SOME METHODS OF DEFINING CLARKE'S CONE†

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It is convenient to study Clarke's tangents by means of infinitesimals. It turns out that the most important properties of such tangents are preserved under a restriction of the set of infinitesimal numbers in question [1]. In this paper we establish that in a finite-dimensional space Clarke's cone admits convenient definitions by means of an arbitrary sequence of scalars that converges to zero.

1. Working in the standard environment, we consider a set F in a finite-dimensional space X, a point x' of F, and a norm $\|\cdot\|$ in X. For an infinitesimal strictly positive number α we put

$$\mathrm{Cl}_\alpha(F, x') := {}^*\{h' \in X\colon\ (\forall x \approx x',\ x \in F)\,(\exists h \approx h')\ x + \alpha h \in F\}.$$

Here, as usual, the symbol $\approx$ denotes infinite closeness in X, and * is the symbol of standardization.

Now if Λ is an (external) set of infinitesimals, we define the set $\mathrm{Cl}_\Lambda(F, x')$ by the relation

$$\mathrm{Cl}_\Lambda(F, x') := {}^*\bigcap_{\alpha \in \Lambda} \mathrm{Cl}_\alpha(F, x').$$

In particular, if Λ is the monad of the filter $\mathscr{F}_\Lambda := {}^*\{A \subset X\colon\ A \supset \Lambda\}$, then

$$\mathrm{Cl}_\Lambda(F, x') = \bigcap_{\varepsilon > 0} \bigcup_{\substack{\delta > 0 \\ A \in \mathscr{F}_\Lambda}} \bigcap_{\substack{x \in F \\ \|x - x'\| \leq \delta \\ \alpha \in A,\, \alpha > 0}} \left(\frac{F - x}{\alpha} + \varepsilon B_X\right),$$

where $B_X := \{\|\cdot\|_X \leqslant 1\}$ is the unit ball in X.

In the case when $\mathscr{F}_\Lambda$ is the filter of a neighborhood of the origin on the semiaxis $\mathbf{R}^+$, we omit the symbol Λ and talk about the usual Clarke cone (the set F at the point x').

2. For a closed not necessarily convex cone F we have

$$\mathrm{Cl}_\alpha(F, 0) = \mathrm{Cl}(F, 0).$$

$\lhd$ First of all we observe that the following relation is satisfied:

$$\mathrm{Cl}_\alpha(F, 0) + F \subset F.$$

In fact, for $f \in F$ we put $x := \alpha f$. Clearly, $x \approx 0$ and $x \in F$. Thus, by definition, for some h infinitely close to h' we have $x + \alpha h \in F$ on condition that $h' \in \mathrm{Cl}_\alpha(F, 0)$. Since $\alpha(f + h) \in F$ and F is a cone, we conclude that $f + h \in F$. Hence $h \in F - f$ and therefore h' lies in the microclosure of F − f. Since F is closed and standard, we see that $f + h' \in F$.

We now take $h' \in \mathrm{Cl}_\alpha(F, 0)$, and suppose that $x \in F$ and $x \approx 0$. For any infinitesimal $\beta > 0$ we have

$$x + \beta h' = \beta(x/\beta + h') \in \beta(F + \mathrm{Cl}_\alpha(F, 0)) \subset \beta F \subset F$$

by what we have already proved. In other words, $h' \in \mathrm{Cl}(F, 0)$. Since $\mathrm{Cl}(F, 0) \subset \mathrm{Cl}_\alpha(F, 0)$, from obvious circumstances we conclude that the sets in question are equal. $\rhd$

3. In the condition of Proposition 2 $\mathrm{Cl}_\alpha(F, 0)$ is the maximal convex cone H such that $H + F \subset F$ (cf. [12]). We note that in the proof of Proposition 2 we did not use the fact that X is finite-dimensional. In a finite-dimensional space a much stronger assertion holds.

4. THEOREM. Let Λ be an (exterior) set of strictly positive infinitesimals containing an (interior) sequence that converges to zero. Then

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$$\mathrm{Cl}_\Lambda(F, x') = \mathrm{Cl}(F, x').$$

◁ By Leibniz' principle we can work in the standard environment. Since the inclusion $\mathrm{Cl}_\Lambda(F, x') \supset \mathrm{Cl}(F, x')$ is obvious, we take a standard point h' from $\mathrm{Cl}_\Lambda(F, x')$ and establish that h' lies in the Clarke cone Cl(F, x'). By a theorem of Cornet [3, 4] we have

$$\mathrm{Cl}(F, x') = \mathrm{Li}_{\substack{x \to x' \\ x \in F}} K(F, x),$$

where Li symbolizes the lower Kuratowski limit, and K(F, x') is the contingency of the set F at the point x', that is,

$$K(F, x') = \bigcap_{\varepsilon > 0} \mathrm{cl}\left(\bigcup_{0 < \delta \leqslant \varepsilon} \frac{F - x'}{\delta} \right).$$

Since

$$\mathrm{Li}_{\substack{x \to x' \\ x \in F}} K(F, x) = {}^*\{h' : (\forall x \approx x', x \in F)(\exists h \approx h')\ h \in K(F, x)\},$$

we verify that when $x \approx x'$ and $x \in F$ we have $h \in K(F, x)$ for some element h infinitely close to h'.

