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ON COMBINING NON-STANDARD METHODS. I. MONADOLOGY

A. G. Kusraev and S. S. Kutateladze

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The non-standard methods of analysis at this time are grouped into two main approaches, the infinitesimal analysis (Robinson's non-standard analysis) and the Boolean-valued analysis (see [1] and literature cited there).

Robinson's non-standard analysis leads to a number of simplifications through an actualization of constructions which under the standard approach are considered potentially realizable. The Boolean-valued analysis leads to a significant advancement in the theory of operators by increasing the number of classical facts relating to functionals. We emphasize that this describes only a small portion of the typical properties of these approaches.

When studying problems of functional analysis motivated by the theory of Kantorovich spaces, it is necessary to combine theoretical and technical methods offered by non-standards models of set theory. It should be specially noted that the Boolean-valued and infinitesimal variations of analysis along with the generality of the "nonstandard approach" have principal differences in their content and method. Therefore, there are many variations in their simultaneous application. One productive approach consists of studying the standard Boolean-valued model in a universe of internal Nelson sets (or more generally, external Kavai sets). In this case the specific methods related to the predicate of standard are used in a world external to the Boolean-valued universe, the world of descents. To be specific, by using a neoclassical statement of the non-standard analysis, a standard K-space E with a basis B is studied by applying a standard separable Boolean-valued universe $V^{(B)}$. This variation of the study of cyclical extensional filters and related applications to the theory of K-spaces is described in [2, 3].

In some cases it is better to use a different approach, i.e., an application of infinitesimal methods to objects contained in $V^{(B)}$. Indeed, suppose we begin an analysis of an operator T acting on E by transforming it to a functional $T\uparrow$ through a lift to $T^{(B)}$. Then by interpreting results of infinitesimal functional analysis and performing a subsequent descent, we obtain the desired properties of the original T. This method has been realized in practice in several of our works and forms a basis for our subsequent analysis.

The aim of the study initiated in this article is to develop and simplify certain formal possibilities which appear along the described path of combining the procedures of lifting and descending with an intermediate application of the apparatus of non-standard analysis in Robinson's case. Hereafter our main attention is devoted to the arising technical methods which are useful in analysis problems. Many interesting mathematical facts and problems connected with the formalization of processes of transfer and interpretation are consciously

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left out by us for future works. In this first article we have collected the simplest facts related to the monadology which will be needed in the future.

1. Preliminary Results

Throughout the following discussion we fix a complete Boolean algebra B and a separable Boolean-valued universe $V^{(B)}$. The truth estimate of a formula φ of the Tsermelo–Frenkel set theory is denoted by [φ]. Other terminology and symbolics of Boolean-valued analysis used below also agree with those of [1].

Applying methods of the infinitesimal analysis, we use the classical approach by A. Robinson realized inside $V^{(B)}$ [1]. In other words, in concrete situations we use standard and non-standard superstructures and the corresponding *-mapping (Robinson's standardization) represented by elements of $V^{(B)}$. Here the non-standard extension is assumed to be appropriately saturated. Here descent standardization is by definition a descent of a *-mapping. Along with the notion of a descent standardization we also use the following expressions: "B-standardization," "prostandardization." Robinson's standardization of a B-set A is denoted by *A. Correspondingly, a descent standardization of a set A with B-structure (i.e., a subset of $V^{(B)}$), by definition represented as $(^*(A\uparrow))\downarrow$, is denoted by $_*A$ (here A↑ is an element of a standard superstructure of "classical" elements in $V^{(B)}$). Thus, $^*a \in {_*A} \leftrightarrow a \in A\uparrow\downarrow$. We can also give a natural definition of a descent standardization $_*\Phi$ of an extensional relation Φ. When necessary to consider descent standardization of standard names of elements in the von Neuman universe V, for convenience we use abbreviations $^*x := {^*(x^\wedge)}$ and, respectively, $_*x := (^*x)\downarrow$ for $x \in V$. Rules regarding the placement and omittance of stars when using descent standardizations without special stipulations are as free as those used for Robinson's *-mapping.

As an illustration, we cite several simple rules of placement of stars and variations of principles of transfer and idealization in the described situation.

1.1. Given sets $F, G \in \mathscr{P}(V^{(B)})$ and an extensional relation Φ, we have

$$_*(G \cap F) = {_*G} \cap {_*F},\ {_*(G \cup F)} = ({_*G} \cup {_*F})\uparrow\downarrow;$$
$$_*\Phi({_*G}) = {_*\Phi}(G).$$

<u>Proof.</u> The proof consists of successive application of rules of descending and lifting and properties of *-mappings. For example, the second formula is derived using the following:

$$_*(G \cup F) = (^*((G \cup F)\uparrow))\downarrow = (^*(G\uparrow \cup F\downarrow)) = (^*(G\uparrow) \cup {^*(F\uparrow)})\downarrow = ((^*(G\uparrow))\downarrow \cup (^*(F\uparrow))\downarrow)\uparrow\downarrow = ({_*G} \cup {_*F})\uparrow\downarrow.$$

1.2. Suppose $(A_\xi)_{\xi\in\Xi}$ is a family of non-empty sets with a B-structure and $(b_\xi)_{\xi\in\Xi}$ is a decomposition of the unit. Then we have the following for an intermixing:

$$_*\Big(\sum_{\xi\in\Xi} b_\xi A_\xi\Big) = \sum_{\xi\in\Xi} b_\xi {_*A_\xi}.$$

<u>Proof.</u> Let $A := \Sigma_{\xi\in\Xi} b_\xi A_\xi$. Clearly, for all $\xi \in \Xi$ we have $b_\xi \leqslant [A_\xi\uparrow = A\uparrow] = [^*(A_\xi\uparrow) = {^*(A\uparrow)}]$. Therefore, $({_*A})\uparrow$ is an intermixing of $(({_*A_\xi})\uparrow)_{\xi\in\Xi}$ with probabilities $(b_\xi)_{\xi\in\Xi}$. In this case [2] we have $_*A = \sum_{\xi\in\Xi} b_\xi {_*A_\xi}$, as desired.

