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THE FARKAS LEMMA REVISITED

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Abstract: Boolean valued analysis is applied to deriving operator versions of the classical Farkas Lemma in the theory of simultaneous linear inequalities.

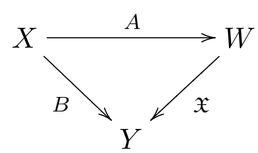
Keywords: Dedekind complete vector lattice, linear programming, linear inequalities, sublinear polyhedral inequalities, interval equations, alternative theorem, Boolean valued model

The Farkas Lemma, also known as the Farkas–Minkowski Lemma, plays a key role in linear programming and the relevant areas of optimization (cp. [1, 2]).

Using Boolean valued analysis [3] and subdifferential calculus [4], we establish some rather general properties of simultaneous operator inequalities. This note appeared as a short comment on [5].

Assume that X is a real vector space, Y is a *Kantorovich space* also known as a Dedekind complete vector lattice or a complete Riesz space. Denote by $L(X, Y)$ the space of linear operators from X to Y. In case X is furnished with some Y-seminorm on X, by $L^{(m)}(X, Y)$ we mean the *space of dominated operators* from X to Y. Given $T : X \to Y$ and $y \in Y$, put $\{T \leq y\} := \{x \in X \mid Tx \leq y\}$ and $\ker(T) := \{T = 0\} := T^{-1}(0)$.

Consider another real vector space W and the diagram



It is well known that

(i) $(\exists\, \mathfrak{X})\ \mathfrak{X}A = B \leftrightarrow \ker(A) \subset \ker(B)$;

(ii)[1] *Let W be ordered by some positive cone W^+ and $A(X) - W^+ = W^+ - A(X) = W$, i.e., $A(X)$ is coinitial in W. Then $(\exists \mathfrak{X} \geq 0)\ \mathfrak{X}A = B \leftrightarrow \{A \leq 0\} \subset \{B \leq 0\}$.*

1. Simultaneous Linear Inequalities

Let $\mathbb{B} := \mathbb{B}(Y)$ be the *base* of Y, i.e., the complete Boolean algebra of positive projections in Y; and let $m(Y)$ be the universal completion of Y. Supposing that $W = Y$, we have the following operator version of the Farkas Lemma.

Theorem 1.1. *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A_1, \dots, A_N$ and B belong to $L^{(m)}(X, Y)$. The following are equivalent:*

(1) *For all $b \in \mathbb{B}$, the operator inequality $bBx \leq 0$ is a consequence of the simultaneous linear operator inequalities $bA_1x \leq 0, \dots, bA_Nx \leq 0$, i.e.,*

$$\{bB \leq 0\} \supset \{bA_1 \leq 0\} \cap \dots \cap \{bA_N \leq 0\}.$$

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[1]This is the Kantorovich Theorem (cp. [4, p. 44]).

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(2) *There are positive orthomorphisms* $\alpha_1, \dots, \alpha_N \in \operatorname{Orth}(m(Y))$ *such that*

$$B = \sum_{k=1}^{N} \alpha_k A_k;$$

i.e., B lies in the operator convex conic hull of $A_1, \dots, A_N$.

We start with some two lemmas presenting versions of the Farkas Lemma for vector spaces over a subfield of the reals. We provide proofs for the sake of completeness, since the main results appear by way of the direct Boolean valued interpretation of the version.

Lemma 1.1. *Let X be a vector space over some subfield R of the reals $\mathbb{R}$. Assume that f and g are R-linear functionals on X; in symbols, $f, g \in X^{\#} := L(X, \mathbb{R})$. For the inclusion $\{g \le 0\} \supset \{f \le 0\}$ to hold it is necessary and sufficient that there be $\alpha \in \mathbb{R}_+$ satisfying $g = \alpha f$.*

Proof. We will prove only necessity since sufficiency is pretty obvious.

The case of $f = 0$ is trivial. If $f \ne 0$ then there is some $x \in X$ such that $f(x) \in \mathbb{R}$ and $f(x) > 0$. Denote the image $f(X)$ of X under f by R_0. Put $h := g \circ f^{-1}$, i.e. $h \in R_0^{\#}$ is the only solution for $h \circ f = g$. By hypothesis, h is a positive R-linear functional on R_0. By the Bigard Theorem [4, p. 108] h can be extended to a positive homomorphism $\bar{h} : \mathbb{R} \to \mathbb{R}$, since R_0 is coinitial in R. Each positive automorphism of $\mathbb{R}$ is multiplication by a positive real. As the sought α we may take $\bar{h}(1)$.

The proof of the lemma is complete.

