FRAGMENTS OF POSITIVE OPERATORS*

S. S. Kutateladze UDC 517.98

Let X be a vector lattice, Y be a K-space, and $T \in \mathscr{L}^+(X, Y)$ be a positive linear operator. An operator $S \in \mathscr{L}^+(X, Y)$ is called a fragment (more rarely, a comment) of T, and we write $S \in \mathbf{B}(T)$, if $S \wedge (T - S) = 0$. It is clear that $\mathbf{B}(T)$ is the set of extreme points of the order interval [0, T], serving as the subdifferential of the sublinear operator $x \to Tx^+$ $(x \in X)$. It is also obvious that the fragments arise as a result of application of the order projection to T in the K-space $\mathscr{L}^r(X, Y) := \mathscr{L}^+(X, Y) - \mathscr{L}^+(X, Y)$ of regular operators. The construction of the fragment algebra $\mathbf{B}(T)$ has sense for the investigation of the K-space Y) as well as for the general theory of operators. Important results have been obtained in this direction in recent years [1-5]. Facts, available in the most complete form, about projections in a operator space often use sometimes-inconvenient supposition about the existence of projections on principal components in the domain of definition.

The aim of the present article is to fill the arising gaps. The main object of investigation is the set $\mathscr{P}$ of the order projections in $\mathscr{L}^r(X, Y)$ that generates the fragments of T and for which, by definition, $Tx^+ = \sup\{PTx: P \in \mathscr{P}\}$ for $x \in X$. If the last condition is fulfilled for each $T \in \mathscr{L}^+(X, Y)$, then the set $\mathscr{P}$ is said to be generating.

With the aim of economy of words in formulations, in the sequel the space Y will be assumed to be extended. In addition, its filter of (weak order) unities is denoted by E and the basis is denoted by $\mathbf{B}$ or, more precisely, by $\mathbf{B}(Y)$. In the use of the technique of lowerings and lifts we bear in mind a separated universe $V^{(\mathbf{B})}$, constructed over $\mathbf{B}$. The terminology and notation, used here, of the Boolean-valued analysis are consistent with the ones adopted in [6, 7]. Let us observe only that the lift $Y\uparrow$ is considered, according to the Gordon theorem, as the canonical realization $\mathscr{R}$ of the field of real numbers $\mathbf{R}$ in $V^{(\mathbf{B})}$. Thus, $Y = \mathscr{R}\downarrow$. The sign $[\![\varphi]\!]$ symbolizes in the sequel the truth value of the formula φ in the Zermelo–Frankel theory.

Attracting the device of infinitesimal analysis, we use a nonclassical formulation, going back to E. Nelson (details and bibliography are given in [8]). Moreover, without special stipulations, we accept the convention on the standardness of entourage, i.e., in the use of the theory of inner sets all disconnected variables in a formal expression of the treatment are assumed to be standard. The sign $\approx$ has the usual meaning in the K-space Y: $x \approx y$ for $x, y \in Y$ means that $(\forall^{st} \varepsilon \in E)\ |x - y| \leqslant \varepsilon$. It is clear that for $Y = \mathbf{R}$ we mean infinitesimality of the number $(x - y)$.

The plan of the work is as follows. With the help of the analogue of the Milman theorem for a dense subfield of $\mathbf{R}$ we describe methods of generation of fragments of functionals and give connected (with it) representations of projections on principal components. Then the general case is analyzed with the help of the lift in the Boolean-value universe $V^{(\mathbf{B})}$ and of lowering of the results obtained for functionals. It is clear that we can give direct proofs of the statements, to be established, without appealing to the transfer principles. In this connection, it is necessary to use the operator version of the so-called Milman inversion – [7, Theorem 2.4.11] and the piecewise uniform approximation of the precise boundaries (cf. [4, Proposition 2.1]). The selected path seems to no less justified, but is more principled in a certain sense.

1. GENERATION OF FRAGMENTS OF FUNCTIONALS

1.1. Let X be a vector space over a dense subfield $\widehat{\mathbf{R}}$ of the field $\mathbf{R}$. Further, let $p: X \to \mathbf{R}$ be a sublinear functional and U be a generating set for p, i.e., $p(x) = \sup\{l(x): l \in U\}$ $(x \in X)$. Let τ denote the topology of pointwise convergence in $X^{\#} := \mathscr{L}(X, \mathbf{R})$ on elements of X. By the classical Milman theorem, for $\widehat{\mathbf{R}} = \mathbf{R}$ the inclusion $\mathrm{Ch}(p) \subset \mathrm{cl}_\tau(U)$ is

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fulfilled for the set Ch(p) of extreme points of the subdifferential ∂p. This inclusion (the Milman inversion of the Krein–Milman theorem) is valid also in the case, being considered presently by us.

1.2. THEOREM. Each extreme point of the subdifferential ∂p belongs to the τ-closure of the generating set for p.

◁ It is clear that τ is a locally convex topology in the vector space $X^{\#}$ over the field **R**. In addition, by the Tikhonov theorem, ∂p is τ-compact. Let V denote the τ-closure of the convex hull of U. It is obvious that $V := \mathrm{cl}_\tau(\mathrm{co}(U))$ is a convex τ-compact set. Let us suppose that a certain element $\bar{l} \in \partial p$ does not belong to V. By the separatedness theorem, there exists a τ-continuous linear functional F over $X^{\#}$ such that $\sup\{F(l): l \in V\} = F(l_0) < r < F(\bar{l})$ for $l_0 \in V$ and $r \in \mathbf{R}$. Since F is continuous, we have $|F(l)| \leqslant t|l(x_1)| \vee \ldots \vee |l(x_n)|$ for certain $x_1, \ldots, x_n \in X$, $t \in \mathbf{R}$ and all $l \in X^{\#}$. By the same token, for suitable numbers $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$ we have $F(l) = \alpha_1 l(x_1) + \ldots + \alpha_n l(x_n)$. Working in the standard entourage, we choose numbers $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_n \in {}^{\circ}\mathbf{R}$, infinitely near to $\alpha_1, \ldots, \alpha_n$, respectively. Let us also observe that by virtue of the hypothesis of standardness of x_k and the inequality $-p(-x_k) \leqslant l(x_k) \leqslant p(x_k)$ we have $l(x_k) \in {}^{\mathrm{fin}}\mathbf{R}$, i.e., $l(x_k)$ is a finite number for each $l \in \partial p$ and k := 1,...,n. Let us set $x := \sum_{k=1}^{n} \widehat{\alpha}_k x_k$. Then