<u>1.3. Principle of Transfer.</u> Let $\varphi = \varphi(x, y)$ be an equation of the Tsermelo–Frankel theory (which does not contain any free variables besides x and y). For every non-empty element F of $V^{(B)}$ and every z we have

$$(\exists x \in {_*F})\ [\varphi(x, {^*z})] = 1 \leftrightarrow (\exists x \in F\downarrow)\ [\varphi(x, z)] = 1;$$
$$(\forall x \in {_*F})\ [\varphi(x, {^*z})] = 1 \leftrightarrow (\forall x \in F\downarrow)\ [\varphi(x, z)] = 1.$$

If G is a subset of $V^{(B)}$ then we have

$$(\exists x \in {_*G})\ [\varphi(x, {^*z})] = 1 \leftrightarrow (\exists x \in G\uparrow\downarrow)\ [\varphi(x, z)] = 1;$$
$$(\forall x \in {_*G})\ [\varphi(x, {^*z})] = 1 \leftrightarrow (\forall x \in G)\ [\varphi(x, z)] = 1.$$

<u>Proof.</u> Using the necessary definitions and consecutively applying both the maximum principle of the Boolean-valued analysis and the transfer principle of Robinson's standardization (Leibnitz' principle), we obtain the following:

$$(\exists x \in {_*F})\ [\varphi(x, {^*z})] = 1 \leftrightarrow [(\exists x \in {^*F})\ \varphi(x, {^*z})] = 1 \leftrightarrow [(\exists x \in F)\ \varphi(x, z)] = 1 \leftrightarrow (\exists x \in F\downarrow)\ [\varphi(x, z)] = 1;$$

$$(\forall x \in {_*G})\ [\varphi(x, {^*z})] = 1 \leftrightarrow [(\forall x \in ({_*G})\uparrow)\ \varphi(x, {^*z})] = 1 \leftrightarrow$$

$$\leftrightarrow [(\forall x \in {}^*(G\uparrow))\ \varphi(x, {}^*z)] = 1 \leftrightarrow [(\forall x \in G\uparrow)\ \varphi(x, z)] = 1 \leftrightarrow \bigwedge_{x \in G} [\varphi(x, z)] = 1 \leftrightarrow (\forall x \in G)\,[\varphi(x, z)] = 1.$$

The other two equivalences are partial cases of the proven ones.

1.4. Principle of Idealization. Suppose $X\uparrow$ and Y are (classical) elements of $V^{(B)}$ and $\varphi = \varphi(x, y, z)$ is a formula of the Tsermelo–Frankel theory. Then every element internal to $V^{(B)}$ satisfies

$$(\forall^{\mathrm{fin}} A \subset X)(\exists y \in {}_*Y)\ (\forall x \in A)\ [\varphi({}^*x, y, z)] = 1 \leftrightarrow (\exists y \in {}_*Y)(\forall x \in X)\ [\varphi({}^*x, y, z)] = 1.$$

Proof. Indeed, applying the principle of transfer of Boolean-valued analysis, we obtain

$$[(\forall A \in \mathscr{P}_{\mathrm{fin}}(X\uparrow))\quad (\exists y \in {}^*Y)\quad (\forall x \in A)\quad \varphi({}^*x, y, z)] = [(\exists y \in_* Y)\quad (\forall x \in X)\quad \varphi({}^*x, y, z)].$$

It remains to show that $\mathscr{P}_{\mathrm{fin}}(X\uparrow) = \mathscr{P}_{\mathrm{fin}}(X)^{\uparrow\uparrow} := \{A\uparrow\colon\ A \in \mathscr{P}_{\mathrm{fin}}(X)\}\uparrow$ and apply rules of estimation in $V^{(B)}$.

2. Descents of Monads

In this paragraph we state fragments of monadology necessary for the representation and comparison of filters in the Boolean-valued model. The presentation is conducted analogously to the theory of cyclic monads in [2, 3].

2.1. Let $\mathscr{G}$ be a basis of a filter in some subset of $V^{(B)}$ and $\mathscr{G}' := \{G\uparrow\colon G \in \mathscr{G}\}$. As usual, a symbol $\mathscr{G}^{\uparrow}$ denotes a filter inside $V^{(B)}$ with a basis $\mathscr{G}'\uparrow$. An element $\mathscr{G}^{\uparrow}$ is called the lift of a filter $\mathscr{G}$. Given a filter $\mathscr{F}$ inside $V^{(B)}$, its descent $\mathscr{F}^{\downarrow}$ is defined as a set of supersets of a basis $\{F\downarrow\colon F \in \mathscr{F}\downarrow\}$. Finally, the cyclic envelope $\mathscr{G}^{\uparrow\downarrow}$ of the basis of the filter $\mathscr{G}$ is defined as a filter with a basis $\{G\uparrow\downarrow\colon G \in \mathscr{G}\}$.

2.2. Given a filter $\mathscr{F}$ of sets with B-structure, we define its descent monad $m(\mathscr{F})$ to be

$$m(\mathscr{F}) := \bigcap_{F \in \mathscr{F}} {}_*F.$$

2.3. 1) The descent monad of a filter is the descent of the monad of its lift;

2) the lift of a descent monad is a monad of the lift of a filter;

3) every set F of elements of $V^{(B)}$ satisfies $F \in \mathscr{F}^{\uparrow\downarrow} \leftrightarrow {}_*F \supset m(\mathscr{F})$;

4) descent monads of a filter $\mathscr{F}$ and its cyclic envelope $\mathscr{F}^{\uparrow\downarrow}$ are equal to each other.

