EPIDERIVATIVES DEFINED BY A SET OF INFINITESIMALS*

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In [1] we have outlined a method for the application of nonstandard methods of analysis for the study of various types of tangents, used in the theory of extremal problems. We know that the corresponding approximating cones for epigraphs are, in their own turn, the epigraphs of certain mappings. These mappings are called epiderivatives. With them, side by side with the classical differentials, are associated the contingent derivative, the Dubinskii–Milyutin derivative, the Clarke and the Rockafellar derivatives, and also their modifications. The aim of the present article is to perfect the existing methods for the construction of epiderivatives and for obtaining the rule for the estimation of the derivative of a sum (cf. [2, 3]). In addition, we isolate the spectrum of the new analogs of the approximating mappings with regard for the attraction of a basically new possibility – an arbitrary choice of the defining family of infinitesimal numbers.

1°. In the sequel, we will use without specifically mentioning the neoclassical installation of the nonstandard analysis, in which the hypothesis of standardness of the entourage is, as a rule, not specifically mentioned. By the same token it is understood that the free variables in the formal expression in the following text in the framework of some theory of inner and outer sets denote standard objects.

2°. Let us consider a real vector space X, equipped with linear topology σ and the almost vector topology τ. Further, let F be a set in X and x' be a point of F. According to the stipulation of 1°, these objects are assumed to be standard sets.

We fix a certain infinitesimal – a real number α, such that $\alpha > 0$ and $\alpha \approx 0$. Let us set

$$\mathrm{Ha}_\alpha(F, x') := {}^*\{h' \in X :\ (\forall x \approx_\sigma x',\ x \in F)\,(\forall h \approx_\tau h')\, x + \alpha h \in F\};$$
$$\mathrm{In}_\alpha(F, x') := {}^*\{h' \in X :\ (\exists h \approx_\tau h')\,(\forall x \approx_\sigma x',\ x \in F)\, x + \alpha h \in F\};$$
$$\mathrm{Cl}_\alpha(F, x') := {}^*\{h' \in X :\ (\forall x \approx_\sigma x',\ x \in F)\,(\exists h \approx_\tau h')\, x + \alpha h \in F\},$$

where, as usual, * is the symbol of standardization of an outer set.

Let us now consider a nonempty, in general, outer set Λ of infinitesimals and set

$$\mathrm{Ha}_\Lambda(F, x') := {}^* \bigcap_{\alpha \in \Lambda} \mathrm{Ha}_\alpha(F, x');$$
$$\mathrm{In}_\Lambda(F, x') := {}^* \bigcap_{\alpha \in \Lambda} \mathrm{In}_\alpha(F, x');$$
$$\mathrm{Cl}_\Lambda(F, x') := {}^* \bigcap_{\alpha \in \Lambda} \mathrm{Cl}_\alpha(F, x').$$

We adopt an analogous policy of notation also for other introduced types of approximations. As an example, it is worth emphasizing that, by virtue of definitions, for a standard h' from X

$$h' \in \mathrm{In}_\Lambda(F, x') \leftrightarrow (\forall \alpha \in \Lambda)\,(\exists h \approx_\tau h')\,(\forall x \approx_\sigma x',\ x \in F)\, x + \alpha h \in F.$$

It is useful to note that in the case where Λ is the monad of the corresponding standard filter $\mathscr{F}_\Lambda$, where $\mathscr{F}_\Lambda := {}^*\{A \subset \mathbf{R} : A \supset \Lambda\}$, we have, e.g., for $\mathrm{Cl}_\Lambda(F, x')$

$$\mathrm{Cl}_\Lambda(F, x') = \bigcap_{V \in \mathscr{N}_\tau} \ \bigcup_{\substack{U \in \sigma(x') \\ A \in \mathscr{F}_\Lambda}} \ \bigcap_{\substack{x \in F \cap U \\ \alpha \in A,\, \alpha > 0}} \left(\frac{F - x}{\alpha} + V \right).$$

Here $\mathscr{N}_\tau$ is the neighborhood filter of the origin in the topology τ and $\sigma(x')$ is the neighborhood filter of the point x' in the topology σ. If Λ is not the monad, then the concrete representations of the approximations, in which we are interested, $\mathrm{Ha}_\Lambda(F, x')$, $\mathrm{Cl}_\Lambda(F, x')$,

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etc. in the general case are connected with the model of analysis in which the investigation is actually carried out. It should be especially mentioned that the ultrafilter $\mathcal{U}(\alpha) := \mathcal{F}_{\{\alpha\}}$ has monad, not reducing to the initial infinitesimal α; i.e., the investigated set with index α is, in general, wider than the analogous set with index $\mu(\mathcal{U}(\alpha))$, where, as usual, $\mu(\mathcal{F})$ is the monad of the filter $\mathcal{F}$.