$$F(l) = l(x) + \sum_{k=1}^{n} (\alpha_k - \widehat{\alpha}_k)\, l(x_k) \approx l(x),$$

for $l \in \partial p$, since $(\alpha_k - \widehat{\alpha}_k)$ is an infinitesimal quantity for k := 1,...,n. Hence $F(l) + \varepsilon \geqslant l(x)$ for each standard ε > 0. Therefore, for such an ε > 0

$$p(x) = \sup\{l(x): l \in U\} \leqslant \sup\{F(l) + \varepsilon: l \in U\} \leqslant F(l_0) + \varepsilon.$$

Hence ${}^{\circ}p(x) \leqslant F(l_0) < r$. On the other hand,

$$(\forall^{\mathrm{st}} \varepsilon > 0) \quad r \leqslant F(\bar{l}) \leqslant \bar{l}(x) + \varepsilon \leqslant p(x) + \varepsilon.$$

Consequently, ${}^{\circ}p(x) \geqslant r > {}^{\circ}p(x)$. We have obtained a contradiction, implying that V = ∂p, and therefore, on the basis of the Milman theorem, mentioned above, $\mathrm{cl}_\tau(U) \supset \mathrm{Ch}(p)$. ▷

1.3. Let us recall that for a set A in a K-space the symbol $A^{\vee}$ denotes the result of addition to A of the suprema of its finite subsets. The symbol $A^{(\uparrow)}$ denotes the result of addition to A of the suprema of increasing nets of elements of A. The symbols $A^{(\uparrow\downarrow)}$ and $A^{(\uparrow\downarrow\uparrow)}$ are interpreted in a natural manner.

Now we fix a set $\mathscr{P}$ of positive projections and the corresponding set $\mathscr{P}(f) := \{pf: p \in \mathscr{P}\}$ of the fragments of a positive functional f in a vector lattice X over a dense subfield $\widehat{\mathbf{R}}$ (with unity).

1.4. The following statements are equivalent:

(1) $\mathscr{P}(f)^{\vee(\uparrow\downarrow\uparrow)} = \mathbf{B}(f)$;

(2) $\mathscr{P}$ generates the fragments of f.

(3) $(\forall x \in {}^{\circ}X)(\exists p \in \mathscr{P})\quad pf(x) \approx f(x^+)$;

(4) A functional g from [0, f] is a fragment of f if and only if for each $x \in X^+$

$$\inf_{p \in \mathscr{P}} ((p^d g)(x) + p(f - g)(x)) = 0;$$

(5) $(\forall g \in {}^{\circ}\mathbf{B}(f))(\forall x \in {}^{\circ}X^+)(\exists p \in \mathscr{P})\quad |pf - g|(x) \approx 0$;

(6) $\inf\{|pf - g|(x): p \in \mathscr{P}\} = 0$ for each fragment $g \in \mathbf{B}(f)$ and each positive element x.

(7) For $x \in X^+$ and $g \in \mathbf{B}(f)$ there exists an element $p \in \mathscr{P}(f)^{\vee(\uparrow\downarrow)}$, ensuring the equality |pf − g|(x) = 0.

◁ Implications (1) → (2) → (3) are straightforward.

(3) → (4). We will work in the standard entourage. First of all, let us observe that the fulfillment of the equality, in which we are interested, for some functionals f and g such that $0 \leqslant g \leqslant f$, ensures for a standard $x \geqslant 0$ the existence of a $p \in \mathscr{P}$, for which $p^d g(x) \approx 0$ and $p(f - g)(x) \approx 0$. Therefore, ${}^{\circ}p(g \wedge (f - g))(x) \leqslant {}^{\circ}p(f - g)(x) = 0$ and ${}^{\circ}p^d((f - g) \wedge g)(x) \leqslant {}^{\circ}p^d g(x) = 0$, i.e., $g \wedge (f - g) = 0$.

Let us now establish that under condition (3) the equality, needed by us, is ensured by the usual criterion for disjunctness:

$$\inf_{\substack{x_1 \geqslant 0, x_2 \geqslant 0 \\ x_1 + x_2 = x}} (g(x_1) + (f - g)(x_2)) = 0.$$

For a fixed standard x we find inner positive x_1 and x_2 such that $x_1 + x_2 = x$ and, moreover, $g(x_1) \approx 0$ and $f(x_2) \approx g(x_2)$. By virtue of condition (3), on the basis of Sec. 1.2 the fragment g belongs to the weak closure of $\mathscr{P}(f)$. In particular, there exists an element $p \in \mathscr{P}$, such that $g(x_1) \approx pf(x_1)$ and $g(x_2) \approx pf(x_2)$. Therefore, $p^d g(x_2) \approx 0$, since $p^d g \leqslant p^d f$. Finally, $p^d g(x) \approx 0$. Hence $p(f-g)(x) = pf(x_2) + pf(x_1) - pg(x) \approx g(x_2) + g(x_1) - pg(x) \approx p^d g(x) \approx 0$. This ensures the needed equality.

(4) → (5). By virtue of the identity $|pf - g|(x) = p^d g(x) + p(f-g)(x)$, choosing $p \in \mathscr{P}$ such that $p^d g(x) \approx 0$ and $p(f-g)(x) \approx 0$, we get the desired result.

The equivalence (5) ↔ (6) is obvious. The implications (5) → (7) → (1) are proved with the help of devices, set forth in [2-4].

1.5. The set of projections $\mathscr{P}$ is a generating set if and only if the following representations are valid for arbitrary positive functionals f and g and a point $x \geqslant 0$:

$$f \vee g(x) = \sup_{p \in \mathscr{P}} (pf(x) + p^d g(x));$$

$$f \wedge g(x) = \inf_{p \in \mathscr{P}} (p^d f(x) + pg(x)).$$

◁ This is a direct consequence of 1.4.