Lemma 1.2. *Let X be an $\mathbb{R}$-seminormed vector space over some subfield R of $\mathbb{R}$. Assume that $f_1, \dots, f_N$ and g are bounded R-linear functionals on X; in symbols, $f_1, \dots, f_N, g \in X^* := L^{(m)}(X, \mathbb{R})$.*

For the inclusion

$$\{g \le 0\} \supset \bigcap_{k=1}^{N} \{f_k \le 0\}$$

to hold it is necessary and sufficient that there be $\alpha_1, \dots, \alpha_N \in \mathbb{R}_+$ satisfying

$$g = \sum_{k=1}^{N} \alpha_k f_k.$$

Proof. Let us induct on N. In other words, we have assumed that the claim is demonstrated for every collection of N functionals over each real vector space X.

To make the induction step, consider the pointwise suprema $q := f_1 \vee \dots \vee f_{N+1}$ and $p := q \vee (-g)$. Clearly, $p(x) \ge 0$ for all $x \in X$. Indeed, if one of the reals $f_k(x)$ is strictly greater than zero then so is $q(x)$. If all $f_1(x), \dots, f_{N+1}$ are negative then so is $g(x)$. Hence, $p(x) \ge -g(x) \ge 0$.

The field $\mathbb{R}$ over R admits convex analysis (cp. [4, p. 119] and [6, p. 259]). Consequently, there are positive reals γ_1 and γ_2 such that $\gamma_1 + \gamma_2 = 1$ and $\gamma_1 f - \gamma_2 g = 0$ for some f belonging to the subdifferential $\partial(q)$ of q.

If $\gamma_2 > 0$ then we are done since $\partial(q) = \operatorname{co}\{f_1, \dots, f_{N+1}\}$. If $\gamma_2 = 0$ then

$$\sum_{k=1}^{N+1} t_k f_k = 0$$

for some convex combination $t_1, \dots, t_{N+1}$ of reals. One of the coefficients $t_1, \dots, t_{N+1}$ is other than zero. For definiteness, we may and will assume that $t_{N+1} \ne 0$. Thus,

$$-f_{N+1} = \sum_{k=1}^{N} \bar{t}_k f_k$$

for some positive reals $\bar{t}_k$, $k := 1, \dots, N$.

Put $X_0 := \{f_{N+1} = 0\} = \ker(f_{N+1})$. If $x_0 \in X_0$ and $f_k(x_0) \le 0$ for all $k := 1, \dots, N$ then $g(x_0) \le 0$ by hypothesis. Therefore, by the induction assumption there are positive reals $\beta_1, \dots, \beta_N$ satisfying $h|_{X_0} = 0$ where

$$h := g - \sum_{k=1}^{N} \beta_k f_k.$$

The functionals h and f_{N+1} are bounded by hypothesis and so may be viewed as $\mathbb{R}$-linear over the completion of X. Therefore,

$$g - \sum_{k=1}^{N} \beta_k f_k = \gamma f_{N+1}$$

for some $\gamma \in \mathbb{R}$. If $\gamma \ge 0$ then we have the sought presentation for g. If $\gamma < 0$ then

$$g = \sum_{k=1}^{N} (\beta_k + |\gamma| \bar{t}_k) f_k.$$

The proof of the lemma is complete.

Note that in case of $R = \mathbb{R}$ the requirement is superfluous that all functionals under consideration are bounded.

Proof of Theorem 1.1.

(2)$\to$(1): If $B = \sum_{k=1}^{N} \alpha_k A_k$ for some positive $\alpha_1, \dots, \alpha_N$ in $\mathrm{Orth}(m(Y))$ while $bA_k x \le 0$ for $b \in \mathbb{B}$ and $x \in X$, then

$$bBx = b \sum_{k=1}^{N} \alpha_k A_k x = \sum_{k=1}^{N} \alpha_k b A_k x \le 0$$

since orthomorphisms commute and projections are orthomorphisms of $m(Y)$.

(1)$\to$(2): Consider the separated Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ over the base $\mathbb{B}$ of Y. By the Gordon Theorem [4, p. 496] the ascent $Y{\uparrow}$ of Y is $\mathscr{R}$, the field of reals inside $\mathbb{V}^{(\mathbb{B})}$.

Using the canonical embedding, we see that $X^{\wedge}$ is an $\mathscr{R}$-seminormed vector space over the standard name $\mathbb{R}^{\wedge}$ of the reals $\mathbb{R}$. Moreover, $\mathbb{R}^{\wedge}$ is a subfield and sublattice of $\mathscr{R} = Y{\uparrow}$ inside $\mathbb{V}^{(\mathbb{B})}$.

Put $f_k := A_k{\uparrow}$ for all $k := 1, \dots, N$ and $g := B{\uparrow}$. Clearly, all $f_1, \dots, f_N, g$ belong to $(X^{\wedge})^*$ inside $\mathbb{V}^{\mathbb{B}}$.

