta) \in \Gamma$ for all $\eta \in H$.

2.5. The element y lies in the $\exists\exists$-limit of Γ if and only if there exist a net $(x_\xi)_{\xi\in\Xi}$ of elements of $\operatorname{dom}\Gamma$, subordinate to $\mathcal{F}$, and a net $(y_\xi)_{\xi\in\Xi}$, converging to y, for which $(x_\xi, y_\xi) \in \Gamma$ for any $\xi \in \Xi$.

2.6. For any inner correspondence Γ one has

$$\forall\forall(\Gamma) \subset \exists\forall(\Gamma) \subset \forall\exists(\Gamma) \subset \exists\exists(\Gamma).$$

Here $\exists\exists(\Gamma)$, $\forall\exists(\Gamma)$ are closed and $\forall\forall(\Gamma)$ and $\exists\forall(\Gamma)$, respectively, open sets.

$\lhd$ The inclusions sought are self-evident. Thus, using duality considerations we establish to be definite that the $\forall\exists$-limit is closed.

If V is a standard open neighborhood of the point y' from $\mathrm{cl}\,\forall\exists(\Gamma)$, then there is a $y\in\forall\exists(\Gamma)$, for which $y\in V$. For $x\in\mu(\mathscr{F})$, we seek a y" such that $y''\in\mu(\tau(y))$ and $(x, y'')\in\Gamma$. It is clear that $y''\in V$, because V is a neighborhood of y. Thus,

$$(\forall x\in\mu(\mathscr{F}))(\forall V\in{}^{\circ}\tau(y'))(\exists y''\in V)\quad (x, y'')\in\Gamma.$$

Using the principle of idealization, we deduce that $y'\in\forall\exists(\Gamma)$. ▷

2.7. Let X and Y be topological spaces, $\Gamma\subset X\times Y$. The following assertions are equivalent:

(1) Γ is an open correspondence (= preserves open sets);

(2) Γ is open at each point $(x, y)\in\Gamma$;

(3) if $y_\xi\to y$, then $\Gamma^{-1}(y)\subset\liminf_{\xi\in\Xi}\Gamma^{-1}(y_\xi)$;

(4) if $y_\xi\to y$, then $\Gamma^{-1}(y)\subset\limsup_{\xi\in\Xi}\Gamma^{-1}(y_\xi)$.

◁ (1) → (2). Obvious.

(2) → (3). First of all we note that the condition of openness at the standard point $(x, y)\in\Gamma$ means the following: $(\forall y'\approx y)(\exists x'\approx x)\ (x', y')\in\Gamma$. Hence, for $x\in\Gamma^{-1}(y)$ and any $\xi\approx+\infty$ for some $x'\approx x$ one has $(x', y_\xi)\in\Gamma$, i.e., $x\in\liminf_{\xi\in\Xi}\Gamma^{-1}(y_\xi)$.

(3) → (4). Obvious.

(4) → (1). If Γ is not open, then there is an open U in X, for which $Y\setminus\Gamma(U)$ is not closed. Thus, there exists a net $(y_\xi)\subset Y\setminus\Gamma(U)$ such that $y_\xi\to y$ and $y\in\Gamma(U)$. For any $x\in U$ one can find (cf. 2.3) a net $(x_\xi)_{\xi\in\Xi}$, converging to x and having the property that $x_\xi\in\Gamma^{-1}(y_\xi)$. It is clear that for $\xi\approx+\infty$ one will have $x_\xi\in U$. We have found a contradiction. ▷

2.8. Let X and Y be topological spaces and $\Gamma\subset X\times Y$. The following assertions are equivalent:

(1) Γ is a closed set in X × Y;

(2) if $y_\xi\to y$, then $\limsup_{\xi\in\Xi}\Gamma^{-1}(y_\xi)\subset\Gamma^{-1}(y)$;

(3) if $y_\xi\to y$, then $\liminf_{\xi\in\Xi}\Gamma^{-1}(y_\xi)\subset\Gamma^{-1}(y)$.

