Interaction of Order and Convexity

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Alexandrov (1912-1999)

- Alexandr Danilovich Alexandrov was the first and foremost Russian geometer of the twentieth century.
- He contributed to mathematics under the slogan:
- “Retreat to Euclid.”

His Contribution

The works of Alexandrov made tremendous progress in the theory of mixed volumes of convex figures. He proved some fundamental theorems on convex polyhedra that ranked his name alongside the names of Euclid and Cauchy.

Alexandrov enriched the methods of differential geometry by the tools of functional analysis and measure theory, driving mathematics to its universal status of the epoch of Euclid. The mathematics of the ancients was geometry (there were no other instances of mathematics at all). Synthesizing geometry with the remaining areas of the today's mathematics, Alexandrov climbed to the antique ideal of the universal science incarnated in mathematics. Return to the synthetic methods of *mathesis universalis* was inevitable and unavoidable as well as challenging and fruitful.

Abstract Convexity

Minkowski Duality

A *convex figure* is a compact convex set. A *convex body* is a solid convex figure. The *Minkowski duality* identifies a convex figure S in $\mathbb{R}^N$ and its *support function* $S(z) := \sup\{(x,z) \mid x \in S\}$ for $z \in \mathbb{R}^N$. Considering the members of $\mathbb{R}^N$ as singletons, we assume that $\mathbb{R}^N$ lies in the set $\mathscr{V}_N$ of all compact convex subsets of $\mathbb{R}^N$.

The classical concept of support function gives rise to abstract convexity which focuses on the order background of convex sets.

H-Convexity

Let $\overline{E}$ be a complete lattice E with the adjoint top $\top := +\infty$ and bottom $\bot := -\infty$. Unless otherwise stated, Y is usually a *Kantorovich space* which is a Dedekind complete vector lattice in another terminology. Assume further that H is some subset of E which is by implication a (convex) cone in E, and so the bottom of E lies beyond H. A subset U of H is *convex relative to* H or H*-convex*, in symbols $U \in \mathscr{V}(H, \overline{E})$, provided that U is the H*-support set* $U_p^H := \{h \in H \mid h \leq p\}$ of some element p of $\overline{E}$.

Alongside the H-convex sets we consider the so-called H-convex elements. An element $p \in \overline{E}$ is H*-convex* provided that $p = \sup U_p^H$; i.e., p represents the supremum of the H-support set of p. The H-convex elements comprise the cone which is denoted by $\mathscr{C}nv(H, \overline{E})$. The convex elements and sets are "glued together" by the *Minkowski duality* $\varphi : p \mapsto U_p^H$.

Hyperspaces

Positive Functionals over Convex Objects

The Minkowski duality makes $\mathscr{V}_N$ into a cone in the space $C(S_{N-1})$ of continuous functions on the Euclidean unit sphere S_{N-1}, the boundary of the unit ball $\mathfrak{z}_N$. This yields the so-called *Minkowski structure* on $\mathscr{V}_N$. Addition of the support functions of convex figures amounts to taking their algebraic sum, also called the *Minkowski addition*. It is worth observing that the *linear span* $[\mathscr{V}_N]$ of $\mathscr{V}_N$ is dense in $C(S_{N-1})$, bears a natural structure of a vector lattice and is usually referred to as the *space of convex sets*. The study of this space stems from the pioneering breakthrough of Alexandrov in 1937 and the further insights of Radström, Hörmander, and Pinsker.

Choquet Theory

A similar idea was suggested by Loomis in 1962 within Choquet theory:

A measure μ *affinely majorizes* or *dominates* a measure ν, both given on a compact convex subset Q of a locally convex space X, provided that to each decomposition of ν into finitely many summands $\nu_1, \ldots, \nu_m$ there are measures $\mu_1, \ldots, \mu_m$ whose sum is μ and for which every difference $\mu_k - \nu_k$ annihilates all restrictions to Q of affine functionals over X. In symbols, $\mu \gg_{\mathrm{Aff}(Q)} \nu$.