If (α_n) is a sequence of elements of Λ that converges to zero, then by hypothesis we have

$$(\forall n \in \mathbf{N})(\exists h_n) \quad x + \alpha_n h_n \in F \wedge h_n \approx h'.$$

For any standard $\varepsilon > 0$ we have $\|h_n - h'\| \leqslant \varepsilon$. Therefore, since X is finite-dimensional, we can try to find sequences $(\bar{\alpha}_n)$ and $(\bar{h}_n)$ such that

$$\bar{\alpha}_n \to 0,\ \bar{h}_n \to \bar{h},\ \|\bar{h} - h'\| \leqslant \varepsilon,\ x + \bar{\alpha}_n \bar{h}_n \in F\ (n \in \mathbf{N}).$$

Using the principle of idealization in the strong form, we conclude that there are sequences $(\bar{\alpha}_n)$ and $(\bar{h}_n)$ that simultaneously serve all standard positive numbers ε. Clearly, the corresponding limiting vector h is infinitely close to h' and at the same time $h \in K(F, x)$ by the definition of contingency. ▷

5. For the set Λ in the theorem we can take the monad of any filter converging to zero, for example, the filter of the tails of a fixed standard sequence (α_n) formed from strictly positive numbers and tending to zero. We give characterizations of Clarke's cone related to this case and additional to those given in [4]. For the formulation let us agree to denote by $d_F(x)$ the distance from a point x of X to the set F.

6. THEOREM. For a sequence (α_n) of strictly positive numbers that converges to zero, the following assertions are equivalent:

(1) $h' \in \mathrm{Cl}(F, x')$;

(2) $\limsup\limits_{\substack{x \to x' \\ n \to \infty}} \dfrac{d_F(x + \alpha_n h') - d_F(x)}{\alpha_n} \leqslant 0$;

(3) $\limsup\limits_{x \to x'} \limsup\limits_{n \to \infty} \alpha_n^{-1}(d_F(x + \alpha_n h') - d_F(x)) \leqslant 0$;

(4) $\lim\limits_{\substack{x \to x' \\ x \in F}} \limsup\limits_{n \to \infty} \alpha_n^{-1} d_F(x + \alpha_n h') = 0$;

(5) $\limsup\limits_{x \to x'} \liminf\limits_{n \to \infty} \alpha_n^{-1}(d_F(x + \alpha_n h') - d_F(x)) \leqslant 0$;

(6) $\lim\limits_{\substack{x \to x' \\ x \in F}} \liminf\limits_{n \to \infty} \dfrac{d_F(x + \alpha_n h')}{\alpha_n} = 0$.

◁ First of all we observe that when $\alpha > 0$ we have the equivalence

$${}^\circ\alpha^{-1} d_F(x + \alpha h') = 0 \leftrightarrow (\exists h \approx h')\ x + \alpha h \in F,$$

where °t is, as usual, the standard part of the number t.

In fact, to establish the implication to the left we put $y := x + \alpha h'$. Then

$$d_F(x + \alpha h')/\alpha \leqslant \|x + \alpha h' - y\|/\alpha \leqslant \|h - h'\|.$$

To verify the opposite implication, invoking the principle of idealization in the strong form, we obtain successively

$$°\alpha^{-1}d_F(x+\alpha h')=0 \to (\forall^{st}\varepsilon>0)\ d_F(x+\alpha h')/\alpha<\varepsilon \to (\forall^{st}\varepsilon>0)(\exists y\in F)\ \|x+\alpha h'-y\|/\alpha<\varepsilon$$
$$\to(\exists y\in F)(\forall^{st}\varepsilon>0)\ \|h'-(y-x)/\alpha\|<\varepsilon\to(\exists y\in F)\ \|h'-(y-x)/\alpha\|\approx 0.$$

Putting $h:=(y-x)/\alpha$, we see that $h \approx h'$ and $x+\alpha h\in F$.

We now turn to the proof of the required equivalences.

Since the implications (3) → (4) → (6) and (3) → (5) → (6) are obvious, we merely establish that (1) → (2) → (3) and (6) → (1).

