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RESHETNYAK'S WORLDLINE AND MEMES

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Abstract—This is a brief overview of the worldline and memes of Yurii Reshetnyak (1929–1921), the Russian leader of geometric analysis.

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Main Achievements

The scientific contributions by Reshetnyak embrace an exceptionally broad area of modern mathematics and his creative style is marked with remarkable depth and originality. His papers are rich in ideas and amazing methods of penetration into the core and essence of the problems under study.

Reshethyak authored many fundamental results in geometry, function theory, variational calculus, and related mathematical trends that reside at the juncture of analysis and geometry. He was a founding father of a few new trends in modern mathematics. One of these is the theory of mappings with bounded distortion which present a multidimensional real analog of analytic functions and a "non-single-list" generalization of conformal space mappings.

Reshetnyak laid grounds of nonlinear potential theory and suggested a toolkit for the theory which includes the concept of (l, p)-capacity. This theory led to significant progress in the theory of functions with distributional derivatives. The ideas and results by Reshetnyak became the basis of research of his scientific school that contains a few dozens of highly qualified researchers.

The authority of Siberian mathematics in analysis and geometry is to a large extent due to the personal contributions of Reshetnyak.

Life's Itinerary

Reshetnyak was born in Leningrad on September 26, 1929. After finishing a secondary school in 1947, he was admitted to the Mathematics and Mechanics Department of Leningrad State University. He graduated in four years and was promoted to a postgraduate. Aleksandr Alexandrov became the supervisor of Reshetnyak. It was the years that determined the fruitful cooperation of life-lasting scientific cooperation between Alexandrov and Reshetnyak.

In 1954 Reshetnyak maintained his kandidat thesis *On the Length and Swerve of a Curve and on the Area of a Surface*. He then was appointed to the Leningrad Department of the Steklov Mathematical Institute.

The year of 1957 is the date when the USSR authorities agreed with the suggestion of Russian scientists to arrange the new scientific organization in the center of Russian, the Siberian Division of the Academy of Sciences. Reshetnyak was among the first young scientists who decided to joint the new epochal venture of Mikhail Lavrentiev, Sergei Sobolev, and Sergei Khristianovich. It was already at the end of 1957 that Reshetnyak and his family moved to Novosibirsk. He joined the staff of the new Institute of Mathematics, now the Sobolev Institute of Mathematics. Here he wrote most of his fundamental papers and books and passed the way from a novice to an established full member of the Russian Academy of Sciences. Here he developed his original style of research at the juncture of analysis and geometry. Here,

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in Siberia, Reshetnyak elaborated his virtuoso mathematical techniques. Here Reshetnyak defended his doctoral thesis on *Isothermal Coordinates in Two Dimensional Manifolds of Bounded Curvature* at the United Scientific Council of the Siberian Division of the Academy of Sciences in 1960. Here he held the Chair of Mathematical Analysis of Novosibirsk State University for more than a half-century. Here he passed away on December 17, 2021.

Contribution to Geometry

Reshetnyak started his research within the areas headed by Alexandrov who plunged in convexity and geometry in the large. The most impressive results of Reshetnyak in the extrinsic geometry of convex surfaces are connected with the Lyapunov Convexity Theorem. From the scratch this theorem had raised the problem of describing the compact convex sets in finite-dimensional real spaces which serve as the ranges of diffuse measures. These compacta are known in the modern geometrical literature as *zonoids*. The Minkowski sums of finitely many straight line segments are distinguished among zonoids. These sets, called *zonotopes*, fill a convex cone in the space of compact convex sets, and the cone of zonotopes is dense in the closed cone of all zonoids. The first description of the ranges of diffuse vector measures in the Lyapunov Convexity Theorem was firstly found by Chuikina practically in the modern terms; see [1, 2]. Soon after that her result was somewhat supplemented and simplified by Glivenko in [3]. The zonotopes of the present epoch were called *parallelohedra* those days.