◁ (1) → (2). If $y_\xi\to y$ and x lies in the corresponding upper limit, then one can find an index $\xi\approx+\infty$ and an element $x'\approx x$, for which $(x', y_\xi)\in\Gamma$. Since $y_\xi\approx y$, we see that $(x, y)\in\Gamma$ (because in Γ there occurs an element which is infinitely close to this point).

(2) → (3). Obvious.

(3) → (1). If $(x_\xi, y_\xi)\in\Gamma$ and $x_\xi\to x$, $y_\xi\to y$, hen considering the theorem of 1.9 we will have $x\in\liminf_{\xi\in\Xi}\Gamma^{-1}(y_\xi)$. Hence $(x, y)\in\Gamma$. ▷

2.9. The propositions of 2.7 and 2.8 actually are due to Kuratowsky [6], cf. also the theorems of 1.2 and 6.2 in [3].

3. Compactness and Subcontinuity

3.0. In this section we give standard and nonstandard tests for compactness and analogous concepts for filters, treating in more detail classical facts of nonstandard general topology [5, 6]. Applications are given to the theory of subcontinuous correspondences developed in [3, 4].

3.1. A filter $\mathscr{F}$ (in a topological space X) is called compact (cf. [2]), if each filter finer than $\mathscr{F}$ has a point of adherence in X. Correspondingly a net is called compact, if each subnet of it has a convergent subnet.

3.2. A standard filter $\mathscr{F}$ in X is compact if and only if each point of its monad is nearly standard: $\mu(\mathscr{F})\subset\mathrm{nst}(X)$.

◁ → Let $x\in\mu(\mathscr{F})$. We consider the ultrafilter $(x) := {}^*\{U\subset X: x\in U\}$ in the original space X. It is clear that $(x)\supset\mathscr{F}$ and hence there is a standard point x' such that $x\approx x'$. In other words, x is a nearly standard point.

← If $\mathscr{G}\supset\mathscr{F}$, then $\mu(\mathscr{G})\subset\mu(\mathscr{F})$. Let $x\in\mu(\mathscr{G})$. Then $x\in\mathrm{nst}(X)$, i.e., for some $x'\in{}^{\circ}X$ we will have $x\approx x'$. The latter means that x' is a point of adherence of $\mathscr{F}$. ▷

3.3. A filter $\mathscr{F}$ in X is compact if and only if for any open covering of the set X one can find a finite subcovering of some element of $\mathscr{F}$.

$\lhd$ $\to$ It is sufficient to work in a standard entourage. Thus, if $\mathscr{F}$ is compact, then $\mu(\mathscr{F}) \subset \mathrm{nst}(X)$. Considering that nst(X) lies in the monad of any standard covering $\mathscr{E}$, we deduce that $(\exists F \in \mathscr{F})(\forall x \in F)(\exists E \in {}^{\circ}\mathscr{E})\, x \in E$. As the F sought one can take any infinitely small element of $\mathscr{F}$. Applying the principles of idealization and transfer successively, we get what is required.

$\leftarrow$ Let $\mathscr{E}$ be an open covering of X and $\mu(\mathscr{E})$ be the union of the standard elements of $\mathscr{E}$, i.e., the monad of $\mathscr{E}$. By the transfer principle there are a standard $F \in \mathscr{F}$ and a finite standard subset $\mathscr{E}_0$ of $\mathscr{E}$ such that $\cup\mathscr{E}_0 \supset F \supset \mu(\mathscr{F})$. Hence, $\mu(\mathscr{F}) \subset \mu(\mathscr{E})$. It remains to recall that nst(X) is precisely the intersection of the monads of the standard open coverings of X. $\rhd$

3.4. The test formulated in 3.3 makes the search for an analog of the Hausdorff test for filters natural. In this connection we shall consider a uniform space $(x, \mathscr{U})$.

