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CONVEX OPERATORS

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Contents

Introduction

Let Y be a real vector space, $G\colon Y \to R$ a convex function, and $Y' = L(Y, R)$ its algebraic dual. We assign to G its *subdifferential* at a point $\bar{y}$, that is, the set

$$\partial_{\bar{y}}(G) = \{l \in Y'\colon\ l(y) - l(\bar{y}) \leqslant G(y) - G(\bar{y}) \quad (y \in Y)\},$$

and its *Young transform* $G^*\colon Y' \to R \cup \{+\infty\}$, given by

$$G^*(l) = \sup_{y \in Y} (l(y) - G(y)).$$

Subdifferentials and Young transforms play a crucial role in the theory of extremum problems. The reasons for this are plain to see: for it is clear that a subdifferential is a non-smooth analogue to a differential and that, just as in the smooth case, Fermat's criterion for an optimum holds:

$$0 \in \partial_{\bar{y}}(G) \Longleftrightarrow G(\bar{y}) = \inf\{G(y)\colon\ y \in Y\}.$$

The role of the Young transform is equally clear, since $-G^*(l)$ is a solution of the extremum problem

$$y \in Y, \quad G(y) - l(y) \to \inf.$$

In particular, $-G^*(0) = \inf\{G(y)\colon\ y \in Y\}$.

This motivates our main theme, which is one of the central problems of local convex analysis, namely, that of finding rules for computing the subdifferentials and Young transforms of complicated convex functions. It is immediately clear that for an adequate formulation of this problem, that is, of describing the subdifferential $\partial_{\bar{x}}(G \circ F)$ and Young transform $(G \circ F)^*$ of a composition with some mapping $F: X \to Y$, we require a reasonable extension of ideas, at least to the presence of an order relation on Y consistent with the vector structure. Thus, the need arises for an apparatus of the local investigation of convex operators acting on arbitrary ordered vector spaces. An account of such a device constitutes the main substance of this article.

It must be pointed out that the construction of a subdifferential calculus for sufficiently general operators necessitates the rejection of standard geometrical schemes of convex analysis, since arguments based on functional separability are on principle unsuitable, say, for operators acting on the space of all measurable functions. More precisely, to associate with a convex operator $F: X \to Y$ the family of convex functions $l \circ F: X \to R$, where l is a positive functional on Y, though useful in a number of cases, can scarcely be regarded as a universal method of analysis, for the simple reason that the only such functional is $l = 0$.

In this context we take in the paper as our starting point another purely analytic approach, based on the theory of Kantorovich spaces. Namely, we observe that, from a technical point of view, there are not all that many essentially "non-linear" convex operators. More precisely, with each cardinality and each K-space of images we can associate only one sublinear operator, the canonical one. Then it turns out that all the other sublinear operators are compositions of this one with linear operators. In the case of convex operators, roughly the same situation prevails. Thus, all questions in the subdifferential calculus reduce to the calculation of a single simply constructed operator. On this basis we solve the problem of the subdifferential of a composition of convex operators at interior points of the domains of definition. In other words, we obtain a universal formula that plays the same role as the chain rule in the differential calculus of smooth functions. This result can be reformulated at will to yield a formal generalization of the chain rule for non-convex smooth mappings.

The case of subdifferentials at boundary points is more complicated. Here we must employ more special arguments, based on the Mazur–Orlicz theorem. Nevertheless, the most important formulae for applications, those of Moreau–Rockafellar, Dubovitzkii–Milyutin, and Gol'shtein, together with the decomposition formula, all admit perfect operator analogues. With the aid of these formulae we obtain not only the basic rules for a change of variables in the Young transform, that is, the formulae of Moreau and Ioffe–Tikhomirov and the minimax formula for the vector case, but also some fundamental new facts, such as a rule for computing $(A \circ F)^*$, where A is a positive linear operator.

This article can be used as an introduction to the foundations of local convex analysis. In particular, this means that proofs are given for all the essential facts. Furthermore, we do not assume familiarity with the theory of ordered vector spaces. True, a reader unacquainted with this theory will have to disregard a few unfamiliar words and unintelligible proofs. In extreme cases, any K-space that occurs should be thought of as the space R^n, or even R. We point out that the method of reasoning will, as a rule, be non-traditional. At the same time, it is absolutely necessary to emphasize that the introduction of K-spaces is not an intermediate stage in the development of local convex analysis. The fact is that if one of the essential principles of local analysis (such as the theorem on the subdifferential of a sum) is valid when formulated en masse for an arbitrary ordered vector space Y, then the bounded subsets of Y necessarily have least upper bounds. Hence, a familiarity with the elements of the theory of K-spaces is desirable.

In conclusion, I express my profound gratitude to G. P. Akilov, L. V. Kantorovich, S. L. Sobolev, and V. M. Tikhomirov for their unfailing interest in the research embodied in this article.

§1. The Hahn–Banach–Kantorovich theorem

The main purpose of this section is to prove the Hahn–Banach–Kantorovich theorem on the existence of a majorized extension of a linear operator. This theorem plays a fundamental role in several parts of analysis and its applications. In particular, it forms the basis of all the methods used in the investigation of convex operators.

1.1. Let X be a vector space and Y an *ordered vector space*, that is, a space with a distinguished (convex) cone Y^+, the cone of positive elements. We adjoin to Y a largest element $+\infty$ (to be denoted more precisely by $+\infty_Y$, though no ambiguity is ever caused by the abbreviated notation). It is convenient to assume that $y + (+\infty) = +\infty$ and $\alpha(+\infty) = +\infty$ for any $y \in Y \cup \{+\infty\}$ and any positive scalar α.

A mapping $F: X \to Y \cup \{+\infty\}$ is called a *convex operator* if for any vectors $x_1, x_2 \in X$ and any scalars $\alpha_1, \alpha_2 \geqslant 0$ with $\alpha_1 + \alpha_2 = 1$, the *Jensen inequality* holds:

$$F(\alpha_1 x_1 + \alpha_2 x_2) \leqslant \alpha_1 F x_1 + \alpha_2 F x_2.$$

It can be checked directly that the set

$$\operatorname{dom}(F) = \{x \in X \colon Fx < +\infty\}$$

is convex; it is called the *effective set* of F. It is often convenient in applications to treat a convex operator $F: X \to Y \cup \{+\infty\}$ as a mapping $F: X \to Y$, defined on the convex set $\operatorname{dom}(F)$ and satisfying the Jensen inequality there.

Now let Z be another ordered vector space and G a mapping from Y to $Z \cup \{+\infty\}$. Without saying so explicitly, we assume in what follows that G is extended to a mapping from $Y \cup \{+\infty\}$ to $Z \cup \{+\infty\}$ by setting $G(+\infty) = +\infty$.

1.1.1. *Let $F: X \to Y \cup \{+\infty\}$ be a convex operator and $G: Y \to Z \cup \{+\infty\}$*

an increasing convex operator. Then the mapping $G \circ F: X \to Z \cup \{+\infty\}$ is also convex.

This obvious proposition provides a convenient method of constructing convex operators. In particular, if $F_1, \ldots, F_n : X \to Y \cup \{+\infty\}$ are convex operators, then it makes sense to speak of their *sum* $F_1 + \ldots + F_n$, namely, the operator $x \mapsto F_1x + \ldots + F_nx$. We now suppose that Y is a *vector lattice*, that is, $Y^+ \cap (-Y^+) = \{0\}$, and any $y_1, y_2 \in Y$ have an upper bound $y_1 \vee y_2$ and a lower bound $y_1 \wedge y_2$. We can then speak of the *maximum* $F_1 \vee \ldots \vee F_n$ of $F_1, \ldots, F_n$,

$$F_1 \vee \ldots \vee F_n \colon x \mapsto F_1x \vee \ldots \vee F_nx.$$

In what follows we need two important classes of convex operators. The first of these consists of the simplest positive convex operators $F: X \to Y \cup \{+\infty\}$, namely, those that assume only the two values 0 and $+\infty$; they are called *indicators*. This term stems from the fact than an indicator plays a role analogous to that of the characteristic function of the effective set dom(F). Furthermore, every convex set U in X is the effective set of a unique indicator operator mapping into $Y \cup \{+\infty\}$, which we denote by $\delta_Y(U)$. Thus, $\delta_Y(U)x = 0$ for $x \in U$ and $\delta_Y(U)x = +\infty$ otherwise. With the aid of indicators we can form cut-offs of convex operators: if $F: X \to Y \cup \{+\infty\}$ is a convex operator and U is a convex set in X, then $F + \delta_Y(U)$ is called the *cut-off* or *restriction* of F to U and is denoted by F_U.

The second important class consists of the sublinear operators. A convex operator $P: X \to Y \cup \{+\infty\}$ is called *sublinear* if it is positively homogeneous. Note that the effective set of a sublinear operator is a cone. In what follows we adopt freedom of expression in reserving the term sublinear for those operators P for which dom$(P) = X$. An arbitrary sublinear operator P will, as a rule, be referred to as a sublinear operator defined on the cone dom(P).

A most important part in the theory of convex operators is played by the concept of a subdifferential. Let $F: X \to Y \cup \{+\infty\}$ be a convex operator and let $L(X, Y)$ denote the set of linear operators from X to Y. The *subdifferential of F at a point $\bar{x}$* in the effective set dom(F) is defined to be the set

$$\partial_{\bar{x}}(F) = \{A \in L(X, Y) \colon Ax - A\bar{x} \leqslant Fx - F\bar{x} \ (x \in X)\}.$$

Its elements are often called the *subdifferentials* of F at $\bar{x}$. When P is a sublinear operator, its subdifferential at zero is called the *support set* of P and is denoted by $\partial(P)$. Thus,

$$\partial(P) = \{A \in L(X, Y) \colon Ax \leqslant Px \ (x \in X)\}.$$

The elements of the support set are called *support operators*. The analogous concepts for a sublinear operator defined on a cone also make sense. In this case we also speak of operators with support on a cone.

Subdifferentials of convex operators generalize the notion of a differential to the case of non-differentiable convex mappings. In this connection the role that subdifferentials can play (and, as we shall see, do play) in various problems of analysis becomes obvious. Equally clear are the questions that arise in the investigation of subdifferentials. We have to clarify conditions for the existence of subgradients, for a subdifferential to be non-empty, we have to know to what extent a subdifferential determines the behaviour of the original mapping, and finally, we must find the subdifferentials of specific operators. In the case of arbitrary ordered vector spaces, many of these questions are unexplored (apparently because of their hopeless complexity, in fact, the theory of arbitrary ordered spaces is not very rich in substantial results). The situation is different in the case of ordered vector spaces having the most restricted order structure, that of a K-space. We recall that a vector lattice Y is called a *Kantorovich space*, or K-space, if any subset of Y that is bounded above has a least upper bound. In the case of K-spaces it is possible to develop a theory of convex operators that is fairly complete and satisfactory from the point of view of applications. It is a fortuitous (but by no means accidental) fact that the classical spaces of analysis (such as the spaces on the Lebesgue scale) possess these remarkable structural properties.

The main tools in the investigation of convex operators are the theorems on qualified extension of linear operators, and we now turn our attention to these results.

1.2. First of all we establish *Kantorovich's theorem* on the extension of a positive operator. We recall that an operator A from an ordered vector space X to an ordered vector space Y is called *positive* if $A[X^+] \subset Y^+$. The set of all positive linear operators from X to Y is denoted by $L^+(X, Y)$.

1.2.1. THEOREM. *Let Y be a K-space, X an ordered vector space, and X_0 a subspace of X with the property that $X_0 + X^+ = X$. Then any positive linear operator $A_0 : X_0 \to Y$ admits a positive linear extension $A : X \to Y$.*

PROOF. We first assume that X_0 is a hypersubspace of X, in other words, that for some $x' \in X \setminus X_0$, we have

$$X = \{x_0 + \alpha x' : x_0 \in X_0, \alpha \in R\}.$$

The set $U_{x'} = \{x_0 \in X_0 : x_0 \leqslant x'\}$ is non-empty, by hypothesis, and is bounded above by an element of X_0. Thus, the set $A\ [U_{x'}]$ has a least upper bound, y' say. Putting $A(x_0 + x') = A_0x_0 + y'$, we see that this is a well-defined linear operator from X to Y, which agrees with A_0 on X_0.

We claim that this operator is positive. If $x_0 + \alpha x' \geqslant 0$ and $\alpha = 0$, then $A(x_0 + \alpha x') = A_0x_0 \geqslant 0$. But if $\alpha > 0$, then $-x_0/\alpha \in U_{x'}$ that is, $y' \geqslant -A_0x_0/\alpha$. Hence $A(x_0 + \alpha x') \geqslant 0$. If $\alpha < 0$, then $x' \leqslant x_0/|\alpha|$, that is, $y' \leqslant -A_0x_0/\alpha$, and again $A(x_0 + \alpha x') \geqslant 0$.

The proof of the theorem is completed by using a standard argument based on Zorn's lemma.

1.2.2. REMARK. It is clear that in this proof we have not invoked the full strength of the condition that Y be a K-space. In fact, we have only required that a set that is bounded above has a least upper bound. The uniqueness of this bound and the boundedness of finite sets, both of which properties hold in K-spaces, were not needed. In the light of this remark it is both curious and significant that Kantorovich's theorem cannot be improved (see 1.3).

