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SIMULTANEOUS LINEAR INEQUALITIES: YESTERDAY AND TODAY

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Это небольшой обзор теории линейных неравенств, одного из псевдонимов выпуклого анализа, с особым вниманием на булевозначную интерпретацию некоторых следствий леммы Фаркаша.

1. Agenda

Linear inequality implies linearity and order. When combined, the two produce an ordered vector space. Each linear inequality in the simplest environment of the sort is some half-space. Simultaneity implies many instances and so yields intersections of half-spaces. This yields polyhedra as well as arbitrary convex sets, identifying the theory of linear inequalities with convexity.

Convexity stems from the remote ages and reigns in the federation of geometry, optimization, and functional analysis. Convexity feeds generation, separation, calculus, and approximation. Generation appears as duality; separation, as optimality; calculus, as representation; and approximation, as stability.

This talk addresses the origin and the state of the art of the relevant areas with a particular emphasis on the Farkas Lemma. Our aim is to demonstrate how Boolean valued analysis may be applied to simultaneous linear inequalities with operators. This particular theme is another illustration of the deep and powerful technique of “stratified validity” which is characteristic of Boolean valued analysis.

2. Environment

Assume that X is a real vector space, Y is a *Kantorovich space* also known as a complete vector lattice or a Dedekind complete Riesz space. Let $\mathbb{B} := \mathbb{B}(Y)$ be the *base* of Y, i.e., the complete Boolean algebras of positive projections in Y; and let $m(Y)$ be the universal completion of Y.

Denote by $L(X, Y)$ the space of linear operators from X to Y. In case X is furnished with some Y-seminorm on X, by $L^{(m)}(X, Y)$ we mean the *space of dominated operators* from X to Y. As usual, $\{T \le 0\} := \{x \in X : Tx \le 0\}$; $\ker(T) = T^{-1}(0)$ for $T \in L(X, Y)$. In the rest of notation we will follow [1] without further specification.

3. Kantorovich's Theorem

The Hahn–Banach Theorem amounts to existence of positive extensions as visualized by the diagram

$$\begin{tikzcd} X \arrow[r, "A"] \arrow[dr, "B"'] & W \arrow[d, dotted, "\mathfrak{X}"] \\ & Y \end{tikzcd}$$

Theorem. *If W is ordered by W_+ and $A(X) - W_+ = W_+ - A(X) = W$, then*

$$(\exists \mathfrak{X} \ge 0)\ \mathfrak{X}A = B \leftrightarrow \{A \le 0\} \subset \{B \le 0\}.$$

4. The Alternative

Theorem. *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A_1, \dots, A_N$ and B belong to $L^{(m)}(X, Y)$.*

Then one and only one of the following holds:

(1) *There are $x \in X$ and $b, b' \in \mathbb{B}$ such that $b' \le b$ and*

$$b'Bx > 0, bA_1x \le 0, \dots, bA_Nx \le 0.$$

(2) *There are $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))_+$ such that $B = \sum_{k=1}^N \alpha_k A_k$.*

5. Inhomogeneous Inequalities

Theorem. *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume given some dominated operators $A_1, \dots, A_N, B \in L^{(m)}(X, Y)$ and elements $u_1, \dots, u_N, v \in Y$. The following are equivalent:*

(1) *For all $b \in \mathbb{B}$ the inhomogeneous operator inequality $bBx \leq bv$ is a consequence of the consistent simultaneous inhomogeneous operator inequalities $bA_1x \leq bu_1, \dots, bA_Nx \leq bu_N$, i.e.,*

$$\{bB \leq bv\} \supset \{bA_1 \leq bu_1\} \cap \dots \cap \{bA_N \leq bu_N\}.$$

(2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \operatorname{Orth}(m(Y))$ satisfying*

$$B = \sum_{k=1}^{N} \alpha_k A_k; \quad v \geq \sum_{k=1}^{N} \alpha_k u_k.$$

6. Inhomogeneous Sublinear Inequalities

Lemma. *Let X be a real vector space. Assume that $p_1, \dots, p_N \in \operatorname{PSub}(X) := \operatorname{PSub}(X, \mathbb{R})$ and $p \in \operatorname{Sub}(X)$. Assume further that $v, u_1, \dots, u_N \in \mathbb{R}$ make consistent the simultaneous sublinear inequalities $p_k(x) \leq u_k$, with $k := 1, \dots, N$.*

The following are equivalent:

(1) $\{p \geq v\} \supset \bigcap\limits_{k=1}^{N} \{p_k \leq u_k\}$;

(2) *there are $\alpha_1, \dots, \alpha_N \in \mathbb{R}_+$ satisfying*

$$(\forall x \in X)\ p(x) + \sum_{k=1}^{N} \alpha_k p_k(x) \geq 0, \quad \sum_{k=1}^{N} \alpha_k u_k \leq -v.$$