Define the finite sequence

$$f : \{1, \dots, N\}^{\wedge} \to (X^{\wedge})^*$$

as the ascent of $(f_1, \dots, f_N)$. In other words, the truth values are as follows:

$$[\![f_{k^{\wedge}}(x^{\wedge}) = A_k x]\!] = \mathbb{1}, \quad [\![g(x^{\wedge}) = Bx]\!] = \mathbb{1}$$

for all $x \in X$ and $k := 1, \dots, N$.

Put

$$b := [\![A_1 x \le 0^{\wedge}]\!] \wedge \dots \wedge [\![A_N x \le 0^{\wedge}]\!].$$

Then $bA_k x \le 0$ for all $k := 1, \dots, N$ and $bBx \le 0$ by (1).

Therefore,

$$[\![A_1 x \le 0^{\wedge}]\!] \wedge \dots \wedge [\![A_N x \le 0^{\wedge}]\!] \le [\![Bx \le 0^{\wedge}]\!].$$

In other words,

$$[\![(\forall k := 1^{\wedge}, \dots, N^{\wedge}) f_k(x^{\wedge}) \le 0^{\wedge}]\!] = \bigwedge_{k:=1,\dots,N} [\![f_{k^{\wedge}}(x^{\wedge}) \le 0^{\wedge}]\!] \le [\![g(x^{\wedge}) \le 0^{\wedge}]\!].$$

This yields

$$[\![(\forall x \in X)((\forall k := 1^\wedge, \dots, N^\wedge)\ f_k(x) \le 0^\wedge) \to g(x) \le 0^\wedge)]\!]$$
$$= \bigwedge_{x \in X} [\![((\forall k := 1^\wedge, \dots, N^\wedge)\ f_k(x) \le 0^\wedge) \to g(x^\wedge) \le 0^\wedge]\!] = \mathbb{1}.$$

Using Lemma 1.2 inside $\mathbb{V}^{(\mathbb{B})}$ and appealing to the maximum principle of Boolean valued analysis, we infer that there is a finite sequence $\alpha : \{1^\wedge, \dots, N^\wedge\} \to \mathscr{R}_+$ inside $\mathbb{V}^{(\mathbb{B})}$ satisfying

$$[\![(\forall x \in X^\wedge)\ g(x) = \sum_{k=1^\wedge}^{N^\wedge} \alpha(k) f_k(x)]\!] = \mathbb{1}.$$

Put $\alpha_k := \alpha(k^\wedge) \in \mathscr{R}_+\!\!\downarrow$ for $k := 1, \dots, N$. Multiplication by an element in $\mathscr{R}\!\!\downarrow$ is an orthomorphism of $m(Y)$. Moreover,

$$B = \sum_{k=1}^{N} \alpha_k A_k,$$

which completes the proof.

Lemma 1.1, describing the consequences of a single inequality, does not restrict the class of functionals under consideration. The analogous version of the Farkas Lemma simply fails for two simultaneous inequalities in general. Indeed, the inclusion $\{f = 0\} \subset \{g \le 0\}$ equivalent to the inclusion $\{f = 0\} \subset \{g = 0\}$ does not imply that f and g are proportional for an arbitrary subfield of $\mathbb{R}$. It suffices to look at $\mathbb{R}$ over the rationals $\mathbb{Q}$, take some discontinuous $\mathbb{Q}$-linear functional on $\mathbb{R}$ and the identity automorphism of $\mathbb{R}$. This gives grounds for the next result.

Theorem 1.2. *Let X be a real vector space and let Y be a Kantorovich space. Take A and B in $L(X, Y)$. The following are equivalent:*

(1) $(\exists \alpha \in \mathrm{Orth}(m(Y)))\ B = \alpha A$;

(2) *There is a projection $\varkappa \in \mathbb{B}$ such that*[2]

$$\{\varkappa b B \le 0\} \supset \{\varkappa b A \le 0\}; \quad \{\neg\varkappa b B \le 0\} \supset \{\neg\varkappa b A \ge 0\}$$

for all $b \in \mathbb{B}$.

Proof. Boolean valued analysis reduces the claim to the scalar case. Applying Lemma 1.1 twice and writing down the truth values, complete the proof.

In case of domination it is possible to obtain some analogs of Theorem 1.2 for multilinear forms.

Theorem 1.3. *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Given $N \in \mathbb{N}$, consider two dominated Y-valued N-linear forms A and B on X.*

There is $\alpha \in \mathrm{Orth}(Y)_+$ such that $B = \alpha A$ if and only if $\{bA \le 0\} \subset \{bB \le 0\}$ for all $b \in \mathbb{B}$.

Proof. In the scalar case this theorem is deduced in [7] as a simple corollary to the main result of [8]. These articles address the multilinear forms from a vector space over some field into the same field. The presence of domination enables us to complete the proof directly by Boolean valued interpretation along the lines of the proof of Theorem 1.1.