Proof. Assertions 1) and 2) are proven simultaneously as follows:

$$z \in m(\mathscr{F}) \leftrightarrow (\forall F \in \mathscr{F})\ z \in {}_*F \leftrightarrow (\forall F \in \mathscr{F})\ [z \in {}^*(F\uparrow)] = 1 \leftrightarrow$$
$$\leftrightarrow \bigwedge_{F \in \mathscr{F}} [z \in {}^*(F\uparrow)] = 1 \leftrightarrow [(\forall F \in \mathscr{F}^{\uparrow})\ z \in {}^*F] = 1 \leftrightarrow [z \in \mu(\mathscr{F}^{\uparrow})] = 1 \leftrightarrow z \in \mu(\mathscr{F}^{\uparrow})\downarrow.$$

To prove assertion 3), we use the fact that a set belongs to a filter if and only if its *-mapping contains a monad of this filter. Then, calculating the truth estimates, we obtain the following (compare with Sec. 1.3):

$$F \in \mathscr{F}^{\uparrow\downarrow} \leftrightarrow (\exists G \in \mathscr{F}^{\uparrow}\downarrow)\ F \supset G\downarrow \leftrightarrow [(\exists G \in \mathscr{F}^{\uparrow})\ G \subset F\uparrow] = 1 \leftrightarrow$$
$$\leftrightarrow [F\uparrow \in \mathscr{F}^{\uparrow}] = 1 \leftrightarrow [{}^*(F\uparrow) \supset \mu(\mathscr{F}^{\uparrow})] = 1 \leftrightarrow {}_*F \supset \mu(\mathscr{F}^{\uparrow})\downarrow = m(\mathscr{F}).$$

Assertion 4) is satisfied since $\mathscr{F}^{\uparrow\downarrow\uparrow} = \mathscr{F}^{\uparrow}$ inside $V^{(B)}$.

2.4. Suppose $(\mathscr{F}_\xi)_{\xi \in \Xi}$ is a family of filters, $(b_\xi)_{\xi \in \Xi}$ a decomposition of the unit, and $\mathscr{F} := \sum_{\xi \in \Xi} b_\xi \mathscr{F}_\xi^{\uparrow}$ is a mixing of $(\mathscr{F}_\xi^{\uparrow})_{\xi \in \Xi}$ with probabilities $(b_\xi)_{\xi \in \Xi}$. Then

$$m(\mathscr{F}^{\downarrow}) = \sum_{\xi \in \Xi} b_\xi m(\mathscr{F}_\xi).$$

Proof. By the definition of mixing we have $[\mathscr{F} = \mathscr{F}_\xi^{\uparrow}] \geqslant b_\xi$ for $\xi \in \Xi$. Since filters inside $V^{(B)}$ are equal if and only if their monads are equal, we have

$$(\forall \xi \in \Xi)\ [\mathscr{F} = \mathscr{F}_\xi^{\uparrow}] \geqslant b_\xi \leftrightarrow (\forall \xi \in \Xi)\ [\mu(\mathscr{F}) = \mu(\mathscr{F}_\xi^{\uparrow})] \geqslant b_\xi \leftrightarrow$$
$$\leftrightarrow (\forall \xi \in \Xi)\ b_\xi \mu(\mathscr{F})\downarrow = b_\xi \mu(\mathscr{F}_\xi^{\uparrow})\downarrow \leftrightarrow m(\mathscr{F}^{\downarrow}) = \sum_{\xi \in \Xi} b_\xi m(\mathscr{F}_\xi).$$

2.5. Suppose Φ is an extensional relation and a filter $\mathscr{F}$ touches the effective domain of definition of Φ. Then the descent monad of the image of the filter is the image of the descent monad, i.e.,

$$m(\Phi(\mathscr{F})) = ({}_*\Phi)(m(\mathscr{F})).$$

Proof. Considering the fact that inside $V^{(B)}$ the monad of the image is the monad of the filter $\mathscr{F}^{\uparrow}$, we compute as follows (compare with Sec. 1.1):

$$({}_*\Phi)(m(\mathscr{F})) = (^*(\Phi\uparrow))\downarrow(\mu(\mathscr{F}^{\uparrow})\downarrow) = ((^*\Phi\uparrow))(\mu(\mathscr{F}^{\uparrow})))\downarrow = \mu(\Phi\uparrow(\mathscr{F}^{\uparrow}))\downarrow = \mu(\Phi(\mathscr{F})^{\uparrow})\downarrow = m(\Phi)).$$

Here we used Sec. 2.3 (2) and a rule $\Phi\uparrow(\mathscr{F}^{\uparrow}) = \Phi(\mathscr{F})^{\uparrow}$.

2.6. The following statements are equivalent:

1) filters $\mathscr{F}^{\uparrow}$ and $\mathscr{G}^{\uparrow}$ intersect inside $V^{(B)}$;

2) for all $F \in \mathscr{F}$ and $G \in \mathscr{G}$ we have $F\uparrow\downarrow \cap G\uparrow\downarrow \neq \varnothing$;

3) descent monads $m(\mathscr{F})$ and $m(\mathscr{G})$ have a common point. When equivalent conditions (1)-(3) are satisfied, we have

$$m(\mathscr{F}) \cap m(\mathscr{G}) = m(\mathscr{F}\uparrow\downarrow \vee \mathscr{G}\uparrow\downarrow);$$
$$\mathscr{F}^{\uparrow} \vee \mathscr{G}^{\uparrow} = (\mathscr{F}\uparrow\downarrow \vee \mathscr{G}\uparrow\downarrow)^{\uparrow}.$$

Proof. We have

$$[\exists\mathscr{F}^{\uparrow} \vee \mathscr{G}^{\uparrow}] = [(\forall F \in \mathscr{F}^{\uparrow})\ (\forall G \in \mathscr{G}^{\uparrow})\ F \cap G \neq \varnothing]$$
$$= \bigwedge_{F\in\mathscr{F}} \bigwedge_{G\in\mathscr{G}} [F\uparrow \cap G\uparrow \neq \varnothing] = [m(\mathscr{F})\uparrow \cap m(\mathscr{G})\uparrow \neq \varnothing],$$

from which the desired conclusions follow.