1.3. THEOREM. *Let Y be a K-space. Then for any vector space X, any sublinear operator P: $X \to Y$, and any subspace X_0 of X we have*

$$\partial(P + \delta_Y (X_0)) = \partial(P) + \partial(\delta_Y (X_0)).$$

PROOF. It is obvious that $\partial(P_{X_0}) \supset \partial(P) + \partial(\delta_Y (X_0))$. To prove the reverse inclusion we take an operator $A \in \partial(P_{X_0})$ and define an operator $\mathfrak{A}_0$ on the subspace $X_0 \times Y$ of $X \times Y$ by the rule

$$\mathfrak{A}_0(x_0, y) = -Ax_0 + y.$$

If we order $X \times Y$ by means of the cone

$$Z = \{(x, y) \in X \times Y: Px \leqslant y\},$$

then the operator $\mathfrak{A}_0$: $X_0 \times Y \to Y$ becomes positive, because $A \in \partial(P_{X_0})$. Moreover, $X_0 \times Y + Z = X \times Y$. Thus, it follows from Kantorovich's theorem that $\mathfrak{A}_0$ has a positive extension $\mathfrak{A}$: $X \times Y \to Y$. We set $A_1 x = \mathfrak{A}(-x, 0)$. Since elements of the form $(0, y)$ belong to $X_0 \times Y$, we see that $\mathfrak{A}(0, y) = \mathfrak{A}_0(0, y) = y$. It follows at once that $A_1 \in \partial(P)$. Besides, for $x_0 \in X_0$ we have, $A_1 x_0 = -\mathfrak{A}(x_0, 0) = Ax_0$, so that $A - A_1 \in \partial(\delta_Y (X_0))$, and this completes the proof.

1.3.2. THEOREM. *Let Y be an ordered vector space with the following property: for any vector space X, any subspace X_0 of X, and any sublinear operator P: $X \to Y$,*

$$\partial(P_{X_0}) = \partial(P) + \partial(\delta_Y(X_0)).$$

Then any subset of Y that is bounded above has a least upper bound.

This remarkable fact, known as the *Bonnice–Silverman–To theorem*, indicates the range of applicability of the fundamental formula 1.3.1 of the subdifferential calculus. Formally speaking, we do not need it in what follows. Since the standard proof is not distinguished by its brevity, we omit it.

1.4. In this subsection we discuss those consequences of the Hahn–Banach–Kantorovich theorem that are technically the simplest and are also very important from the point of view of applications. Throughout we fix the following notation: X is a vector space, Y is a K-space, and P: $X \to Y$ is a sublinear operator.

1.4.1. *The support set $\partial(P)$ is non-empty.*

For the subdifferential of the cut-off $P_{\{0\}}$ is non-empty. Furthermore,

$$\partial(P_{\{0\}}) = \partial(P) + \partial(\delta_Y (\{0\})).$$

1.4.2. *For each point $\bar{x} \in X$ the subdifferential $\partial_{\bar{x}}(P)$ is non-empty, and*

$$\partial_{\bar{x}}(P) = \{A \in \partial(P): A\bar{x} = P\bar{x}\}.$$

The validity of the last relation is verified by direct calculation. Thus, we only have to show that $\partial_{\bar{x}}(P)$ is non-empty. To this end we consider the subspace $X_0 = \{\alpha\bar{x}: \alpha \in R\}$ and any operator $A \in \partial(P_{X_0})$ such that $A\bar{x} = P\bar{x}$. By the Hahn–Banach–Kantorovich theorem, there is an $A_1 \in \partial(P)$ for which $A - A_1 \in \partial(\delta_Y(X_0))$. It is clear that A_1 is the required operator.

1.4.3. *For each* $x \in X$

$$Px = \sup\{Ax: A \in \partial(P)\}.$$

This follows immediately from 1.4.2.

1.4.4. REMARK. By modifying slightly the proof of 1.4.2., we can show that the set

$$\partial(P)x = \{Ax: A \in \partial(P)\}$$

coincides with the order interval $\{y \in Y: -P(-x) \leqslant y \leqslant Px\}$.

The following important result is known as the *Levin–Rockafellar lemma.*

1.4.5. *Let* X_1 *be another vector space and* $A \in L(X_1, X)$. *Then*

$$\partial(P \circ A) = \partial(P) \circ A.$$

The fact that $\partial(P) \circ A \subset \partial(P \circ A)$ is obvious. But if $B \in \partial(P \circ A)$, then $\mathrm{Ker}(B) = B^{-1}[0]$ contains the kernel of A, and by an algebraic lemma on triples, there is a $C \in L(X, Y)$ for which $B = C \circ A$. It is clear that $C \in \partial(P_{A[X_1]})$. By 1.3.1., there is a $C_1 \in \partial(P)$ such that $C - C_1 \in \partial(\delta_y(A[X_1]))$. Since, by definition,

$$\partial(\delta_Y(A[X_1])) = \{D \in L(X, Y): \mathrm{Ker}(D) \supset A[X_1]\},$$

it follows that $B = C_1 \circ A$, that is, $B \in \partial(P) \circ A$.

The following proposition is known as the Mazur–Orlicz lemma and is a special case of a theorem to be proved in full later on.

1.4.6. *For every cone H in X,*

$$\partial(P_H) = \partial(P) + \partial(\delta_Y(H)).$$

§2. Support sets of sublinear operators

In this section we study the structure of support sets of sublinear operators and, in particular, we derive an explicit representation of the support set of a composition of such operators.

2.1. We consider a vector space X, a K-space Y, and a subset $\mathfrak{A}$ of $L(X, Y)$. Such a set $\mathfrak{A}$ is called *weakly order-bounded* if for each $x \in X$ the set $\{Ax: A \in \mathfrak{A}\}$ is bounded. We can thus associate with any weakly order-bounded set $\mathfrak{A}$ the sublinear operator

$$P_{\mathfrak{A}} : x \mapsto \sup\{Ax: A \in \mathfrak{A}\}.$$

Then $\partial(P_{\mathfrak{A}})$ is called the *support hull* of $\mathfrak{A}$ and is denoted by $\mathrm{cop}(\mathfrak{A})$.

First of all, we use the results of the previous section to show that sublinear operators are not very numerous. More precisely, every sublinear operator can be represented as a composition of some linear operator and a universal canonical operator. For a formal description of the situation we need the following subsidiary construction.

Let Y be a K-space and $\mathfrak{A}$ an arbitrary set. We consider the K-space $Y^{\mathfrak{A}}$, the product of the corresponding number of copies of Y, endowed with the natural order structure. Let $\Delta_{\mathfrak{A}, Y}$ or $\Delta_{\mathfrak{A}}$ denote the operator embedding Y in the diagonal subspace of $Y^{\mathfrak{A}}$, namely,

$$\Delta_{\mathfrak{A}}\colon y \mapsto (y)_{A\in\mathfrak{A}}.$$

We consider the following set:

$$(Y^{\mathfrak{A}})_\infty = (\Delta_{\mathfrak{A}}[Y] + (Y^+)^{\mathfrak{A}}) \cap (\Delta_{\mathfrak{A}}[Y] - (Y^+)^{\mathfrak{A}}),$$

that is, the subspace of $Y^{\mathfrak{A}}$ consisting of the bounded Y-valued functions on $\mathfrak{A}$. There is a *canonical sublinear operator*, denoted by $\varepsilon_{\mathfrak{A}, Y}$ or $\varepsilon_{\mathfrak{A}}$, sending $(Y^{\mathfrak{A}})_\infty$ to Y, which is defined by the rule

$$\varepsilon_{\mathfrak{A}}\colon (y_A)_{A\in\mathfrak{A}} \mapsto \sup\{y_A\colon A\in\mathfrak{A}\}.$$

When A is a weakly order-bounded set in $L(X, Y)$, there is a natural linear operator

$$\langle\mathfrak{A}\rangle\colon X \to (Y^{\mathfrak{A}})_\infty, \quad \langle\mathfrak{A}\rangle x = (Ax)_{A\in\mathfrak{A}}.$$

As an immediate consequence of this definition we have the following *fundamental lemma.*

2.1.1. *Let X be a vector space, Y a K-space, and $P\colon X\to Y$ a sublinear operator such that* $\partial(P) = \operatorname{cop}(\mathfrak{A})$. *Then* $P = \varepsilon_{\mathfrak{A}} \circ \langle\mathfrak{A}\rangle$.

2.2. With the aid of this lemma we can find natural representations of support hulls.

2.2.1. *Let Z be a K-space and $P\colon Y\to Z$ an increasing sublinear operator. Then*

$$\partial(P\circ\varepsilon_{\mathfrak{A}}) = \{A\in L^+((Y^{\mathfrak{A}})_\infty, Z)\colon A\circ\Delta_{\mathfrak{A}}\in\partial(P)\}.$$

First let $Ay \leqslant P\circ\varepsilon_{\mathfrak{A}}y$ for all $y\in(Y^{\mathfrak{A}})_\infty$. If $y\leqslant 0$, then $\varepsilon_{\mathfrak{A}}y\leqslant 0$, and so $A\in L^+((Y^{\mathfrak{A}})_\infty, Z)$. But if $y=\Delta_{\mathfrak{A}}x$, then

$$A\circ\Delta_{\mathfrak{A}}x = Ay \leqslant P\circ\varepsilon_{\mathfrak{A}}y = (P\circ\varepsilon_{\mathfrak{A}})\,\Delta_{\mathfrak{A}}x = Px,$$

so that $A\circ\Delta_{\mathfrak{A}}\in\partial(P)$.

Suppose now that A is known to be a positive operator with $A\circ\Delta_{\mathfrak{A}}\in\partial(P)$. Then

$$Ay \leqslant A\circ\Delta_{\mathfrak{A}}\circ\varepsilon_{\mathfrak{A}}y \leqslant P\circ\varepsilon_{\mathfrak{A}}y.$$

as required.

2.2.2. *We have*

$$\partial(\varepsilon_{\mathfrak{A}}) = \{\alpha\in L^+((Y^{\mathfrak{A}})_\infty, Y)\colon \alpha\circ\Delta_{\mathfrak{A}} = I_Y\},$$

where I_Y denotes the identity mapping of Y onto itself.

To prove this we just apply 2.2.1 with $P = I_Y$.

2.2.3. *For any weakly order-bounded set* $\mathfrak{A}$ *we have* $\mathrm{cop}(\mathfrak{A}) = \partial(\varepsilon_{\mathfrak{A}}) \circ \langle\mathfrak{A}\rangle$.

For we see at once from 1.4.5 that

$$\mathrm{cop}\,(\mathfrak{A}) = \partial\,(P_{\mathfrak{A}}) = \partial\,(\varepsilon_{\mathfrak{A}} \circ \langle\mathfrak{A}\rangle) = \partial\,(\varepsilon_{\mathfrak{A}}) \circ \langle\mathfrak{A}\rangle.$$

Note that Proposition 2.2.3 is an analogue to a theorem of Choquet type on integral representations.

2.2.4. EXAMPLE. Let $Y = R$. Then $(Y^{\mathfrak{A}})_\infty$ is the space $l_\infty(\mathfrak{A})$ of bounded numerical functions on $\mathfrak{A}$. Thus, if f_0 is a linear functional on a vector space X and $\mathfrak{A}$ is a weakly bounded set of such functionals, then 2.2.3 asserts that

$$f_0(x) \leqslant \sup\{f(x)\colon f \in \mathfrak{A}\} \qquad (x \in X)$$

if and only if

$$f_0\,(x) = \int\limits_{\mathfrak{A}} x\,(\cdot)\,d\alpha \qquad (x \in X)$$

for some finitely additive probability measure α on the algebra of subsets of $\mathfrak{A}$. (The functional $x(\cdot)$ is given by the rule $x(\cdot)\colon f \mapsto f(x)$.)

In applications we are interested in those cases when the ranges of values in Y of the relevant sublinear operators are equipped with a separable linear topology τ. Then each space of operators $L(X, Y)$ acquires the so-called *τ-operator topology*, that is, the topology induced on $L(X, Y)$ by the embedding of this set in the space Y^X endowed with the Tychonoff topology. A topology τ is called *admissible* if the support set of any sublinear operator is compact in the τ-operator topology. The following proposition shows that admissible topologies are unfortunately few in number.

2.2.5. *A topology on Y is admissible if and only if the order intervals in Y are compact in this topology.*

If the intervals are compact, then by Tychonoff's theorem and 1.4.4, $\partial(P)$ is compact for any sublinear operator P, and this gives the sufficiency of the condition. The necessity follows from the fact that the image of the support set $\partial(z \mapsto z \vee 0)$ under the continuous mapping $A \mapsto Ay$ coincides with the order interval $\{y' \in Y\colon 0 \leqslant y' \leqslant y\}$.

2.3. An important role in what follows is played by some special operators on a K-space Y, whose properties are set forth in this subsection.

An operator $\alpha \in L^+(Y, Y)$ is called a *multiplier* if $\alpha \leqslant I_Y$. The set of multipliers on Y is denoted by $\Lambda(Y)$.

2.3.1. *For any multiplier α and any subset U of Y that is bounded above, we have* $\alpha \sup U = \sup \alpha[U]$. *In particular, α is (o)-continuous (that is, if* $y_\gamma \downarrow y$, *then* $\alpha y_\gamma \downarrow \alpha y$).

For since α and I_Y are positive, α has the bounds

$$\alpha \sup U \geqslant \sup \alpha[U], \qquad (I_Y - \alpha) \sup U \geqslant \sup(I_Y - \alpha)[U].$$

Also $\alpha + (I_Y - \alpha) = I_Y$, so that

$$\sup U = \alpha \sup U + (I_Y - \alpha) \sup U \geqslant \sup \alpha[U] + \sup(I_Y - \alpha)[U] = \sup U.$$

2.3.2. *Any two multipliers commute.*

We use the realization of a K-space Y as a foundation[1] of the K-space $C_\infty(Q)$ for some external compactum Q (see [9]). In other words, for every positive $z \in C_\infty(Q)$ we have

$$z = \sup\{y \in Y \colon 0 \leqslant y \leqslant z\}.$$

Thus, any multiplier $\gamma \in \Lambda(Y)$ admits an extension to a multiplier $\widetilde{\gamma} \in \Lambda(C_\infty(Q))$, given by the obvious relation

$$\widetilde{\gamma} z = \sup\{\gamma y \colon 0 \leqslant y \leqslant z\}.$$

It is clear that this extension is unique.