Reshetnyak and Zalgaller described zonoids as the results of mixing the linear elements of a rectifiable curve in a finite-dimensional space in 1954; see [4]. In the same paper they suggested a new proof of the Lyapunov Convexity Theorem and demonstrated that zonotopes are precisely those convex polyhedra whose two-dimensional faces have centers of symmetry. Analogous results were obtained by Bolker only fifteen years later in 1969; see [5]. Unfortunately, Reshetnyak and Zalgaller's paper appeared in English only in 2012. As regards zonoids, also see [6].

Reshetnyak's ideas in combining measure theory with geometry extended the approach by Alexandrov and led to their joint book [7]. The plot of the book belongs to Alexandrov who developed the methods in convexity which utilize the approximation of surfaces by polyhedra. He suggested that the same approach would lead to the theory of irregular curves. The main concepts of the theory of curves in differential geometry are Alexandrov's theory of curves with integral curvature and integral torsion that in the classical case are the integrals of curvature and torsion over the arclength. The irregular case starts with inscribed broken lines and passage to the limit, which involves obstacles in proving the existence of the due limits. The integral curvature of a curve was defined as the upper limit of the inscribed broken lines but no proof was available that this is the usual limit, Reshetnyak obviated the difficulty on using some analytical relations from integral geometry. The latter relations enabled him to solve other Alexandrov's problems of the theory of curves. The book [7] has an intriguing history. The official bibliography [8, p. 1117] contained the reference to *The General Theory of Irregular Curves in Three-Dimensional Space* published in 1962, but in fact the book was still a manuscript. Reshetnyak told that he had prepared five drafts to Alexandrov who disapproved them all...

Another bulk of Reshetnyak's contribution to geometry is connected with studying two-dimensional manifolds of bounded curvature. The classical differential geometry addresses sufficiently regular objects, i.e., those given by some functions having a good deal of continuous derivatives. In many situations this constraint distorts the geometrical essence of an object. Alexandrov became aware of that predicament while plunging into convexity. So the problem had arisen to propound some geometry of irregular curves and surfaces that will accompany the theory of curves and surfaces in differential geometry. In particular, Alexandrov posed and solved the problem of constructing the intrinsic geometry of two-dimensional irregular surfaces at the beginning of the 1950s.

The main result by Reshetnyak in this area was obtained in 1953 as the theorem of conformal presentation of two-dimensional manifolds of bounded curvature. He proved that it is possible to introduce a coordinate system in a neighborhood of each point of a manifold under study such that the metric of the manifold will be given by the linear element of the form

$$ds^2 = \lambda(x, y)(dx^2 + dy^2),$$

which is an *isothermal coordinate system*, where λ presents the logarithm of the difference of two subharmonic functions. Reshetnyak continued the study of the class of manifolds by Alexandrov on using their conformal presentation in Novosibirsk. He proved in particular how to express the geometric characteristics of such manifolds through λ. Rather recently, the papers by Reshetnyak on these matters have appeared as the book [9], where the term *submetric* was suggested for the metric discovered by Reshetnyak.

Contribution to Analysis

Reshetnyak opened up new directions in mathematics on the boundary between analysis and geometry. One of them is the theory of nonlinear capacity whose central concept of (l, p)-capacity is his invention which led to essential advances in the theory of functions with generalized derivatives. He proved the general theorem of almost everywhere differentiability of the functions of Sobolev spaces which surpassed all available results in this matter; see [10]. Also he established rather popular theorem on semicontinuity in variational calculus in [11].