2.7. Filters $\mathscr{F}$ and $\mathscr{G}$ satisfy

$$m(\mathscr{F} \wedge \mathscr{G}) = (m(\mathscr{F}) \cup m(\mathscr{G}))\uparrow\downarrow;$$
$$\mathscr{F}^{\uparrow} \wedge \mathscr{G}^{\uparrow} = (\mathscr{F} \wedge \mathscr{G})^{\uparrow}.$$

Proof. By definition, a filter $(\mathscr{F} \wedge \mathscr{G})^{\uparrow}$ inside $V^{(B)}$ has a basis $\{(F \cup G)\uparrow : F \in \mathscr{F},\ G \in \mathscr{G}\}\uparrow$. Therefore, by using an equation $(F \cup G)\uparrow = F\uparrow \cup G\uparrow$, we conclude that $(\mathscr{F} \wedge \mathscr{G})^{\uparrow} = \mathscr{F}^{\uparrow} \wedge \mathscr{G}^{\uparrow}$. Thus, Sec. 2.3 implies that

$$m(\mathscr{F} \wedge \mathscr{G})\uparrow = \mu((\mathscr{F} \wedge \mathscr{G})^{\uparrow}) = \mu(\mathscr{F}^{\uparrow} \wedge \mathscr{G}^{\uparrow}) = \mu(\mathscr{F}^{\uparrow}) \cup \mu(\mathscr{G}^{\uparrow}) = m(\mathscr{F})\uparrow \cup m(\mathscr{G})\uparrow.$$

It remains to recall that $(m(\mathscr{F})\uparrow \cup m(\mathscr{G})\uparrow)\downarrow = (m(\mathscr{F}) \cup m(\mathscr{G}))\uparrow\downarrow$.

2.8. THEOREM. Let $\mathscr{E}$ be a set of filters and $\mathscr{E}^{\uparrow} := \{\mathscr{F}^{\uparrow} : \mathscr{F} \in \mathscr{E}\}\uparrow$ its lift to $V^{(B)}$. Then the following statements are equivalent:

1) the set of cyclic envelopes $\mathscr{E}\uparrow\downarrow := \{\mathscr{F}\uparrow\downarrow : \mathscr{F} \in \mathscr{E}\}$ is bounded from above;

2) the set $\mathscr{E}^{\uparrow}$ is bounded from above in $V^{(B)}$;

3) $\cap\{m(\mathscr{F}) : \mathscr{F} \in \mathscr{E}\} \neq \varnothing$;

When the equivalent conditions (1)-(3) are satisfied, we have

$$m(\sup \mathscr{E}\uparrow\downarrow) = \cap\{m(\mathscr{F}) : \mathscr{F} \in \mathscr{E}\};$$
$$\sup \mathscr{E}^{\uparrow} = (\sup \mathscr{E})^{\uparrow}.$$

Proof. If $\mathscr{E}\uparrow\downarrow$ is bounded from above then for $\mathscr{G} := \sup \mathscr{E}\uparrow\downarrow$ and $\mathscr{F} \in \mathscr{E}$ we have $\mathscr{G} \geqslant \mathscr{F}\uparrow\downarrow$, and therefore $m(\mathscr{G}) \subset m(\mathscr{F}\uparrow\downarrow) = m(\mathscr{F})$, as implies by Sec. 2.3 (4). Thus, $m(\mathscr{G}) \subset \cap\{m(\mathscr{F}) : \mathscr{F} \in \mathscr{E}\}$. Now suppose that $F \in \sup \mathscr{E}\uparrow\downarrow$. There exists a finite set $\mathscr{E}_0 \subset \mathscr{E}$ such that $F \in \sup(\mathscr{E}_0\uparrow\downarrow)$. Therefore, ${}_*F \supset m(\sup(\mathscr{E}_0\uparrow\downarrow))$. Section 2.5 then implies that ${}_*F \supset \cap\{m(\mathscr{F}) : \mathscr{F} = \mathscr{E}_0\}$. Finally, we obtain

$$m(\sup \mathscr{E}\uparrow\downarrow) \supset \cap\{m(\mathscr{F}) : \mathscr{F} \in \mathscr{E}_0,\ \mathscr{E}_0 \in \mathscr{P}_{\text{fin}}(\mathscr{E})\} = \cap\{m(\mathscr{F}) : \mathscr{F} \in \mathscr{E}\}.$$

The above arguments prove implications (1) → (2) → (3) and the desired equation for descent monads. If (3) holds then for any finite subset $\mathscr{E}_0$ of $\mathscr{E}$ an intersection $\cap\{m(\mathscr{F}) : \mathscr{F} \in \mathscr{E}_0\}$ is non-empty. Therefore, Sec. 2.6 implies that there exists an exact upper bound $\sup(\mathscr{E}_0\uparrow\downarrow)$ and therefore a bound $\sup \mathscr{E}\uparrow\downarrow$.

The proof is completed by the following calculation of truth estimates:

$$[\sup \mathscr{E}^{\uparrow} = (\sup \mathscr{E})^{\uparrow}] = [\mu((\sup \mathscr{E})^{\uparrow}) = \cap\{\mu(\mathscr{F}) : \mathscr{F} \in \{\mathscr{F}^{\uparrow} : \mathscr{F} \in \mathscr{E}\}\uparrow\}]$$
$$= [m(\sup \mathscr{E}\uparrow\downarrow)\uparrow = \cap\{m(\mathscr{F})\uparrow : \mathscr{F} \in \mathscr{E}\}\uparrow] = [m(\sup \mathscr{E}\uparrow\downarrow)\uparrow = \cap\{m(\mathscr{F}) : \mathscr{F} \in \mathscr{E}\}\uparrow].$$

2.9. It is useful to note that the union of an infinite set of descent monads and even the cyclic envelope of this union are not in general descent monads. This situation is similar to that of usual monads.

