# SYMMETRY MEASURES

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The reconstruction of a superminimal measure of symmetry is considered on the basis of its values on indecomposable polyhedra.

0°. Grünbaum has formulated the problem of describing symmetry measures that satisfy the so-called condition of superminimality [1]. In this note we establish a connection between this type of problem and Choquet's theory, i.e., the theory of integral representations of compact sets [2]. In particular, we present a complete (in a certain sense) description of classes of symmetry measures that satisfy a condition that is stronger than superminimality, namely the condition of superadditivity with respect to Blaschke addition. The choice of this addition, and not of standard addition of surfaces in the sense of Minkowski, is due primarily to Shephard's theorem [3] which asserts the absence of nontrivial approximating classes in a cone of more than two-dimensional convex compact sets. This signifies that it is hopeless to try and classify symmetry measures on the basis of the values assumed by them on the set of indecomposable polyhedra.

1°. Let $\mathfrak{B}_n$ be a set of convex compact sets in $R^n$, and let $\mathfrak{B}_n/R^n$ be a factor of $\mathfrak{B}_n$ with respect to equivalence defined as equality of surfaces to within parallel translation. Next let $Z_n$ be a unit Euclidean sphere in $R^n$, and $\mathfrak{A}_n$ a set of positive regular Borel measures on $Z_n$ that are orthogonal to the traces of linear functions on $Z_n$. This property of measures is called invariance under shifts.

Between $\mathfrak{B}_n/R^n$ and $\mathfrak{A}_n$ there exists a natural bijection. More precisely, a class of points is identified with the measure zero. To a class that contains a segment with ends x and y we assign a measure $|x-y|\,(\varepsilon_{(x-y)/|x-y|}+\varepsilon_{(y-x)/|x-y|})$, where $|\cdot|$ is the Euclidean length, and $\varepsilon_x$ is Dirac's measure, i.e., unit mass at the point x. Finally, if the dimension of the affine hull $A(\mathfrak{x})$ of the representative $\mathfrak{x}$ of the class of surfaces in $\mathfrak{B}_n/R^n$ is larger than unity, we shall assume that $A(\mathfrak{x})$ is a subspace of $R^n$ and that a class is identified with a surface function $\mathfrak{x}$ in $A(\mathfrak{x})$ that is in this case a measure on $Z_n \cap A(\mathfrak{x})$. By continuing this measure in a trivial manner to a measure on $Z_n$, we obtain a measure in $\mathfrak{A}_n$ that corresponds to the class generated by $\mathfrak{x}$. The bijectiveness of such a correspondence can be established with the aid of Aleksandrov's theorem on the reconstruction of a surface on the basis of its surface function [4].

The structure of a vector space in the space of regular Borel measures induces in $\mathfrak{A}_n$, and hence, also in $\mathfrak{B}_n/R^n$, the structure of a cone (more precisely, the structure of an $R_+$-operator commutative semigroup with cancellation). This structure in $\mathfrak{B}_n/R^n$ is called a Blaschke structure [5].

In the following we shall usually not distinguish between a convex compact set corresponding to a class in $\mathfrak{B}_n/R^n$ and a measure in $\mathfrak{A}_n$ that corresponds to this class.

2°. Now let $C(Z_n)/R^n$ be a factor space of the space $C(Z_n)$ of continuous functions on $Z_n$ with respect to the subspace of traces of linear functions on $Z_n$, and let $\mathfrak{A}_n$ be a space of measures that are invariant under shifts. These spaces are in a canonical duality relation. Let us endow $\mathfrak{A}_n$ with a topology induced by the weak topology $\sigma(\mathfrak{A}_n - \mathfrak{A}_n, C(Z_n)/R^n)$ of the space $\mathfrak{A}_n - \mathfrak{A}_n$.