We now take two multipliers $\alpha, \beta \in \Lambda(Y)$ and consider their extensions $\widetilde{\alpha}, \widetilde{\beta} \in \Lambda(C_\infty(Q))$. Note that $\widetilde{\alpha}[C(Q)] \subset C(Q)$ and $\widetilde{\beta}[C(Q)] \subset C(Q)$, where $C(Q)$ is the space of continuous functions on Q. It follows at once that the restrictions of $\widetilde{\alpha}$ and $\widetilde{\beta}$ to $C(Q)$ belong to $\Lambda(C(Q))$, hence they commute on $C(Q)$, since their action there is simply that of multiplication by suitable functions. To conclude that $\widetilde{\alpha}$ and $\widetilde{\beta}$ commute on $C_\infty(Q)$, it now remains to point out that $C(Q)$ is a foundation of $C_\infty(Q)$. And so their traces α and β on Y also commute.

2.3.3. *If a multiplier α is invertible, then $\alpha[Y]$ is a foundation of Y, and α is an order isomorphism between Y and $\alpha[Y]$.*

First of all, let us verify that $\alpha[Y]$ is a normal subspace of Y. For if $0 \leqslant z \leqslant \alpha y$, then bearing in mind that

$$\Lambda(Y) = \partial(z \mapsto z \vee 0)$$

and 1.4.4, we can find a multiplier $\beta \in \Lambda(Y)$ such that $z = \beta \circ \alpha y$. Since $\beta \circ \alpha = \alpha \circ \beta$, by 2.3.2, it follows that $z \in \alpha[Y]$. Now if $z \in Y$ is such that $|z| \wedge |\alpha y| = 0$ for all $y \in Y$, then

$$0 = |z| \wedge |\alpha y| \geqslant \alpha |z| \wedge |\alpha y| = \alpha(|z| \wedge |y|) \geqslant 0$$

by 2.3.1. By hypothesis, $\mathrm{Ker}(\alpha) = \{0\}$, and this means that $|z| \vee |y| = 0$ for all $y \in Y$. Thus, $z = 0$, and $\alpha[Y]$ is a foundation of Y.

It remains to check that if $\alpha y > 0$, then $y > 0$. If this were not so, we could find a projection (that is, an idempotent multiplier) Pr on Y such that $\mathrm{Pr}\, y < 0$ (this follows, for example, from the above theorem on realization). Applying 2.3.2, we would then have

$$\alpha \circ \mathrm{Pr}\, y \leqslant 0 \leqslant \mathrm{Pr} \circ \alpha y = \alpha \circ \mathrm{Pr}\, y,$$

so that $\alpha \circ \mathrm{Pr}\, y = 0$. Hence, $\mathrm{Pr}\, y = 0$, and this is a contradiction.

2.4. This subsection is a mark of respect for the geometrical concepts of duality. Here we discuss the question of an intrinsic characterization of support sets. Formally speaking no use is made in what follows of the facts cited below, therefore, we present them in a concise form.

[1] In the Western literature a foundation is sometimes called a quasi-order dense ideal (see [23]). (Transl.)

A weakly order-bounded set $\mathfrak{A}$ in the space $L(X, Y)$ of linear operators is called *operator convex* if for any $A_1, A_2 \in \mathfrak{A}$ and any multipliers $\alpha_1, \alpha_2 \in \Lambda(Y)$ with $\alpha_1 + \alpha_2 = I_Y$

$$\alpha_1 \circ A_1 + \alpha_2 \circ A_2 \in \mathfrak{A}.$$

A weakly order-bounded set $\mathfrak{A}$ in $L(X, Y)$ is called *strongly operator convex* if $\Sigma \alpha_\xi \circ A_\xi \in \mathfrak{A}$, for any (o)-summable family (α_ξ) of multipliers with $\Sigma \alpha_\xi = I_Y$ and any family (A_ξ) of members of $\mathfrak{A}$ where the summation is pointwise, that is, for each $x \in X$ we take the (o)-sum of the family $(\alpha_\xi \circ A_\xi x)$.

2.4.1. *The support set of any sublinear operator is strongly operator convex.*

We include a proof of the next proposition to illustrate the technique of inversion by multipliers.

2.4.2. *For every weakly order bounded set $\mathfrak{A}$ there is a smallest operator convex set* $\mathrm{op}(\mathfrak{A})$ *containing* $\mathfrak{A}$, *namely,*

$$\mathrm{op}(\mathfrak{A}) = \{\sum_{k=1}^{n} \alpha_k \circ A_k \colon A_k \in \mathfrak{A};\ \alpha_k \in \Lambda(Y),\ \sum_{k=1}^{n} \alpha_k = I_Y\}.$$

By 2.4.1, the existence of the set $\mathrm{op}(\mathfrak{A})$ is clear, because $\mathfrak{A} \subset \mathrm{cop}(\mathfrak{A})$ and $\mathrm{cop}(\mathfrak{A}) = \partial(P_{\mathfrak{A}})$. Thus, we only have to check that if a weakly order-bounded set $\mathfrak{A}'$ is operator convex, then for any set of multipliers $\alpha_1, \ldots, \alpha_n$ with

$$\sum_{k=1}^{n} \alpha_k = I_Y,$$

and any $A_1, \ldots, A_n \in \mathfrak{A}'$ we have

$$\sum_{k=1}^{n} \alpha_k \circ A_k \in \mathfrak{A}'.$$

Suppose that this is true for some $n \geqslant 2$, and consider the operators

$$B = \sum_{k=1}^{n} \alpha_k \circ A_k + \alpha_{n+1} \circ A_{n+1},$$

$$\alpha_1, \ldots, \alpha_{n+1} \in \Lambda(Y) \qquad \sum_{k=1}^{n+1} \alpha_k = I_Y,\ A_1, \ldots, A_{n+1} \in \mathfrak{A}'.$$

It is clear that

$$B - \alpha_{n+1} \circ A_{n+1} \in \partial\left(\left(\sum_{k=1}^{n} \alpha_k\right) \circ A_1 \vee \ldots \vee \left(\sum_{k=1}^{n} \alpha_k\right) \circ A_n\right).$$

By invoking 2.2.2 and 2.2.3, we find multipliers $\beta_s \in \Lambda(Y)$ such that

$$\sum_{k=1}^{n} \beta_k = I_Y, \qquad B - \alpha_{n+1} \circ A_{n+1} = \sum_{s=1}^{n} \beta_s \circ \left(\sum_{k=1}^{n} \alpha_k\right) \circ A_s.$$

Applying Proposition 2.3.2, we obtain

$$B = \left(\sum_{k=1}^{n} \alpha_k\right) \circ \sum_{s=1}^{n} \beta_s \circ A_s + \alpha_{n+1} \circ A_{n+1}.$$

Hence, $B \in \mathfrak{A}'$ since $\mathfrak{A}'$ is operator convex.

2.4.3. *For any weakly order-bounded set $\mathfrak{A}$ there is a smallest strongly*

$$\sup U = \alpha \sup U + (I_Y - \alpha) \sup U \geqslant \sup \alpha[U] + \sup(I_Y - \alpha)[U] = \sup U.$$

2.3.2. *Any two multipliers commute.*

We use the realization of a K-space Y as a foundation[1] of the K-space $C_\infty(Q)$ for some external compactum Q (see [9]). In other words, for every positive $z \in C_\infty(Q)$ we have

$$z = \sup\{y \in Y: 0 \leqslant y \leqslant z\}.$$

Thus, any multiplier $\gamma \in \Lambda(Y)$ admits an extension to a multiplier $\widetilde{\gamma} \in \Lambda(C_\infty(Q))$, given by the obvious relation

$$\widetilde{\gamma} z = \sup\{\gamma y: 0 \leqslant y \leqslant z\}.$$

It is clear that this extension is unique.

We now take two multipliers $\alpha, \beta \in \Lambda(Y)$ and consider their extensions $\widetilde{\alpha}, \widetilde{\beta} \in \Lambda(C_\infty(Q))$. Note that $\widetilde{\alpha}[C(Q)] \subset C(Q)$ and $\widetilde{\beta}[C(Q)] \subset C(Q)$, where $C(Q)$ is the space of continuous functions on Q. It follows at once that the restrictions of $\widetilde{\alpha}$ and $\widetilde{\beta}$ to $C(Q)$ belong to $\Lambda(C(Q))$, hence they commute on $C(Q)$, since their action there is simply that of multiplication by suitable functions. To conclude that $\widetilde{\alpha}$ and $\widetilde{\beta}$ commute on $C_\infty(Q)$, it now remains to point out that $C(Q)$ is a foundation of $C_\infty(Q)$. And so their traces α and β on Y also commute.

2.3.3. *If a multiplier α is invertible, then $\alpha[Y]$ is a foundation of Y, and α is an order isomorphism between Y and $\alpha[Y]$.*

First of all, let us verify that $\alpha[Y]$ is a normal subspace of Y. For if $0 \leqslant z \leqslant \alpha y$, then bearing in mind that

$$\Lambda(Y) = \partial(z \mapsto z \vee 0)$$

and 1.4.4, we can find a multiplier $\beta \in \Lambda(Y)$ such that $z = \beta \circ \alpha y$. Since $\beta \circ \alpha = \alpha \circ \beta$, by 2.3.2, it follows that $z \in \alpha[Y]$. Now if $z \in Y$ is such that $|z| \wedge |\alpha y| = 0$ for all $y \in Y$, then

$$0 = |z| \wedge |\alpha y| \geqslant \alpha |z| \wedge |\alpha y| = \alpha(|z| \wedge |y|) \geqslant 0$$

by 2.3.1. By hypothesis. $\mathrm{Ker}(\alpha) = \{0\}$, and this means that $|z| \vee |y| = 0$ for all $y \in Y$. Thus, $z = 0$, and $\alpha[Y]$ is a foundation of Y.

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so that $\alpha \circ \mathrm{Pr}\, y = 0$. Hence, $\mathrm{Pr}\, y = 0$, and this is a contradiction.

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operator convex set stop $(\mathfrak{A})$ *containing* $\mathfrak{A}$, *namely*,

$$\operatorname{stop}(\mathfrak{A})=\{\sum \alpha_\xi \circ A_\xi:\ A_\xi\in\mathfrak{A};\ \alpha_\xi\in\Lambda(Y),\ \sum\alpha_\xi=I_Y\}.$$

Thus, we have

$$\mathfrak{A}\subset\operatorname{op}(\mathfrak{A})\subset\operatorname{stop}(\mathfrak{A})\subset\operatorname{cop}(\mathfrak{A}),$$

and these inclusions are proper, in general. So far little of interest is known in situations when the operator cop admits a representation, say, in terms of op and some "good" operator, for instance, the topological closure operator. We confine ourselves here to the simplest model situations illustrating the direction of research into such questions and the nature of the difficulties that arise.

2.4.4. EXAMPLE. Let Y be a discrete K-space, that is, a foundation in a product $R^{\mathfrak{B}}$ of lines. Note that

$$(Y^{\mathfrak{A}})_\infty=\{y\in R^{\mathfrak{A}\times\mathfrak{B}}:\ \sup_{A\in\mathfrak{A}}|y(A,\ \cdot)|\in Y\}.$$

We consider the set $\mathfrak{P}=\{p_A:\ A\in\mathfrak{A}\}$, where

$$p_A:\ (y_A)_{A\in\mathfrak{A}}\mapsto y_A$$

is a coordinate projection. It is clear that $\varepsilon_{\mathfrak{A}}=P_{\mathfrak{P}}$. An operator $\alpha:\ (Y^{\mathfrak{A}})_\infty\to Y$ belongs to $\partial(\varepsilon_{\mathfrak{A}})$ if and only if

$$\alpha y(B)=\int_{\mathfrak{A}} y(\cdot,\ B)\,d\mu_B,$$

where μ_B is a finitely additive probability measure on the algebra of subsets of $\mathfrak{A}$. Thus, $\alpha_0\in\operatorname{stop}(\mathfrak{P})$ if and only if there exists numbers $\alpha_{A,B}$ for which

$$0\leqslant\alpha_{A,B}\leqslant 1,\qquad \sum_{A\in\mathfrak{A}}\alpha_{A,B}=1,$$

$$\alpha_0 y(B)=\sum_{A\in\mathfrak{A}}\alpha_{A,B}\,y(A,\ B).$$

At the same time, to the elements of $\operatorname{op}(\mathfrak{P})$ there correspond families $(\alpha_{A,B})$ such that $\alpha_{A,B}=0$ for all $B\in\mathfrak{B}$ and all but finitely many $A\in\mathfrak{A}$.

For any vector space X the space $L(X,\ Y)$ can be endowed with the *simple operator topology*, which is defined as being induced by pointwise convergence in $R^{\mathfrak{B}}$. But if Y is a subspace of $l_\infty(\mathfrak{B})$, then it makes sense to endow $L(X,\ Y)$ with the *strong operator topology*, which is defined as being induced by uniform convergence on $\mathfrak{B}$. By considering the closuresop$(\mathfrak{P})$and stop$(\mathfrak{P})$ in the corresponding topologies and invoking 2.2.3, we obtain the following results.

2.4.5. *If Y is a foundation in $R^{\mathfrak{B}}$, then a set of operators is a support set if and only if it is weakly order-bounded, operator convex, and closed in the simple operator topology.*

2.4.6. *If Y is a foundation in $l_\infty(\mathfrak{B})$, then a set of operators is a support set if and only if it is weakly order-bounded, strongly operator convex, and closed in the strong operator topology.*

The case of arbitrary K-spaces has not been studied thoroughly (see [36]).