Cartier, Fell, and Meyer proved in 1964 that

$$\int_Q f d\mu \geq \int_Q f d\nu$$

for each continuous convex function f on Q if and only if $\mu \gg_{\mathrm{Aff}(Q)} \nu$. An analogous necessity part for linear majorization was published in 1970.

Majorization

Majorization is a vast subject. We only site one of the relevant abstract claims of subdifferential calculus:

Theorem. Assume that $H_1, \dots, H_N$ are cones in a Riesz space X. Assume further that f and g are positive functionals on X. The inequality

$$f(h_1 \vee \cdots \vee h_N) \geq g(h_1 \vee \cdots \vee h_N)$$

holds for all $h_k \in H_k$ $(k := 1, \dots, N)$ if and only if to each decomposition of g into a sum of N positive terms $g = g_1 + \cdots + g_N$ there is a decomposition of f into a sum of N positive terms $f = f_1 + \cdots + f_N$ such that

$$f_k(h_k) \geq g_k(h_k) \quad (h_k \in H_k;\ k := 1, \dots, N).$$

Alexandrov's Theorem

Alexandrov Measures and the Blaschke Structure

The celebrated *Alexandrov Theorem* proves the unique existence of a translate of a convex body given its surface area function. Each surface area function is an *Alexandrov measure*. So we call a positive measure on the unit sphere which is supported by no great hypersphere and which annihilates singletons. Thus, each Alexandrov measure is a translation-invariant additive functional over the cone $\mathscr{V}_N$. This yields some abstract cone structure that results from identifying the coset of translates $\{z + \mathfrak{x} \mid z \in \mathbb{R}^N\}$ of a convex body $\mathfrak{x}$ the corresponding Alexandrov measure on the unit sphere which we call the *surface area function* of the coset of $\mathfrak{x}$ and denote by $\mu(\mathfrak{x})$. The cone of positive translation-invariant measures in the dual $C'(S_{N-1})$ of $C(S_{N-1})$ is denoted by $\mathscr{A}_N$.

Dual Hyperspace

Given $\mathfrak{x}, \mathfrak{y} \in \mathscr{V}_N$, we let the record $\mathfrak{x} =_{\mathbb{R}^N} \mathfrak{y}$ mean that $\mathfrak{x}$ and $\mathfrak{y}$ are equal up to translation or, in other words, are translates of one another. We may say that $=_{\mathbb{R}^N}$ is the associate equivalence of the preorder $\geq_{\mathbb{R}^N}$ on $\mathscr{V}_N$ which symbolizes the possibility of inserting one figure into the other by translation.

Arrange the factor set $\mathscr{V}_N/\mathbb{R}^N$ which consists of the cosets of translates of the members of $\mathscr{V}_N$. Clearly, $\mathscr{V}_N/\mathbb{R}^N$ is a cone in the factor space $[\mathscr{V}_N]/\mathbb{R}^N$ of the vector space $[\mathscr{V}_N]$ by the subspace $\mathbb{R}^N$.

Identification

There is a natural bijection between $\mathscr{V}_N/\mathbb{R}^N$ and $\mathscr{A}_N$. Namely, we identify the coset of singletons with the zero measure. To the straight line segment with endpoints x and y, we assign the measure $|x-y|(\varepsilon_{(x-y)/|x-y|}+\varepsilon_{(y-x)/|x-y|})$, where $|\cdot|$ stands for the Euclidean norm and the symbol ε_z for $z \in S_{N-1}$ stands for the *Dirac measure* supported at z. If the dimension of the affine span $\mathrm{Aff}(\mathfrak{x})$ of a representative $\mathfrak{x}$ of a coset in $\mathscr{V}_N/\mathbb{R}^N$ is greater than unity, then we assume that $\mathrm{Aff}(\mathfrak{x})$ is a subspace of $\mathbb{R}^N$ and identify this class with the surface area function of $\mathfrak{x}$ in $\mathrm{Aff}(\mathfrak{x})$ which is some measure on $S_{N-1}\cap\mathrm{Aff}(\mathfrak{x})$ in this event. Extending the measure by zero to a measure on S_{N-1}, we obtain the member of $\mathscr{A}_N$ that we assign to the coset of all translates of $\mathfrak{x}$. The fact that this correspondence is one-to-one follows easily from the Alexandrov Theorem.