2. PROJECTIONS OF FUNCTIONALS OF PRINCIPAL COMPONENTS

2.1. The following statements are equivalent for positive functionals f and g and a generating set of projections $\mathscr{P}$:

(1) $g \in \{f\}^{dd}$;

(2) For each finite $x \in {}^{\text{fin}}X := \{x \in X \colon (\exists \bar{x} \in {}^\circ X) |x| \leqslant \bar{x}\}$ the relation $pg(x) \approx 0$, is valid, provided $pf(x) \approx 0$ for $p \in \mathscr{P}$;

(3) $(\forall x \in X^+)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall p \in \mathscr{P})\ \ pf(x) \leqslant \delta \to pg(x) \leqslant \varepsilon$.

(1) → (2). Using, e.g., the classical Robinson lemma, we choose an infinitely large natural number $N \approx +\infty$ such that $Npf(x) \approx 0$ for a finite positive vector x. Let us observe that $g(x) \approx (g \wedge Nf)(x)$, for this N since g coincides with its projection on $\{f\}^{dd}$. Hence, considering the relations

$$pg(x) = p(g - g \wedge Nf)(x) + p(g \wedge Nf)(x) \leqslant (g - g \wedge Nf)(x) + Npf(x),$$

we conclude that $pg(x) \approx 0$.

Applying the Nelson algorithm, we see that (3) is equivalent to the condition

(2') $(\forall^{st} x \in X^+)(\forall p \in \mathscr{P})\ \ pf(x) \approx 0 \to pg(x) \approx 0$.

Thus, since (2) → (2'), it remains only to establish that (2') → (1).

(2') → (1). We take a functional h such that $h \wedge f = 0$. By virtue of 1.4(4), for a standard $x \in X^+$ there exists an element $p \in \mathscr{P}$, such that $ph(x) \approx 0$ and $p^d f(x) \approx 0$. By condition (2'), $p^d g(x) \approx 0$. Consequently, $h \wedge g)(x) \leqslant {}^\circ(ph(x) + p^d g(x)) = 0$. On the basis of 1.4(4), we conclude that $h \wedge g = 0$, i.e., $g \in \{f\}^{dd}$, since h is arbitrary.

2.2. THEOREM. Let f and g be positive functionals on X and x be a positive element of X. The following representations hold for the projection P_f on the component $\{f\}^{dd}$:

(1) $P_f g(x) \doteq \inf^* \{{}^\circ pg(x) \colon p^d f(x) \approx 0,\ p \in \mathscr{P}\}$ (the sign $\doteq$ symbolizes the sharpness of a formula, i.e., the attainability of equality).

(2) $P_f g(x) = \sup_{\varepsilon > 0} \inf \{pg(x) \colon p^d f(x) \leqslant \varepsilon,\ p \in \mathscr{P}\}$;

(3) $P_f g(x) \doteq \inf^* \{{}^\circ g(y) \colon f(x - y) \approx 0,\ 0 \leqslant y \leqslant x\}$;

(4) $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall p \in \mathscr{P})\ \ pf(x) \leqslant \delta \to P_f g(x) \leqslant p^d g(x) + \varepsilon$,
$(\forall \varepsilon > 0)(\forall \delta > 0)(\exists p \in \mathscr{P})\ \ pf(x) \leqslant \delta \wedge p^d g(x) \leqslant P_f g(x) + \varepsilon$;

(5) $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall 0 \leqslant y \leqslant x)\ \ f(x - y) \leqslant \delta \to P_f g(x) \leqslant g(y) + \varepsilon$,
$(\forall \varepsilon > 0)(\forall \delta > 0)(\exists 0 \leqslant y \leqslant x)\ \ f(x - y) \leqslant \delta \wedge g(y) \leqslant P_f g(x) + \varepsilon$.

◁ For brevity, let us set $h := P_f g$. It is clear that $h(x) \leqslant g(x)$ and $pg(x) \geqslant ph(x)$. If $p^d f(x) \approx 0$, then $p^d h(x) \approx 0$ and, therefore, $h(x) = {}^\circ ph(x) \leqslant {}^\circ pg(x)$. Consequently, each standard element of the outer set $\{{}^\circ pg(x)\colon p \in \mathscr{P},\ p^d f(x) \approx 0\}$ majorizes h(x). By the transfer principle, we conclude that the left-hand side of (1) does not exceed the corresponding right-hand side. To prove the attainability of equality in (1) we observe that $f \wedge (g - h) = 0$. Therefore, by virtue of 1.5, $p^d f(x) \approx 0$ and $pg(x) \approx ph(x)$ for a certain $p \in \mathscr{P}$. Since $h \in \{f\}^{dd}$, on the basis of 2.1(2) we deduce that $p^d h(x) \approx 0$. Finally, $pg(x) \approx ph(x) + p^d h(x) = h(x)$. By the same token, h(x) = °pg(x), and (1) is established.

To prove (2), taking δ > 0, in the standard entourage we get

$$\inf\{pg(x)\colon p^d f(x) \leqslant \varepsilon\} \leqslant \inf{}^*\{pg(x) + \delta\colon p^d f(x) \leqslant \varepsilon\} \leqslant$$
$$\leqslant \inf{}^*\{{}^\circ pg(x)\colon p^d f(x) \approx 0\} + \delta = h(x) + \delta.$$

Since δ is arbitrary, we have

$$h(x) \geqslant \sup_{\varepsilon > 0} \inf\{pg(x)\colon p^d f(x) \leqslant \varepsilon\}.$$

Again, fixing a standard number δ > 0 for each infinitesimal ε > 0, on the basis of (1) we conclude that the following inner condition is fulfilled:

$$\inf\{pg(x)\colon p^d f(x) \leqslant \varepsilon\} + \delta \geqslant h(x).$$

Indeed, the inequality $p^d f(x) \leqslant \varepsilon$ implies the relation $p^d f(x) \approx 0$ and, therefore, $pg(x) + \delta \geqslant {}^\circ pg(x) \geqslant h(x)$. By the Cauchy principle, the above-mentioned inner condition is fulfilled for a certain standard strictly positive number ε. Using the transfer principle, we finally deduce that

$$(\forall \delta > 0)\,(\exists \varepsilon > 0)\quad h(x) - \delta \leqslant \inf\{pg(x)\colon\ p^d f(x) \leqslant \varepsilon\},$$

which completes the proof of validity of (2).