2.5. We now turn our attention to the determination of the support sets of composites of sublinear operators.

2.5.1. THEOREM. *Let $P_1 : X \to Y$ be a sublinear operator and $P_2 : Y \to Z$ an increasing sublinear operator, where Y and Z are K-spaces. Then*

$$\partial (P_2 \circ P_1) = \{A \circ \langle \partial (P_1) \rangle : \ A \circ \Delta_{\partial (P_1)} \in \partial (P_2),\ A \in L^+ ((Y^{\partial (P_1)})_\infty, Z)\}.$$

If $\partial (P_1) = \operatorname{cop} (\mathfrak{A}_1)$ and $\partial (P_2) = \operatorname{cop} (\mathfrak{A}_2)$, then

$$\partial (P_2 \circ P_1) = \{A \circ \langle \mathfrak{A}_1 \rangle : \ \exists \alpha_2 \in \partial (\varepsilon_{\mathfrak{A}_2})\ A \circ \Delta_{\mathfrak{A}_1} = \alpha_2 \circ \langle \mathfrak{A}_2 \rangle,\ A \in L^+ ((Y^{\mathfrak{A}_1})_\infty, Z)\}.$$

PROOF. We know from 2.1.1 that

$$P_2 \circ P_1 = P_2 \circ \varepsilon_{\mathfrak{A}_1} \circ \langle \mathfrak{A}_1 \rangle.$$

Appealing to the results of 2.2 we have in succession

$$\partial (P_2 \circ P_1) = \partial (P_2 \circ \varepsilon_{\mathfrak{A}_1} \circ \langle \mathfrak{A}_1 \rangle) = \partial (P_2 \circ \varepsilon_{\mathfrak{A}_1}) \circ \langle \mathfrak{A}_1 \rangle =$$
$$= \{A \in L^+ ((Y^{\mathfrak{A}_1})_\infty, Z) : \ A \circ \Delta_{\mathfrak{A}_1} \in \partial (\varepsilon_{\mathfrak{A}_2}) \circ \langle \mathfrak{A}_2 \rangle\} \circ \langle \mathfrak{A}_1 \rangle,$$

as required.

2.5.2. *For any projection* Pr *into a K-space Y we have*

$$\partial (P_2 \circ P_1) = \bigcup_{A \in \partial (P_2)} (\partial (A \circ \mathrm{Pr} \circ P_1) + \partial (A \circ \mathrm{Pr}^d \circ P_1)),$$

where $\mathrm{Pr}^d = I_Y - \mathrm{Pr}$ is the complementary projection. In particular,

$$\partial (P_2 \circ P_1) = \bigcup_{A \in \partial (P_2)} \partial (A \circ P_1).$$

We define $Y_0 = \mathrm{Pr}[Y]$. A direct verification shows that the subspace $(Y_0^{\partial (P_1)})_\infty$ of $(Y^{\partial (P_1)})_\infty$ is a component, that is, is the range of values of same projection Pr_0. Here

$$\mathrm{Pr}_0 \circ \Delta_{\partial (P_1)} = \Delta_{\partial (P_1)} \circ \mathrm{Pr}.$$

By 2.5.1, each element of $\partial(P_2 \circ P_1)$ has the form $B \circ \langle \partial(P_1) \rangle$, where

$$B \in L^+ ((Y^{\partial (P_1)})_\infty, Z) \text{ and } B \circ \Delta_{\partial (P_1)} \in \partial (P_2).$$

Defining $A = B \circ \Delta_{\partial P_1}$, we have

$$B \circ \mathrm{Pr}_0 \circ \Delta_{\partial (P_1)} = B \circ \Delta_{\partial (P_1)} \circ \mathrm{Pr} = A \circ \mathrm{Pr},$$
$$B \circ \mathrm{Pr}_0^d \circ \Delta_{\partial (P_1)} = B \circ \Delta_{\partial (P_1)} \circ \mathrm{Pr}^d = A \circ \mathrm{Pr}^d.$$

Again applying 2.5.1 and using the definition, we obtain

$$B \circ \mathrm{Pr}_0 \circ \langle \partial (P_1) \rangle \in \partial (A \circ \mathrm{Pr} \circ P_1),$$
$$B \circ \mathrm{Pr}_0^d \circ \langle \partial (P_1) \rangle \in \partial (A \circ \mathrm{Pr}^d \circ P_1),$$
$$B \circ \langle \partial (P_1) \rangle = B \circ \mathrm{Pr}_0 \circ \langle \partial (P_1) \rangle + B \circ \mathrm{Pr}_0^d \circ \langle \partial (P_1) \rangle.$$

Thus, we have

$$\partial (P_2 \circ P_1) \subset \{\partial (A \circ \mathrm{Pr} \circ P_1) + \partial (A \circ \mathrm{Pr}^d \circ P_1) : \ A \in \partial (P_2)\}.$$

The reverse inclusion obviously holds.

2.5.3. *If $P_1, \ldots, P_n: X \to Y$ are sublinear operators, then*

$$\partial(P_1 + \ldots + P_n) = \partial(P_1) + \ldots + \partial(P_n),$$

$$\partial(P_1 \vee \ldots \vee P_n) = \bigcup_{\substack{\alpha_1, \ldots, \alpha_n \in \Lambda(Y) \\ \alpha_1 + \ldots + \alpha_n = I_Y}} (\alpha_1 \circ \partial(P_1) + \ldots + \alpha_n \circ \partial(P_n)).$$

It is enough to prove this in the case $n = 2$. If we write $P_1 + P_2 = + \circ (P_1, P_2)$, where $+: Y \times Y \to Y$ denotes the operator of addition and (P_1, P_2) is defined by

$$(P_1, P_2)x = (P_1 x, P_2 x),$$

then by defining Pr: $(y_1, y_2) \mapsto (y_1, 0)$ and applying 2.5.2, we obtain the formula for the sum. Furthermore,

$$\partial(P_1 \vee P_2) = \partial(\varepsilon_2 \circ (P_1, P_2)) = \bigcup_{\alpha \in \partial(\varepsilon_2)} \partial(\alpha \circ (P_1, P_2)) =$$

$$= \bigcup_{\substack{\alpha_1 + \alpha_2 = I_Y \\ \alpha_1, \alpha_2 \in \Lambda(Y)}} \partial(\alpha_1 \circ P_1 + \alpha_2 \circ P_2) = \bigcup_{\substack{\alpha_1 + \alpha_2 = I_Y \\ \alpha_1, \alpha_2 \in \Lambda(Y)}} \partial(\alpha_1 \circ P_1) + \partial(\alpha_2 \circ P_2).$$

Thus, to complete the proof, we only have to show that $\partial(\alpha \circ P) = \alpha \circ \partial(P)$ for any sublinear operator P and any multiplier $\alpha \in \Lambda(Y)$. Since it is obvious that $\alpha \circ \partial(P) \subset \partial(\alpha \circ P)$, we take an $A \in \partial(\alpha \circ P)$ and prove that $A \in \alpha \circ \partial(P)$.

By 2.3.3, $A[X] \subset \alpha[Y]$, therefore, as is easily seen from the results on properties of multipliers, it is sufficient to consider the case when $\mathrm{Ker}(\alpha) = \{0\}$. But then we can again apply 2.3.3 to find that $\alpha^{-1} \circ A$ is in the support of P.

To conclude this section we give an interesting characterization of *lattice homomorphisms.*

2.5.4. *Let X be a vector lattice, Y a K-space, and $T \in L^+(X, Y)$. Then the following conditions are equivalent*:

(1) *T preserves the bounds of finite sets*;

(2) *for any $T' \in L^+(X, Y)$ such that $T' \leqslant T$ there is a multiplier $\alpha \in \Lambda(Y)$ for which $T' = \alpha \circ T$.*

We define sublinear operators $P_1, P_2: X \times X \to Y$ by the rules

$$P_1(x_1, x_2) = T(x_1 \vee x_2) \quad \text{and} \quad P_2(x_1, x_2) = Tx_1 \vee Tx_2.$$

Condition (1) means that these operators or, what is the same, their support sets are equal. By calculating $\partial(P_1)$ and $\partial(P_2)$, we obtain the required result.

§3. Subdifferentials of convex operators

The main aim of this section is to evaluate the subdifferentials of convex operators in an explicit form. We first give a method for studying such operators at interior points of their effective sets, and then we discuss the general case, where we have to employ specific methods. From this we derive rules for change of variables in the Young transform.

3.1. Let U be a convex set in a vector space X. An element $h \in X$ is called an *admissible direction* at a point $\bar{x} \in U$ if $\bar{x} + \bar{\alpha}h \in U$ for some $\bar{\alpha} > 0$. The set of all admissible directions at $\bar{x}$ forms a cone $Fd_{\bar{x}}(U)$, called the *cone of admissible directions.*

Now consider a K-space Y, a convex operator $F\colon X \to Y \cup \{+\infty\}$, and a point $\bar{x} \in \mathrm{dom}(F)$. For each

$$h \in \mathrm{Fd}_{\bar{x}}(\mathrm{dom}(F))$$

we put

$$z_{\bar{x},h}(\alpha) = (F(\bar{x} + \alpha h) - F\bar{x})/\alpha.$$

A simple calculation based on Jensen's inequality shows that the function $\alpha \mapsto Z_{\bar{x},h}(\alpha)$, defined on some interval $(0, \bar{\alpha})$, is decreasing. Thus, if this function is bounded below (for example, when $\partial_{\bar{x}}(F) \neq \emptyset$), then

$$F'(\bar{x})h = (o)\text{-}\lim_{\alpha \downarrow 0} \frac{F(\bar{x} + \alpha h) - F\bar{x}}{\alpha} = \inf_{\alpha} z_{\bar{x},h}(\alpha)$$

is defined and is called the *derivative of F in the admissible direction h at $\bar{x}$.* We need some properties of these derivatives.

3.1.1. *If $\partial_{\bar{x}}(F) \neq \emptyset$, then $F'(\bar{x})$ is a sublinear operator defined on the cone $Fd_{\bar{x}}(\mathrm{dom}(F))$, and*

$$\partial_{\bar{x}}(F) = \partial(F'(\bar{x})).$$

It is clear that $F'(\bar{x})$ is sublinear. If A is an operator such that $A(x - \bar{x}) \leqslant F\bar{x} - Fx$ for all $x \in X$, then for any admissible direction h and any sufficiently small $\alpha > 0$,

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But if $A \in \partial(F'(\bar{x}))$, then

$$Ax - A\bar{x} \leqslant F'(\bar{x})(x - \bar{x}) \leqslant Fx - F\bar{x}$$

for $x \in \mathrm{dom}(F)$.

3.1.2. *If $\bar{x}$ is an interior point of* $\mathrm{dom}(F)$ (*that is, the set* $\mathrm{dom}(F) - \bar{x}$ *is absorbing*), *then $\partial_{\bar{x}}(F) \neq \emptyset$ and* $\mathrm{dom}(F'(\bar{x})) = X$. *If, in addition, X is a K-space,* $\mathrm{dom}(F) = X$, *and F is (o)-continuous at $\bar{x}$, then $F'(\bar{x})$ is (o)-continuous on X.*

For it follows from the definition that for any $h \in X$ and any sufficiently small $\alpha > 0$

$$-z_{\bar{x},-h}(\alpha) \leqslant -F'(\bar{x})(-h) \leqslant F'(\bar{x})h \leqslant z_{\bar{x},h}(\alpha),$$

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for $x \in \mathrm{dom}(F)$.

3.1.2. *If $\bar{x}$ is an interior point of* $\mathrm{dom}(F)$ *(that is, the set* $\mathrm{dom}(F) - \bar{x}$ *is absorbing), then $\partial_{\bar{x}}(F) \neq \emptyset$ and* $\mathrm{dom}(F'(\bar{x})) = X$. *If, in addition, X is a K-space,* $\mathrm{dom}(F) = X$, *and F is (o)-continuous at $\bar{x}$, then $F'(\bar{x})$ is (o)-continuous on X.*

For it follows from the definition that for any $h \in X$ and any sufficiently small $\alpha > 0$

$$-z_{\bar{x},-h}(\alpha) \leqslant -F'(\bar{x})(-h) \leqslant F'(\bar{x})\,h \leqslant z_{\bar{x},h}(\alpha),$$

and this guarantees that $F'(\bar{x})$ is (o)-continuous at zero. It remains to point out that for any sublinear operator P

$$|\,Px - Py\,| \leqslant P(x - y) \vee P(y - x).$$

3.1.3. *Let X_1 and X be vector spaces, Y a K-space, $A \in L(X_1, X)$, and $F: X \to Y \cup \{+\infty\}$ a convex operator such that $A\bar{x} + y$ for some $y \in X$ is an interior point of the effective set* dom(F). *We consider the affine operator $A_y : x \mapsto Ax + y$. Then*

$$\partial_{\bar{x}}(F \circ A_y) = \partial_{A_y\bar{x}}(F) \circ A.$$

Direct calculation shows that $(F \circ A_y)'(\bar{x}) = F'(A_y\bar{x}) \circ A$, and it merely remains to apply 3.1.1. and 1.4.5.

