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MONADS OF PRO-ULTRAFILTERS AND EXTENSIONAL FILTERS

S. S. Kutateladze UDC 517.11

In [1] we proposed an approach to the application of monadology, a branch of nonstandard analysis, to the study of cyclic filters, which arise in the context of Boolean-valued models. In this paper we characterize the monads of pro-ultrafilters and extensional filters and discuss some relevant properties of these objects. To save space, we shall use the notation and conventions described in detail in [1] without further explanation. We emphasize only that henceforth B will denote a fixed complete Boolean algebra and $V^{(B)}$ the corresponding separable Boolean-valued universe. The truth value of a formula φ of Zermelo—Fraenkel set theory will be denoted by $[\![\varphi]\!]$. When monadology is used, the neoclassical formulation is assumed. We shall generally adopt the hypothesis that the entourage is standard, without further mention.

1. Let X be a cyclic set (= descent of some B-set). As usual, the symbol μ_d will denote the operation producing the (discrete) monadic hull. In other words, $\mu_d(\varnothing) := \varnothing$, and if U is a nonempty set in X then $\mu_d(U)$ is the monad of the standardization of the external filter of supersets of U, i.e.,

$$x \in \mu_d(U) \leftrightarrow ((\forall^{st} V \subset X)\ U \subset V \to x \in U).$$

By analogy, we define the <u>cyclic monadic hull</u> μ_c as follows:

$$x \in \mu_c(U) \leftrightarrow (\forall^{st} V)(V = V\uparrow\downarrow \wedge V \subset X \wedge U \subset V \to x \in V).$$

Thus, if U is not empty, the cyclic monadic hull $\mu_c(U)$ is the monad of the cyclic closure of the standardization of the filter of supersets of U.

2. The cyclic monadic hull of a set is the cyclic closure of its monadic hull:

$$\mu_c(U) = \mathrm{mix}(\mu_d(U))$$

for every U.

◁ Let $U \neq \varnothing$ and let V be a standard set such that $V \supset \mathrm{mix}(\mu_d(U))$. By Theorem 2.3 of [1], there exists W in the filter $^*\{U_1 \subset X\colon U_1 \supset U\}$ such that $V \supset W\uparrow\downarrow$ and so $V \supset \mu_c(U)$. Thus $\mu_c(U) \subset \mathrm{mix}(\mu_d(U))$, since the set on the right is a monad. Conversely, if $V \supset \mu_c(U)$ and V is standard, then V contains the cyclic closure of a superset of U and thus V ⊃ U. Hence $V \supset \mu((^*\{W\colon W \supset U\})\uparrow\downarrow)$ and it remains to appeal again to Theorem 2.3 of [1]. ▷

3. Cyclic filters in X that are maximal with respect to inclusion will be called <u>pro-ultrafilters</u> in X. An <u>essential point</u> in X is defined to be an element of the monad of a standard pro-ultrafilter. The external set of all essential points of X will be denoted by ^eX. It is useful to emphasize that the pro-ultrafilters in X are precisely the descents of the ultrafilters in the ascent X↑ of X.

4. <u>Nonstandard Pro-Ultrafilter Criterion</u>. A filter is a pro-ultrafilter if and only if, first, its monad is cyclic, and, second, it is easily captured by a standard cyclic set.

◁ Let $\mathscr{F}$ be a filter. We have to prove that the following proposition is valid:

$$(\mathscr{F} \text{ is a pro-ultrafilter}) \leftrightarrow \mu(\mathscr{F}) = \mathrm{mix}(\mu(\mathscr{F})) \wedge (\forall^{st} V)(V = V\uparrow\downarrow \to \mu(\mathscr{F}) \subset V \vee \mu(\mathscr{F}) \subset V').$$

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If V is standard, then either $\mu(\mathcal{F}) \cap V = \varnothing$, or $\mu(\mathcal{F}) \cap V \neq \varnothing$. In the first case $V' := X \setminus V \in \mathcal{F}$. In the second, we have a filter $\mathcal{G}$ with monad $\mu(\mathcal{F}) \cap V$. Clearly, if $\mathcal{F}$ is a pro-ultrafilter and V a cyclic set, then $\mathcal{G} = \mathcal{F}$ (by the criterion for cyclic filters contained in Theorem 2.2 of [1]). Thus $V \in \mathcal{F}$, proving the implication $\rightarrow$.