To verify (3), we proceed according to the pattern of proof of (1). Namely, if $0 \leqslant y \leqslant x$ and $f(x-y) \approx 0$, then, since $h(x) = h(y) + h(x-y) \leqslant g(y) + h(x-y)$ and $h \in \{f\}^{dd}$, we see that $h(x - y) \approx 0$, and therefore $h(x) \leqslant {}^\circ g(y)$. To establish the sharpness in (3), let us note the equation $f \wedge (g - h) = 0$. This equation ensures the validity of the following statement: $f(x-y) \approx 0$ and $h(y) \approx g(y)$ for a certain y from the interval [0, x]. Since $h \in \{f\}^{dd}$, on the basis of statements 2.1 we have $h(x) \approx h(y)$. Therefore, h(x) = °g(y).

Statements (4) and (5) are verified in a similar manner by the application of the Nelson algorithm. We give the appropriate computations, e.g., for (5). To this end, we decipher the statement established in (3). In the first place, a certain inequality and, in the second place, the sharpness of this inequality constitute the content of (3). Analyzing the inequality, we deduce that

$$(\forall 0 \leqslant y \leqslant x)\quad f(x-y) \approx 0 \to h(x) \leqslant {}^\circ g(y) \leftrightarrow$$
$$\leftrightarrow (\forall^{st} \varepsilon > 0)\,(\forall 0 \leqslant y \leqslant x)\quad f(x-y) \approx 0 \to h(x) \leqslant g(y) + \varepsilon \leftrightarrow$$
$$\leftrightarrow (\forall^{st} \varepsilon > 0)\,(\forall 0 \leqslant y \leqslant x)\,(\exists^{st} \delta > 0)\quad (f(x-y) \leqslant \delta \to h(x) \leqslant g(y) + \varepsilon) \leftrightarrow$$
$$\leftrightarrow (\forall^{st} \varepsilon > 0)\,(\exists^{st} \delta > 0)\,(\forall 0 \leqslant y \leqslant x)\quad f(x-y) \leqslant \delta \to h(x) \leqslant g(y) + \varepsilon.$$

Considering the statement about sharpness, we have

$$(\exists y)\,(0 \leqslant y \leqslant x) \wedge f(x-y) \approx 0 \wedge h(x) = {}^\circ g(y) \leftrightarrow (\exists y)\,(0 \leqslant y \leqslant x) \wedge (\forall^{st} \delta > 0)\, f(x-y) \leqslant$$
$$\leqslant \delta \wedge (\forall^{st} \varepsilon > 0)\, |h(x) - g(y)| \leqslant \varepsilon \leftrightarrow (\forall^{st} \varepsilon > 0)\,(\forall^{st} \delta > 0)\,(\exists y)$$
$$(0 \leqslant y \leqslant x \wedge f(x-y) \leqslant \delta \wedge |h(x) - g(y)| \leqslant \varepsilon).$$

Applying the transfer principle twice, we complete the proof.

3. GENERATION OF FRAGMENTS AND PROJECTIONS OF OPERATORS ON PRINCIPAL COMPONENTS

3.1. Let $f\colon A \times B \to Y$ be an extension mapping and $f_V(a) := \sup f(a, V)$ for $a \in A$ and $V \subset B$. Then the mapping $f_V\colon A \to Y$ is also extensional, and $f_V{\uparrow} = f{\uparrow}_{V{\uparrow}}$.

By virtue of the general rules of lifting, for $a \in A$ we successively deduce that

$$f_V(a) = \sup f(a, V) = \sup f(a, V){\uparrow} = \sup f(\{a\} \times V){\uparrow} =$$
$$= \sup f{\uparrow}((\{a\} \times V){\uparrow}) = \sup f{\uparrow}(\{a\}{\uparrow} \times V{\uparrow}) = \sup f{\uparrow}(\{a\} \times V{\uparrow}) = \sup f{\uparrow}(a, V{\uparrow}) = f{\uparrow}_{V{\uparrow}}(a).$$

Since f↑ is a function in the considered Boolean-valued universe, with the help of the above-proved relation, for a_1, and a_2 in A we have

$$[a_1 = a_2] \leqslant [f\uparrow_{V\uparrow}(a_1) = f\uparrow_{V\uparrow}(a_2)] = [\sup f\uparrow(a_1, V\uparrow) = \sup f\uparrow(a_2, V\uparrow)] = [f_V(a_1) = f_V(a_2)].$$

Thus, f_V is an extensional mapping. Moreover,

$$[f_V\uparrow(a) = f\uparrow_{V\uparrow}(a)] = [f_V(a) = f\uparrow_{V\uparrow}(a)] = 1 \quad (a \in A)$$

by virtue of what we have already established.

3.2. The following statements are equivalent for a set of projections $\mathscr{P}$ in $\mathscr{L}^r(X, Y)$ and $T \in \mathscr{L}^+(X, Y)$:

(1) $\mathscr{P}(T)\uparrow\downarrow^{V(\uparrow\downarrow\uparrow)} = \mathbf{B}(T)$;

(2) $\mathscr{P}$ generates the fragments of T.

(3) An operator $S \in [0, T]$ is a fragment of T if and only if for each $x \in X^+$

$$\inf_{P \in \mathscr{P}} \left(P^d Sx + P(T - S)x\right) = 0;$$

(4) $(\forall x \in {}^\circ X)(\exists P \in \mathscr{P}\uparrow\downarrow) \quad PTx \approx Tx^+$.