3°. We need certain facts about the introduced approximations (cf. [1]). We formulate them under the assumption of continuity of the mapping $(x, \beta, h) \to x + \beta h$ at the origin, which in the standard entourage appears as $\mu(\sigma) + \mu(\mathbf{R}_+)\mu(\tau) \subset \mu(\sigma)$, where $\mu(\mathbf{R}_+)$ is the monad composed of positive infinitesimal numbers.

(1) The sets $\mathrm{Ha}_\Lambda(F, x')$, $\mathrm{In}_\Lambda(F, x')$, and $\mathrm{Cl}_\Lambda(F, x')$ are semigroups, and

$$\mathrm{Ha}(F, x') \subset \mathrm{Ha}_\Lambda(F, x') \subset \mathrm{In}_\Lambda(F, x') \subset \mathrm{Cl}_\Lambda(F, x') \subset K(F, x'),$$
$$\mathrm{Cl}(F, x') \subset \mathrm{Cl}_\Lambda(F, x'),$$

where $\mathrm{Ha}(F, x') := \mathrm{Ha}_{\mu(\mathbf{R}_+)}(F, x')$ is the Hadamard cone, $\mathrm{Cl}(F, x') := \mathrm{Cl}_{\mu(\mathbf{R}_+)}(F, x')$ is the Clarke cone, and $K(F, x') := {}^*\{h' \in X : (\exists h \approx_\tau h')(\exists \alpha \in \mu(\mathbf{R}_+))\, x' + \alpha h \in F\}$ is the contingency of the set F at the point x'.

(2) If Λ is an inner set or a monad, then $\mathrm{Ha}_\Lambda(F, x')$ is τ-open.

(3) $\mathrm{Cl}_\Lambda(F, x')$ is a τ-closed set, and $K(F, x') = \mathrm{Cl}_\Lambda(F, x')$ for convex F, provided $\sigma = \tau$.

(4) If $\sigma = \tau$, then

$$\mathrm{Cl}_\Lambda(F, x') = \mathrm{Cl}_\Lambda(\mathrm{cl}\, F, x').$$

(5) The Rockafellar formula

$$\mathrm{Ha}_\Lambda(F, x') + \mathrm{Cl}_\Lambda(F, x') \subset \mathrm{Ha}_\Lambda(F, x')$$

holds.

(6) If x' is a τ-boundary point of F, then for $F' := (X \setminus F) \cup \{x'\}$

$$\mathrm{Ha}_\Lambda(F, x') = -\mathrm{Ha}_\Lambda(F', x').$$

(7) Let τ be a vector topology and $t\Lambda \subset \Lambda$ for a certain standard $t \in [0, 1]$. Then $\mathrm{Cl}_\Lambda(F, x')$ is a convex cone.

(8) Let $t\Lambda \subset \Lambda$ for each standard $t \in [0, 1]$. Then the sets $\mathrm{Cl}_\Lambda(F, x')$, $\mathrm{In}_\Lambda(F, x')$, and $\mathrm{Ha}_\Lambda(F, x')$ are convex cones.

The set Λ is said to be representative if $\mathrm{Ha}_\Lambda(F, x')$ and $\mathrm{Cl}_\Lambda(F, x')$ are (convex) cones. Propositions (7) and (8) give examples of representative Λ.