We now prove the implication $\leftarrow$. Let $\mathcal{G}$ be a cyclic filter finer than $\mathcal{F}$. Clearly, $G \in \mathcal{G} \rightarrow G' \notin \mathcal{F}$ [otherwise $G' \supset \mu(\mathcal{F}) \supset \mu(\mathcal{G})$]. Thus $G \in \mathcal{F}$. Consequently, $\mathcal{G} = \mathcal{F}$. ▷

5. <u>Standard Pro-Ultrafilter Criteria.</u> Let $\mathcal{F}$ be a cyclic filter in X. Then the following statements are equivalent:

(1) $\mathcal{F}$ is a pro-ultrafilter;

(2) for any finite set $\mathcal{E}$ of subsets of X, either $(\cup \mathcal{E})' \in \mathcal{F}$ or $E{\uparrow\downarrow} \in \mathcal{F}$ for some $E \in \mathcal{E}$;

(3) for every finite sequence of cyclic sets, $\mathcal{F}$ contains either one of them or the complement of each of them;

(4) if U is an arbitrary set, then either $U{\uparrow\downarrow} \in \mathcal{F}$, or $U' \in \mathcal{F}$;

(5) for every cyclic V, either $V \in \mathcal{F}$ or $V' \in \mathcal{F}$.

◁ To prove (1) → (2), we use the transfer principle and the nonstandard pro-ultrafilter criterion of Sec. 4. Thus, let $\mathcal{F}$ be a standard filter and $\mathcal{E}$ a standard finite set of standard subsets of X. There are two possibilities: either $\mu(\mathcal{F}) \cap \cup \mathcal{E} = \varnothing$ or $\mu(\mathcal{F}) \cap \cup \mathcal{E} \neq \varnothing$. In the first case, the set $(\cup \mathcal{E})'$ is obviously in $\mathcal{F}$. In the second, there exists $E \in \mathcal{E}$ such that $E \cap \mu(\mathcal{F}) \neq \varnothing$. Thus $E{\uparrow\downarrow} \cap \mu(\mathcal{F}) \neq \varnothing$. Since E↑↓ is standard, it follows from Sec. 4 that $E{\uparrow\downarrow} \supset \mu(\mathcal{F})$ and therefore $E{\uparrow\downarrow} \in \mathcal{F}$.

The implications (2) → (3) → (4) → (5) are evident. The fact that (5) implies (1) follows from Sec. 4 and the transfer principle. ▷

6. <u>COROLLARY.</u> Let $\mathcal{F}$ be a filter in X. The filter $\mathcal{F}{\uparrow\downarrow}$ is a pro-ultrafilter if and only if, for every subset U of X, either $U{\uparrow\downarrow} \in \mathcal{F}$ or there exists F in $\mathcal{F}$ such that $F{\uparrow\downarrow} \subset U'$.

7. <u>COROLLARY.</u> Let $\mathcal{F}$ be a filter in X. Then $\mathcal{F}$ is a pro-ultrafilter if and only if $\mathcal{F} = (\ddot{\mathcal{F}}){\uparrow\downarrow}$, where $\ddot{\mathcal{F}}$ is the grill of $\mathcal{F}$, defined by

$$U \in \ddot{\mathcal{F}} \leftrightarrow (\forall F \in \mathcal{F})\ U \cap F \neq \varnothing.$$

◁ Suppose it is known that $\mathcal{F}$ is a pro-ultrafilter. Clearly, $\mathcal{F} \subset \ddot{\mathcal{F}}$ and so $\mathcal{F} = \mathcal{F}{\uparrow\downarrow} \subset (\ddot{\mathcal{F}}){\uparrow\downarrow}$. If $V \in (\ddot{\mathcal{F}}){\uparrow\downarrow}$, there exists U in $\ddot{\mathcal{F}}$, such that $V \supset U{\uparrow\downarrow}$. Then U↑↓ is surely an element of $\mathcal{F}$, by (4). Thus also $V \in \mathcal{F}$.