Blaschke's Sum

The vector space structure on the set of regular Borel measures induces in $\mathscr{A}_N$ and, hence, in $\mathscr{V}_N/\mathbb{R}^N$ the structure of an abstract cone or, strictly speaking, the structure of a commutative $\mathbb{R}_+$-operator semigroup with cancellation. This structure on $\mathscr{V}_N/\mathbb{R}^N$ is called the *Blaschke structure* and the references therein). Note that the sum of the surface area functions of $\mathfrak{x}$ and $\mathfrak{y}$ generates a unique class $\mathfrak{x}\#\mathfrak{y}$ which is referred to as the *Blaschke sum* of $\mathfrak{x}$ and $\mathfrak{y}$.

Let $C(S_{N-1})/\mathbb{R}^N$ stand for the factor space of $C(S_{N-1})$ by the subspace of all restrictions of linear functionals on $\mathbb{R}^N$ to S_{N-1}. Denote by $[\mathscr{A}_N]$ the space $\mathscr{A}_N - \mathscr{A}_N$ of translation-invariant measures. It is easy to see that $[\mathscr{A}_N]$ is also the linear span of the set of Alexandrov measures.

Mixed Volume as Duality

The spaces $C(S_{N-1})/\mathbb{R}^N$ and $[\mathscr{A}_N]$ are made dual by the canonical bilinear form

$$\langle f, \mu \rangle = \frac{1}{N} \int_{S_{N-1}} f d\mu \ (f \in C(S_{N-1})/\mathbb{R}^N, \ \mu \in [\mathscr{A}_N]).$$

For $\mathfrak{x} \in \mathscr{V}_N/\mathbb{R}^N$ and $\mathfrak{y} \in \mathscr{A}_N$, the quantity $\langle \mathfrak{x}, \mathfrak{y} \rangle$ coincides with the *mixed volume* $V_1(\mathfrak{y}, \mathfrak{x})$. The space $[\mathscr{A}_N]$ is usually furnished with the weak topology induced by the above indicated duality with $C(S_{N-1})/\mathbb{R}^N$.

Dual Cones

Cones of Feasible Directions

By the *dual* K^* of a given cone K in a vector space X in duality with another vector space Y, we mean the set of all positive linear functionals on K; i.e., $K^* := \{y \in Y \mid (\forall x \in K)\ \langle x, y\rangle \geq 0\}$. Recall also that to a convex subset U of X and a point $\bar{x}$ in U there corresponds the cone

$$U_{\bar{x}} := \mathrm{Fd}(U, \bar{x}) := \{h \in X \mid (\exists \alpha \geq 0)\ \bar{x} + \alpha h \in U\}$$

which is called the *cone of feasible directions* of U at $\bar{x}$. Fortunately, description is available for all dual cones we need.

Let $\bar{\mathfrak{x}} \in \mathscr{A}_N$. Then the dual $\mathscr{A}^*_{N,\bar{\mathfrak{x}}}$ of the cone of feasible directions of $\mathscr{A}_N n$ at $\bar{\mathfrak{x}}$ may be represented as follows

$$\mathscr{A}^*_{N,\bar{\mathfrak{x}}} = \{f \in \mathscr{A}^*_N \mid \langle \bar{\mathfrak{x}}, f\rangle = 0\}.$$