4°. Let $f : X \to \bar{\mathbf{R}}$ be a function that acts into the extended number line. For an infinitesimal α, a point x' in $\mathrm{dom}(f)$, and a vector $h' \in X$ we set

$$f(\mathrm{Ha}_\alpha)(x')(h') := \inf\{t \in \mathbf{R} : (h', t) \in \mathrm{Ha}_\alpha(\mathrm{epi}(f), (x', f(x')))\};$$
$$f(\mathrm{In}_\alpha)(x')(h') := \inf\{t \in \mathbf{R} : (h', t) \in \mathrm{In}_\alpha(\mathrm{epi}(f), (x', f(x')))\};$$
$$f(\mathrm{Cl}_\alpha)(x')(h') := \inf\{t \in \mathbf{R} : (h', t) \in \mathrm{Cl}_\alpha(\mathrm{epi}(f), (x', f(x')))\}.$$

The derivatives $f(\mathrm{Ha}_\Lambda)$, $f(\mathrm{In}_\Lambda)$, and $f(\mathrm{Cl}_\Lambda)$ are introduced in the natural manner. Let us observe that the derivative $f(\mathrm{Cl}) := f(\mathrm{Cl}_{\mu(\mathbf{R}_+)})$ is called the Rockafellar derivative and is denoted by the symbol $f^\uparrow$. In this connection, we write

$$f^\uparrow_\alpha(x') := f(\mathrm{Cl}_\alpha)(x'), \quad f^\uparrow_\Lambda(x') := f(\mathrm{Cl}_\Lambda)(x').$$

If τ is the discrete topology, then $\mathrm{Ha}_\Lambda(F, x') = \mathrm{In}_\Lambda(F, x') = \mathrm{Cl}_\Lambda(F, x')$. Here the Rockafellar derivative is called the Clarke derivative and we use the following notation:

$$f^\circ_\alpha(x') := f^\uparrow_\alpha(x'), \quad f^\circ_\Lambda(x') := f^\uparrow_\Lambda(x').$$

For $\Lambda = \mu(\mathbf{R}_+)$ the index Λ is omitted.

In the consideration of epiderivatives we suppose that the space $X \times \mathbf{R}$ is equipped with the usual products of topologies $\sigma \times \tau_{\mathbf{R}}$ and $\tau \times \tau_{\mathbf{R}}$, where $\tau_{\mathbf{R}}$ is the standard topology of $\mathbf{R}$. It is sometimes convenient to equip $X \times \mathbf{R}$ with the pair of the topologies $\sigma \times \tau_0$ and $\tau \times \tau_{\mathbf{R}}$, where τ_0 is the trivial topology in $\mathbf{R}$. While using these topologies, we speak about the

Clarke and the Rockafellar derivatives along the effective domain dom(f) and add the index d to the symbols: f^{0}_{d}, $f^{\uparrow}_{\Lambda,d}$, etc.

5°. The following statements are valid:

$$f^{\uparrow}_{\alpha}(x')(h') \leqslant t' \leftrightarrow (\forall x \approx_{\sigma} x',\ t \approx f(x'),\ t \geqslant f(x)) \\ (\exists h \approx_{\tau} h')^{\circ}((f(x+\alpha h)-t)/\alpha) \leqslant t';$$

$$f^{\circ}_{\alpha}(x')(h') < t' \leftrightarrow (\forall x \approx_{\sigma} x',\ t \approx f(x'),\ t \geqslant f(x))\,(\forall h \approx_{\tau} h')^{\circ}((f(x+\alpha h)-t)/\alpha) < t';$$

$$f^{\uparrow}_{\alpha,d}(x')(h') \leqslant t' \leftrightarrow (\forall x \approx_{\sigma} x',\ x \in \mathrm{dom}(f))\,(\exists h \approx_{\tau} h')\,{}^{\circ}((f(x+\alpha h)-t)/\alpha) \leqslant t';$$

$$f^{\circ}_{\alpha,d}(x')(h') < t' \leftrightarrow (\forall x \approx_{\sigma} x',\ x \in \mathrm{dom}(f))\,(\forall h \approx_{\tau} h')^{\circ}((f(x+\alpha h)-t)/\alpha) < t'.$$

◁ To prove the above statements, it is necessary to observe only that the following relations are valid for numbers $t,\ s \in \overline{\mathbf{R}}$:

$$(\exists t' \approx t)\ \ t' \geqslant s \leftrightarrow {}^{\circ}s \leqslant {}^{\circ}t \leftrightarrow (\forall^{st}\varepsilon > 0)\quad s \leqslant {}^{\circ}t + \varepsilon;$$
$$(\forall t' \approx t)\ \ t' \geqslant s \leftrightarrow {}^{\circ}s < {}^{\circ}t \quad (t \in {}^{*}\mathbf{R}).$$

Indeed, in the first equivalence the right implication is obvious and the reverse implication is ensured by the fact that $s \leqslant {}^{\circ}t + s - {}^{\circ}s$ for ${}^{\circ}s \leqslant {}^{\circ}t$. The second equivalence in the upper row has been noted in [1].