2.10. Suppose $(\mathcal{F}_\xi)_{\xi\in\Xi}$ is a family of cyclic filters. Then the descent monads satisfy

$$m\Big(\prod_{\xi\in\Xi}\mathcal{F}_\xi\Big)=\prod_{\xi\in\Xi}m(\mathcal{F}_\xi).$$

Proof. Let $\mathcal{F}$ be the product of these filters. Then, using the obvious notation, we have $\mathcal{F}=\sup\{\mathrm{Pr}_\xi^{-1}(\mathcal{F}_\xi):\ \xi\in\Xi\}$. Using a symbol $(\mathcal{F}_\xi^\updownarrow)_{\xi\in\Xi^\frown}$ for a lift of a mapping $\xi\mapsto\mathcal{F}_\xi^\updownarrow$ and using Secs. 2.3 and 2.8, we obtain

$$m(\mathcal{F})=\bigcap\{m(\mathrm{Pr}_\xi^{-1}(\mathcal{F}_\xi)):\ \xi\in\Xi\}=(\bigcap\{\mu(\mathrm{Pr}_\xi^{-1}(\mathcal{F}_\xi^\updownarrow)):\ \xi\in\Xi\}\uparrow)\downarrow$$
$$=\mu(\sup\{\mathrm{Pr}_\xi^{-1}(\mathcal{F}_\xi^\updownarrow):\ \xi\in\Xi^\wedge\})\downarrow=\mu\Big(\prod_{\xi\in\Xi^\frown}\mathcal{F}_\xi^\updownarrow\Big)\downarrow=\Big(\prod_{\xi\in\Xi^\frown}\mu(\mathcal{F}_\xi^\updownarrow)\Big)\downarrow.$$

Thus,

$$x\in m(\mathcal{F})\leftrightarrow\Big[x\in\prod_{\xi\in\Xi^\frown}\mu(\mathcal{F}_\xi^\updownarrow)\Big]=1\leftrightarrow[(\forall\xi\in\Xi^\wedge)\ \ x_\xi\in\mu(\mathcal{F}_\xi^\updownarrow)]=1$$
$$\leftrightarrow(\forall\xi\in\Xi)\ \Big[x_{\xi^\frown}\in\mu\big(\mathcal{F}_{\xi^\frown}^\updownarrow\big)\Big]=1\leftrightarrow(\forall\xi\in\Xi)\ x_\xi\in\mu(\mathcal{F}_\xi^\updownarrow)\downarrow$$
$$\leftrightarrow(\forall\xi\in\Xi)\ x_\xi\in m(\mathcal{F}_\xi)\leftrightarrow x\in\prod_{\xi\in\Xi}m(\mathcal{F}_\xi).\ \ \triangleright$$

3. Descent Monads of Proultrafilters

In this paragraph we characterize proultrafilters, i.e., maximal cyclic filters whose lifts are ultrafilters in the Boolean-valued universe.

3.1. The following statements hold:

(1) given a cyclic filter $\mathcal{F}$ and a cyclic set U, we either have $_*U\cap m(\mathcal{F})\neq\varnothing$, or $U'\in\mathcal{F}$;

(2) given an extensional filter $\mathcal{F}$ and a cyclic set U, we either have $_*U\cap m(\mathcal{F})\neq\varnothing$, or $_*(U')\supset m(\mathcal{F})$.

Proof. Applying 2.3 (1), we see that it suffices to prove statement (1). To accomplish this, we ascertain that there exist two mutually exclusive possibilities:

$$(\forall F\in\mathcal{F})\ \ U\cap F\uparrow\downarrow\neq\varnothing,\qquad(\exists F\in\mathcal{F})\ \ U\cap F=\varnothing.$$

In the second case we have $U'\in\mathcal{F}$. In the first one, using Sec. 2.5, we obtain $_*U\cap m(\mathcal{F})\neq\varnothing$.

3.2. Non-Standard Criteria of Proultrafilters. The following statements are equivalent:

(1) $\mathfrak{A}$ is a proultrafilter;

(2) $\mathfrak{A}$ is an extensional filter whose descent monad is minimal with respect to inclusion;

(3) we have

$$\mathfrak{A}=(x)^\downarrow:=\{A\uparrow\downarrow:\ x\in{}_*A\}$$

for every point x in the descent monad $m(\mathfrak{A})$;

(4) $\mathfrak{A}$ is an extensional filter whose descent monad is easily caught by a cyclic set, i.e., every $U=U\uparrow\downarrow$ satisfies either $m(\mathfrak{A})\subset{}_*U$, or $m(\mathfrak{A})\subset{}_*(U')$;

(5) $\mathfrak{A}$ is a cyclic filter such that every cyclic U such that $_*U\cap m(\mathcal{F})\neq\varnothing$ satisfies $U\in\mathfrak{A}$.

Proof. It is convenient for us to carry out an excessively detailed proof using a scheme (1) → (2) → (3) → (4) → (1) and (1) ↔ (5).

(1) → (2). If $m(\mathfrak{A})\supset m(\mathfrak{B})$, then $\mathfrak{B}^{\uparrow\downarrow}\supset\mathfrak{A}^{\uparrow\downarrow}=\mathfrak{A}$. Therefore, $\mathfrak{B}^{\uparrow\downarrow}=\mathfrak{A}$ and $m(\mathfrak{A})=m(\mathfrak{B}^{\uparrow\downarrow})=m(\mathfrak{B})$.

(2) → (3). Consider $\mathfrak{A}^\uparrow$. If in $V^{(B)}$ we have $\mu(\mathfrak{A}^\uparrow)\subset\mu(\mathfrak{B})$, then $m(\mathfrak{A})=m(\mathfrak{B}^\downarrow)$. Therefore, $\mu(\mathfrak{A}^\uparrow)=m(\mathfrak{A})^\uparrow=m(\mathfrak{B}^\downarrow)\uparrow=m(\mathfrak{B})$. Thus, $\mathfrak{A}^\uparrow$ is an ultrafilter in $V^{(B)}$. In other words,

$$[(\forall x\in\mu(\mathfrak{A}^\uparrow))\ \ A\in\mathfrak{A}^\uparrow\leftrightarrow x\in{}^*A]=1.$$

Using rules of truth calculation, we conclude that $\mathfrak{A}=\mathfrak{A}^{\uparrow\downarrow}=(x)^\downarrow$ for all $x\in m(\mathfrak{A})=\mu(\mathfrak{A}^\uparrow)\downarrow$.