Another direction of research Reshetnyak opened up was the theory of spatial mappings with bounded distortion. These mappings represent a deep and far-reaching generalization of quasiconformal space mappings; see [12]. The latter can be characterizes as homomorphisms of the Sobolev space W_n^1 such that

$$\|f'(x)\|^n \leq K \det(f'(x)),$$

with $K \geq 1$ the quasiconformality coefficient of f. Here $f'(x)$ stands for the Jacobian of f, while the left-hand side contains the matrix norm

$$\|f'(x)\| = \sup_{|h|\leq 1} |f'(x)h|.$$

Reshetnyak proved in 1950 that every quasiconformal mapping is Hölder continuous by elaborating the Nirenberg techniques on using the isoperimetric property of a ball, which yielded the exact value of K.

The Liouville Theorem asserts that a quasiconformal mapping with conformality coefficient 1 is a Möbius transformation, i.e., a composition of translations, similarities, orthogonal transformations, and inversions. The classical prof of this theorem assumes that the mapping is of class C^3. Reshetnyak demonstrated that if f is 1-quasiconformal then

$$\lambda(x) := \|f'(x)\|^n$$

is a subharmonic function, hence, upper bounded. Therefore, λ has square integrable first distributional derivatives. Since λ is a combination of first derivatives, Reshetnyak proved in 1959 that f has second derivatives, implying that f is of class C^3, which reduces the matter to the classical situation.

Lavrentiev posed the problem of stability in the Liouville Theorem, which required to demonstrate that if the conformality coefficient K of f is close to 1 then f is close to a Möbius transformation. Using the above proof of the Liouville Theorem, Reshetnyak provided a partial solution to Lavrentiev's problem. The further research addressed mappings with bounded distortion. Reshetnyak introduced the concepts by annihilating the requirement that the mappings are homomorphisms.

Note that the distinction between a quasiconformal mapping and a mapping with bounded distortion are approximately similar to the distinction between one-sheeted and arbitrary analytic functions. Reshetnyak inspected the topological properties of the mappings with bounded distortion. He proved that such mapping f is a discrete open mapping; i.e., each open set is sent to an open set, and for each $x \in \mathrm{dom}(f)$ there is $\delta > 0$ for which from $|x' - x| < \delta$ it follows that

$$f(x') \neq f(x).$$

Recall that $x \in \operatorname{dom}(f)$ is a branching point of f provided that f is not a homomorphism in any neighborhood of x. The $(n-2)$-dimensional set of branching points of f is negligible as showed by Reshetnyak.

Reshetnyak involved differential equations to studying functions with bounded distortion. The components of an analytic function of a single variable enjoy the Laplace equation. Reshetnyak's approach presents the components of a mapping with bounded distortion as solutions to some differential equations of elliptic type. Reshetnyak's method was developed in the end of the 1960s and revealed the connection between the theory of mappings with bounded distortion and nonlinear potential theory. Recall that the theory of quasiconformal mappings is closely tied with the theory of functions with distributional derivatives. Capacity is one of the principal tools of the theory of quasiconformal mappings. The conception of capacity proves to be very efficient in inspecting the properties of the functions in the Sobolev space W_p^l in the wake of real function theory. The great progress in using capacity for studying W_p^l is due to Reshetnyak who proposed nonlinear potential theory as a generalization of the classical version. The main results on these matters appeared in the Reshetnyak book [13].

One of the most important results by Reshetnyak was his solution of the Lavrentiev problem of stability of quasiconformal space mappings in the Liouville Theorem. The solution was given in 1975. Reshetnyak proved that if the quasiconformality coefficient K of f tends to 1 then the summability exponent of the derivatives of f is

$$p \geq \frac{C}{K-1},$$

and this estimate is optimal. He also established that if K of f is close to 1, then not only f is closed to a Möbius transformation but also the derivatives of f are close to the respective derivatives of the transformation and their deviation in the appropriate integral norm has order $C(K-1)$. Thus, Reshetnyak obtained not only the solution to the Lavrentiev problem but also discovered new properties of quasiconformal mappings as K tends to 1. The main results on these matters appeared in the Reshetnyak book [14].