Suppose now that $\mathcal{F} = (\ddot{\mathcal{F}}){\uparrow\downarrow}$. Since by definition every element of the set on the right of this equality is a superset of a cyclic set, $\mathcal{F}$ is a cyclic filter. Let U be an arbitrary cyclic set. If $U \cap F = \varnothing$ for some $F \in \mathcal{F}$, then $U' \in \mathcal{F}$. But if $U \cap F \neq \varnothing$ for every $F \in \mathcal{F}$, then U is an element of $(\ddot{\mathcal{F}}){\uparrow\downarrow}$ and therefore $U \in \mathcal{F}$. It now follows from (5) that $\mathcal{F}$ is a pro-ultrafilter. ▷

8. The family $(\ddot{\mathcal{F}}){\uparrow\downarrow}$, figuring in Sec. 7 is called the <u>cyclic grill</u> of $\mathcal{F}$ of (rarely) a <u>pro-grill</u>. The meaning of this concept and its mode of application are clarified in connection with the technique of descents and ascents. Recall that if $\mathcal{E}$ is a family of nonempty subsets of X↑ then the descent $\mathcal{E}{\downarrow}$ is defined by

$$U \in \mathcal{E}{\downarrow} \leftrightarrow U \subset X \wedge (\exists E \in \mathcal{E}{\downarrow})\ U \supset E{\downarrow}.$$

9. If $\mathcal{F}$ is a filter and $\ddot{\mathcal{F}}$ its grill, then within $V^{(B)}$

$$(\ddot{\mathcal{F}}){\downarrow} = (\ddot{\mathcal{F}}{\downarrow}){\uparrow\downarrow}.$$

◁ To prove this, it suffices to observe that by the rules for evaluation of true values in $V^{(B)}$,

$$[\![U{\uparrow} \in \ddot{\mathcal{F}}]\!] = \bigwedge_{F \in \mathcal{F}{\downarrow}} [\![A{\uparrow} \cap F{\uparrow} \neq \varnothing]\!],$$

where A is a subset of X. ▷

10. An extensional filter is a pro-ultrafilter if and only if its cyclic grill is a filter.

◁ Clearly, a filter $\mathscr{F}$ is a pro-ultrafilter if and only if $\mathscr{F}^{\uparrow}$ coincides with its grill within $V^{(B)}$. But this happens if and only if the grill of $\mathscr{F}^{\uparrow}$ is a filter within $V^{(B)}$. It remains to appeal to Sec. 9.

11. <u>Essentiality Criterion.</u> A point is essential if and only if it can be separated by a standard cyclic set from any standard set not containing it.

◁ Written symbolically, our assertion is

$$x \in {}^{e}X \leftrightarrow (\forall U = U\uparrow\downarrow)\ x \notin U \rightarrow (\exists V = V\uparrow\downarrow)\ \ x \in V \wedge U \cap V = \varnothing.$$

Suppose first that x is an essential point and let U be a cyclic set such that $x \notin U$. By Sec. 3, the complement U' is an element of the filter (x)↑↓ generated by the cyclic supersets of x [because (x)↑↓ is a pro-ultrafilter by assumption]. Thus, for some V we have $x \in V$ and $V\uparrow\downarrow \cap U \neq \varnothing$.

If the separability condition is fulfilled, then the cyclic filter (x)↑↓ satisfies the conditions of the pro-ultrafilter criterion of Sec. 4. In fact, let U = U↑↓ be an arbitrary cyclic set. We must show that either U or U' occurs in (x)↑↓. If $x \in U$, then by definition $U \in (x)\uparrow\downarrow$. But if $x \in U'$, then by assumption there exists some $V \in (x)\uparrow\downarrow$ such that $V \cap U = \varnothing$, i.e., $V \subset U'$ and $U' \in (x)\uparrow\downarrow$. ▷

12. <u>COROLLARY.</u> If the monad of an ultrafilter $\mathscr{F}$ contains an essential point, then $\mu(\mathscr{F}) \subset {}^{e}X$ and, in addition, $\mathscr{F}\uparrow\downarrow$ is a pro-ultrafilter.