2. Simultaneous Sublinear Inequalities

We now turn to the Farkas Lemma for sublinear operators. Denote the set of sublinear operators from X to Y by $\mathrm{Sub}(X, Y)$. An element $P \in \mathrm{Sub}(X, Y)$ is *polyhedral*, in symbols $P \in \mathrm{PSub}(X, Y)$, provided that P is the upper envelope of finitely many linear operators; i.e. there is a finite set $\Lambda \subset L(X, Y)$ such that $P(x) = P_\Lambda(x) := \sup\{Ax \mid A \in \Lambda\}$. In case X is furnished with some Y-seminorm, we consider the set of dominated sublinear operators $\mathrm{Sub}^{(m)}(X, Y)$ and the set of dominated polyhedral sublinear operators $\mathrm{PSub}^{(m)}(X, Y)$, implying the operators whose support sets lie in $L^{(m)}(X, Y)$.

We start with two lemmas in the scalar case, the second generalizing the main result of [9].

[2] As usual, $\neg\varkappa := \mathbb{1} - \varkappa$.

Lemma 2.1. *Let X be a real vector space. Assume that $f_1, \dots, f_N \in X^{\#}$ and $p \in \operatorname{Sub}(X) := \operatorname{Sub}(X, \mathbb{R})$. For the inclusion*

$$\{p \geq 0\} \supset \bigcap_{k=1}^{N} \{f_k \leq 0\}$$

to hold it is necessary and sufficient that there be $\alpha_1, \dots, \alpha_N \in \mathbb{R}_+$ satisfying

$$(\forall x \in X)\ p(x) + \sum_{k=1}^{N} \alpha_k f_k(x) \geq 0.$$

Proof. Sufficiency is obvious and we will prove necessity. To this end put $H := \bigcap_{k=1}^{N} \{f_k \leq 0\}$. Clearly, H is a (convex) cone in X. By hypothesis, $p(x) \geq 0$ for all $x \in H$. By separation (cp. [4, 3.2.16]), there is $l \in \partial(p)$ such that $l(h) \geq 0$ for all $h \in H$. By the Farkas Lemma $-l = \sum_{k=1}^{N} \alpha_k f_k$ for some positive $\alpha_1, \dots, \alpha_N$, which completes the proof.

Lemma 2.2. *Let X be a real vector space. Assume that $p_1, \dots, p_N \in \operatorname{PSub}(X) := \operatorname{PSub}(X, \mathbb{R})$ and $p \in \operatorname{Sub}(X)$. The following are equivalent:*

(1) $\{p \geq 0\} \supset \bigcap_{k=1}^{N} \{p_k \leq 0\}$;

(2) *There are $\alpha_1, \dots, \alpha_N \in \mathbb{R}_+$ such that*

$$(\forall x \in X)\ p(x) + \sum_{k=1}^{N} \alpha_k p_k(x) \geq 0.$$

Proof. By hypothesis there are finite subsets $\Lambda_1, \dots, \Lambda_N$ of $X^{\#}$ such that $p_k = p_{\Lambda_k}$ for $k := 1, \dots, N$. Let Λ be the disjoint union of all Λ_k for $k := 1, \dots, N$. Clearly,

$$\bigcap_{k=1}^{N} \{p_k \leq 0\} = \bigcap_{\lambda \in \Lambda} \{\lambda \leq 0\}.$$

By Lemma 2.1 there are some $(\beta_\lambda)_{\lambda \in \Lambda} \subset \mathbb{R}_+$ such that for all $x \in X$ we have

$$\begin{aligned} 0 \leq p(x) + \sum_{\lambda \in \Lambda} \beta_\lambda \lambda(x) &= p(x) + \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k} \beta_\lambda \lambda(x) \\ &\leq p(x) + \sum_{k=1}^{N} \sum_{\lambda \in \Lambda_k} \beta_\lambda p_k(x) = p(x) + \sum_{k=1}^{N} \Bigl(\sum_{\lambda \in \Lambda_k} \beta_\lambda \Bigr) p_k(x). \end{aligned}$$

Putting $\alpha_k := \sum_{\lambda \in \Lambda_k} \beta_\lambda$ for $k := 1, \dots, N$, we complete the proof.

We proceed now to the operator case.

Lemma 2.3. *Let X be a vector space over some subfield R of the reals $\mathbb{R}$. Assume that $f \in X^{\#}$ and $p \in \operatorname{Sub}(X)$. For the inclusion $\{p \geq 0\} \supset \{f \leq 0\}$ to hold it is necessary and sufficient that there be $\alpha \in \mathbb{R}_+$ satisfying $(\forall x \in X)\ p(x) + \alpha f(x) \geq 0$.*

Proof. We argue as in Lemma 2.1 appealing to Lemma 1.1 instead of the Farkas Lemma.