Let us consider $X^\wedge$ be the standard name of X in the separate Boolean-value universe $V^{(\mathbf{B})}$, constructed over $\mathbf{B} = \mathbf{B}(Y)$. Let us observe that $X^\wedge$ is a vector lattice over the standard name $\mathbf{R}^\wedge$ of the field $\mathbf{R}$. In addition, $\mathbf{R}^\wedge$ is a dense subfield of $\mathscr{R}$ in $V^{(\mathbf{B})}$. Here, as usual, $\mathscr{R} = Y\uparrow$ is an element that plays the role of the field of real numbers in $V^{(\mathbf{B})}$. Let us realize by the general rules the liftings of mappings of X into Y to the mappings of $X^\wedge$ into $\mathscr{R}$ in $V^{(\mathbf{B})}$. Then it is easily verified that $X^{\wedge r}\downarrow := \mathscr{L}^r(X^\wedge, \mathscr{R})\downarrow = \{R\uparrow : R \in \mathscr{L}^r(X, Y)\}$.

The lowered-down structures turn $X^{\wedge r}\downarrow$ into a K-space and even into an extended module over the algebra of orthomorphisms [6, 9]. Furthermore, we find ourselves, in essence, in the scalar situation, studied earlier. For the sake of completeness, we elucidate certain typical moments, needed by us.

Let us recall that for $R \in \mathscr{L}^r(X, Y)$ the lift R↑ is defined by the rule $[R\uparrow x^\wedge = Rx] = 1$ for $x \in X$. In addition, R↑ becomes a regular form on $X^\wedge$ – an element of $X^{\wedge r}$ in $V^{(\mathbf{B})}$. For $P \in \mathscr{P}$ the mapping $R\uparrow \to (PR)\uparrow \quad (R \in \mathscr{L}^r(X, Y))$ is extensional. Indeed, for $\pi \in \mathbf{B}$ we have

$$\pi \leqslant [R_1\uparrow = R_2\uparrow] \to (\forall x \in X) \quad \pi \leqslant [R_1\uparrow x^\wedge = R_2\uparrow x^\wedge] \to (\forall x \in X)$$
$$\pi R_1 x = \pi R_2 x \to (\forall x \in X) \quad \pi P R_1 x = \pi P R_2 x \to \pi \leqslant [(PR_1)\uparrow = (PR_2)\uparrow].$$

Thus, the lift P↑ – a projection in $X^{\wedge r}$ in $V^{(\mathbf{B})}$ – is defined. The rule of action of P↑ is as follows: $P\uparrow R\uparrow = (PR)\uparrow$ for $R \in \mathscr{L}^r(X, Y)$.

It is useful to observe that for $S, T \in \mathscr{L}^+(X, Y)$ we have $(S \wedge T)\uparrow = S\uparrow \wedge T\uparrow$ in $V^{(\mathbf{B})}$. Indeed, since $[(S\wedge T)\uparrow \leqslant S\uparrow \wedge T\uparrow] = 1$, we deduce that

$$[(S\wedge T)\uparrow = S\uparrow \wedge T\uparrow] = [S\uparrow \wedge T\uparrow \leqslant (S\wedge T)\uparrow] =$$
$$= [(\forall R \in X^{\wedge r+})\ R \leqslant S\uparrow \wedge R \leqslant T\uparrow \to R \leqslant (S\wedge T)\uparrow] = \bigwedge_{R \in \mathscr{L}^+(X,Y)} [R\uparrow \leqslant S\uparrow \wedge R\uparrow \leqslant T\uparrow \to R\uparrow \leqslant (S\wedge T)\uparrow].$$

Let us set $\pi := [R\uparrow \leqslant S\uparrow] \wedge [R\uparrow \leqslant T\uparrow]$. It is certain that $\pi R \leqslant \pi S$ and $\pi R \leqslant \pi T$. Therefore, $\pi R \leqslant \pi(S\wedge T)$. Hence

$$[R\uparrow \leqslant (S\wedge T)\uparrow] = [(\forall x \in X^{\wedge +})\ R\uparrow x \leqslant (S\wedge)T\uparrow x] = \bigwedge_{x \in X^+} [Rx \leqslant (S\wedge T)x] \geqslant \pi,$$

i.e., the estimate, in which we are interested, is equal to one. In other words, the mapping $R \in \mathscr{L}^r(X, Y) \to R\uparrow \in X^{\wedge r}\downarrow$ realizes an isomorphism of the structures $\mathscr{L}^r(X, Y)$ and $X^{\wedge r}\downarrow$. By the same token, we can assert that S is a fragment of T if and only if S↑ is a fragment of T↑ in $V^{(\mathbf{B})}$.

Let us now consider the lift $\mathscr{P}\uparrow$, defined by the rule $\mathscr{P}\uparrow := \{P\uparrow : P \in \mathscr{P}\}\uparrow$. Using statement 3.1 and what we have just noted, we see that $\mathscr{P}$ generates the fragments of T if and only if generates the fragments of T↑ in $V^{(\mathbf{B})}$. This proves in essence the equivalences (1) ↔ (2) ↔ (3).

Finally, let us verify the equivalence (2) ↔ (4). Using the definitions and the estimation rule, for $x \in X$ we successively deduce that

$$[T\uparrow x^{\wedge +} = \sup\{(PT\uparrow)x^\wedge : P \in \mathscr{P}\uparrow\}] = 1 \leftrightarrow$$

$$\leftrightarrow [(\forall \varepsilon > 0)(\exists P \in \mathscr{P}\uparrow)(PT\uparrow)x^{\wedge} + \varepsilon \geqslant Tx^{+}] = 1 \leftrightarrow$$

$$\leftrightarrow (\forall \varepsilon \in E) \bigvee_{P \in \mathscr{P}} [(Tx^{+} - P\uparrow T\uparrow x^{\wedge}) \leqslant \varepsilon] = 1 \quad \leftrightarrow (\forall \varepsilon \in E) \bigvee_{P \in \mathscr{P}} [(Tx^{+} - PTx) \leqslant \varepsilon] = 1 \leftrightarrow$$

$$\leftrightarrow (\forall \varepsilon \in E)(\exists (P_{\xi}))(\exists (\pi_{\xi}))(\forall \xi) \quad \pi_{\xi}(Tx^{+} - P_{\xi}Tx) \leqslant \varepsilon \leftrightarrow (\forall^{st} \varepsilon \in E)(\exists (P_{\xi}))(\exists (\pi_{\xi}))(\forall \xi) \quad \pi_{\xi}(Tx^{+} - P_{\xi}Tx) \leqslant \varepsilon \leftrightarrow$$

$$\leftrightarrow (\exists (P_{\xi}))(\exists (\pi_{\xi}))(\forall \xi)(\forall^{st} \varepsilon \in E) \quad \pi_{\xi}(Tx^{+} - P_{\xi}Tx) \leqslant \varepsilon \leftrightarrow (\exists (P_{\xi}))(\exists (\pi_{\xi}))(\forall \xi) \quad \pi_{\xi}(Tx^{+} - P_{\xi}Tx) \approx 0.$$