3.1.4. *Let Y and Z be K-spaces, $F: X \to Y \cup \{+\infty\}$ a convex operator, and $G: Y \to Z \cup \{+\infty\}$ an increasing convex operator. Assume that* dom$(G) = Y$ *and that $\bar{x}$ is an interior point of* dom(F). *If G is (o)-continuous at $F\bar{x}$, then*

$$(G \circ F)'(\bar{x}) = G'(F\bar{x}) \circ F'(\bar{x}).$$

First of all we note that for any $x \in X$

$$F(\bar{x} + \alpha x) = F\bar{x} + \alpha F'(\bar{x})x + \alpha z(\alpha, x),$$

where $z(\alpha, x) \downarrow 0$ as $\alpha \downarrow 0$. For we can take

$$z(\alpha, x) = z_{\bar{x}, x}(\alpha) - F'(\bar{x})\, x.$$

For similar reasons we have for each $y \in Y$

$$G(F\bar{x} + \alpha y) = G(F\bar{x}) + \alpha G'(F\bar{x})y + \alpha w(\alpha, y),$$

where $w(\alpha, y) \downarrow 0$ as $\alpha \downarrow 0$. Having fixed a sufficiently small positive number α_0 we deduce from the monotonicity of G that for $0 < \alpha \leqslant \alpha_0$

$$\begin{aligned} G \circ F(\bar{x} + \alpha x) - G \circ F\bar{x} = G(F\bar{x} - \alpha F'(\bar{x})x + \alpha z(\alpha, x)) - G(F\bar{x}) \leqslant \\ \leqslant G(F\bar{x} + \alpha F'(\bar{x})x - \alpha z(\alpha_0, x)) - G(F\bar{x}) = \\ = \alpha G'(F\bar{x})(F'(\bar{x})x - z(\alpha_0, x)) - \alpha w(\alpha, F'(\bar{x})x + z(\alpha_0, x)) \leqslant \\ \leqslant \alpha G'(F\bar{x}) \circ F'(\bar{x})x - \alpha(G'(F\bar{x})z(\alpha_0, x) + w(\alpha, F'(\bar{x})x + z(\alpha_0, x))). \end{aligned}$$

For the same reasons

$$\begin{aligned} G \circ F(\bar{x} + \alpha\bar{x}) - G \circ \bar{F}x \geqslant G(\bar{F}x - \alpha F'(\bar{x})x) - G(F\bar{x}) = \\ = \alpha G'(F\bar{x}) \circ F'(\bar{x})x + \alpha w(\alpha, F'(\bar{x})x). \end{aligned}$$

Finally, we obtain the two-sided bounds

$$\begin{aligned} G'(F\bar{x}) \circ F'(\bar{x})x + w(\alpha, F'(\bar{x})x) \leqslant (G \circ F(\bar{x} - \alpha x) - G \circ F\bar{x})/\alpha \leqslant \\ \leqslant G'(F\bar{x}) \circ F'(\bar{x})x - G'(F\bar{x})z(\alpha_0, x) - w(\alpha, F'(\bar{x})x + z(\alpha_0, x)). \end{aligned}$$

Passing to the limit as $\alpha \downarrow 0$, we have

$$G'(F\bar{x}) \circ F'(\bar{x})x \leqslant (G \circ F)'(\bar{x})x \leqslant G'(F\bar{x}) \circ F'(\bar{x})x + G'(F\bar{x})z(\alpha_0, x).$$

The last inequality holds for any $\alpha_0 > 0$. Hence an application of 3.1.2 yields the required result.

3.1.5. REMARK. It is clear that Proposition 3.1.4 remains true under weaker hypotheses. For example, it is sufficient to assume that $G'(F\bar{x})$ is (o)-continuous. However, it is impossible to abandon the appropriate continuity condition altogether (see 3.2.6).

3.2. We now pass on to the computation of subdifferentials.

3.2.1. THEOREM. *Let $F\colon X \to Y \cup \{+\infty\}$ be a convex operator and $G\colon Y \to Z \cup \{+\infty\}$ an increasing convex operator such that* $\operatorname{dom}(G) \supset F[\operatorname{dom}(F)]$. *If $\bar{x}$ is an interior point of* $\operatorname{dom}(F)$ *and $F\bar{x}$ is an interior point of* $\operatorname{dom}(G)$ *such that $G'(F\bar{x})$ is (o)-continuous, then*

$$\partial_{\bar{x}}(G \circ F) = \{A \circ \langle \partial_{\bar{x}}(F)\rangle\colon\ A \circ \Delta_{\partial_{\bar{x}}(F)} \in \partial_{F\bar{x}}(G);\ A \in L^+((Y^{\partial_{\bar{x}}(F)})_\infty, Z)\}.$$

PROOF. Invoking the results of §3.1 and Theorem 2.5.1. we see at once that

$$\partial_{\bar{x}}(G \circ F) = \partial((G \circ F)'(\bar{x})) = \partial(G'(F\bar{x}) \circ F'(\bar{x})) =$$

$$= \{A \circ \langle \partial(F'(\bar{x}))\rangle\colon\ A \circ \Delta_{\partial(F'(\bar{x}))} \in \partial(G'(F\bar{x}));\ A \in L^+((Y^{\partial(F'(\bar{x}))})_\infty, Z)\}.$$

It remains to point out that $\partial_{\bar{x}}(F) = \partial(F'(\bar{x}))$ and $\partial_{F\bar{x}}(G) = \partial(G'(F\bar{x}))$.

3.2.2. COROLLARY. *For any projection* Pr *and any K-space Y*

$$\partial_{\bar{x}}(G \circ F) = \bigcup_{A \in \partial_{F\bar{x}}(G)} (\partial_{\bar{x}}(A \circ \mathrm{Pr} \circ F) + \partial_{\bar{x}}(A \circ \mathrm{Pr}^d \circ F)).$$

Note that the representation in 3.2.2 is substantially less refined than that in 3.2.1. For this reason the corollary remains true in much more general situations to be discussed below.

For the purpose of applications it is useful to attach a more precise meaning to the formula in 3.2.1.

3.2.3. *Let $F\colon X \to Y \cup \{+\infty\}$ be a convex operator, $G\colon Y \to Z \cup \{+\infty\}$ an increasing convex operator such that* $\operatorname{dom}(G) \supset F[\operatorname{dom}(F)]$, *and $\bar{x}$ an interior point of* $\operatorname{dom}(F)$ *such that $F\bar{x}$ is an interior point of* $\operatorname{dom}(G)$.

We consider the local approximation $F_\varepsilon\colon X \to Y \cup \{+\infty\}$ of F given by

$$F_\varepsilon\colon x \to \sup\{Ax - A\bar{x} + F\bar{x}\colon A \in \partial_{\bar{x}}(F)\},$$

Then F_ε is a convex operator, $\operatorname{dom}(F_\varepsilon) = X$, *and*

$$\partial_{\bar{x}}(F_\varepsilon) = \partial_{\bar{x}}(F), \quad (G \circ F_\varepsilon)'(\bar{x}) = G'(F\bar{x}) \circ F'(\bar{x}),$$

$$\partial_{\bar{x}}(G \circ F_\varepsilon) = \{A \circ \langle \partial_{\bar{x}}(F)\rangle\colon\ A \circ \Delta_{\partial_{\bar{x}}(F)} \in \partial_{F\bar{x}}(G);\ A \in L^+((Y^{\partial_{\bar{x}}(F)})_\infty, Z)\}.$$

First of all we note that the local approximation F_ε has the representation

$$F_\varepsilon x = F\bar{x} + F'(\bar{x})(x - \bar{x}).$$

Thus,

$$(G\circ F_\varepsilon)'(\bar x)\,x=(o)\text{-}\lim_{\alpha\downarrow 0}\alpha^{-1}(G(F_\varepsilon(x+\alpha x))-G\circ F_\varepsilon\bar x)=$$
$$=(o)\text{-}\lim_{\alpha\downarrow 0}\alpha^{-1}(G(F\bar x+F'(\bar x)(\alpha x))-G(F\bar x))=G'(F\bar x)\circ F'(\bar x)\,x.$$

3.2.4. REMARK. Generally speaking, to establish 3.2.3 we do not need to make use of directional derivatives. It is sufficient simply to shorten the expression for the subdifferential of a composition by using the following representations:

$$F_\varepsilon=\varepsilon_{\mathfrak{A}}\circ\langle\mathfrak{A}\rangle_y,\quad \mathfrak{A}=\partial_{\bar x}(F),\quad y=\Delta_{\mathfrak{A}}F\bar x-\langle\mathfrak{A}\rangle\,\bar x.$$

We carry out the corresponding construction in a substantially more important and complicated situation.

Let $\mathfrak{A}$ be a weakly order-bounded set in $L(X, Y)$, and $y\in(Y^{\mathfrak{A}})_\infty$. An operator $F\colon X\to Y\cup\{+\infty\}$ is called *regular* if it has a representation $F=\varepsilon^{\mathfrak{A}}\circ\langle\mathfrak{A}\rangle_y$.

3.2.5. THEOREM. *Let $F\colon X\to Y\cup\{+\infty\}$ be a regular convex operator and $G\colon Y\to Z\cup\{+\infty\}$ an increasing convex operator such that $F\bar x$ is an interior point of* dom(G). *Then*

$$\partial_{\bar x}(G\circ F)=\{A\circ\langle\mathfrak{A}\rangle\colon\ A\in L^+((Y^{\mathfrak{A}})_\infty, Z);\ A\circ\Delta_{\mathfrak{A}}\in\partial_{F\bar x}(G);\ A\circ\langle\mathfrak{A}\rangle_y\,\bar x=A\circ\Delta_{\mathfrak{A}}F\bar x\}.$$

PROOF. By 3.1.3,

$$\partial_{\bar x}(G\circ F)=\partial_{\bar x}(G\circ\varepsilon_{\mathfrak{A}}\circ\langle\mathfrak{A}\rangle_y)=\partial_{\langle\mathfrak{A}\rangle_y\bar x}(G\circ\varepsilon_{\mathfrak{A}})\circ\langle\mathfrak{A}\rangle.$$

We put $\langle\mathfrak{A}\rangle_y\bar x=\bar y$ and claim that

$$\partial_{\bar y}(G\circ\varepsilon_{\mathfrak{A}})=\{A\in L^+((Y^{\mathfrak{A}})_\infty, Z)\colon\ A\circ\Delta_{\mathfrak{A}}\in\partial_{\varepsilon_{\mathfrak{A}}\bar y}(G);\ A\bar y=A\circ\Delta_{\mathfrak{A}}\circ\varepsilon_{\mathfrak{A}}\bar y\}.$$

To see this, we first assume that for all $z\in(Y^{\mathfrak{A}})_\infty$

$$Az-A\bar y\leqslant G\circ\varepsilon_{\mathfrak{A}}z-G\circ\varepsilon_{\mathfrak{A}}\bar y.$$

If $h\in(Y^{\mathfrak{A}})_\infty$, $h\leqslant 0$, then for sufficiently small values of $\alpha>0$ the element $\bar y+\alpha h$ belongs to $\operatorname{dom}(G\circ\varepsilon_{\mathfrak{A}})$, and

$$\alpha Ah\leqslant G\circ\varepsilon_{\mathfrak{A}}(\bar y+\alpha h)-G\circ\varepsilon_{\mathfrak{A}}\bar y\leqslant 0,$$

so that $A\in L^+((Y^{\mathfrak{A}})_\infty, Z)$. Furthermore, since

$$\Delta_{\mathfrak{A}}\circ\varepsilon_{\mathfrak{A}}\geqslant I_{(Y^{\mathfrak{A}})_\infty},\qquad \varepsilon_{\mathfrak{A}}\circ\Delta_{\mathfrak{A}}=I_Y$$

the element $\Delta_{\mathfrak{A}}\circ\varepsilon_{\mathfrak{A}}\bar y$ satisfies

$$0\leqslant A\circ\Delta_{\mathfrak{A}}\circ\varepsilon_{\mathfrak{A}}\bar y-A\bar y\leqslant G(\varepsilon_{\mathfrak{A}}\circ\Delta_{\mathfrak{A}}\circ\varepsilon_{\mathfrak{A}}\bar y)-G\circ\varepsilon_{\mathfrak{A}}\bar y=0.$$

Also, for any $y_0\in Y$

$$A\circ\Delta_{\mathfrak{A}}y_0-A\circ\Delta_{\mathfrak{A}}\circ\varepsilon_{\mathfrak{A}}\bar y=A\circ\Delta_{\mathfrak{A}}y_0-A\bar y\leqslant G\circ\varepsilon_{\mathfrak{A}}\circ\Delta_{\mathfrak{A}}y_0-G\circ\varepsilon_{\mathfrak{A}}\bar y=Gy_0-G\circ\varepsilon_{\mathfrak{A}}\bar y,$$

so that $A \circ \Delta_{\mathfrak{A}} \in \partial_{\varepsilon_{\mathfrak{A}}\bar{y}}(G)$. Conversely, if we know that

$$A \in L^{+}((Y^{\mathfrak{A}})_{\infty}, Z), \quad A \circ \Delta_{\mathfrak{A}} \in \partial_{\varepsilon_{\mathfrak{A}}\bar{y}}(G), \quad A \circ \Delta_{\mathfrak{A}} \circ \varepsilon_{\mathfrak{A}}\bar{y} = A\bar{y},$$

then for any $z \in (Y^{\mathfrak{A}})_{\infty}$

$$Az - A\bar{y} = Az - A \circ \Delta_{\mathfrak{A}} \circ \varepsilon_{\mathfrak{A}}\bar{y} \leqslant A \circ \Delta_{\mathfrak{A}}(\varepsilon_{\mathfrak{A}}z) - A \circ \Delta_{\mathfrak{A}}(\varepsilon_{\mathfrak{A}}\bar{y}) \leqslant G \circ \varepsilon_{\mathfrak{A}}z - G \circ \varepsilon_{\mathfrak{A}}\bar{y},$$

which indicates that $A \in \partial_{\bar{y}}(G \circ \varepsilon_{\mathfrak{A}})$.

Thus, bearing in mind that $\varepsilon_{\mathfrak{A}} \circ \langle \mathfrak{A} \rangle_y \bar{x} = F\bar{x}$, we finally obtain

$$\partial_{\bar{x}}(G \circ F) = \partial_{\mathfrak{A}\langle\rangle_y \bar{x}}(G \circ \varepsilon_{\mathfrak{A}}) \circ \langle \mathfrak{A} \rangle =$$
$$= \{A \in L^{+}((Y^{\mathfrak{A}})_{\infty}, Z)\colon\ A \circ \Delta_{\mathfrak{A}} \in \partial_{F\bar{x}}(G);\ A \circ \langle \mathfrak{A} \rangle_y \bar{x} = A \circ \Delta_{\mathfrak{A}} F\bar{x}\} \circ \langle \mathfrak{A} \rangle.$$

3.2.6. REMARK. Theorem 3.2.5 shows that even the regularity of F does not, in general, guarantee the validity of 3.2.1 without additional conditions on G. By analyzing the formula it is not hard to choose concrete examples. Thus, if $X = l_{\infty}$, $G \in \partial(x \mapsto \overline{\lim_{n}}\, x_n)$ is a *generalized limit*, and F is the operator $Fx = x \vee 0$, then for $\bar{x}_n = -1/n$ and $h_n = 1$ we have $(G \circ F)'(\bar{x})h = 1$, whereas $F'(x)h = 0$.