Decomposition and Inclusion

Let $\mathfrak{x}$ and $\mathfrak{y}$ be convex figures. Then

(1) $\mu(\mathfrak{x}) - \mu(\mathfrak{y}) \in \mathscr{V}_N^* \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$;

(2) If $\mathfrak{x} \geq_{\mathbb{R}^N} \mathfrak{y}$ then $\mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$;

(3) $\mathfrak{x} \geq_{\mathbb{R}^2} \mathfrak{y} \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^2} \mu(\mathfrak{y})$;

(4) If $\mathfrak{y} - \overline{\mathfrak{x}} \in \mathscr{A}_{N,\overline{\mathfrak{x}}}^*$ then $\mathfrak{y} =_{\mathbb{R}^N} \overline{\mathfrak{x}}$;

(5) If $\mu(\mathfrak{y}) - \mu(\overline{\mathfrak{x}}) \in \mathscr{V}_{N,\overline{\mathfrak{x}}}^*$ then $\mathfrak{y} =_{\mathbb{R}^N} \overline{\mathfrak{x}}$.

It stands to reason to avoid discriminating between a convex figure, the respective coset of translates in $\mathscr{V}_N/\mathbb{R}^N$, and the corresponding measure in $\mathscr{A}_N$.

Isoperimetric-Type Problems

Comparison Between the Blaschke and Minkowski Structures

The isoperimetric-type problems with subsidiary constraints on location of convex figures comprise a unique class of the challenging extremal problems with two essentially different parametrizations. The principal features of the latter are seen from the table.

Problem's Ingredients

Object of Parametrization	Minkowski's Structure	Blaschke's Structure
cone of sets	$\mathscr{Y}_N/\mathbb{R}^N$	$\mathscr{A}_N$
dual cone	$\mathscr{Y}_N^*$	$\mathscr{A}_N^*$
positive cone	$\mathscr{A}_N^*$	$\mathscr{A}_N$
typical linear functional	$V_1(\mathfrak{Z}_N, \cdot)$ (width)	$V_1(\cdot, \mathfrak{Z}_N)$ (area)
concave functional (power of volume)	$V^{1/N}(\cdot)$	$V^{(N-1)/N}(\cdot)$
simplest convex program	isoperimetric problem	Urysohn's problem
operator-type constraint	inclusion of figures	inequalities on "curvatures"
Lagrange's multiplier	surface	function
differential of volume at a point $\bar{\mathfrak{x}}$ is proportional to	$V_1(\bar{\mathfrak{x}}, \cdot)$	$V_1(\cdot, \bar{\mathfrak{x}})$

Thus the classical isoperimetric problem is not a convex program in the Minkowski structure for $N \geq 3$. In the Blaschke structure this problem is a convex program whose optimality criterion reads: "Each solution is a ball."

Convex Problems

The problems are challenging that contain some constrains of inclusion type: for instance, the isoperimetric problem or Urysohn problem with the requirement that the solutions lie among the subsets or supersets of a given body. These problems can be solved in a generalized sense "modulo" the Alexandrov Theorem. Clearly, some convex combination of the ball and a tetrahedron is proportional to the solution of the Urysohn problem in this tetrahedron. If we replace the condition on the integral width which is typical of the Urysohn problem by a constraint on the surface area or other mixed volumes of a more general shape then we come to possibly nonconvex programs for which a similar reasoning yields only necessary extremum conditions in general. Recall that in case $N = 2$ the Blaschke sum transforms as usual into the Minkowski sum modulo translates.

Soap Bubbles

The task of choosing an appropriate parametrization for a wide class of problems is practically unstudied in general. In particular, those problems of geometry remain unsolved which combine constraints each of which is linear in one of the two vector structures on the set of convex figures. The simplest example of an unsolved "combined" problem is the internal isoperimetric problem in the space $\mathbb{R}^N$ for $N \geq 3$. The only instance of progress is due to Pogorelov who found in 1995 the form of a soap bubble inside a three-dimensional tetrahedron. This happens to be proportional to the Minkowski convex combination of the ball and the solution to the internal Urysohn problem in the tetrahedron. The above geometric facts make it reasonable to address the general problem of parametrizing the important classes of extremal problems of practical provenance.