Here we have used the natural notation about the family (P_{ξ}) of elements of $\mathscr{P}$ and the partition of unity (π_{ξ}) in $\mathbf{B}$. Choosing as P the mixing of (P_{ξ}) with probabilities (π_{ξ}), we arrive at the desired result.

3.3. A set $\mathscr{P}$ is generating if and only if for arbitrary $S, T \in \mathscr{L}^{+}(X, Y)$ and $x \in X^{+}$

$$(S \vee T)x = \sup_{P \in \mathscr{P}} (PSx + P^{d}Tx);$$

$$(S \wedge T)x = \inf_{P \in \mathscr{P}} (PSx + P^{d}Tx).$$

This is an obvious consequence of 3.2 (or 1.5).

3.4. The following statements are equivalent for positive operators S and T and a generating set $\mathscr{P}$ of projections in $\mathscr{L}^{r}(X, Y)$:

(1) $S \in \{T\}^{dd}$;

(2) $(\forall x \in {}^{\mathrm{fin}}X)(\forall P \in \mathscr{P})(\forall \pi \in \mathbf{B}(Y)) \quad \pi PTx \approx 0 \to \pi PSx \approx 0$;

(3) $(\forall x \in {}^{\mathrm{fin}}X)(\forall \pi \in \mathbf{B}(Y)) \quad \pi Tx \approx 0 \to \pi Sx \approx 0$;

(4) $(\forall x \geqslant 0)(\forall \varepsilon \in E)(\exists \delta \in E)(\forall P \in \mathscr{P})(\forall \pi \in \mathbf{B}(Y)) \quad \pi PTx \leqslant \delta \to \pi PSx \leqslant \varepsilon$;

(5) $(\forall x \geqslant 0)(\forall \varepsilon \in E)(\exists \delta \in E)(\forall \pi \in \mathbf{B}(Y)) \quad \pi Tx \leqslant \delta \to \pi Sx \leqslant \varepsilon$.

◁ It is useful to observe that for $R \in \mathscr{L}^{+}(X, Y)$

$$[R\uparrow = 0] = [(\forall x \in X^{\wedge})\ R\uparrow x = 0] = \bigwedge_{x \in X} [R\uparrow x^{\wedge} = 0] = \bigwedge_{x \in X} [Rx = 0].$$

Thus, for $\pi \in \mathbf{B} := \mathbf{B}(Y)$

$$[R\uparrow = 0] \geqslant \pi \leftrightarrow \pi R = 0 \leftrightarrow \pi R\uparrow = 0.$$

By virtue of the above equivalences, we establish that $S \in \{T\}^{dd}$ if and only if $S\uparrow \in \{T\uparrow\}^{dd}$ in $V^{(\mathbf{B})}$. Indeed,

$$[S\uparrow \in \{T\uparrow\}^{dd}] = 1 \leftrightarrow [(\forall R \in X^{\wedge r+})\ R \wedge T\uparrow = 0 \to R \wedge S\uparrow = 0] = 1 \leftrightarrow$$

$$\leftrightarrow (\forall R \in \mathscr{L}^{+}(X, Y))\ [R\uparrow \wedge T\uparrow = 0 \to R\uparrow \wedge S\uparrow = 0] = 1 \leftrightarrow$$

$$\leftrightarrow (\forall R \in \mathscr{L}^{+}(X, Y)) \quad (\forall \pi \in \mathbf{B}) \quad \pi R \wedge \pi T = 0 \to \pi R \wedge \pi S = 0.$$

The condition in the last line obviously ensures the relation $S \in \{T\}^{dd}$. In its turn, if the last relation is valid and the operator $R \in \mathscr{L}^{+}(X, Y)$ and the projection $\pi \in \mathbf{B}$ are such that $\pi R \wedge \pi T = 0$, then $\pi R \wedge T = 0$ trivially. Therefore, $\pi R \wedge S = 0$, and hence $\pi R \wedge \pi S = 0$. By the same token, $S\uparrow$ belongs to the component $\{T\uparrow\}^{dd}$ in $V^{(\mathbf{B})}$.

The desired equivalences now follow from 2.1 with regard for what we have noted above and the rules of computation of estimates. For example, for (5) we have

$$[S\uparrow \in \{T\uparrow\}^{dd}] = 1 \leftrightarrow [(\forall x \in X^{\wedge +})(\forall \varepsilon > 0)(\exists \delta > 0) \quad S\uparrow x \leqslant \delta \to T\uparrow x \leqslant \varepsilon] = 1 \leftrightarrow$$

$$\leftrightarrow (\forall x \in X^{+})(\forall \varepsilon \in E)(\exists \delta \in E) \quad [S\uparrow x^{\wedge} \leqslant \delta \to T\uparrow x^{\wedge} \leqslant \varepsilon] = 1 \leftrightarrow$$

$$\leftrightarrow (\forall x \in X^{+})(\forall \varepsilon \in E)(\exists \delta \in E) \quad [Sx \leqslant \delta \to Tx \leqslant \varepsilon] = 1 \leftrightarrow$$

$$\leftrightarrow (\forall x \in X^{+})(\forall \varepsilon \in E)(\exists \delta \in E)(\forall \pi \in \mathbf{B}) \quad [Sx \leqslant \delta] \geqslant \pi \to [Tx \leqslant \varepsilon] \geqslant \pi \leftrightarrow$$

$$\leftrightarrow (\forall x \in X^{+})(\forall \varepsilon \in E)(\exists \delta \in E)(\forall \pi \in \mathbf{B}) \quad \pi Sx \leqslant \delta \to \pi Tx \leqslant \varepsilon. \ ▷$$