Let us consider a set $\mathfrak{A}_n^1 = \{\mu \in \mathfrak{A}_n : \mu(Z_n) = 1\}$ of probability measures that are invariant under shifts. Let us note that $\mathfrak{A}_n^1$ is a compact base of the cone $\mathfrak{A}_n$. In geometrical language, $\mathfrak{A}_n^1$ is a set of compact sets whose area (calculated in the corresponding affine hull) is equal to unity.

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A symmetry measure (more precisely, a superadditive symmetry measure) is a function $f:\mathfrak{A}_n^1 \to R$ such that:

(1) f is a concave function;

(2) f is a function that is semicontinuous from above;

(3) $0 \leqslant f(\mathfrak{x}) \leqslant 1$ for any $\mathfrak{x} \in \mathfrak{A}_n^1$;

(4) $f(\mathfrak{x}) = 1$ if and only if $\mathfrak{x}$ is a centrally symmetric surface.

Let us note that the set of centrally symmetric surfaces in $\mathfrak{A}_n$ is realized as a set of symmetrical measures (a measure $\mu$ on $Z_n$ is symmetrical if for any Borel set $e \subset Z_n$ we have $\mu(e) = \mu(-e)$).

Remark 1. Symmetry measures are usually defined without specifying the compact base in $\mathfrak{B}_n/R^n$ or in $\mathfrak{A}_n$. Since this requires invariance of measures under stretching, the selection of the base is in fact not essential.

Example. Let $\mathfrak{x}$ be a compact set (measure) in $\mathfrak{A}_n^1$. Let us consider a set $S(\mathfrak{x})$ of symmetrical compact sets $\mathfrak{y}$ such that $\mathfrak{x} - \mathfrak{y} \in \mathfrak{A}_n$. Let us note that this set is nonempty. Let us write

$$f_{\min}(\mathfrak{x}) = \sup\{\mathfrak{y}(Z_n):\ \mathfrak{y} \in S(\mathfrak{x})\}.$$

If is evident that $f_{\min}$ is a concave function that satisfies the condition (3). It is easy to see that $f_{\min}$ is a symmetry measure. This result can be obtained from the theorem presented below, but it is simpler to verify it directly. We shall check only the validity of condition (4). (Formally this is convenient to use in the proof of the fundamental theorem.) It suffices to show that if $f_{\min}(\mathfrak{x}) = 1$, then $\mathfrak{x}$ will be centrally symmetric. By virtue of compactness of the set $S(\mathfrak{x})$ there exists a $\mathfrak{y} \in S(\mathfrak{x})$ such that $f_{\min}(\mathfrak{x}) = \mathfrak{y}(Z_n) = 1$. The measure $\mathfrak{y}$ is symmetrical, with $\mathfrak{y} \leqslant \mathfrak{x}$. Moreover, the areas of $\mathfrak{x}$ and $\mathfrak{y}$ coincide. Thus, the measure $\mathfrak{x} - \mathfrak{y}$ will belong to the cone which is dual to the cone of feasible directions, to the cone of surfaces in Minkowski's structure, and to the unit sphere. Since the sphere is a regular surface, it follows from Proposition 4.1.1 of [6] that $\mathfrak{y} = \mathfrak{x}$. In conclusion let us note that the symmetry measure $f_{\min}$ is a similarly invariant measure, i.e., for any motion D of the space $R^n$ we have

$$f_{\min}(D\mathfrak{x}) = f_{\min}(\mathfrak{x}).$$

Remark 2. The importance of the symmetry measure $f_{\min}$ is due to the circumstance that (as we can see below) any similarly invariant symmetry measure majorizes $f_{\min}$.

3°. It is well known that in a certain sense concave functions are determined by their values on the Choquet boundary of the space of affine functions, i.e., on the set of extreme points of a compact set. For using this result in the analysis of symmetry measures, let us prove the almost evident proposition.

Proposition. The closure of the set $\operatorname{ex}(\mathfrak{A}_n^1)$ of extreme points of $\mathfrak{A}_n^1$ consists of simplexes (including degenerate ones), i.e., of discrete probability measures on $Z_n$ with a support at not more than n + 1 points. The set of extreme points of symmetrical compact sets in $\mathfrak{A}_n^1$ is closed and coincides with the set of segments.