At present the theory of quasiconformal space mappings has transformed in *geometric analysis* the area studying quasiconformal mappings in the so-called Carnot–Carathéodory spaces with a rather bizarre geometry. These spaces appear naturally in inspecting subelliptic partial differential equations and non-isotropic Sobolev classes. These matters are studied by the new generations of the students and coworkers of Reshetnyak.

Sometimes we hear that the importance of a scientific theory is determined by the numbers of its supporters. This quantitative approach is similar to the bureaucratic cipher games like Hirsch and impact factors. The future of science in developing the system of its concepts. Science, as well as mathematics, does exists as a system od developing concepts. It is concepts that preserve any facts, machines, and technologies as any gadget or program is dead without description. Reshetnyak enriched the treasure trove of science by developing the foundations of nonlinear potential theory and geometric analysis.

Contribution to Education

It is hard to overestimate Reshetnyak's contribution to teaching thee new generations of mathematicians. More than a half-century he held the chair od mathematical analysis of the Mechanics and Mathematics Department of Novosibirsk State University. The course of calculus is the basis of the professional training of a mathematician in every university of the world. Mathematical analysis in Siberia is delivered along the lines drawn by Reshetnyak. The Lebesgue integral, limits and series in metric spaces, as well as line and surface integrations by means of exterior differential forms are the obligatory ingredients of lectures on mathematical analysis irrespective of the lecturer. All these topics were firstly implanted into curriculum by Reshetnyak at the beginning of the 1960s. Reshetnyak was not a rhetorician but all alumni who listened his lectures appraise Reshetnyak as amazing lecturer. Reshetnyak's lectures were filled with the inspiration, beauty, and enchantment of mathematics, which witnessed the unique mathematical gift of Reshetnyak who shared it with his students lavishly.

The contribution of Reshetnyak to education was crowned with his course in four books [15] which remains the main textbook of calculus in the universities of Siberia.

Memes of Academism

Reshetnyak belonged to the exceptionally scarce groups of academicians who can be called peacemakers in the sense of the Sermon on the Mount as he had no hatred to anyone but never complied with injustice, meritocracy, slander, plagiarism, and other abominations of academic life.

A scientist by belief is broader than the narrow frames of speciality, he is an adversary of any violations of the academic standards of freedom and broad-mindedness, a staunch enemy of pseudoscience, a warrior against wouldbe-scientists. Reshetnyak always defended his teachers, forerunners, students, and coworkers from persecution and libel irrespective of any external pressure.

The beginnings of Gorbi's perestroika were marked with unscrupulous attacks against many noble scientists. Reshetnyak valiantly defended his colleagues; see [16–19].

Reshetnyak vigorously struggled with pseudoscience that proliferates in Russia in the 2000s; see [20, 21].

Reshetnyak was a professional scholar of international calibre. There centuries of farewell rituals with the persons of that status. It is customary to recall the signposts of his life, the landmarks of his contribution to science, the awards and best students and followers. The tradition prevails and so makes sense. However, it is not the formalities that remain in the memory of those who were close to and fond of the deceased. The irreplaceable loss and inescapable sorrow neighbor indelible reminiscences and the feelings of the gone love and kindness that warm souls of the alive.

Thousands of persons keep good memories of Reshetnyak, an inspired teacher who revealed the beauty and strength of mathematics to them. Reshetnyak followed the ancient pattern of a genuine scholar: He did for the future generations that what the previous generations had done for him.

Those who were close to Reshetnyak and knew him will always keep in memory his personal particular features. He was resilient rather than soft, steadfast rather than stubborn, rejecting rather than angry, withdrawn rather than spiteful, just rather than kind-heart, a companion rather than a passer-by, a friend, rather than a dude, a professional rather than a volunteer, genuine rather than would-be... Yurii Reshetnyak is an epoch in Siberian science, a moral and spiritual exemplar for all who had happiness and honor to knew him.

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CONFLICT OF INTEREST

As author of this work, I declare that I have no conflicts of interest.

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