Theorem 2.1. *Let X be a real vector space, and let Y be a Kantorovich space. Assume that $A \in L(X, Y)$ and $P \in \operatorname{Sub}(X, Y)$. For the inclusion $\{bP \geq 0\} \supset \{bA \leq 0\}$ to hold for all $b \in \mathbb{B}$ it is necessary and sufficient that there be $\alpha \in \operatorname{Orth}(m(Y))_+$ satisfying $(\forall x \in X)\ P(x) + \alpha Ax \geq 0$.*

Proof. The claim follows from Lemma 2.3 by Boolean valued interpretation.

Theorem 2.2. *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $P_1, \dots, P_N \in \mathrm{PSub}^{(m)}(X, Y)$ and $P \in \mathrm{Sub}^{(m)}(X, Y)$. The following are equivalent:*

(1) *For all $b \in \mathbb{B}$, the sublinear operator inequality $bP(x) \geq 0$ is a consequence of the simultaneous polyhedral sublinear operator inequalities $bP_1(x) \leq 0, \dots, bP_N(x) \leq 0$, i.e.,*

$$\{bP \geq 0\} \supset \{bP_1 \leq 0\} \cap \dots \cap \{bP_N \leq 0\}.$$

(2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))$ such that*

$$(\forall x \in X)\ P(x) + \sum_{k=1}^{N} \alpha_k P_k(x) \geq 0.$$

Proof. The demonstration of the claim proceeds along the lines of Lemma 2.2 on appealing to Theorem 1.1 in place of the Farkas Lemma.

Let us shortly address the case of linear inequalities with inexact data in the spirit of interval analysis.

Assume additionally that X is a vector lattice. Recall that an *interval operator* $\mathbf{T}$ from X to Y is simply an order interval $[\underline{T}, \overline{T}]$ in the space of order bounded operators $L^{(r)}(X, Y)$. By default $\underline{T} \leq \overline{T}$. The interval equation $\mathbf{B} = \mathfrak{X}\mathbf{A}$ has a *weak interval solution* provided that there is a solution to $B = \mathfrak{X}A$ for some $A \in \mathbf{A}$ and $B \in \mathbf{B}$. Other types of solution are also considered. To illustrate the mechanism behind the research into these matters, we will focus only on weak interval solution of interval operator equations, balancing room and ideas. Every relevant detail in finite dimensions can be extracted from [10, Chapters 2 and 3].

To each interval operator $\mathbf{T}$ we associate the sublinear operator $P_{\mathbf{T}}$. Note that $\mathbf{T} = [0, \overline{T} - \underline{T}] + \underline{T}$. Given $x \in X$, we thus have

$$P_{\mathbf{T}}(x) = P_{[0,\overline{T}-\underline{T}]}x + \underline{T}x = (\overline{T} - \underline{T})x_+ + \underline{T}x = \overline{T}x_+ - \underline{T}x_-.$$

Call $\mathbf{T}$ an *adapted* interval operator provided that $P_{\mathbf{T}} \in \mathrm{PSub}(X, Y)$; i.e., $\mathbf{T}$ has finitely many o-extreme points[3] or, in other words, $\overline{T} - \underline{T}$ is the sum of finitely many disjoint addends. Obviously, if X and Y are finite-dimensional then every interval operator from X to Y is adapted. Finally, put $\sim (x) := -x$ for all $x \in X$.

Lemma 2.4. *Let X be a vector lattice. Assume that $\mathbf{f}$ and $\mathbf{g}$ are interval functionals, with $\mathbf{f}$ adapted. The following are equivalent:*

(1) *The interval equation $\mathbf{g} = \alpha\mathbf{f}$ has a weak interval solution $\alpha \in \mathbb{R}_+$.*

(2) *If $\mathfrak{f}^{\sim} := P_{\mathbf{f}} \circ \sim$ and $\mathfrak{g} := P_{\mathbf{g}}$ then $\{\mathfrak{f}^{\sim} \leq 0\} \subset \{\mathfrak{g} \geq 0\}$.*

Proof. The sublinear functional $\mathfrak{f}$ is polyhedral. Therefore, (2) amounts by Lemma 2.2 to the existence of $\alpha \in \mathbb{R}_+$ such that $\mathfrak{g}(x) + \alpha\mathfrak{f}(-x) \geq 0$ for all $x \in X$. A sublinear functional is positive if and only if it has a positive supporting operator. In other words, (2) is equivalent to the existence of some positive α satisfying $0 \in (\mathbf{g} - \alpha\mathbf{f})$.

Theorem 2.3. *Let X be a vector lattice, and let Y be a Kantorovich space. Assume that $\mathbf{A}_1, \dots, \mathbf{A}_N$ are adapted interval operators and $\mathbf{B}$ is an arbitrary interval operator in the space of order bounded operators $L^{(r)}(X, Y)$. The following are equivalent:*

(1) *The interval equation*

$$\mathbf{B} = \sum_{k=1}^{N} \alpha_k \mathbf{A}_k$$

has a weak interval solution $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(Y)_+$.