Proof. The closure of the set of simplexes and segments in the topology $\sigma(\mathfrak{A}_n - \mathfrak{A}_n,\ C(Z_n)/R^n)$ is obvious. Indeed, the set of segments belonging to $\mathfrak{A}_n^1$ is a continuous image of the sphere $Z_n$, whereas the set of simplexes is an image of the compact set

$$\left\{(z_1, \ldots, z_{n+1}) \in (R^n)^{n+1} : \sum_{k=1}^{n+1} z_k = 0,\ \sum_{k=1}^{n+1} |z_k| = 1\right\}$$

under the continuous mapping

$$(z_1, \ldots, z_{n+1}) \mapsto \sum_{k=1}^{n+1} |z_k|\, \varepsilon_{z_k/|z_k|},$$

where it is assumed that $|z|\, \varepsilon_{z/|z|} = 0$ for z = 0.

Now the validity of this proposition follows, for example, from the Mil'man inversion of the Krein–Mil'man theorem and from the theorems of Grünbaum and Fáry on the decomposition of polyhedra [4].

THEOREM. Let $O$ be a set of segments in $\mathfrak{A}_n^1$ and let $T = \operatorname{ex}(\mathfrak{A}_n^1) \setminus O$ be a set of nondegenerate simplexes, and g a function on $\operatorname{ex}(\mathfrak{A}_n^1)$ that is semicontinuous from above; g is positive, it vanishes outside a closed subset of $\operatorname{ex}(\mathfrak{A}_n^1)$, and

$$g(\mathfrak{x}) = 1 \quad \text{for} \quad \mathfrak{x} \in O, \quad g(\mathfrak{x}) \leqslant 1 - \varepsilon \quad \text{for} \quad \mathfrak{x} \in T$$

for a positive $\varepsilon$. Then:

(a) There exists a symmetry measure f such that the restriction $f|_{\mathrm{ex}(\mathfrak{A}_n^1)}$ of the measure f to $\mathrm{ex}(\mathfrak{A}_n^1)$ coincides with g;

(b) among symmetry measures f such that $f|_{\mathrm{ex}(\mathfrak{A}_n^1)} = g$ there exists a minimal symmetry measure $g_{\min}$, with

$$g_{\min}(\mathfrak{x}) = \sup_{\mu} \int_{\mathrm{ex}(\mathfrak{A}_n^1)} g\, d\mu,$$

the supremum being taken over all probability measures on $\mathfrak{A}_n^1$ representing the point $\mathfrak{x}$ (i.e., with their barycenter at $\mathfrak{x}$);

(c) the symmetry measure $f_{\min}$ is a minimal element of the set of all symmetry measures.

Proof. (a) and (b). Let $g_0(\mathfrak{x}) = g(\mathfrak{x})$ for $\mathfrak{x} \in \mathrm{ex}(\mathfrak{A}_n^1)$ and $g_0(\mathfrak{x}) = 0$ for $\mathfrak{x} \in \mathfrak{A}_n^1 \setminus \mathrm{ex}(\mathfrak{A}_n^1)$. It is evident that $g_0$ is a function semicontinuous from above. Let us consider the upper envelope $\hat{g}_0$ of the function $g_0$, i.e.,

$$\hat{g}_0(\mathfrak{x}) = \inf\{h(\mathfrak{x}) : h \in A,\ h \geqslant g_0\},$$

where A is a set of continuous affine functions on $\mathfrak{A}_n^1$. It follows from Erwe's theorem [2] that $\hat{g}_0|_{\mathrm{ex}(\mathfrak{A}_n^1)} = g$ with

$$\hat{g}_0(\mathfrak{x}) = \sup_{\mu} \int_{\mathfrak{A}_n^1} g_0\, d\mu = \sup_{\mu} \int_{\mathrm{ex}(\mathfrak{A}_n^1)} g\, d\mu,$$

where the supremum is taken over all probability measures on $\mathfrak{A}_n^1$, representing the point $\mathfrak{x}$. Since $\hat{g}_0$ is a minimal concave function that is semicontinuous from above and takes the value g at the boundary of $\mathrm{ex}(\mathfrak{A}_n^1)$ the proof of (a) and (b) will require only the verification that $\hat{g}_0$ is a symmetry measure. We must show that the last relation is valid.