3.3. As applications of the above theorems we can obtain various results on the representation of subdifferentials, and we mention a few of them now.

3.3.1. *Let Y be a K-space, $F_1, \ldots, F_n\colon X \to Y \cup \{+\infty\}$ convex operators acting in Y, and $\bar{x}$ an interior point of a set $U \subset \operatorname{dom}(F_1) \cap \ldots \cap \operatorname{dom}(F_n)$. Then*

$$\partial_{\bar{x}}(F_1 + \ldots + F_n) = \partial_{\bar{x}}(F_1) + \ldots + \partial_{\bar{x}}(F_n),$$

$$\partial_{\bar{x}}(F_1 \vee \ldots \vee F_n) = \bigcup_{(\alpha_1, \ldots, \alpha_n) \in \Gamma(\bar{x})} (\alpha_1 \circ \partial_{\bar{x}}(F_1) + \ldots + \alpha_n \circ \partial_{\bar{x}}(F_n)),$$

where the union ranges over the following set:

$$\Gamma(\bar{x}) = \{(\alpha_1, \ldots, \alpha_n) \in \Lambda(Y)^n\colon\ \sum_{k=1}^{n} \alpha_k = I_Y;\ \sum_{k=1}^{n} \alpha_k \circ F_k\bar{x} = F_1\bar{x} \vee \ldots \vee F_n\bar{x}\}.$$

The next proposition determines the form of the positive operators on $(Y^{\mathfrak{A}})_{\infty}$ that appear in the formulae above.

3.3.2. *Let Y and Z be K-spaces. For any set $\mathfrak{A}$ and any operator $A \in L^{+}((Y^{\mathfrak{A}})_{\infty}, Z)$ the following system of conditions is compatible*:

$$\alpha \in \partial(\varepsilon_{\mathfrak{A}, Z}), \quad A = \alpha \circ \langle A \circ \Delta_{\mathfrak{A}, Y} \circ \partial(\varepsilon_{\mathfrak{A}, Y}) \rangle.$$

For let $B \doteq A \circ \Delta_{\mathfrak{A}, Y}$. Then $B \in L^{+}(Y, Z)$ and, by 2.2.1,

$$\partial(B \circ \varepsilon_{\mathfrak{A}, Y}) = \{C \in L^{+}((Y^{\mathfrak{A}})_{\infty}, Z)\colon\ C \circ \Delta_{\mathfrak{A}, Y} = B\}.$$

Furthermore, by applying 1.4.2 we show by direct calculation

$$\partial(B \circ \varepsilon_{\mathfrak{A}, Y}) = \operatorname{cop}(B \circ \partial(\varepsilon_{\mathfrak{A}, Y})).$$

This result enables us to find other representations of the subdifferentials of a composition.

3.3.3. *Let $F: X \to Y \cup \{+\infty\}$ be a convex operator, and*

$$G: Y \to Z \cup \{+\infty\}$$

an increasing convex operator such that

$$\operatorname{dom}(G) \supset F[\operatorname{dom}(F)].$$

If $\bar{x}$ is an interior point of dom(*F*) *and $F\bar{x}$ an interior point of* dom(*G*), *then*

$$\partial_{\bar{x}}(G \circ F_\varepsilon) = \{\alpha \circ \langle A \circ \partial_{\bar{x}}(F) \rangle : \alpha \in \partial(\varepsilon_{\partial_{\bar{x}}(F)}, z);\ A \in \partial_{F\bar{x}}(G)\},$$

$$\partial_{\bar{x}}(G \circ F_\varepsilon) = \{\alpha \circ \langle \partial_{F\bar{x}}(G) \circ \partial_{\bar{x}}(F) \rangle : \alpha \in \partial(\varepsilon_{\partial_{F\bar{x}}(G) \circ \partial_{\bar{x}}(F)}, z)\}.$$

Both of these representations follow immediately from 3.2.3 and 3.3.2.

3.4. We now concern ourselves with the study of the subdifferentials of a composition at arbitrary points of the effective sets. The theory of sublinear operators that are not defined everywhere has by no means yet reached an adequate state, therefore, definitive results of the quality of those obtained above are still conspicuous by their absence. However, the situation is far from hopeless. In particular, it turns out that the formulae for calculating subdifferentials most commonly used in applications carry over to the case of boundary points under conditions that are not too burdensome. It is helpful to bear in mind that the approach described here, based on the Mazur–Orlicz theorem, yields new proofs of some of the results described in the preceding subsection.

3.4.1. THEOREM. *Let X be a vector space, Y a K-space, $P: X \to Y \cup \{+\infty\}$ a sublinear operator, and $(x_\xi)_{\xi \in \Xi}$ and $(y_\xi)_{\xi \in \Xi}$ subsets of X and Y, respectively. If the smallest cone in X containing the elements $(-x_\xi)$ and the effective set* dom(*P*) *is a subspace, then the following assertions are equivalent*:

(1) *There is an operator $A \in L(X, Y)$ such that*

$$A \in \partial(P), \quad Ax_\xi \geqslant y_\xi \qquad (\xi \in \Xi).$$

(2) *For any $\lambda_1, \ldots, \lambda_n \in R^+$ and any $\xi_1, \ldots, \xi_n \in \Xi$ such that*

$$\sum_{i=1}^{n} \lambda_i x_{\xi_i} \in \operatorname{dom}(P)$$

we have

$$P\Big(\sum_{i=1}^{n} \lambda_i x_{\xi_i}\Big) \geqslant \sum_{i=1}^{n} \lambda_i y_{\xi_i}.$$

PROOF. We have to verify that (2) ⇒ (1). To this end we consider the subspace X_1, the direct sum of the lines $(R)_{\xi \in \Xi}$, and we define operators B and C by

$$Bz = \sum_{\xi \in \Xi} z(\xi)\, x_\xi, \qquad B \in L(X_1, X),$$

$$Cz = \sum_{\xi \in \Xi} z(\xi)\, y_\xi, \qquad C \in L(X_1, Y).$$

Here $z \in X_1$ is regarded as a function: $\Xi \to R$ that is non-zero only on finitely many points of Ξ.

Let X_0 denote the smallest cone containing both dom(P) and the family $(-x_\xi)_{\xi\in\Xi}$. By hypothesis, we have

$$X_0 = B[X_1^+] - \mathrm{dom}(P) = \mathrm{dom}(P) - B[X_1^+],$$
$$P(Bh) \geqslant Ch \qquad (h \in X_1^+ \cap B^{-1}[\mathrm{dom}(P)]).$$

Thus, for every $x_0 \in X_0$ there are elements $z_1, z_2 \in X_1^+$ such that

$$x_0 + Bz_1 \in \mathrm{dom}(P), \quad -x_0 + Bz_2 \in \mathrm{dom}(P), \quad P(Bz_1 + Bz_2) \geqslant C(z_1 + z_2).$$

Consequently, we have the bounds

$$P(x_0 + Bz_1) - Cz_1 \geqslant - P(-x_0 + Bz_2) + P(Bz_1 + Bz_2) - Cz_1 \geqslant$$
$$\geqslant C(z_1 + z_2) - Cz_1 - P(-x_0 + Bz_2) = -P(-x_0 + Bz_2) + Cz_2.$$

In other words, each $x_0 \in X_0$ determines an element

$$P_1x_0 = \inf\{P(x_0 + Bz_1) - Cz_1 \colon z_1 \in X_1^+,\ x_0 + Bz_1 \in \mathrm{dom}(P)\}.$$

It is not hard to check that the resulting operator P_1 defined on X_0 is sublinear. Appealing to 1.4.1, we take any element of the support set of P_1 and extend it somehow to a linear operator defined on X. A routine calculation shows that the resulting operator has the required properties.

3.5. The Mazur–Orlicz theorem furnishes a suitable apparatus for the calculation of subdifferentials. First of all we apply it to describe support sets. We need the following definitions.

Two cones H_1 and H_2 (not necessarily distinct) in a vector space X are said to be *in general position* if $H_1 - H_2 \doteq H_2 - H_1$, in other words, if the smallest cone containing H_1 and $-H_2$ is a subspace of X. The cones $H_1, \ldots, H_n$ are *in general position* if for any rearrangement $\{i_1, \ldots, i_n\}$of the indexing set $\{1, \ldots, n\}$ the cones H_{i_k} and $\bigcap_{s=k+1}^{n} H_{i_s}$ are in general position for all $k = 1, \ldots, n-1$.

Let us list some of the most frequently encountered cases of cones in general position.

3.5.1. *If $H_1, \ldots, H_n$ are subspaces of X, then they are in general position.*

3.5.2. *If the cones $H_1, \ldots, H_n$ coincide, then they are in general position.*

3.5.3. *If the cone $H_1 \cap \ldots \cap H_n$ contains an interior point of each of $H_1, \ldots, H_n$ with the possible exception of one, then these cones are in general position.*

We only have to verify 3.5.3. To do this it is clearly sufficient to show that if H_1 and H_2 are cones such that $H_1 \cap H_2$ contains an interior point $\bar{x}$ of one of them (say H_2, for the sake of definiteness), then H_1 and H_2 are in general position. This follows at once from the fact that the cone $H_2 - H_1$ must be an absorbing set, because $H_2 - H_1 \supset H_2 - \bar{x}$. Hence, this cone is the whole space X.

3.5.4. *Let* $P_1, \ldots, P_n : X \to Y \cup \{+\infty\}$ *be sublinear operators whose effective sets* $\operatorname{dom}(P_1), \ldots, \operatorname{dom}(P_n)$ *are in general position. Then.*

$$\partial(P_1 + \ldots + P_n) = \partial(P_1) + \ldots + \partial(P_n).$$

It is sufficient to consider the case $n = 2$ and to verify that

$$\partial(P_1 + P_2) \subset \partial(P_1) + \partial(P_2).$$

To this end we take $A \in \partial(P_1 + P_2)$. Then

$$P_1 x \geqslant Ax - P_2 x$$

for all

$$x \in \operatorname{dom}(P_2) \cap \operatorname{dom}(P_1).$$

Hence, if

$$\sum_{i=1}^{n} \lambda_i x_i \in \operatorname{dom}(P_1), \quad \text{where } \lambda_i \in R^+ \text{ and } x_i \in \operatorname{dom}(P_2),$$

then

$$\sum_{i=1}^{n} \lambda_i x_i \in \operatorname{dom}(P_1) \cap \operatorname{dom}(P_2)$$

and we obtain the bound

$$P_1\Big(\sum_{i=1}^{n} \lambda_i x_i\Big) \geqslant -P_2\Big(\sum_{i=1}^{n} \lambda_i x_i\Big) + A\Big(\sum_{i=1}^{n} \lambda_i x_i\Big) \geqslant -\sum_{i=1}^{n} \lambda_i (P_2 x_i - A x_i).$$

Thus, applying 3.4.1 we find an $A_1 \in L(X, Y)$ for which

$$A_1 \in \partial(P_1), \quad A_1 x \geqslant -P_2 x + Ax \qquad (x \in \operatorname{dom}(P_2)).$$

It merely remains to point out that $A - A_1 \in \partial(P_2)$.

3.5.5. *Let* Y *be an ordered vector space and* $P_1 : X \to Y \cup \{+\infty\}$ *a sublinear operator. Also, let* Z *be a K-space and* $P_2 : Y \to Z \cup \{+\infty\}$ *an increasing sublinear operator. If the cone* $\operatorname{dom}(P_2)$ *and the support hull of the set* $P_1[\operatorname{dom}(P_1)]$ *are in general position, then*

$$\partial(P_2 \circ P_1) = \bigcup_{A \in \partial(P_2)} \partial(A \circ P_1).$$

This proposition is proved in just the same way as 3.5.4.

We give some simple consequences of 3.5.4. and 3.5.5.

3.5.6. *If* Y *is a K-space and* Pr *is any projection in* Y, *then*

$$\partial(P_2 \circ P_1) = \bigcup_{A \in \partial(P_2)} (\partial(A \circ \Pr \circ P_1) + \partial(A \circ \Pr^d \circ P_2)).$$

3.5.7. *Let* Y *be a K-space and* $P_1, \ldots, P_n : X \to Y \cup \{+\infty\}$ *sublinear operators whose effective sets* $\operatorname{dom}(P_1), \ldots, \operatorname{dom}(P_n)$ *are in general position. Then*

$$\partial (P_1 \vee \dots \vee P_n) = \bigcup_{\substack{\alpha_1+\dots+\alpha_n=I_Y \\ \alpha_1,\dots,\alpha_n \in \Lambda(Y)}} (\partial (\alpha_1 \circ P_1) + \dots + \partial (\alpha_n \circ P_n)).$$

3.6. We now establish the basic formulae for calculating subdifferentials at arbitrary points.

3.6.1. THEOREM. *Let Y be a K-space and $F_1, \dots, F_n: X \to Y \cup \{+\infty\}$ convex operators such that the cones of admissible directions*

$$\mathrm{Fd}_{\bar{x}}(\mathrm{dom}(F_1)), \dots, \mathrm{Fd}_{\bar{x}}(\mathrm{dom}(F_n))$$

are in general position. Then the Moreau–Rockafellar formula bolds:

$$\partial_{\bar{x}}(F_1 + \dots + F_n) = \partial_{\bar{x}}(F_1) + \dots + \partial_{\bar{x}}(F_n).$$

PROOF. We only have to show that when

$$\partial_{\bar{x}}(F_1 + \dots + F_n) \neq \varnothing$$

then each of the operators $F_1, \dots, F_n$ is differentiable in all directions of the corresponding cone of admissible directions. For by 3.1.4 and 3.1.5, in this case

$$(F_1 + \dots + F_n)'(\bar{x}) = F_1'(\bar{x}) + \dots + F_n'(\bar{x}),$$

and it remains to use 3.5.4.