(2) *For all $b \in \mathbb{B}$ we have*

$$\{b\mathfrak{B} \geq 0\} \supset \{b\mathfrak{A}_1^{\sim} \leq 0\} \cap \dots \cap \{b\mathfrak{A}_N^{\sim} \leq 0\},$$

where $\mathfrak{A}_k^{\sim} := P_{\mathbf{A}_k} \circ \sim$ for $k := 1, \dots, N$ and $\mathfrak{B} := P_{\mathbf{B}}$.

Proof. It suffices to repeat the argument of Lemma 2.4 and appeal to Theorem 2.2.

[3] Cp. [4, p. 95].

3. Simultaneous Inhomogeneous Inequalities

We turn now to inhomogeneous inequalities.

Lemma 3.1. *Let X be a vector space over a subfield R of $\mathbb{R}$, while $f, g \in X^{\#}$ and $u, v \in \mathbb{R}$. Assume that the inhomogeneous inequality $f(x) \le u$ is consistent.*

The inclusion $\{g \le v\} \supset \{f \le u\}$ holds if and only if there is $\alpha \in \mathbb{R}_+$ satisfying $g = \alpha f$ and $v \ge \alpha u$.

Proof. Demonstrate necessity since sufficiency is obvious.

Put $p(x) := (f(x) - u) \vee (v - g(x))$ for all $x \in X$. By hypothesis $(\forall x \in X)\ p(x) \ge 0$. Consequently, there are positive reals γ and δ such that $\gamma + \delta = 1$ while $\gamma g - \delta f = 0$ and $\gamma v \ge \delta u$. If $\gamma > 0$ then we put $\alpha := \delta/\gamma$. If $\gamma = 0$ then $\delta = 1$. Therefore, $f = 0$. Considering consistency, we see that $v \ge 0$ and $g = 0$. So we may put $\alpha := 0$.

Lemma 3.2. *Let X be an $\mathbb{R}$-seminormed vector space over a subfield R of $\mathbb{R}$. Assume further that $f_1, \dots, f_N, g \in X^*$ and $u_1, \dots, u_N, v \in \mathbb{R}$, while the simultaneous inhomogeneous inequalities $\{f_k \le u_k\}$, with $k := 1, \dots, N$, are consistent.*

The inhomogeneous inequality $g(x) \le v$ is a consequence of the simultaneous inhomogeneous inequalities under study if and only if there are $\alpha_1, \dots, \alpha_N \in \mathbb{R}_+$ satisfying $g = \sum_{k=1}^N \alpha_k f_k$ and $v \ge \sum_{k=1}^N \alpha_k u_k$.

Proof. As above we need to show only the necessity of the condition.

As usual, we will use the Hörmander transform [4, p. 28]. Consider the vector space $X \times \mathbb{R}$ over R and furnish it with the natural product seminorm. Given $(x, t) \in X \times \mathbb{R}$ put $\bar{f}_k(x, t) := f_k(x) - t u_k$, $\bar{g}(x, t) := g(x) - t v$, and $\tau(x, t) := -t$. Take

$$(x, t) \in \{\tau \le 0\} \cap \bigcap_{k=1}^{N} \{\bar{f}_k \le 0\}.$$

If $t > 0$ then $u_k \ge f_k(x/t)$ for $k := 1, \dots, N$. Hence, $g(x/t) \le v$ by hypothesis. In other words, $(x, t) \in \{\bar{g} \le 0\}$. If $t = 0$ then take some solution $\bar{x}$ of the simultaneous inhomogeneous inequalities such that $g(\bar{x}) \le v$. Take $x \in K := \bigcap_{k=1}^N \{f_k \le 0\}$. Then $x + \bar{x} \in \bigcap_{k=1}^N \{f_k \le u_k\}$. Therefore, $x \in \{g \le v - g(\bar{x})\}$; i.e., the R-linear functional g is bounded above on the convex cone K. Hence, g assumes negative values on K. By Lemma 1.2 there are positive reals $\alpha_1, \dots, \alpha_N, \beta$ satisfying

$$\bar{g} = \beta\tau + \sum_{k=1}^{N} \alpha_k \bar{f}_k.$$

Clearly, we are done with these $\alpha_1, \dots, \alpha_N$, completing the proof of the lemma.

Note that the proof of Lemma 3.2 preserves verbatim if X is a real vector space and all functionals under study belong to $X^{\#}$.

Theorem 3.1. *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume given some dominated operators $A_1, \dots, A_N, B \in L^{(m)}(X, Y)$ and elements $u_1, \dots, u_N, v \in Y$. Assume further that the simultaneous inhomogeneous operator inequalities $A_1 x \le u_1, \dots, A_N x \le u_N$ are consistent.*[4] *Then the following are equivalent:*

(1) *For all $b \in \mathbb{B}$ the inhomogeneous operator inequality $bBx \le bv$ is a consequence of the simultaneous inhomogeneous operator inequalities under consideration; i.e.,*

$$\{bB \le bv\} \supset \{bA_1 \le bu_1\} \cap \dots \cap \{bA_N \le bu_N\}.$$

(2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))$ satisfying*

$$B = \sum_{k=1}^{N} \alpha_k A_k; \quad v \ge \sum_{k=1}^{N} \alpha_k u_k.$$

[4]The term *feasible* is also in common parlance.