◁ Let V be an arbitrary cyclic set and $x \in \mu(\mathscr{F}) \cap {}^{e}X$. If $x \in V$, then $V \cap \mu(\mathscr{F}) \neq \varnothing$ and so $V \in \mathscr{F}$; therefore also $V \in \mathscr{F}\uparrow\downarrow$. If $x \notin V$, then by Sec. 11 there exists a cyclic U such that $x \in U$ and $U \cap V = \varnothing$. Clearly, $U \in \mathscr{F}\uparrow\downarrow$. Hence it follows that $V' \in \mathscr{F}\uparrow\downarrow$. It remains to refer to Sec. 5, to conclude that $\mathscr{F}\uparrow\downarrow$ is a pro-ultrafilter. As already remarked, $\mu(\mathscr{F})\uparrow\downarrow \subset {}^{e}X$ in this case. Since $(\mathscr{F}\uparrow\downarrow) = \mathrm{mix}(\mu(\mathscr{F}))$, our assertion follows from Proposition 2.3 of [1].

13. <u>Extensional Filter Criterion.</u> A filter is extensional if and only if its monad is the cyclic monadic hull of the set of its essential points.

◁ Symbolically written, our assertion is

$$(\mathscr{F} \text{ is extensional}) \leftrightarrow \mu(\mathscr{F}) = \mathrm{mix}(\mu_{d}({}^{e}\mu(\mathscr{F}))).$$

The condition that $\mathscr{F}$ be extensional may be written [$\mathscr{F}^{\uparrow}$ <u>is a filter in</u> X↑] = 1. Using the transfer principle of Boolean-valued analysis, we see that for some set $\mathfrak{A}$ of pro-ultrafilters in X

$$[F \in \mathscr{F}^{\uparrow}] = \bigwedge_{\mathscr{A} \in \mathfrak{A}} [F \in \mathscr{A}^{\uparrow}].$$

Hence it follows that for a cyclic set F in X

$$F \in \mathscr{F}^{\uparrow\downarrow} \leftrightarrow F \in \bigcap_{\mathscr{A} \in \mathfrak{A}} \mathscr{A}^{\uparrow\downarrow}.$$

Hence, for a standard cyclic F,

$$F \supset \mu(\mathscr{F}^{\uparrow\downarrow}) \leftrightarrow F \supset \mu_{d}\Big(\bigcup_{\mathscr{A} \in {}^{\circ}\mathfrak{A}} \mu(\mathscr{A}^{\uparrow\downarrow})\Big),$$

where ${}^{\circ}\mathfrak{A}$ is the standard kernel of $\mathfrak{A}$, i.e., the external set of standard elements of $\mathfrak{A}$. By Sec. 2, this can be rewritten as

$$\mu(\mathscr{F}^{\uparrow\downarrow}) = \mathrm{mix}\Big(\mu\Big(\bigcup_{\mathscr{A} \in {}^{\circ}\mathfrak{A}} \mu(\mathscr{A}^{\uparrow\downarrow})\Big)\Big).$$

It remains to observe that by Proposition 2.5 of [1] monads of pro-ultrafilters consist only of essential points and the set $\mathfrak{A}$ is the collection of pro-ultrafilters majorizing $\mathscr{F}$. ▷

14. <u>COROLLARY.</u> A standard set is cyclic if and only if it is the cyclic monadic hull of its essential points.

15. Let $\mathscr{F}$ be a filter in X and b an element of the Boolean algebra B. Let $b\mathscr{F}$ be the image of $\mathscr{F}$ under multiplication by b. Then

$$b(b\mathscr{F})^{\uparrow} = b\mathscr{F}^{\uparrow}.$$

◁ Indeed, we have the estimate

$$[(b\mathscr{F})^{\uparrow} = \mathscr{F}^{\uparrow}] \geqslant \bigwedge_{F\in\mathscr{F}} [(bF)\uparrow \in \mathscr{F}^{\uparrow}] \geqslant \bigwedge_{F\in\mathscr{F}} [(bF)\uparrow = F\uparrow] \geqslant \bigwedge_{F\in\mathscr{F}} \bigwedge_{x\in X} [bx \in F\uparrow] \geqslant \bigwedge_{F\in\mathscr{F}} \bigwedge_{x\in X} [bx = x] \geqslant b.$$

This implies our assertion. ▷

16. Let $\mathscr{F}, \mathscr{G}$ be filters within $V^{(B)}$ and $b \in B$. Then

$$b\mathscr{F} = b\mathscr{G} \leftrightarrow b\mathscr{F}^{\downarrow} = b\mathscr{G}^{\downarrow}.$$

◁ If $[\mathscr{F} \subset \mathscr{G}] \geqslant b$, then by the maximum principle, for every $F \in \mathscr{F}^{\downarrow}$ there exists $G \in \mathscr{G}^{\downarrow}$ such that

$$[F\uparrow \supset G\uparrow] = [\mathscr{F} \supset \mathscr{G}] \geqslant b.$$

In other words, $bF\uparrow \supset bG\uparrow$, and so if F and G are cyclic then bF ⊃ bG. Thus $b\mathscr{F}^{\downarrow} \subset b\mathscr{G}^{\downarrow}$.