(3) → (4). Using 3.1 (2), we see that a cyclic U either satisfies $m(\mathfrak{A}) \subset {}_*(U')$, or ${}_*U \cap m(\mathfrak{A}) \neq \varnothing$. In the second case an element $x \in m(\mathscr{F}) \cap {}_*U$ by definition satisfies $x \in {}_*U$, and $U\uparrow\downarrow = U \in \mathfrak{A}$. Therefore, ${}_*U \supset m(\mathfrak{A})$.

(4) → (1). Suppose $\mathfrak{B}$ is a cyclic filter such that $\mathfrak{B} \supset \mathfrak{A}$. Choose a cyclic set $U \in \mathfrak{B}$. If $m(\mathfrak{A}) \subset {}_*(U')$, then 2.3 (1)-(3) imply that $U' \in \mathfrak{A}^{\uparrow\downarrow} = \mathfrak{A} \subset \mathfrak{B}$. This is a contradiction. Thus, we have a case where ${}_*U \supset m(\mathfrak{A})$. Again using 2.3 (1)-(3), we conclude that $U \in \mathfrak{A}$. Consequently, $\mathfrak{A}$ is a maximal cyclic filter, i.e., a proultrafilter.

(1) ↔ (5). If $\mathfrak{A}$ is a proultrafilter and a cyclic U satisfies ${}_*U \cap m(\mathfrak{A}) \neq \varnothing$, then statement (3) implies that $U \in \mathfrak{A}$. On the other hand, if $U \in \mathfrak{B}$, where $U = U\uparrow\downarrow$ and $\mathfrak{B}$ is a cyclic filter majorizing $\mathfrak{A}$, then ${}_*U \cap m(\mathfrak{A}) \supset {}_*U \cap m(\mathfrak{B}) = {}_*U \neq \varnothing$. Therefore, $U \in \mathfrak{A}$ by condition of (5). Thus, $\mathfrak{B} = \mathfrak{A}$.

3.3. It is useful to compare the derived criteria for proultrafilters with those cited in [3]. To feel the difference in the formalism of the two, we emphasize that descent monads are obviously unions of descent monads of proultrafilters. On the other hand, the standard monads of extensional filters do not reduce to unions of their essential points.

4. Quantification on Descent Monads

In this paragraph we cite some basic facts related to the deciphering of statements containing quantors extended to descent monads.

4.1. Suppose $\varphi = \varphi(x)$ is a formula of the Tsermelo-Frankel theory. The truth estimate of φ is the same on any descent monad of any given proultrafilter $\mathfrak{A}$, i.e.,

$$(\forall x,\ y \in m(\mathfrak{A}))\quad [\varphi(x)] = [\varphi(y)].$$

Proof. It is known [4] that the existence and universality are equivalent for ultramonads, i.e., in $V^{(B)}$ we have $(\forall x \in \mu(\mathfrak{A}^{\uparrow}))\ \varphi(x) \leftrightarrow (\exists y \in \mu(\mathfrak{A}^{\uparrow}))\ \varphi(y)$. Using 2.3 (2) to calculate the truth estimates, we obtain

$$\bigwedge_{x \in m(\mathfrak{A})} [\varphi(x)] = [(\forall x \in m(\mathfrak{A})\uparrow)\ \varphi(x)] = \left[(\forall x \in \mu(\mathfrak{A}^{\uparrow}))\ \varphi(x)\right]$$
$$= \left[(\exists y \in \mu(\mathfrak{A}^{\uparrow}))\ \varphi(y)\right] = [(\exists y \in m(\mathfrak{A})\uparrow)\ \varphi(y)] = \bigvee_{y \in m(\mathfrak{A})} [\varphi(y)],$$

which implies the desired result.

4.2. THEOREM. Suppose $\varphi = \varphi(x, y, z)$ is a formula of the Tsermelo-Frankel theory and $\mathscr{F}$, $\mathscr{G}$ are filters of sets with B-structure. Then the following rules of quantification hold (where y and z are internal in $V^{(B)}$):

(1) $(\exists x \in m(\mathscr{F}))\,[\varphi(x, y, z)] = 1 \leftrightarrow (\forall F \in \mathscr{F})(\exists x \in {}_*F)\,[\varphi(x, y, z)] = 1$;

(2) $(\forall x \in m(\mathscr{F}))\,[\varphi(x, y, z)] = 1 \leftrightarrow (\exists F \in \mathscr{F}^{\uparrow\downarrow})(\forall x \in {}_*F)\,[\varphi(x, y, z)] = 1$;

(3) $(\forall x \in m(\mathscr{F}))(\exists y \in m(\mathscr{G}))\,[\varphi(x, y, z)] = 1 \leftrightarrow (\forall G \in \mathscr{G})(\exists F \in \mathscr{F}^{\uparrow\downarrow})(\forall x \in {}_*F)(\exists y \in {}_*G)\,[\varphi(x, y, z)] = 1$;

(4) $(\exists x \in m(\mathscr{F}))(\forall y \in m(\mathscr{G}))\,[\varphi(x, y, z)] = 1 \leftrightarrow (\exists G \in \mathscr{G}^{\uparrow\downarrow})(\forall F \in \mathscr{F})(\exists x \in {}_*F)(\forall y \in {}_*G)\,[\varphi(x, y, z)] = 1$.