Proof. As above, check only the implication (1)→(2). Repeating the proof of Theorem 1.1, put $f_k := A_k{\uparrow}$ for $k := 1, \dots, N$ and $g := B{\uparrow}$. Clearly, $f_1, \dots, f_N, g \in (X^{\wedge})^*$ inside $\mathbb{V}^{\mathbb{B}}$. Define the finite sequences

$$f : \{1, \dots, N\}^{\wedge} \to (X^{\wedge})^*; \quad u : \{1, \dots, N\}^{\wedge} \to \mathscr{R}$$

as the ascents of the families $(f_1, \dots, f_N)$ and $(u_1, \dots, u_N)$.

Obviously, $\{g \le v\}$ is a consequence of the simultaneous inhomogeneous inequalities $\{f(k^{\wedge}) \le u(k^{\wedge})\}$ for $k \in \{1, \dots, N\}^{\wedge}$ inside $\mathbb{V}^{\mathbb{B}}$. It is easy that all inequalities under consideration are consistent inside $\mathbb{V}^{\mathbb{B}}$. Therefore, by Lemma 3.2 there is a sequence $\alpha : \{1^{\wedge}, \dots, N^{\wedge}\} \to \mathscr{R}_+$ inside $\mathbb{V}^{(\mathbb{B})}$ satisfying

$$[\![(\forall x \in X^{\wedge})\ g(x) = \sum_{k=1^{\wedge}}^{N^{\wedge}} \alpha(k) f(k)(x)]\!] = \mathbb{1};$$

$$[\![v \ge \sum_{k=1^{\wedge}}^{N^{\wedge}} \alpha(k) u(k)]\!] = \mathbb{1}.$$

Putting $\alpha_k := \alpha(k^{\wedge}) \in \mathscr{R}_+{\downarrow}$ for $k := 1, \dots, N$, we complete the proof.

Inhomogeneous polyhedral inequalities are now in order. By way of illustration we will consider only two particular cases.

Theorem 3.2. *Let X be a real vector space, and let Y be a Kantorovich space. Assume that $u, v \in Y$ and $A, B \in L(X, Y)$. Assume further that the inequality $Ax \le u$ is consistent.*

For the inclusion $\{bB \ge bv\} \supset \{bA \le bu\}$ to hold for all $b \in \mathbb{B}$ it is necessary and sufficient that there be $\alpha \in \mathrm{Orth}(m(Y))_+$ satisfying $B = \alpha A$ and $v \ge \alpha u$.

Proof. This is a straightforward Boolean valued interpretation of Lemma 3.1.

In applications we encounter inhomogeneous matrix inequalities over various finite-dimensional spaces (cp. [11, Proposition 2.1]).

Theorem 3.3. *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A \in L^{(m)}(X, Y^s)$, $B \in L^{(m)}(X, Y^t)$, $u \in Y^s$ and $v \in Y^t$, where s and t are some naturals, while the inhomogeneous inequality $Ax \le u$ is consistent.*

The following are equivalent:

(1) The inclusion $\{bB \le bv\} \supset \{bA \le bu\}$ holds for all $b \in \mathbb{B}$; i.e., the inhomogeneous operator inequality $bBx \le bv$ is a consequemce of the inhomogeneous inequality $bAx \le bu$.

(2) There is some $s \times t$ matrix with entries positive orthomorphisms of $m(Y)$ such that $B = \mathfrak{X}A$ and $\mathfrak{X}u \le v$ for the corresponding linear operator $\mathfrak{X} \in L_+(Y^s, Y^t)$.

Proof. We will check that (1)→(2). To this end, put $A_k := \mathrm{Pr}_k\, A$, $u_k := \mathrm{Pr}_k\, u$, $B_l := \mathrm{Pr}_l\, B$, and $v_l := \mathrm{Pr}_l\, v$ for the appropriate coordinate projections. For all $l := 1, \dots, t$ and $b \in \mathbb{B}$ we then have

$$\{bB_l \le bv_l\} \supset \{bB \le bu\} \supset \bigcap_{k=1}^{s} \{bA_k \le bu_k\}.$$

By Theorem 3.1 there are positive $\alpha_{lk} \in \mathrm{Orth}(m(Y))$ satisfying

$$B_l = \sum_{k=1}^{s} \alpha_{lk} A_k; \quad v_l \ge \sum_{k=1}^{s} \alpha_{lk} u_k.$$

Balancing between completeness and conciseness, we will finish with the inhomogeneous operator inequalities with complex scalars.