Suppose now that $b\mathscr{F}^{\downarrow} \subset b\mathscr{G}^{\downarrow}$. Then, successively applying Sec. 15, we obtain

$$b\mathscr{F}^{\downarrow} \subset b\mathscr{G}^{\downarrow} \to (b\mathscr{F}^{\downarrow})^{\uparrow} \subset (b\mathscr{G}^{\downarrow})^{\uparrow} \to b(b\mathscr{F}^{\downarrow})^{\uparrow} \subset b(b\mathscr{G}^{\downarrow})^{\uparrow} \to b(b\mathscr{F}^{\downarrow\uparrow}) \subset b(b\mathscr{G}^{\downarrow\uparrow}) \to b\mathscr{F} \subset b\mathscr{G}.$$

Finally, we conclude that $[\mathscr{F} \subset \mathscr{G}] \geqslant b \leftrightarrow b\mathscr{F}^{\downarrow} \subset b\mathscr{G}^{\downarrow}$. This gives the required equivalence.

17. <u>Nonstandard Criterion for Mixing of Filters.</u> Let $(\mathscr{F}_{\xi})_{\xi\in\Xi}$ be a standard family of extensional filters and $(b_{\xi})_{\xi\in\Xi}$ a standard partition of the identity. A filter $\mathscr{F}$ is a mixture of $(\mathscr{F}_{\xi})_{\xi\in\Xi}$ with probabilities $(b_{\xi})_{\xi\in\Xi}$ if and only if

$$(\forall^{st}\xi \in \Xi)\ b_{\xi}\mu(\mathscr{F}) = b_{\xi}\mu(\mathscr{F}_{\xi}).$$

◁ By the general definition, F is an element of a mixture $\Sigma_{\xi\in\Xi} b_{\xi}\mathscr{F}_{\xi}$, if there exists a family $(F_{\xi})_{\xi\in\Xi}$ such that $F_{\xi} \in \mathscr{F}_{\xi}$ $(\xi \in \Xi)$ and at the same time $F \supset \Sigma_{\xi\in\Xi} b_{\xi}F_{\xi}$. Applying rules 1.1 and 1.2 of [1] and using the extensionality of the filters in the family $(\mathscr{F}_{\xi})_{\xi\in\Xi}$, we conclude that, first, $\mathscr{F}$ is also extensional, and, second, that the ascent $\mathscr{F}^{\uparrow}$ is a mixture of $(\mathscr{F}_{\xi}^{\uparrow})_{\xi\in\Xi}$ with the same probabilities. Using the separability of $V^{(B)}$ and Sec. 16, we successively obtain

$$\mathscr{F}^{\uparrow} = \sum_{\xi\in\Xi} b_{\xi}\mathscr{F}_{\xi}^{\uparrow} \leftrightarrow (\forall^{st}\xi \in \Xi)\ b_{\xi}\mathscr{F}^{\uparrow} = b_{\xi}\mathscr{F}_{\xi}^{\uparrow} \leftrightarrow$$

$$\leftrightarrow (\forall^{st}\xi \in \Xi)\ b_{\xi}\mathscr{F}^{\uparrow\downarrow} = b_{\xi}\mathscr{F}_{\xi}^{\uparrow\downarrow} \leftrightarrow$$

$$\leftrightarrow (\forall^{st}\xi \in \Xi)\ b_{\xi}\mathscr{F} = b_{\xi}\mathscr{F}_{\xi} \leftrightarrow (\forall^{st}\xi \in \Xi)\ \mu(b_{\xi}\mathscr{F}) = \mu(b_{\xi}\mathscr{F}_{\xi}) \leftrightarrow (\forall^{st}\xi \in \Xi)\ b_{\xi}\mu(\mathscr{F}) = b_{\xi}\mu(\mathscr{F}_{\xi}).$$

We have here used the fact that the monad of the image under a standard map is the image of that monad.

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