In addition, for standardized free variables we have

(1') $(\exists x \in m(\mathscr{F}))\,[\varphi(x, {}^*y, {}^*z)] = 1 \leftrightarrow (\forall F \in \mathscr{F})(\exists x \in F\uparrow\downarrow)\,[\varphi(x, y, z)] = 1$;

(2') $(\forall x \in m(\mathscr{F}))\,[\varphi(x, {}^*y, {}^*z)] = 1 \leftrightarrow (\exists F \in \mathscr{F}^{\uparrow\downarrow})(\forall x \in F)\,[\varphi(x, y, z)] = 1$;

(3') $(\forall x \in m(\mathscr{F}))(\exists y \in m(\mathscr{G}))\quad [\varphi(x, y, {}^*z)] = 1 \leftrightarrow (\forall G \in \mathscr{G})(\exists F \in \mathscr{F}^{\uparrow\downarrow})(\forall x \in F)(\exists y \in G\uparrow\downarrow)\,[\varphi(x, y, z)] = 1$;

(4') $(\exists x \in m(\mathscr{F}))(\forall y \in m(\mathscr{G}))\quad [\varphi(x, y, {}^*z)] = 1 \leftrightarrow (\exists G \in \mathscr{G}^{\uparrow\downarrow})(\forall F \in \mathscr{F})(\exists x \in F\uparrow\downarrow)(\forall y \in G)\,[\varphi(x, y, z)] = 1$.

Proof. We begin by proving rules (1) and (2). Applying Nelson's algorithm in $V^{(B)}$, we deduce that

$$(\exists x \in m(\mathscr{F}))\,[\varphi(x, y, z)] = 1 \leftrightarrow \left[(\exists x \in \mu(\mathscr{F}^{\uparrow}))\ \varphi(x, y, z)\right] = 1$$
$$\leftrightarrow \left[(\forall F \in \mathscr{F}^{\uparrow})(\exists x \in {}^*\mathscr{F})\ \varphi(x, y, z)\right] = 1 \leftrightarrow (\forall F \in \mathscr{F})(\exists x \in {}_*F)\,[\varphi(x, y, z)] = 1;$$
$$(\forall x \in m(\mathscr{F}))\,[\varphi(x, y, z)] = 1 \leftrightarrow \left[(\forall x \in \mu(\mathscr{F}^{\uparrow}))\ \varphi(x, y, z)\right] = 1$$
$$\leftrightarrow \left[(\exists F \in \mathscr{F}^{\uparrow})(\forall x \in {}^*F)\ \varphi(x, y, z)\right] = 1 \leftrightarrow (\exists F \in \mathscr{F}^{\uparrow\downarrow})(\forall x \in {}_*F)\,[\varphi(x, y, z)] = 1.$$

We now prove statement (3). Applying (2) and then (1), we have

$$(\forall x \in m(\mathscr{F}))(\exists y \in m(\mathscr{S}))\ [\varphi(x, y, z)] = 1$$
$$\leftrightarrow (\forall x \in m(\mathscr{F}))(\forall G \in \mathscr{S})(\exists y \in {}_*G)\ [\varphi(x, y, z)] = 1$$
$$\leftrightarrow (\forall G \in \mathscr{S})(\forall x \in m(\mathscr{F}))\ [(\exists y \in {}^*G)\ \varphi(x, y, z)] = 1$$
$$\leftrightarrow (\forall G \in \mathscr{S})(\exists F \in \mathscr{F}^{\uparrow\downarrow})(\forall x \in {}_*F)(\exists y \in {}_*G)\ [\varphi(x, y, z)] = 1.$$

Statement (4) is proven similarly. The equivalences contained in (1')-(4') follow from the already established ones by applying the transfer principle 1.2.

5. Procompact Filters and Mappings of Compact Spaces

In the future we will need criteria of procompact (cyclically compact) sets and filters (see [5]). To save space, as in [2], we restrict our attention to the case of uniform spaces.

Thus, let $(X, \mathscr{U})$ be a uniform space in $V^{(B)}$ and $(X\downarrow, \mathscr{U}\downarrow)$ its descent. Clearly, a descent monad $m(\mathscr{U}\downarrow)$ is an equivalence relation on ${}_*X$. For its notation we use the same symbol of infinite closeness $\approx$ as that used for $\mu(\mathscr{U})$ in $V^{(B)}$. Thus, ${}^{\approx}x = m(\mathscr{U}\downarrow(x))$.

5.1. An element x is a contact point of the cyclic envelope of a filter $\mathscr{F}$ if and only if ${}^{\approx}x \cap m(\mathscr{F}) \neq \varnothing$, i.e., if x is micro-limiting for $m(\mathscr{F})$.

Proof. It suffices to use Sec. 2.6, since $x \in \mathrm{cl}\ (\mathscr{F}\uparrow\downarrow) \leftrightarrow \exists \mathscr{F}\uparrow\downarrow \vee \mathscr{U}\downarrow(x) \leftrightarrow m(\mathscr{F}) \cap m(\mathscr{U}\downarrow(x)) \neq \varnothing$.

5.2. If U is internal in $V^{(B)}$ then let $\mathrm{cl}_{\approx}(U\downarrow)$ be the set of all micro-limiting points of U↓, i.e., the microclosure of U↓. The set $\mathrm{cl}_{\approx}(U\downarrow)$ is closed. Every subset U of X satisfies $\mathrm{cl}_{\approx}({}_*U) = \mathrm{cl}\,(U\uparrow\downarrow)$.

Proof. For $y \in \mathrm{cl}_{\approx}(U\downarrow)$ and a cyclic open neighborhood $\mathscr{O}$ of a point y there exists x such that ${}^{\approx}x \cap U\downarrow \neq \varnothing$ and $x \in \mathscr{O}$. Since ${}_*\mathscr{O} \supset {}^{\approx}x$, we have $(\forall \mathscr{O} \in \mathscr{U}\downarrow(y))\ {}_*\mathscr{O} \cap U\downarrow \neq \varnothing$. The idealization principle dictates that ${}^{\approx}y \cap U\downarrow \neq \varnothing$, i.e., $y \in \mathrm{cl}_{\approx}(U\downarrow)$. Clearly, $\mathrm{cl}_{\approx}(U\downarrow) = (\mathrm{cl}_{\approx}(U))\downarrow$. The latter observation concludes the proof.

5.3. A point y in ${}_*X$ is called descent nearly standard, or simply nearly standard if there is no danger of confusion, if for some $x \in X\downarrow$ we have ${}^*x \approx y$. Thus, the descent nearly standard points of ${}_*X$ are elements which are infinitely close to the points of the standard kernel of the set ${}^*X$ in $V^{(B)}$.