At first let $\mathfrak{x}$ be a symmetric compact set on $\mathfrak{A}_n^1$. By virtue of Choquet's theorem there exists a maximal measure on the set of symmetrical compact sets representing the point $\mathfrak{x}$. Since the set of extreme points $O$ is closed, it follows that $\mu$ is concentrated on $O$.

Thus,

$$\hat{g}_0(\mathfrak{x}) \geqslant \int_{\mathfrak{A}_n^1} g_0\, d\mu = \int_O g\, d\mu = \mu(O) = 1.$$

Since, evidently,

$$\hat{g}_0(\mathfrak{x}) \leqslant 1, \text{ it follows that } \hat{g}_0(\mathfrak{x}) = 1.$$

Conversely, let $\hat{g}_0(\mathfrak{x}) = 1$ for any $\mathfrak{x} \in \mathfrak{A}_n^1$. It is well known that there exists a probability measure $\mu$ with its barycenter at the point $\mathfrak{x}$ and such that $\hat{g}_0(\mathfrak{x}) = \mu(g_0)$.

We have

$$1 = \int_{\mathfrak{A}_n^1} g_0\, d\mu = \int_{\mathrm{ex}(\mathfrak{A}_n^1)} g_0\, d\mu = \int_T g\, d\mu + \int_O g\, d\mu \leqslant$$
$$\leqslant (1-\varepsilon)\mu(T) + \mu(O) = \mu(\mathrm{ex}(\mathfrak{A}_n^1)) - \varepsilon\mu(T) \leqslant 1 - \varepsilon\mu(T) \leqslant 1.$$

Hence, the restriction of $\mu$ to $O$ will be a probability measure that represents the point $\mathfrak{x}$, and from the above proposition it follows that $\mathfrak{x}$ is a centrally symmetric set.

(c) Let $(\chi_O)_{\min}$ be a symmetry measure constructed for the characteristic function $\chi_O$ of a set of segments with the aid of assertion (b) of the theorem. If f is a symmetry measure and $\mathfrak{x}$ a point belonging to $\mathfrak{A}_n^1$, then the measure $\mu$ representing $\mathfrak{x}$ will satisfy the relation

$$\mu(f) \leqslant f(\mathfrak{x}).$$

Hence follows that

$$\mu(O) \leqslant f(\mathfrak{x}).$$

Thus,

$$(\chi_O)_{\min}(\mathfrak{x}) = \sup_{\mu} \int_{\mathrm{ex}(\mathfrak{A}_n^1)} \chi_O \, d\mu = \sup_{\mu} \mu(O) \leqslant f(\mathfrak{x}),$$

i.e., the measure $(\chi_O)_{\min}$ is a minimal symmetry measure.

It remains to show that $(\chi_O)_{\min}$ coincides with the symmetry measure $f_{\min}$ constructed in the example.

Let $f_{\min}(\mathfrak{x}) = t$. If $t = 1$, then $\mathfrak{x}$ will be symmetrical and $(\chi_O)_{\min}(\mathfrak{x}) = 1$. Let us assume that $1 > t > 0$. There exists an element $\mathfrak{y} \in S(\mathfrak{x})$ such that $f_{\min}(\mathfrak{x}) = \mathfrak{y}(Z_n)$. Let us write $\mathfrak{z} = \mathfrak{x} - \mathfrak{y}$, and let $\mu_1$ be a measure concentrated on $O$ and representing $\mathfrak{y}/t$, whereas $\mu_2$ is a measure representing $\mathfrak{z}/(1-t)$. Since $\mathfrak{x} = t(\mathfrak{y}/t) + (1-t)(\mathfrak{z}/(1-t))$, it follows that the measure $t\mu_1 + (1-t)\mu_2$ represents $\mathfrak{x}$. Hence,

$$(\chi_O)_{\min}(\mathfrak{x}) \geqslant (t\mu_1 + (1-t)\mu_2)(O) \geqslant t\mu_1(O) = t = f_{\min}(\mathfrak{x}).$$