External
I Problems

The *external Urysohn problem*: Among the convex figures, circumscribing $\mathfrak{x}_0$ and having integral width fixed, find a convex body of greatest volume.

Theorem. A feasible convex body $\bar{\mathfrak{x}}$ is a solution to the external Urysohn problem if and only if there are a positive measure μ and a positive real $\bar{\alpha} \in \mathbb{R}_+$ satisfying

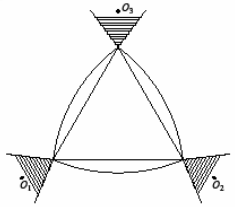
(1) $\bar{\alpha}\mu(\mathfrak{z}_N) \gg_{\mathbb{R}^N} \mu(\bar{\mathfrak{x}}) + \mu$;

(2) $V(\bar{\mathfrak{x}}) + \frac{1}{N}\int_{S_{N-1}} \bar{\mathfrak{x}} d\mu = \bar{\alpha} V_1(\mathfrak{z}_N, \bar{\mathfrak{x}})$;

(3) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ for all z in the support of μ.

Uryhson-Type Problems

If, in particular, $\mathfrak{x}_0 = \mathfrak{z}_{N-1}$ then the sought body is a *spherical lens*, that is, the intersection of two balls of the same radius; while the critical measure μ is the restriction of the surface area function of the ball of radius $\bar{\alpha}^{1/(N-1)}$ to the complement of the support of the lens to S_{N-1}. If $\mathfrak{x}_0$ is an equilateral triangle then the solution $\bar{\mathfrak{x}}$ looks as follows:



Note that $\bar{\mathfrak{x}}$ is the union of $\mathfrak{x}_0$ and three congruent slices of a circle of radius $\bar{\alpha}$ and centers O_1–O_3. The critical measure μ is the restriction of $\mu(\mathfrak{z}_2)$ to the subset of S_1 comprising the endpoints of the unit vectors of the shaded zone.

Current Hyperplanes

The *internal Urysohn problem with a current hyperplane*: Find two convex figures $\bar{\mathfrak{x}}$ and $\bar{\mathfrak{y}}$ lying in a given convex body $\mathfrak{x}_o$, separated by a hyperplane with the unit outer normal z_0, and having the greatest total volume of $\bar{\mathfrak{x}}$ and $\bar{\mathfrak{y}}$ given the sum of their integral widths.

Theorem. A feasible pair of convex bodies $\bar{\mathfrak{x}}$ and $\bar{\mathfrak{y}}$ solves the internal Urysohn problem with a current hyperplane if and only if there are convex figures $\mathfrak{x}$ and $\mathfrak{y}$ and positive reals $\bar{\alpha}$ and $\bar{\beta}$ satisfying (1) $\bar{\mathfrak{x}} = \mathfrak{x}\#\bar{\alpha}\mathfrak{z}_N$; (2) $\bar{\mathfrak{y}} = \mathfrak{y}\#\bar{\alpha}\mathfrak{z}_N$; (3) $\mu(\mathfrak{x}) \geq \bar{\beta}\varepsilon_{z_0}$, $\mu(\mathfrak{y}) \geq \bar{\beta}\varepsilon_{-z_0}$; (4) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ for all $z \in \operatorname{supp}(\mathfrak{x}) \setminus \{z_0\}$; (5) $\bar{\mathfrak{y}}(z) = \mathfrak{x}_0(z)$ for all $z \in \operatorname{supp}(\mathfrak{x}) \setminus \{-z_0\}$, with $\operatorname{supp}(\mathfrak{x})$ standing for the *support* of $\mathfrak{x}$, i.e. the support of the surface area measure $\mu(\mathfrak{x})$ of $\mathfrak{x}$.