5.4. Criteria of Procompactness of Filters. Given a filter $\mathscr{F}$, the following statements are equivalent:

(1) $\mathscr{F}$ is procompact (cyclically compact), i.e., every cyclic filter which is thinner than $\mathscr{F}$ has a contact point;

(2) every proultrafilter majorizing $\mathscr{F}\uparrow\downarrow$ converges;

(3) a descent monad $m(\mathscr{F})$ consists of nearly standard points;

(4) $\mathscr{F}^{\uparrow}$ is compact in $V^{(B)}$;

(5) $\mathscr{F}^{\uparrow\downarrow}$ is cyclically compact.

Proof. (1)→(2). If x is a contact point of a proultrafilter $\mathfrak{A}$, majorizing $\mathscr{F}\uparrow\downarrow$, then ${}^{\approx}x \cap m(\mathfrak{A}) \neq \varnothing$, so then 3.2 (2) implies that $m(\mathfrak{A}) \supset {}^{\approx}x$. Therefore, $\mathfrak{A} \to x$.

(2) → (3). Suppose $y \in m(\mathscr{F})$. Then $(y)^{\downarrow}$ is a proultrafilter. Clearly, y is infinitely close to the limit of $(y)^{\downarrow}$.

(3) → (4). Using the rules of lifting, we conclude that a monad $\mu(\mathscr{F}^{\uparrow})$ consists of nearly standard points in $V^{(B)}$.

(4) → (5). If $\mathscr{S}$ is a cyclic filter thinner than $\mathscr{F}^{\uparrow\downarrow}$, then $\mu(\mathscr{S}) \subset \mu(\mathscr{F}^{\uparrow\downarrow})$ in $V^{(B)}$. Using Sec. 5.1, we see that $\mathscr{S}$ has a contact point.

(5) → (1). It suffices to note that $\mathscr{F}\uparrow\downarrow$ and $\mathscr{F}^{\uparrow\downarrow}$ are majorized by the same proultrafilter.

5.5. A Criterion of Procompactness. A set A↑↓ is precompact (cyclically compact) if and only if every point of ${}_*A$ is descent nearly standard.

Proof. It suffices to apply Sec. 5.4 to a filter with a basis {A↑↓}.

5.6. It is useful to compare criterion 5.4 with criterion 3.2 in [2] for the nearly standardness of essential points of Robinson's standardization of a given cyclic set.

5.7. A point y of a set $_{*}X$ is called descent pre-standard, or simply pre-standard if there is no fear of confusion, if a microhalo $\approx y$ contains a descent monad. Thus, the descent prestandard points of $_{*}X$ represent the pre-standard points of $^{*}X$ in $V^{(B)}$.

5.8. Given a filter $\mathscr{F}$, the following statements are equivalent:

(1) every descent pre-standard point of $m(\mathscr{F})$ is descent nearly standard;

(2) every cyclic Cauchy filter thinner than $\mathscr{F}\uparrow\downarrow$ converges;

(3) the filter $\mathscr{F}\uparrow\downarrow$ is complete;

(4) $\mathscr{F}^{\uparrow}$ is complete in $V^{(B)}$.

Proof. The equivalence (1) $\leftrightarrow$ (4) follows from the usual criterion of completeness of the filter $\mathscr{F}^{\uparrow}$, which consists of a requirement that $\mathrm{pst}\,(^{*}X)\cap\mu(\mathscr{F}^{\uparrow})\subset\mathrm{nst}\,(^{*}X)$ in $V^{(B)}$ (see [6]).

(1) $\to$ (2). Suppose $\mathscr{G}$ is a cyclic Cauchy filter $\mathscr{G}\subset\mathscr{F}^{\uparrow\downarrow}$. If $x\in m(\mathscr{G})$, then $x\in\mu(\mathscr{G}^{\uparrow})\downarrow$, so therefore x is a descent pre-standard point. Thus, x is descent nearly standard, and there exists $y\in X$ such that $^{\approx}y\cap m(\mathscr{G})\neq\varnothing$. Using Sec. 5.1, we see that y is a contact point of $\mathscr{G}$. Since $\mathscr{G}$ is a Cauchy filter, we have $\mathscr{G}\to y$.

(2) $\to$ (1). Suppose $x\in m(\mathscr{F})$ and $^{\approx}x=m(\mathscr{G})$. Clearly, $\mathscr{G}^{\uparrow\downarrow}$ is a cyclic filter which is a Cauchy filter. Section 2.6 implies that we have a filter $\mathscr{F}\uparrow\downarrow\vee\mathscr{G}^{\uparrow\downarrow}$. By assumption, it has a limit y. We see that $x\in m(\mathscr{F})\cap m(\mathscr{G})\subset m(\mathscr{U}^{\downarrow}(y))$. Therefore, x is a descent nearly standard point.

Finally, it remains to show that (2) $\to$ (3), since the implication (3) $\to$ (2) follows directly from the definition.

(2) $\to$ (3). Thus, suppose $\mathscr{G}$ is a Cauchy filter which is thinner than $\mathscr{F}\uparrow\downarrow$. Clearly, $\mathscr{G}\subset\mathscr{G}\uparrow\downarrow\subset\mathscr{F}\uparrow\downarrow$. Since the uniformity of $\mathscr{U}^{\downarrow}$ is cyclic, $\mathscr{G}\uparrow\downarrow$ is also a Cauchy filter. Since statement (2) implies that $\mathscr{G}\uparrow\downarrow$ converges, $\mathscr{G}$ also converges.

5.9. A filter $\mathscr{F}$ is called pro-pre-compact (cyclically completely bounded) if $\mathscr{F}^{\uparrow}$ is completely bounded in $V^{(B)}$. The following results are direct consequences of the introduced definitions and known criteria [6].

5.10. A filter is pro-pre-compact if and only if every point of its descent monad is descent pre-standard.

5.11. Hausdorff's Criterion. A cyclic filter is procompact if and only if it is complete and pro-pre-compact.

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