It is sufficient to consider the case $n = 2$. We may assume without loss of generality that $\bar{x} = 0$ and $F_1\bar{x} = F_2\bar{x} = 0$, since otherwise, we can take the operators

$$G_1: x \mapsto F_1(\bar{x} + x) - F_1\bar{x}, \quad G_2: x \mapsto F_2(\bar{x} + x) - F_2x,$$

for which clearly

$$G_10 = G_20 = 0,$$

$$\partial_0(G_1) = \partial_{\bar{x}}(F_1), \quad \partial_0(G_2) = \partial_{\bar{x}}(F_2), \quad \partial_0(G_1 + G_2) = \partial_{\bar{x}}(F_1 + F_2).$$

Let $h \in Fd_{\bar{x}}(\mathrm{dom}(F_1))$. By hypothesis, we can find elements

$$h_1 \in \mathrm{Fd}_{\bar{x}}(\mathrm{dom}(F_1)) \text{ and } h_2 \in \mathrm{Fd}_{\bar{x}}(\mathrm{dom}(F_2))$$

such that $h_2 = h + h_1$. We may assume that $3h, 3h_1 \in \mathrm{dom}(F_1)$ and $h_2 \in \mathrm{dom}(F_2)$. Then for sufficiently small positive α

$$F_1(\alpha h_1) \leqslant \alpha F_1 h_1, \quad F_2(\alpha h_2) \leqslant \alpha F_2 h_2,$$
$$F_1(\alpha h_2) \geqslant A(\alpha h_2) - F_2(\alpha h_2)$$

for any operator A in $\partial_{\bar{x}}(F_1 + F_2)$. Furthermore, because of the convexity of F_1 and the simplifying assumptions we have made,

$$F_1(\alpha h_2) = F_1(3\alpha h/3 + 3\alpha h_1/3 + 0/3) \leqslant F_1(3\alpha h)/3 + F_1(3\alpha h_1)/3 \leqslant$$
$$\leqslant F_1(3\alpha h)/3 + \alpha F_1 h_1.$$

Finally, we obtain

$$F_1(3\alpha h)/3 \geqslant \alpha(Ah_2 - F_2h_2 - F_1h_1).$$

Thus, the function $\alpha \mapsto F_1(\alpha h)/\alpha$, which is defined for sufficiently small values of α, is bounded below, and this means that F_1 is differentiable at zero in the direction h. For analogous reasons, F_2 is differentiable in all directions belonging to $Fd_{\bar{x}}(\mathrm{dom}(F_2))$. This completes the proof.

In fact, the method described in the proof of 3.6.1 yields more. For example, the following version of the Mazur–Orlicz theorem holds.

3.6.2. *Let Y be a K-space and suppose that we are given a convex operator*

$$F: X \to Y \cup \{+\infty\}$$

and families

$$(x_\xi)_{\xi \in \Xi} \subset X \text{ and } (y_\xi)_{\xi \in \Xi} \subset Y.$$

If the conical hull of $(x_\xi)_{\xi \in \Xi}$ and the cone $Fd_{\bar{x}}(\mathrm{dom}(F))$ of admissible directions are in general position, then the following assertions are equivalent:

(1) *There is an operator $A \in L(X, Y)$ such that*

$$A \in \partial_{\bar{x}}(F), \quad Ax_\xi \geqslant y_\xi \quad (\xi \in \Xi).$$

(2) *For any $\lambda_1, \ldots, \lambda_n \in R^+$ and $\xi_1, \ldots, \xi_n \in \Xi$ such that*

$$\sum_{i=1}^{n} \lambda_i < 1 \text{ and } \bar{x} + \sum_{i=1}^{n} \lambda_i x_{\xi_i} \in \mathrm{dom}(F),$$

we have

$$F\left(\bar{x} + \sum_{i=1}^{n} \lambda_i x_{\xi_i}\right) - F\bar{x} \geqslant \sum_{i=1}^{n} \lambda_i y_{\xi_i}.$$

This modification of 3.4.1 is also used frequently in applications. All the same, in concrete situations it is usually simpler to pass directly to directional derivatives, thereby reducing the matter to the case of sublinear operators.

3.6.3. *Let Y be a K-space and $F, G: X \to Y \cup \{+\infty\}$ be convex operators such that*

$$\mathrm{dom}(F) = \mathrm{dom}(G).$$

If $F + G \geqslant 0$, then there is an $A \in L(X, Y)$ and a $y \in Y$ such that

$$F - A_y \geqslant 0 \text{ and } A_y + G \geqslant 0.$$

This representation is of independent interest and is called the *proposition on the separation of operators.* Since it is not used in what follows, we omit the proof.

3.6.4. THEOREM. *Let X be a vector space, Y an ordered vector space, and Z a K-space. Also, let*

$$F\colon X \to Y \cup \{+\infty\}$$

be a convex operator and

$$G\colon Y \to Z \cup \{+\infty\}:$$

be an increasing convex operator such that $\operatorname{dom}(G) \supset F[\operatorname{dom}(F)]$, *and that* $F\bar{x}$ *for some* $\bar{x} \in \operatorname{dom}(F)$ *is an interior point of* $\operatorname{dom}(G)$. *Then Gol'shtein's formula holds*:

$$\partial_{\bar{x}}(G \circ F) = \bigcup_{A \in \partial_{F\bar{x}}(G)} \partial_{\bar{x}}(A \circ F).$$

PROOF. If $C \in \partial_{\bar{x}}(A \circ F)$ for some $A \in \partial_{F\bar{x}}(G)$, then

$$Cx - C\bar{x} \leqslant A \circ Fx - A \circ F\bar{x} \leqslant G(Fx) - G(F\bar{x}),$$

so that $C \in \partial_{\bar{x}}(G \circ F)$.

Let us now assume that $C \in \partial_{\bar{x}}(G \circ F)$. Note first of all that for any $x \in \operatorname{dom}(F)$

$$Cx - C\bar{x} \leqslant G'(F\bar{x})(Fx - F\bar{x}).$$

In fact, for all α such that $0 < \alpha < 1$ we have by definition

$$\begin{aligned}\alpha C\,(x-\bar{x}) &= C\,(\alpha x+(1-\alpha)\,\bar{x}) - C\bar{x} \leqslant G\,(F\,(\alpha x+(1-\alpha)\,\bar{x})) - G\,(F\bar{x}) \leqslant \\ &\leqslant G\,(\alpha Fx+(1-\alpha)\,F\bar{x}) - G\,(F\bar{x}) = G\,(F\bar{x}+\alpha\,(Fx-F\bar{x})) - G\,(F\bar{x}) = \\ &= \alpha G'\,(F\bar{x})\,(Fx-F\bar{x}) + \alpha w\,(\alpha),\end{aligned}$$

where $w(\alpha) \downarrow 0$ as $\alpha \downarrow 0$.

The question of interest to us is whether there is an operator $A \in \partial_{F\bar{x}}(G)$ such that

$$Cx - C\bar{x} \leqslant A(Fx - F\bar{x}) \qquad (x \in \operatorname{dom}(F)).$$

Using the above bound, we see that for any $\lambda_1, \ldots, \lambda_n \in R^+$ and any $x_1, \ldots, x_n \in \operatorname{dom}(F)$

$$\sum_{i=1}^{n} \lambda_i\,(Cx_i - C\bar{x}) \leqslant G'\,(F\bar{x})\left(\sum_{i=1}^{n} \lambda_i\,(Fx_i - F\bar{x})\right).$$

For we may assume without loss of generality that

$$\lambda_0 = \sum_{i=1}^{n} \lambda_i$$

is strictly positive. Then the element

$$x_0 = \frac{1}{\lambda_0} \sum_{i=1}^{n} \lambda_i x_i$$

belongs to dom(F), hence

$$\sum_{i=1}^{n} \lambda_i (Cx_i - C\overline{x}) = \lambda_0 (Cx_0 - C\overline{x}) \leqslant \lambda_0 G(F\overline{x})(Fx_0 - F\overline{x}) =$$

$$= G(F\overline{x})(\lambda_0 (Fx_0 - F\overline{x})) \leqslant G'(F\overline{x}) \left(\sum_{i=1}^{n} \lambda_i F x_i - \lambda_0 F\overline{x}\right).$$

Consequently, the existence of the required operator A is assured by the Mazur–Orlicz theorem. Thus, $C \in \partial_{\overline{x}}(A \circ F)$, and this completes the proof.

3.6.5. THEOREM. *Let Y be a K-space and let*

$$F_1, \ldots, F_n \colon X \to Y \cup \{+\infty\}$$

be convex operators such that the cones of admissible directions

$$\mathrm{Fd}_{\overline{x}}(\mathrm{dom}(F_1)), \ldots, \mathrm{Fd}_{\overline{x}}(\mathrm{dom}(F_n))$$

are in general position. Then the Dubovitskii–Milyutin formula holds:

$$\partial_{\overline{x}}(F_1 \vee \ldots \vee F_n) = \bigcup_{(\alpha_1, \ldots, \alpha_n) \in \Gamma(\overline{x})} (\partial_{\overline{x}}(\alpha_1 \circ F_1) + \ldots + \partial_{\overline{x}}(\alpha_n \circ F_n)),$$

where the union is taken over the following set:

$$\Gamma(\overline{x}) = \{(\alpha_1, \ldots, \alpha_n) \in \Lambda(Y)^n \colon \sum_{k=1}^{n} \alpha_k = I_Y;\ \sum_{k=1}^{n} \alpha_k \circ F_k \overline{x} = F_1 \overline{x} \vee \ldots \vee F_n \overline{x}\}.$$

We obtain this fact as a consequence of the following more general assertion.

3.6.6. *Let Y be a vector lattice, Z a K-space, and $A \in L^+(Y, Z)$. Let*

$$F_1, \ldots, F_n \colon X \to Y \cup \{+\infty\}$$

be convex operators such that the cones of admissible directions

$$\mathrm{Fd}_{\overline{x}}(\mathrm{dom}(F_1)), \ldots, \mathrm{Fd}_{\overline{x}}(\mathrm{dom}(F_n))$$

are in general position. Then the following decomposition formula holds:

$$\partial_{\overline{x}}(A \circ F_1 \vee \ldots \vee F_n) =$$

$$= \{\sum_{k=1}^{n} \partial_{\overline{x}}(A_k \circ F_k) \colon A_k \in L^+(Y, Z), \sum_{k=1}^{n} A_k = A;\ \sum_{k=1}^{n} A_k \circ F_k \overline{x} = A(F_1 \overline{x} \vee \ldots \vee F_n \overline{x})\}.$$

We consider the convex operators

$$\varepsilon \colon Y^n \to Y, \quad \varepsilon(y_1, \ldots, y_n) = y_1 \vee \ldots \vee y_n,$$
$$(F_1, \ldots, F_n) \colon X \to Y^n, \quad (F_1, \ldots, F_n)x = (F_1 x, \ldots, F_n x).$$

Then it is clear that we have the following representation:

$$A \circ F_1 \vee \ldots \vee F_n = A \circ \varepsilon \circ (F_1, \ldots, F_n).$$

Thus, by 3.6.4,

$$\partial_{\bar{x}} (A \circ F_1 \vee \ldots \vee F_n) = \bigcup_{B \in \partial_{(F_1 \bar{x}, \ldots, F_n \bar{x})}(A \circ \varepsilon)} \partial_{\bar{x}} (B \circ (F_1, \ldots, F_n)).$$

A direct calculation (see 2.2.1) shows that

$$\partial (A \circ \varepsilon) = \{(y_1, \ldots, y_n) \mapsto \sum_{k=1}^{n} A_k y_k \colon A_k \in L^+(Y, Z), \ \sum_{k=1}^{n} A_k = A\}.$$

Thus, the required result follows by appealing to 1.4.2 and 3.6.1.

3.7. In this subsection, we apply the above results to the study of the Young transform.

Let Y be a K-space and $F \colon X \to Y \cup \{+\infty\}$ a convex operator such that $\operatorname{dom}(F) \neq \varnothing$. For $A \in L(X, Y)$ we define

$$F^* A = \sup_{x \in \operatorname{dom}(F)} (Ax - Fx).$$

The operator F^* is called the *Young transform* of F. As a rule, we study the Young transform with the aid of the following construction.

For $(x, t) \in X \times R$ we define

$$H_F(x, t) = \begin{cases} tF(x/t), & t > 0, \ x/t \in \operatorname{dom}(F), \\ 0, & t = 0, \ x = 0, \\ +\infty, & \text{otherwise}. \end{cases}$$

Thus, the resulting sublinear operator $H_F \colon X \times R \to Y \cup \{+\infty\}$ is such that

$$\operatorname{dom}(H_F) = \{(x, t) \in X \times R^+ \colon x \in t \operatorname{dom}(F)\}.$$

This H_F is called the *Hörmander transform of F*. Its importance clearly stems from the fact that affine operators majorized by F can be identified with linear operators supporting H_F.

3.7.1. THEOREM. *Let X be a vector space, Y an ordered vector space, and Z a K-space. Also, let* $F \colon X \to Y \cup \{+\infty\}$ *be a convex operator and* $G \colon Y \to Z \cup \{+\infty\}$ *an increasing convex operator. If* $F[\operatorname{dom}(F)]$ *contains an interior point of* $\operatorname{dom}(G)$, *then for any* $A \in L(X, Z)$

$$(G \circ F)^* A = \inf\{(B \circ F)^* A + G^* B \colon B \in L^+(Y, Z)\}.$$

This formula is exact, that is, the infinum on the right-hand side is attained.