To verify the validity of the arrow towards right in the lower equivalence, we observe that s does not belong to the monad $\mu(t)$ of the number t. Therefore, the whole monad of s lies to the left of the monad of t, i.e., $\mu(s) < \mu(t)$. Consequently, $°s < °t$. Finally, to establish the remaining implication, we observe that $s < t$ for $°s = -\infty$ since $t \in {}^{*}\mathbf{R}$. But if ${}^{\circ}s \in {}^{\circ}\mathbf{R}$, then $\mu(°s) < °t$. Therefore, $t' \geq s$ for $t' \approx t$. ▷

6°. If f is a lower semicontinuous function, then

$$f^{\uparrow}_{\alpha}(x')(h') \leqslant t' \leftrightarrow (\forall x \approx_{\sigma} x',\ f(x) \approx f(x'))\,(\exists h \approx_{\tau} h')^{\circ}\left(\frac{f(x+\alpha h)-f(x)}{\alpha}\right) \leqslant t';$$

$$f^{\circ}_{\alpha}(x')(h') < t' \leftrightarrow (\forall x \approx_{\sigma} x',\ f(x) \approx f(x'))\,(\forall h \approx_{\tau} h')^{\circ}\left(\frac{f(x+\alpha h)-f(x)}{\alpha}\right) < t'.$$

◁ We need verify only the implication to the right. Since such verifications are identical, we realize the first of them. On the basis of the lower semicontinuity of f we deduce that $x' \approx_{\sigma} x \to {}^{\circ}f(x) \geqslant f(x')$. Therefore, ${}^{\circ}t \geqslant {}^{\circ}f(x) \geqslant f(x') = {}^{\circ}t$ for x, t such that $t \approx f(x')$ and $t \geqslant f(x)$. In other words, $°f(x) = f(x')$ and $f(x) \approx f(x')$. Choosing a suitable h with the help of the conditions, we see that

$${}^{\circ}(\alpha^{-1}(f(x+\alpha h)-t)) \leqslant {}^{\circ}(\alpha^{-1}(f(x+\alpha h)-f(x))) \leqslant t',$$

which ensures the desired result. ▷

7°. The following equations are valid for each continuous function f:

$$f^{\uparrow}_{\Lambda,d}(x') = f^{\uparrow}_{\Lambda}(x'),\quad f^{\circ}_{\Lambda,d}(x') = f^{\circ}_{\Lambda}(x').$$

◁ It is sufficient to observe that the continuity of f at a standard point implies that $(x \approx_{\sigma} x',\ x \in \mathrm{dom}(f)) \to f(x) \approx f(x')$. ▷

8°. THEOREM. Let Λ be a monad. Then the following representations are valid:

(1) If f is a lower semicontinuous function, then

$$f^{\uparrow}_{\Lambda}(x')(h') = \limsup_{\substack{x \to_f x' \\ \alpha \in \mathscr{F}_{\Lambda}}} \inf_{h \to h'} \frac{f(x+\alpha h)-f(x)}{\alpha};$$

$$f^{\circ}_{\Lambda}(x')(h') = \limsup_{\substack{x \to_f x' \\ \alpha \in \mathscr{F}_{\Lambda}}} \frac{f(x+\alpha h')-f(x)}{\alpha},$$

where $x \to_f x'$ means that $x \to_{\sigma} x'$ and $f(x) \to f(x')$ and lim sup inf is the Rockafellar limit.

(2) The following equations are valid for an arbitrary function f:

$$f^{\uparrow}_{\Lambda,d}(x')(h') = \limsup_{\substack{x \to x' \\ \alpha \in \mathscr{F}_{\Lambda}}} \inf_{h \to h'} \frac{f(x+\alpha h)-f(x)}{\alpha};$$

$$f^{\circ}_{\Lambda,d}(x')(h') = \limsup_{\substack{x \to_f x' \\ \alpha \in \mathscr{F}_\Lambda}} \frac{f(x+\alpha h') - f(x)}{\alpha}.$$