3.5. A filter $\mathscr{F}$ on X is called completely bounded, if for each surrounding $U \in \mathscr{U}$ there is a finite U-net of some element F of the filter $\mathscr{F}$.

3.6. A filter $\mathscr{F}$ on X is called complete, if each Cauchy filter, finer than $\mathscr{F}$, converges in X.

3.7. A standard filter is complete if and only if each prestandard point of its monad is nearly standard.

$\lhd$ $\to$ Let $\mathscr{F}$ be a complete filter and $x \in \mathrm{pst}(X) \cap \mu(\mathscr{F})$ be a prestandard point of the monad of $\mathscr{F}$. The prestandardness of x means that x lies in the monad of some Cauchy filter $\mathscr{G}$. Here $\mu(\mathscr{F}) \cap \mu(\mathscr{G}) \neq \varnothing$. It is clear that the least upper bound of $\mathscr{G}$ and $\mathscr{F}$ is a Cauchy filter and hence there is a point $x' \in {}^{\circ}X$, for which $x' \in \mu(\mathscr{G}) \cap \mu(\mathscr{F})$. Hence $x' \approx x$ and $x \in \mathrm{nst}(X)$.

$\leftarrow$ Let $\mathscr{G} \supset \mathscr{F}$ and $\mathscr{G}$ be a Cauchy filter. If $x \in \mu(\mathscr{G})$, then $x \in \mu(\mathscr{F}) \subset \mathrm{nst}(X)$. Hence $\mathscr{G}$ has a point of adherence. $\rhd$

3.8. A standard filter is completely bounded if and only if each point of its monad is prestandard.

$\lhd$ $\to$ By the transfer principle, for each standard surrounding U from $\mathscr{U}$ there are a standard element F of the filter $\mathscr{F}$ and a finite standard set E such that $U(E) \supset F$. Hence, $\mu(\mathscr{F}) \subset U(E)$. Thus for $x \in \mu(\mathscr{F})$ and any $U \in {}^{\circ}\mathscr{U}$ we will have $x \in U(x')$ for a suitable standard x'. We set $\mathscr{G} := {}^{*}\{U(x') : U \in \mathscr{U},\ x \in U(x')\}$. It is clear that $\mathscr{G}$ is a base of a Cauchy filter and $x \in \mu(\mathscr{G})$ by construction. Consequently, $\mu(\mathscr{F}) \subset \mathrm{pst}(X)$.

$\leftarrow$ Let us assume that the filter $\mathscr{F}$ considered is such that $\mu(\mathscr{F}) \subset \mathrm{pst}(X)$, and nevertheless $\mathscr{F}$ is not completely bounded. By the transfer principle there is a standard surrounding U from ${}^{\circ}\mathscr{U}$ such that for any $F \in {}^{\circ}\mathscr{F}$ and any standard finite set E one can find an $x \in F$, which does not land in U(E). By the idealization principle there is an element $x \in \mu(\mathscr{F})$ such that $x \notin U(y)$ for each standard $y \in X$. By hypothesis $x \in \mu(\mathscr{G})$, where $\mathscr{G}$ is Cauchy filter. We take $G \in {}^{\circ}\mathscr{G}$ such that $G \times G \subset U$. Then for any $y \in G$ one has $x \in \mu(\mathscr{G}) \subset U(y)$ contrary to the original assumption. $\rhd$

3.9. Hausdorff Test for Filters. A filter is compact if and only if it is complete and completely bounded.

$\rhd$ $\to$ It is sufficient to work in a standard entourage. If $\mathscr{F}$ is compact, then $\mu(\mathscr{F}) \subset \mathrm{nst}(X)$ by 3.2. Considering that $\mathrm{nst}(X) \subset \mathrm{pst}(X)$, we conclude: $\mathscr{F}$ is complete and completely bounded.