3.5. THEOREM. Let X be a vector lattice and Y be a K-space with the filter of unities E and basis $\mathbf{B}(Y)$. Further, let S and T be positive operators from $\mathscr{L}^{r}(X, Y)$ and R be the projection of S on the component $\{T\}^{dd}$. The following representations are valid for a positive $x \in X$:

(1) $Rx = \sup_{\varepsilon \in E} \inf \{\pi Sy + \pi^{d}Sx\colon 0 \leqslant y \leqslant x,\ \pi \in \mathbf{B}(Y),\ \pi T(x - y) \leqslant \varepsilon\}$;

(2) $Rx = \sup_{\varepsilon \in E} \inf \{(\pi P)^{d}Sx\colon \pi PTx \leqslant \varepsilon,\ P \in \mathscr{P},\ \pi \in \mathbf{B}(Y)\}$,

where $\mathscr{P}$ is a generating set of projections in $\mathscr{L}^r(X, Y)$.

◁ We pass to the Boolean-valued universe $V^{(\mathbf{B})}$ over the Boolean algebra $\mathbf{B} := \mathbf{B}(Y)$. Since the component $\{T\uparrow\}^{dd}$ in $V^{(\mathbf{B})}$ coincides with the lift of the component $\{T\uparrow\}^{dd}$ in the lowering $X^{\wedge r}\downarrow$ – with the image of the component $\{T\}^{dd}$ under the lift of mappings, by virtue of 3.1 we see that $R\uparrow$ is the projection of $S\uparrow$ on the principal component $\{T\uparrow\}^{dd}$ in $X^{\wedge r}$ in $V^{(\mathbf{B})}$. Now, using the first part of 2.2(5) and working in $V^{(\mathbf{B})}$, for $x \in X^+$ we deduce that

$$[(\forall\varepsilon > 0)(\exists\delta > 0)(\forall y \in X^{\wedge})\ (0 \leqslant y \leqslant x^{\wedge} \wedge T\uparrow(x^{\wedge} - y) \leqslant \delta) \to$$
$$\to R\uparrow x^{\wedge} \leqslant S\uparrow y + \varepsilon] = 1 \leftrightarrow (\forall\varepsilon \in E)(\exists\delta \in E)(\forall y \in X)$$
$$[0 \leqslant y^{\wedge} \leqslant x^{\wedge} \wedge T\uparrow(x^{\wedge} - y^{\wedge}) \leqslant \delta \to R\uparrow x^{\vee} \leqslant S\uparrow y^{\wedge} + \varepsilon] = 1 \leftrightarrow$$
$$\leftrightarrow (\forall\varepsilon \in E)(\exists\delta \in E)(\forall 0 \leqslant y \leqslant x)\quad [T(x - y) \leqslant \delta \to Rx \leqslant Sy + \varepsilon] = 1 \leftrightarrow$$
$$\leftrightarrow (\forall\varepsilon \in E)(\exists\delta \in E)(\forall 0 \leqslant y \leqslant x)\quad [T(x - y) \leqslant \delta] \leqslant [Rx \leqslant Sy + \varepsilon] \leftrightarrow$$
$$\leftrightarrow (\forall\varepsilon \in E)(\exists\delta \in E)(\forall 0 \leqslant y \leqslant x)(\forall\pi \in \mathbf{B})\quad [T(x - y) \leqslant \delta] \geqslant \pi \to$$
$$\to [Rx \leqslant Sy + \varepsilon] \geqslant \pi \leftrightarrow (\forall\varepsilon \in E)(\exists\delta \in E)(\forall 0 \leqslant y \leqslant x)$$
$$(\forall\pi \in \mathbf{B})\quad \pi T(x - y) \leqslant \delta \to \pi Rx \leqslant \pi Sy + \varepsilon \to$$
$$\to (\forall\varepsilon \in E)(\exists\delta \in E)(\forall 0 \leqslant y \leqslant x)(\forall\pi \in \mathbf{B})\quad \pi T(x - y) \leqslant \delta \to$$
$$\to Rx \leqslant \pi Sy + \pi^d Sx + \varepsilon.$$

Let us set $r(\delta) := \inf\{\pi Sy + \pi^d Sx\colon\ \pi T(x - y) \leqslant \delta,\ \pi \in \mathbf{B},\ 0 \leqslant y \leqslant x\}$. It is clear that in this notation

$$(\forall\varepsilon \in E)(\exists\delta \in E)\quad Rx \leqslant r(\delta) + \varepsilon \to Rx \leqslant \sup\{r(\delta)\colon\ \delta \in E\}.$$

In a similar manner, from the second part of 2.2(5) we conclude that

$$[(\forall\varepsilon > 0)(\forall\delta > 0)(\exists 0 \leqslant y \leqslant x^{\wedge})\ T\uparrow(x^{\wedge} - y) \leqslant \delta \wedge S\uparrow y \leqslant$$
$$\leqslant R\uparrow x^{\wedge} + \varepsilon] = 1 \leftrightarrow (\forall\varepsilon \in E)(\forall\delta \in E) \bigvee_{0 \leqslant y \leqslant x} [T(x - y) \leqslant \delta \wedge Sy \leqslant$$
$$\leqslant Rx + \varepsilon] = 1 \leftrightarrow (\forall\varepsilon \in E)(\forall\delta \in E)(\exists(y_\xi))(\exists(\pi_\xi))(\forall_\xi)\, \pi_\xi T(x - y_\xi) \leqslant \delta \wedge \pi_\xi S y_\xi \leqslant \pi_\xi Rx + \varepsilon$$

for a certain family (y_ξ) of elements of the interval [0, x] and a certain partition of unity (π_ξ) in the algebra $\mathbf{B}$. It is obvious that $r(\delta) \leqslant \pi_\xi S y_\xi + \pi_\xi^d Sx$ for all the named parameters. Hence $\pi_\xi r(\delta) \leqslant \pi_\xi S y_\xi \leqslant \pi_\xi Rx + \varepsilon$ for each ξ and, therefore, $r(\delta) \leqslant Rx + \varepsilon$. Since ε is arbitrary, we have $\sup\{r(\delta)\colon\ \delta \in E\} \leqslant Rx$. Together with the earlier-established reverse inequality, this ensures the vality of (1).