Thus, by writing $(\chi_O)_{\min}(\mathfrak{y}) = t_1$, we obtain $1 > t_1 > 0$. Now let $\mu$ be a probability measure representing $\mathfrak{x}$ and such that $\mu(O) = t_1$. Let us consider two measures $\mu_1$ and $\mu_2$ which are the traces of $\mu$ on $O$ and $\mathfrak{A}_n^1 \setminus O$ respectively. We have

$$\mu = t_1(\mu_1/t_1) + (1-t_1)(\mu_2/(1-t_1)).$$

Let $\mathfrak{y}$ be a centrally symmetric surface represented by the measure $\mu_1/t_1$. Then the body

$$t_1\mathfrak{y} \in S(\mathfrak{x}).$$

Hence

$$f_{\min}(\mathfrak{x}) \geqslant t_1\mathfrak{y}(Z_n) = t_1 = (\chi_O)_{\min}(\mathfrak{x}).$$

Thus, we have proved in fact that if one of the numbers $f_{\min}(\mathfrak{x})$ and $(\chi_O)_{\min}(\mathfrak{x})$ is nonzero, these numbers will coincide. This completes the proof of the theorem.

Remark 3. Let us endow $\mathfrak{V}_n/R^n$ with a standard Minkowski structure generated by algebraic addition of sets, and with a topology generated by a Blaschke metric (see, e.g., [5]). As a compact base $Q_n$ of a locally compact cone $\mathfrak{V}_n/R^n$ we shall take a set of surfaces of unit integral width. By replacing $\mathfrak{A}_n^1$ by $Q_n$ in the definition of a symmetry measure, and by interpreting concavity in a Minkowski structure, we also obtain a nonempty class of symmetry measures that satisfy the superminimality condition. Indeed, let $\hat{\mathfrak{x}}$ be the largest centrally symmetric figure contained in $\mathfrak{x}$ (a Kovner–Bezikovich body). Let us write

$$f : \mathfrak{x} \mapsto (\text{the integral width of } \hat{\mathfrak{x}}).$$

It is evident that f is a similarly invariant symmetry measure. This function is also called an $F_\sigma^*$ measure of symmetry of Fáry [1].

Let us note that the measure f is evidently not minimal. In the case $n \geq 3$, the problem of minimality of a symmetry measure (solved by the above theorem for $n = 2$) cannot be analyzed in the language of Choquet's theory. As a matter of fact, in this case the Shilov boundary of the compact set $Q_n$ coincides with $Q_n$; in other words, the extreme points are dense in $Q_n$ (see [3]).

## LITERATURE CITED


1. B. Grünbaum, Studies in Combinatorial Geometry and the Theory of Convex Bodies [Russian translation], Nauka, Moscow (1971).
2. E. Alfsen, Compact Convex Sets and Boundary Integrals, Springer, Berlin–Heidelberg–New York (1971).
3. G. Shephard, "Approximation problems for convex polyhedra," Mathematika, 11, No. 1, 9-18 (1964).
4. W. Fáry and B. Grünbaum, "Addition and decomposition of convex polytopes," Israel J. Math., 2, 91-100 (1964).
5. S. S. Kutateladze, "Blaschke structures in the programming of isoperimetric problems," Matem. Zametki, 14, No. 5, 767-775 (1973).
6. S. S. Kutateladze and A. M. Rubinov, "Minkowski duality and its applications," Usp. Matem. Nauk, 27, No. 3, 127-176 (1972).