$\leftarrow$ If $\mathscr{F}$ is completely bounded, then by 3.8, $\mu(\mathscr{F}) \subset \mathrm{pst}(X)$. If $\mathscr{F}$ is complete, then $\mathrm{pst}(X) \subset \mathrm{nst}(X)$. From this we deduce that $\mu(\mathscr{F}) = \mu(\mathscr{F}) \cap \mathrm{pst}(X) \subset \mathrm{nst}(X)$. It remains to refer to 3.2. $\rhd$

3.10. The tests found can be placed at the foundation of the study of different topological concepts similar to continuity. We dwell here on one of them (cf. [2-4]).

3.11. A correspondence Γ, acting from X to Y, is called subcontinuous at the point x of dom Γ, if the image of the filter of neighborhoods of the point x under Γ is compact in Y. A correspondence Γ, which is subcontinuous at each point of Γ is called subcontinuous.

3.12. A standard correspondence Γ from X to Y is subcontinuous if and only if $\Gamma(\mathrm{nst}(X)) \subset \mathrm{nst}(Y)$.

◁ The proof follows from 1.2 and 3.2, because nst(X) is the union of monads of points of the standard kernel °X. ▷

3.13. A correspondence is subcontinuous if and only if it carries compact filters into compact ones.

◁ Since the filter of neighborhoods of a point is a fortiori compact, the sufficiency of the condition cited is self-evident. Now let it be known in advance that the correspondence is subcontinuous. Without loss of generality one can work in a standard entourage. Drawing on 3.12 and 3.2, we see that in the present situation the image of a standard compact filter is compact. It remains to refer to the transfer principle. ▷

3.14. In connection with the test of 3.13, subcontinuous correspondences are sometimes called compact (cf. [2]).

3.15. A subcontinuous correspondence acting on a Hausdorff space preserves relative compactness.

◁ If U is a standard relatively compact set in X, then $U \subset \mathrm{nst}(X)$. Hence, $\Gamma(U) \subset \mathrm{nst}(Y)$. It is known [5] that in this case $\Gamma(U)$ is relatively compct. ▷

3.16. Let Γ be a closed subcontinuous correspondence. Then Γ is upper semicontinuous.

◁ By the transfer principle one can work in a standard entourage. Thus, let A be a standard closed set and $x \in \mathrm{cl}\,\Gamma^{-1}(A)$. One has $x' \approx x$, for which for some $a' \in A$ one will have $(x', a') \in \Gamma$. Once $a' \in \Gamma(\mathrm{nst}(X))$, one can find a standard a in the image, for which $a \approx a'$. By the closedness of A we deduce that $a \in A$. Since Γ is closed one has $(x, a) \in \Gamma$. Thus, $x \in \Gamma^{-1}(A)$. ▷

3.17. The proposition of 3.16 is actually established in [4] and generalizes an earlier assertion about functions in [3]. In conclusion, with the help of Theorem 1.9 we give a simple nonstandard proof of a small modification of the continuity test 5.1 of [3].

3.18. Let $f: X \to Y$ be a function, acting on a Hausdorff space. Then f is continuous if and only if for each point x from X there is an element y from Y such that the condition $x_\xi \to x$ implies the existence of a subnet $(x_\eta)_{\eta \in H}$, for which $f(x_\eta) \to y$.

◁ In the verification only the sufficiency of the test formulated is needed. We shall work in the standard entourage. By hypothesis we have

$$(\forall x_\xi \to x)(\exists y_\eta \to y) \quad (x_\eta, y_\eta) \in f.$$

On the basis of Theorem 1.9 the last assertion can be rewritten in the form

$$(\forall x' \approx x)(\exists y' \approx y) \quad (x', y') \in f.$$

In particular, for some $y' \approx y$ one has $y' = f(x)$. Since Y is Hausdorff, we conclude that $y = f(x)$. Moreover, $x' \approx x \to f(x') \approx f(x)$, i.e., f is a continuous function. ▷

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