Equation (2) is deduced in the same manner as (1). We should only note that $\mathscr{P}\uparrow := \{P\uparrow\colon P \in \mathscr{P}\}\uparrow$ is a generating set of projections in $X^{\wedge r}$ in $V^{(\mathbf{B})}$. Let us also note the following useful identities:

$$(\pi P)^d Q = Q - \pi PQ = \pi Q - \pi PQ + \pi^d Q = \pi(Q - PQ) + \pi^d Q = \pi P^d Q + \pi^d Q.$$

Finally, computing the truth values of the variants of 2.2(4), carried over to $V^{(\mathbf{B})}$, for a positive $x \in X$ we get

$$(\forall\varepsilon \in E)(\exists\delta \in E)(\forall P \in \mathscr{P})(\forall\pi \in \mathbf{B})\quad \pi PTx \leqslant \delta \to \pi P^d Sx + \pi^d Sx + \varepsilon \geqslant Rx;$$
$$(\forall\varepsilon \in E)(\forall\delta \in E)(\exists(P_\xi))(\exists(\pi_\xi))\ \pi_\xi P_\xi Tx \leqslant \delta \wedge \pi_\xi P_\xi^d Sx \leqslant \pi_\xi Rx + \varepsilon$$

for a suitable family (P_ξ) of elements of $\mathscr{P}$ and a suitable partition of unity (π_ξ) in the algebra $\mathbf{B}$. ▷

3.6. Equation 3.5(1) is obtained as the solution of a problem, given to me by E. V. Kolesnikov. Meeting him recently, I have come to know that he has successfully found an analogous expression for the projection on the principal component by modifying the argument of Ch. Aliprantis and O. Burkinshaw in [1] (see [10]).

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TANGENTIAL COMPONENT OF A TENSOR FIELD†

I. V. L'vov and V. A. Sharafutdinov UDC 517.9:512.62

The present article deals with the proof of the tangential component theorem used in [1]. The theorem on conjugate tensor fields on a sphere, which arises in the process of this proof, is in our opinion of independent interest. We shall preface a statement of the problem and formulation of results for vector fields to the consideration of tensor fields of arbitrary degree, since in this particular case both theorems mentioned are especially obvious.

An arbitrary vector field $g:\mathbf{R}^n\to\mathbf{R}^n$ expands uniquely into the sum $g(x) = f(x) + \lambda(x)x$, where $\langle f(x), x\rangle=0$, $\lambda:\mathbf{R}^n\to\mathbf{R}$. We shall call the terms of this expansion, respectively, the tangential and radial components of field g, since vector f(x) is tangent to the sphere $|x|$ = const, and vector $\lambda(x)x$ is parallel to x. These components as a rule have singularities at x = 0, even if g(x) is a smooth (i.e., infinitely differentiable) field. In connection with this, the problem arises: let f be a smooth vector field on $\mathbf{R}_0^n=\mathbf{R}^n\setminus\{0\}$ that satisfies $\langle f(x), x\rangle = 0$; what order additional conditions must it satisfy in order that there exist a field that is smooth on all $\mathbf{R}^n$, for which f is a tangential component? The answer to this question is simple. In order to state it, we introduce into consideration the tangent bundle space $T\Omega=\{(x, \xi)\in\mathbf{R}^n\times\mathbf{R}^n|\ \langle x, \xi\rangle=0, |\xi|=1\}$ of the unit sphere $\Omega=\{\xi\in\mathbf{R}^n|\ |\xi|=1\}$ and note that for vector field g and its tangential component f on $T\Omega$ we have

$$\langle f(x), \xi\rangle=\langle g(x), \xi\rangle \quad ((x, \xi)\in T\Omega).$$

For the given field f that is a priori smooth on $\mathbf{R}_0^n$, the function on the left-hand side of this equation is defined and smooth only on an open subset $T_0\Omega=\{(x, \xi)\in T\Omega | x\neq 0\}$ of manifold $T\Omega$. As for the right-hand side of this equation, it belongs to $C^\infty(T\Omega)$ if g is smooth on $\mathbf{R}^n$. Thus, we arrive at the following necessary condition for a positive answer to the problem posed: function $\langle f(x), \xi\rangle\in C^\infty(T_0\Omega)$ must be the restriction to $T_0\Omega$ of some function in $C^\infty(T\Omega)$. The tangential component theorem asserts the sufficiency of this condition for $n\geqslant 3$.

If for field f the answer to the above-posed problem is positive, then a second problem is pertinent: with what degree of uniqueness is field g determined by its tangential component f? From the proof presented below, this answer follows from the problem: the germ $\mathrm{Jet}_\infty g(0)$ [i.e., the values at x = 0 of all partial derivative of field g(x)] is uniquely determined by field f(x), while the radial component of g can be arbitrary outside any neighborhood of zero. The proposed proof consists of successive consideration of all partial derivatives of field f(x) and determination of their singularities at x = 0. In this connection we arrive immediately at the following problem: to describe the structure of pairs of tangent vector fields $a, b:\Omega\to\mathbf{R}^n$ on the sphere satisfying $\langle a(x), y\rangle=\langle x, b(y)\rangle$ for $x, y\in\Omega$, $\langle x, y\rangle = 0$. We shall call these conjugate fields. We introduce one example of conjugate vector fields. For $x\in\mathbf{R}_0^n$, we denote by P_x the orthogonal projection of $\mathbf{R}^n$ onto $x^\perp=\{y\in\mathbf{R}^n|\langle x, y\rangle=0\}$. If $A:\mathbf{R}^n\to\mathbf{R}^n$ is a linear operator and A^* is conjugate to A, then the fields defined

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