◁ To prove the theorem it is sufficient to use the criterion for the Rockafellar limit, found in [1]:

$$\limsup\nolimits_{\mathscr{F}} \inf\nolimits_{\mathscr{G}} f(x, y) := \sup_{G \in \mathscr{G}} \inf_{F \in \mathscr{F}} \sup_{x \in F} \inf_{y \in G} f(x, y) \leqslant t \leftrightarrow (\forall x \in \mu(\mathscr{F}))(\exists y \in \mu(\mathscr{G}))\ {}^{\circ}f(x, y) \leqslant t.$$

A reference to pp. 6° and 7° completes the proof. ▷

9°. THEOREM. Let Λ be a representative set of infinitesimals. The following statements are valid:

(1) If f is a mapping that is directionally Lipschitzian at a point x', i.e., is such that $\mathrm{Ha}(\mathrm{epi}(f), (x', f(x'))) \neq \varnothing$, then

$$f^{\uparrow}_{\Lambda}(x') = f^{\circ}_{\Lambda}(x');$$

if, furthermore, f is continuous at x', then

$$f^{\uparrow}_{\Lambda}(x') = f^{\uparrow}_{\Lambda,d}(x') = f^{\circ}_{\Lambda,d}(x') = f^{\circ}_{\Lambda}(x').$$

(2) If f is an arbitrary mapping and the Hadamard cone of the effective set of f at the point x' is nonempty, i.e., $\mathrm{Ha}(\mathrm{dom}(f), x') \neq \varnothing$, then

$$f^{\uparrow}_{\Lambda,d}(x') = f^{\circ}_{\Lambda,d}(x').$$

◁ The proof of both the desired statements is carried out according to the same pattern, connected with the application of the propositions from 3°. We analyze in detail the case where f is directionally Lipschitzian.

Let us set $\mathscr{A} := \mathrm{epi}(f)$, and $a' := (x', f(x'))$. By virtue of the conditions, $\mathrm{Cl}_\Lambda(\mathscr{A}, a')$ and $\mathrm{Ha}_\Lambda(\mathscr{A}, a')$ are convex conves. In addition, $\mathrm{Ha}_\Lambda(\mathscr{A}, a') \supset \mathrm{Ha}(\mathscr{A}, a')$ and, therefore, $\mathrm{int}_{\tau \times \tau_{\mathbf{R}}} \mathrm{Ha}_\Lambda(\mathscr{A}, a') \neq \varnothing$. On the basis of the Rockafellar formula, we deduce that

$$\mathrm{cl}_{\tau \times \tau_{\mathbf{R}}} \mathrm{Ha}_\Lambda(\mathscr{A}, a') = \mathrm{Cl}_\Lambda(\mathscr{A}, a').$$

Hence the desired statement follows. ▷

10°. THEOREM. Let $f_1, f_2 : X \to \overline{\mathbf{R}}$ be arbitrary functions and $x' \in \mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$. Then

$$(f_1 + f_2)^{\uparrow}_{\Lambda,d}(x') \leqslant (f_1)^{\uparrow}_{\Lambda,d}(x') + (f_2)^{\circ}_{\Lambda,d}(x').$$

If, moreover, f_1 and f_2 are continuous at the point x', then

$$(f_1 + f_2)^{\uparrow}_{\Lambda}(x') \leqslant (f_1)^{\uparrow}_{\Lambda}(x') + (f_2)^{\circ}_{\Lambda}(x').$$

◁ Let the standard element h' be chosen in the following manner:

$$h' \in \mathrm{dom}\left((f_2)^{\circ}_{\Lambda,d}\right) \cap \mathrm{dom}\left((f_1)^{\uparrow}_{\Lambda,d}\right).$$

If there is no such h', then the desired estimates are obvious.

We take $t' \geqslant (f_1)^{\uparrow}_{\Lambda,d}(x')(h')$ and $s' > (f_2)^{\circ}_{\Lambda,d}(x')(h')$. Then, on the basis of 5°, for each $x \approx_\sigma x'$, $x \in \mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$, and an arbitrary $\alpha \in \Lambda$ there exists an h such that $h \approx_\tau h'$ and, moreover,

$$\delta_1 := {}^{\circ}((f_1(x + \alpha h) - f_1(x))/\alpha) \leqslant t';$$
$$\delta_2 := {}^{\circ}((f_2(x + \alpha h) - f_2(x))/\alpha) < s'.$$

Hence we deduce that $\delta_1 + \delta_2 < t' + s'$, which ensures the validity of the statement (1). If f_1 and f_2 are continuous at x', then we should use p. 7°. ▷

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