PROOF. First of all we note that for any $B \in L^+(Y, Z)$

$$(G \circ F)^* A = \sup_{x \in \operatorname{dom}(G \circ F)} (Ax - G \circ Fx) = \sup_{x \in \operatorname{dom}(G \circ F)} (Ax - B \circ Fx + B \circ Fx - G \circ Fx) \leqslant$$

$$\leqslant \sup_{x \in \operatorname{dom}(F)} (Ax - B \circ Fx) + \sup_{x \in \operatorname{dom}(G \circ F)} (B \circ Fx - G \circ Fx) \leqslant$$

$$\leqslant (B \circ F)^* A + \sup_{y \in \operatorname{dom}(G)} (By - Gy) = (B \circ F)^* A + G^* B.$$

Now let $A \in \operatorname{dom}((G \circ F)^*)$. Then for $(x, t) \in X \times R$ we have

$$Ax - t(G \circ F)^*A \leqslant H_{G \circ F}(x, t)$$

If we assume that $(H_F, 1)$: $(x, t) \to (H_F(x, t), t)$, then

$$H_{G \circ F} = H_G \circ (H_F, 1).$$

Note that $\mathrm{dom}((H_F, 1)) = \mathrm{dom}(H_F)$, and furthermore,

$$(H_F, 1)\,[\mathrm{dom}(H_F)] \cap \mathrm{dom}(H_G) \subset (F[\mathrm{dom}(F)] \cap \mathrm{dom}(G)) \times \{1\}.$$

We order $Y \times R$ by means of the cone $Y^+ \times \{0\}$. Then H_G becomes an increasing operator and $(H_F, 1)$ is sublinear. By 3.5.5 we have

$$\partial (H_{G \circ F}) = \bigcup_{C \in \partial(H_G)} \partial (C \circ (H_F, 1)).$$

In other words, there is an operator $B \in L^+(Y, Z)$ and an element $z \in Z$ such that for all $x \in X$, $y \in Y$, and $t \in R^+$

$$Ax - t(G \circ F)^*A \leqslant tB \circ F(x/t) - tz,$$
$$By + tz \leqslant tG(y/t).$$

This yields the inequalities

$$-z \geqslant G^*B, \quad (G \circ F)^*A \geqslant (B \circ F)^*A - z,$$

and the proof is complete.

3.7.2. COROLLARY. *If F is a sublinear operator, then the following exact formula holds*:

$$(G \circ F)^*A = \inf\{G^*B\colon A \in \partial(B \circ F);\quad B \in L^+(Y, Z)\}.$$

We only consider the case when $(G \circ F)^*A < +\infty$. Then for some $B \in L^+(Y, Z)$

$$(G \circ F)^*A = (B \circ F)^*A + G^*B.$$

Here

$$\sup_{x \in \mathrm{dom}\,(F)} (Ax - B \circ Fx) = 0,$$

so that $A \in \partial(B \circ F)$. This yields the required result.

The next two propositions are proved similarly.

3.7.3. *If G is a sublinear operator, then we have the following exact formula*:

$$(G \circ F)^* = \inf_{B \in \partial\,(G)} (B \circ F)^*.$$

3.7.4. *Let X_1 and X be vector spaces, Y a K-space, and $F\colon X \to Y \cup \{+\infty\}$ a convex operator whose effective set contains an interior point belonging to the image of X_1 under some affine mapping A_x, where $A \in L(X_1, X)$ and $x \in X$. Then for any $B \in L(X_1, Y)$ we have the following exact formula*:

$$(F \circ A_x)^*B = \inf\{F^*C - Cx\colon B = C \circ A\}.$$

3.7.5. REMARK. These assertions are *theorems on a vector minimax*. Thus, in the case of 3.7.3 with $\operatorname{dom}(G) = Y$ we have

$$-(G \circ F)^* 0 = \inf_{x \in \operatorname{dom}(F)} \sup_{B \in \partial (G)} B \circ Fx,$$
$$(B \circ F)^* 0 = -\inf_{x \in \operatorname{dom}(F)} B \circ Fx.$$

Hence,

$$\sup_{B \in \partial (G)} \inf_{x \in \operatorname{dom}(F)} B \circ Fx = \inf_{x \in \operatorname{dom}(F)} \sup_{B \in \partial (G)} B \circ Fx.$$

3.7.6. *Let Y be a K-space and $F_1, \ldots, F_n : X \to Y \cup \{+\infty\}$ convex operators such that the effective sets $\operatorname{dom}(H_{F_1}), \ldots, \operatorname{dom}(H_{F_n})$ of their Hörmander transforms are in general position. Then the Moreau formula holds*:

$$(F_1 + \ldots + F_n)^* = F_1^* \oplus \ldots \oplus F_n^*,$$

where $\oplus$ denotes the operation of inf-*convolution, that is,*

$$F_1^* \oplus \ldots \oplus F_n^* A = \inf \{ \sum_{k=1}^{n} F_k^* A_k \colon A_k \in L(X, Y), \ \sum_{k=1}^{n} A_k = A \}.$$

Since $H_{F_1 + \ldots + F_n} = H_{F_1} + \ldots + H_{F_n}$, this proposition follows from 3.5.4.

3.7.7. *Let Y be a vector lattice, and $F_1, \ldots, F_n : X \to Y \cup \{+\infty\}$ convex operators such that the effective sets $\operatorname{dom}(H_{F_1}), \ldots, \operatorname{dom}(H_{F_n})$ of their Hörmander transforms are in general position. If Z is a K-space and $A \in L^+(Y, Z)$, then the following decomposition formula holds*:

$$(A \circ F_1 \vee \ldots \vee F_n)^* = \inf \{ \bigoplus_{k=1}^{n} (A_k \circ F_k)^* \colon A_k \in L^+(Y, Z), \ \sum_{k=1}^{n} A_k = A \}.$$

The proof follows on the same lines as that of 3.6.6, with appeals to 3.7.3 and to Moreau's formula.

3.7.8. REMARK. When $Y = Z$ and $A = I_Y$, this decomposition is called the *Ioffe–Tikhomirov formula*. We also point out that all these formulae are *exact* in the natural sense. For example, the exactness of 3.7.7 means that for any $B \in L(X, Z)$ the following system of conditions is compatible:

$$B_k \in L(X, Z), \quad B = \sum_{k=1}^{n} B_k, \quad A_k \in L^+(Y, Z), \quad A = \sum_{k=1}^{n} A_k,$$
$$(A \circ F_1 \vee \ldots \vee F_n)^* B = \sum_{k=1}^{n} (A_k \circ F_k)^* B_k.$$

It turns out that in a number of cases the Hörmander transform is not entirely suitable; thus, 3.7.1 says nothing when G is linear. Such situations must be handled in a different way.

3.7.9. *Let Y be a K-space and $F\colon X \to Y \cup \{+\infty\}$ a convex operator such that* $\operatorname{dom} F$ *has an interior point $\bar{x}$ for which $F\bar{x} \geqslant 0$. Then for any $h \in X$ there is an element*

$$S(F, \bar{x})h = \inf_{\alpha>0} \frac{F(\bar{x}+\alpha h)}{\alpha},$$

where $S(F, \bar{x})\colon X \to Y$ *is a sublinear operator.*

The existence of $S(F, \bar{x})h \in Y$ is guaranteed by the condition $F\bar{x} \geqslant 0$, together with 3.1.2. The sublinearity of $S(F, \bar{x})$ can be verified by direct calculation.

With the aid of 3.7.9 we obtain a representation of the Young transform of a composition with a regular operator.

3.7.10. THEOREM. *Let* $F = \varepsilon_{\mathfrak{A}} \circ \langle\mathfrak{A}\rangle_y$, *where* $\mathfrak{A}$ *is a weakly order-bounded set in* $L(X, Y)$ *and* $y \in (Y^{\mathfrak{A}})_\infty$. *Also, let* Z *be a K-space and* $G\colon Y \to Z \cup \{+\infty\}$ *an increasing convex operator. If* $F[X]$ *contains an interior point of* dom(G), *then for any* $A \in L(X, Z)$ *the following exact formula holds*:

$$(G \circ F)^* A = \inf\{G^*(B \circ \Delta_{\mathfrak{A}}) - By\colon\ B \circ \langle\mathfrak{A}\rangle = A,\ B \in L^+((Y^{\mathfrak{A}})_\infty, Z)\}.$$

PROOF. First of all we choose $x \in X$ so that Fx is an interior point of dom(G). Then it is clear that $\Delta_{\mathfrak{A}} Fx$ is an interior point of dom$(G \circ \varepsilon_{\mathfrak{A}})$. Hence, since both G and the canonical operator $\varepsilon_{\mathfrak{A}}$ are increasing, $\langle\mathfrak{A}\rangle_y x$ is also an interior point of dom$(G \circ \varepsilon_{\mathfrak{A}})$. Therefore, we can apply 3.7.4, that is, the following exact formula holds:

$$(G \circ F)^* A = \inf\{(G \circ \varepsilon_{\mathfrak{A}})^* B - By\colon\ A = B \circ \langle\mathfrak{A}\rangle,\ B \in L((Y^{\mathfrak{A}})_\infty, Z)\}.$$

If B belongs to $L^+((Y^{\mathfrak{A}})_\infty, Z)$, then

$$(G \circ \varepsilon_{\mathfrak{A}})^* B = \sup_{z \in \mathrm{dom}(G \circ \varepsilon_{\mathfrak{A}})} (Bz - G \circ \varepsilon_{\mathfrak{A}} z) \leqslant$$

$$\leqslant \sup_{z \in \mathrm{dom}(G \circ \varepsilon_{\mathfrak{A}})} (B \circ \Delta_{\mathfrak{A}} \circ \varepsilon_{\mathfrak{A}} z - G \circ \varepsilon_{\mathfrak{A}} z) \leqslant \sup_{u \in \mathrm{dom}(G)} (B \circ \Delta_{\mathfrak{A}} u - Gu) = G^*(B \circ \Delta_{\mathfrak{A}}).$$

On the other hand, if $B \in \mathrm{dom}((G \circ \varepsilon_{\mathfrak{A}})^*)$, then

$$B \circ \Delta_{\mathfrak{A}} Fx + \alpha Bz - (G \circ \varepsilon_{\mathfrak{A}})^* B \leqslant G \circ \varepsilon_{\mathfrak{A}} (\Delta_{\mathfrak{A}} Fx + \alpha z) = G(Fx + \alpha \varepsilon_{\mathfrak{A}} z)$$

for all $\alpha > 0$ and any $Z \in (Y^{\mathfrak{A}})_\infty$. By appealing to 3.7.9, we find that

$$B \in \partial(S(G(\cdot) - B \circ \Delta_{\mathfrak{A}} Fx + (G \circ \varepsilon_{\mathfrak{A}})^* B,\ Fx) \circ \varepsilon_{\mathfrak{A}}).$$

By 2.2.1, this implies that the operator B is positive and that

$$B \circ \Delta_{\mathfrak{A}} \in \partial(S(G(\cdot) - B \circ \Delta_{\mathfrak{A}} Fx + (G \circ \varepsilon_{\mathfrak{A}})^* B,\ Fx)).$$

Hence, for any $u \in Y$ we have the bound

$$B \circ \Delta_{\mathfrak{A}} u \leqslant G(Fx + u) - B \circ \Delta_{\mathfrak{A}} Fx + (G \circ \varepsilon_{\mathfrak{A}})^* B,$$

that is, $(G \circ \varepsilon_{\mathfrak{A}})^* B \geqslant G^*(B \circ \Delta_{\mathfrak{A}})$, and this completes the proof.

GUIDE TO THE LITERATURE

Detailed bibliographies covering the fields of convex analysis and the theory of extremum problems are to be found in [19], [34], and [43] (see. also [6], [39], [12], [32], [46], and [49]), therefore, we confine ourselves here to a bare minimum of literary comments.

§1. The notion of a subdifferential goes back to the geometrical ideas of Minkowski. A systematic study of subdifferentials in relation to the theory of extremum problems began with the papers of Moreau [30] and Dubovitskii and Milyutin [14]. For the theory of ordered vector spaces see [9], [20], [21], and [44], and for examples of convex operators see [2] and [29]. Theorems 1.2.1. and 1.3.1. were discovered in this generality by Kantorovich [21]. For the subsequent research into extension problems see [1], [22], and [44]. A proof of Theorem 1.3.2. is in [7] and [40]. Lemma 1.4.5 was established in the scalar case by Rockafellar (see [18]), and in the general case by Levin [27].

§2. The canonical operator method was proposed in [23]. The support set of a sum was first described in [27]. Proposition 2.4.2 was first established (by a slightly different method) in [35]. A detailed survey of questions in the theory of sublinear operators, other than those related to the calculus, is to be found in [36]. About 2.2.5 see [8] and [18].

§3. A survey of the theory of subdifferentials for numerical functions appears in [18]. The results of §3.1, together with the formula for the subdifferential of a sum of convex operators at an interior point, were obtained by Levin in [18] and [27]. For directional derivatives of non-convex functions see [13]. The version given of the Mazur-Orlicz theorem, the formula for the subdifferential of a sum of convex operators defined on a single set, and Proposition 3.6.3 are all established in [41]–[42]. A bibliography covering this area is given in [22]. The general formulae of the subdifferential calculus were announced in [24]. Theorem 3.2.5 appears here for the first time. The subdifferential calculus is carried over to locally Lipschitz functions in Chapter XI of [49], which also contains a bibliography. A history of the Young transform is given in [19] and [46]. The Young vector transform in an infinite-dimensional situation apparently occurs first in [28]. Proposition 3.7.5 was first discovered by Rubinov. General rules for a change of variables in a Young transform were proposed in [25]. Research into Young transforms borders on work on multiple criteria decision making. A detailed bibliography of articles in this area can be found in [47]. In this context, we also mention [3], [4], [15]–[17], and [37]–[38].

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