(1) → (2). Working in the standard environment, we take $x \approx x'$ and $N \approx +\infty$. We choose $x''\in F$ so that $\|x-x''\|\leqslant d_F(x)+\alpha_N^2$. Since

$$d_F(x+\alpha_N h')-d_F(x''+\alpha_N h')\leqslant \|x-x''\|,$$

we derive the following estimates:

$$(d_F(x+\alpha_N h')-d_F(x))/\alpha_N\leqslant(d_F(x''+\alpha_N h')+\|x-x''\|-d_F(x))/\alpha_N\leqslant d_F(x''+\alpha_N h')/\alpha_N+\alpha_N.$$

Since $h'\in \mathrm{Cl}(F, x')$, taking account of the choice of x'' and N, for some $h \approx h'$ we have $x''+\alpha_N h\in F$. So, on the basis of what we have already proved, $°d_F(x''+\alpha_N h')/\alpha_N=0$. Hence

$$(\forall x\approx x')(\forall N\approx+\infty)\ °\alpha_N^{-1}(d_F(x+\alpha_N h')-d_F(x))\leqslant 0.$$

As we know, this is a nonstandard criterion for the truth of (2).

(2) → (3). It is sufficient to observe that for $f\colon U\times V\to\overline{\mathbb{R}}$ and the filters $\mathscr{F}$ in U and $\mathscr{G}$ in V we have

$$\limsup_{\mathscr{F}}\limsup_{\mathscr{G}} f(x,y)\leqslant t\leftrightarrow(\forall x\in\mu(\mathscr{F}))\,°\limsup_{\mathscr{G}} f(x,y)\leqslant t$$

$$\leftrightarrow(\forall x\in\mu(\mathscr{F}))(\forall^{st}\varepsilon>0)\inf_{G\in\mathscr{G}}\sup_{y\in G} f(x,y)<t+\varepsilon\leftrightarrow(\forall x\in\mu(\mathscr{F}))(\forall^{st}\varepsilon>0)(\exists G\in\mathscr{G})\sup_{y\in G} f(x,y)<t+\varepsilon$$

$$\leftrightarrow(\forall x\in\mu(\mathscr{F}))(\exists G\in\mathscr{G})(\forall^{st}\varepsilon>0)\sup_{y\in G} f(x,y)<t+\varepsilon\leftrightarrow(\forall x\in\mu(\mathscr{F}))(\exists G\in\mathscr{G})(\forall^{st}\varepsilon>0)\sup_{y\in G} f(x,y)\leqslant t+\varepsilon$$

$$\leftrightarrow(\forall x\in\mu(\mathscr{F}))(\exists G\in\mathscr{G})(\forall y\in G)\ °f(x,y)\leqslant t.$$

Here, as usual, $\mu(\mathscr{F})$ is the monad of the filter $\mathscr{F}$, that is, the exterior of standard elements of $\mathscr{F}$.

(6) → (1). First of all, in the notation of the previous fragment of the proof, we have

$$\limsup_{\mathscr{F}}\liminf_{\mathscr{G}} f(x,y)\leqslant t\leftrightarrow(\forall x\in\mu(\mathscr{F}))\,°\sup_{G\in\mathscr{G}}\inf_{y\in G} f(x,y)\leqslant t$$

$$\leftrightarrow(\forall x\in\mu(\mathscr{F}))(\forall^{st}\varepsilon>0)(\forall G\in\mathscr{G})\inf_{y\in G} f(x,y)\leqslant t+\varepsilon\ \leftrightarrow(\forall x\in\mu(\mathscr{F}))(\forall G\in\mathscr{G})(\forall^{st}\varepsilon>0)\inf_{y\in G} f(x,y)<t+\varepsilon$$

$$\leftrightarrow(\forall x\in\mu(\mathscr{F}))(\forall G\in\mathscr{G})(\forall^{st}\varepsilon>0)(\exists y\in G)\ f(x,y)<t+\varepsilon\leftrightarrow(\forall x\in\mu(\mathscr{F}))(\forall G\in\mathscr{G})(\exists y\in G)\ °f(x,y)\leqslant t.$$

Invoking the conditions from the criterion we have established, we deduce that

$$(\forall x\approx x',\, x\in F)(\forall n)(\exists N\geqslant n)\,°\alpha_N^{-1}d_F(x+\alpha_N h')=0.$$

In other words, for some h_N such that $h_N \approx h'$ we have $x+\alpha_N h_N\in F$. On the basis of the previous arguments, as in the proof of Theorem 4, we can deduce that h' lies in the lower Kuratowski limit of the contingencies of the set F at points close to x', that is, in the Clarke cone Cl(F, x'). ▷

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