Theorem 3.4. *Let X be a Y-seminormed complex vector space, with Y a Kantorovich space. Assume given some $u_1, \dots, u_N, v \in Y$ and dominated operators $A_1, \dots, A_N, B \in L^{(m)}(X, Y_{\mathbb{C}})$ from X into the complexification $Y_{\mathbb{C}} := Y \otimes iY$ of Y.*[5] *Assume further that the simultaneous inhomogeneous inequalities $|A_1 x| \le u_1, \dots, |A_N x| \le u_N$ are consistent. Then the following are equivalent:*

(1) *For all $b \in \mathbb{B}$ and $x \in X$ the inhomogeneous inequality $b|Bx| \le bv$ is a consequence of the simultaneous inhomogeneous inequalities under consideration, i.e.,*

$$\{b|B| \le bv\} \supset \{b|A_1| \le bu_1\} \cap \dots \cap \{b|A_N| \le bu_N\}.$$

(2) *There are complex orthomorphisms $c_1, \dots, c_N \in \operatorname{Orth}(m(Y)_{\mathbb{C}})$ satisfying*

$$B = \sum_{k=1}^{N} c_k A_k; \quad v \ge \sum_{k=1}^{N} |c_k| u_k.$$

Proof. As above, we will check the implication (1)→(2).

Arguing as in the proof of Theorem 1.1 while putting $f_k := A_k{\uparrow}$ for $k := 1, \dots, N$ and $g := B{\uparrow}$, reduce the matter to the scalar case. Clearly, $f_1, \dots, f_N, g \in (X^\wedge)^* := L^{(m)}(X^\wedge, \mathscr{C})$, with $\mathscr{C}$ the complexes inside $\mathbb{V}^{\mathbb{B}}$. Define the finite sequences

$$f : \{1, \dots, N\}^\wedge \to (X^\wedge)^*, \quad u : \{1, \dots, N\}^\wedge \to \mathscr{C}$$

as the ascents of the families $(f_1, \dots, f_N)$ and $(u_1, \dots, u_N)$.

By transfer the inequality $\operatorname{Re}(g(x)) \le v$ is a consequence of the consistent simultaneous real inequalities

$$\operatorname{Re}(f(k))(x) \le u(k); \quad \operatorname{Im}(f(k))(x) \le u(k)$$

for all $x \in X^\wedge$ and $k \in \{1, \dots, N\}^\wedge$. By Lemma 3.2 there are sequences

$$\alpha : \{1^\wedge, \dots, N^\wedge\} \to \mathscr{R}_+; \quad \beta : \{1^\wedge, \dots, N^\wedge\} \to \mathscr{R}_+$$

inside $\mathbb{V}^{(\mathbb{B})}$ satisfying

$$[\![(\forall x \in X^\wedge) \ \operatorname{Re}(g(x)) = \sum_{k=1^\wedge}^{N^\wedge} \big(\alpha(k) \operatorname{Re}(f(k)(x)) + \beta(k) \operatorname{Im}(f(k)(x))\big)]\!] = \mathbb{1};$$

$$[\![v \ge \sum_{k=1^\wedge}^{N^\wedge} (\alpha(k) + \beta(k)) u_k]\!] = \mathbb{1}.$$

Put $c_k := \alpha(k^\wedge) - i^\wedge \beta(k^\wedge) \in \mathscr{C}{\downarrow}$ for $k := 1, \dots, N$.

It is easy that, given $a, b \in \mathbb{R}$ and two $\mathbb{C}$-linear functionals l and m on X, for $c := a - ib$ we have $l = cm$ if and only if $\operatorname{Re}(l(x)) = a \operatorname{Re}(m(x)) + b \operatorname{Im}(m(x))$ for all $x \in X$. Moreover, $|c| \le |a| + |b|$. Using this observation and descending, we complete the proof of the theorem.

In closing we note that the theory of linear inequalities favors stating the various versions of the Farkas Lemma in terms of mutually exclusive possibilities (for instance, see [2] and [12, Chapter 4]). By way of illustration we give only the relevant reformulation of Theorem 1.1.

Theorem of the Alternative. *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A_1, \dots, A_N$ and B belong to $L^{(m)}(X, Y)$.*

Then one and only one of the following holds:

(1) *There are $x \in X$ and $b, b' \in \mathbb{B}$ such that $b' \le b$ and*

$$b'Bx > 0, bA_1 x \le 0, \dots, bA_N x \le 0.$$

(2) *There are $\alpha_1, \dots, \alpha_N \in \operatorname{Orth}(m(Y))_+$ such that*

$$B = \sum_{k=1}^{N} \alpha_k A_k.$$

Proof. The inequality $bBx \le 0$ is false if and only if there is $\mathfrak{b} \in \mathbb{B}$ satisfying $\mathfrak{b}bBx > 0$.

[5] Cp. [3, p. 338].

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