Proof. (2) $\to$ (1): If x is a solution to the inhomogeneous simultaneous inequalities $p_k(x) \leq u_k$ with $k := 1, \dots, N$, then

$$0 \leq p(x) + \sum_{k=1}^{N} \alpha_k p_k(x) \leq p(x) + \sum_{k=1}^{N} \alpha_k u_k(x) \leq p(x) - v.$$

(1) $\to$ (2): Given $(x,t) \in X \times \mathbb{R}$, put $\bar{p}_k(x,t) := p_k(x) - tu_k$, $\bar{p}(x,t) := p(x) - tv$ and $\tau(x,t) := -t$. Clearly, $\tau, \bar{p}_1, \dots, \bar{p}_N \in \operatorname{PSub}(X \times \mathbb{R})$ and $\bar{p} \in \operatorname{Sub}(X \times \mathbb{R})$. Take

$$(x,t) \in \{\tau \leq 0\} \cap \bigcap_{k=1}^{N} \{\bar{p}_k \leq 0\}.$$

If, moreover, $t > 0$; then $u_k \geq p_k(x/t)$ для $k := 1, \dots, N$ and so $p(x/t) \leq v$ by hypothesis. In other words $(x,t) \in \{\bar{p} \leq 0\}$. If $t = 0$

then take some solution $\bar{x}$ of the simultaneous inhomogeneous polyhedral inequlities under study. Since $x \in K := \bigcap_{k=1}^{N}\{p_k \le 0\}$; therefore, $p_k(\bar{x}+x) \le p(x)+p_k(x) \le u_k$ for all $k := 1,\dots,N$. Hence, $p(\bar{x}+x) \ge v$ by hypotyhesis. So the sublinear functional p is bounded below on the cone K. Consequently, p assumes only positive values on K. In other words, $(x,0) \in \{\bar{p} \le 0\}$. Thus

$$\{\bar{p} \ge 0\} \supset \bigcap_{k=1}^{N}\{\bar{p}_k \le 0\}$$

and by Lemma 2.2. of [1] there are positive reals $\alpha_1,\dots,\alpha_N,\beta$ sych that for all $(x,t) \in X \times \mathbb{R}$ we have

$$\bar{g}(x) + \beta\tau(x) + \sum_{k=1}^{N}\alpha_k\bar{p}_k(x) \ge 0.$$

Clearly, the so-obtained parameters $\alpha_1,\dots,\alpha_N$ are what we sought for. The proof of the lemma is complete.

Theorem. *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Given are some dominated polyhedral sublinear operators $P_1,\dots,P_N \in \mathrm{PSub}^{(m)}(X,Y)$ and dominated sublinear operator $P \in \mathrm{Sub}^{(m)}(X,Y)$. Assume further that $u_1,\dots,u_N,v \in Y$ make consistent the inhomogeneous simultaneous inequalities $P_1(x) \le u_1,\dots,P_N(x) \le u_N, P(x) \ge v$.*

The following are equivalent:

(1) *for all $b \in \mathbb{B}$ the inhomogeneous sublinear operator inequality $bP(x) \ge v$ is a conseqence of the simultaneous inhomogeneous sublinear operator inequalities $bP_1(x) \le u_1,\dots,bP_N(x) \le u_N$, i. e.,*

$$\{bP \ge v\} \supset \{bP_1 \le u_1\} \cap \dots \cap \{bP_N \le u_N\};$$

(2) *there are positive $\alpha_1,\dots,\alpha_N \in \mathrm{Orth}(m(Y))$ satisfying*

$$(\forall x \in X)\ P(x) + \sum_{k=1}^{N}\alpha_k P_k(x) \ge 0, \quad \sum_{k=1}^{N}\alpha_k u_k \le -v.$$

Proof. The claim is an instance of Boolean valued interpratation of the Lemma.

Remark. The above theorem shows that the Lagrange principle is valid for the extremal proiblem

$$P_1(x) \le u_1, \dots, P_N(x) \le u_N, \quad P(x) \to \inf .$$

In other words, the finite value of the contrained problem is the value of the unconstrained problem for an appropriate Lagrangian. It is worth observing that we do not assume any constraint qualification other that polyhedrality. Recall that the Slater condition allows us to eliminate polyhedrality as well as considering a unique target space. This is well known in a practically unrestricted generality (for instance, see [2]).

7. Freedom and Inequality

Abstraction is the freedom of generalization. Freedom is the loftiest ideal and idea of man, but it is demanding, limited, and vexing. So is abstraction. So are its instances in convexity, hence, in simultaneous inequalities.

Freedom of set theory empowered us with the Boolean-valued models yielding a lot of surprising and unforeseen visualizations of the ingredients of mathematics. Many promising opportunities are open to modeling the powerful habits of reasoning and verification. Convexity, the theory of simultaneous linear inequalities in disguise, is a topical illustration of the wisdom and strength of mathematics, the ever fresh art and science of calculus.

Inequality paves way to freedom.

References

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This is a brief overview of the theory of simultaneous linear inequalities, an alias of convex analysis, with a particular emphasis on the Boolean valued interpretation of some consequences of the Farkas lemma.