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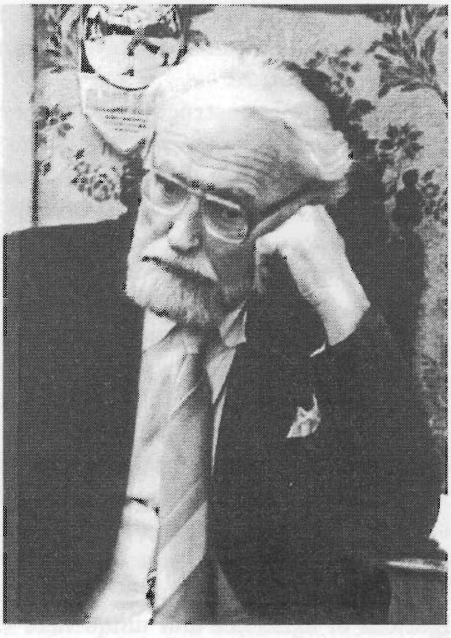
On the 90th anniversary of the birth of A. D. Aleksandrov (1912–1999)



The 4th of August 2002 is the 90th birthday anniversary of an outstanding mathematician of the 20th century, Aleksandr Danilovich Aleksandrov. He was born in the village of Volyn' of the former Ryazan' region. His parents were school teachers.

In 1929 Aleksandrov entered the Physics Department of Leningrad University, from which he graduated in 1933. In 1935 he defended his PhD thesis, and in 1937 his DSc dissertation. In 1946 he was elected a corresponding member of the Academy of Sciences of the USSR, and in 1964 a full member.

From 1952 to 1964 Aleksandrov was the Rector of Leningrad University. In 1964 he moved to Novosibirsk, where he was the head of a department in the Sobolev Mathematical Institute of the Siberian Branch of the Academy of Sciences of the USSR until 1986, and at the same time he was a professor at Novosibirsk University. From April 1986 until his death on 27 July 1999, he worked at the St. Petersburg Branch of the Steklov Mathematical Institute.

His teachers were B. N. Delone, an outstanding geometer and algebraist, and V. A. Fok, a prominent physicist of the last century. The first scientific investigations of Aleksandrov were devoted to certain questions of theoretical physics and geometry. Later, mathematics became his principal field, to which his main achievements belong.

Aleksandrov is the author of nearly 300 published papers, and many monographs and textbooks. The main area of his research activity was geometry, a field in which he founded a major school. Among his students are many respected researchers, two of whom, A. V. Pogorelov and Yu. G. Reshetnyak, are members of the Russian Academy of Sciences.

The mathematical results of Aleksandrov covered a wide circle of questions, including the geometry of convex bodies, measure theory, partial differential equations, and the mathematical foundations of relativity theory.

The work of Aleksandrov gave further development to the theory of mixed volumes of convex bodies. He proved a number of fundamental theorems on convex polyhedra, which can be compared to the theorems of Euler and Cauchy. In particular, working on Weyl's problem, Aleksandrov created a new method of proving existence theorems. The results of this cycle of studies put his name alongside those of Euclid and Cauchy.

One of the main achievements of Aleksandrov in geometry was the creation of the theory of two-dimensional manifolds of bounded curvature, or, which is the same, of the intrinsic geometry of irregular surfaces. In connection with this theory he developed an exceptionally strong and intuitive method of cutting and gluing, which turned out to be very effective in the theory of bending of convex surfaces. Using this method, Aleksandrov solved a number of extremal problems for manifolds of bounded curvature.

Aleksandrov constructed the theory of metric spaces with one-sided bounds on the curvature. At present, this is the only known class of metric spaces that can be regarded as generalised Riemannian spaces, in the sense that there is a notion of curvature for them which is central in Riemannian geometry.

Aleksandrov's papers on the theory of two-dimensional manifolds of bounded curvature and the theory of spaces with one-sided bounds on the curvature developed the geometric concept of space and continued the tradition starting with Lobachevsky, Gauss, Riemann, Poincaré, and Cartan.

Investigations in the theory of convex bodies led Aleksandrov to problems in the general theory of additive set functions. In particular, he carried out a deep study of weak convergence of set functions. His results in this area are included in textbooks on functional analysis and find unexpected applications in geometry as well as in probability theory. He is one of the creators of the theory of irregular curves, which is a continuation and development of ideas that belong to the classics of geometry: Jordan, Peano, and others.

The origin of Aleksandrov's work on differential equations lies in his investigations of existence and uniqueness theorems in the theory of convex bodies. It was essentially these studies that gave rise to the notion of a generalised solution of a partial differential equation, moreover, in the case of hard non-linear problems. He laid the foundations of the geometric theory of Monge–Ampère type equations. He developed a geometric approach to the maximum principle in the theory of partial differential equations. His studies in this area were many years ahead of similar investigations by experts in differential equations. Aleksandrov solved the problem of linearity of maps that preserve cones in the space of the theory of special relativity—a result rediscovered by physicists of various countries decades later; this work started research in chronogeometry.

Questions on methodology and the history of science and the teaching of it were important for him. He is the author of many journalistic articles on science, invariably current and discerning. His papers on the content and role of mathematics are used in teaching philosophy and the history of science. His geometry textbooks have also found their place in the practice of school education.

The task of geometry includes the study of abstract intuitive shapes: curves, surfaces, Riemannian and other manifolds, supplied with various additional structures. In the framework of differential geometry, powerful analytical techniques

were developed that are suitable for investigating and describing mainly local properties of geometric images.

By the beginning of the last century many problems in the theory of surfaces had appeared concerning relationships between various quantities that characterise the structure of geometric images 'in the large', such as the area of a surface, the volume of the enclosed domain, integral curvature, and so on. The classical methods of differential geometry could not be used without restrictive smoothness assumptions. The efforts of outstanding mathematicians such as Steiner, Hilbert, Minkowski, Weyl, Cohn-Vossen, and Liebmann led only to isolated results on geometry in the large. The papers of these geometers contained the formulations of numerous unsolved problems that determined the future development of global geometry for many decades.

The most important of these problems are now solved. Much credit must be given to Aleksandrov himself and his students. Their efforts enriched geometry in the large with many fruitful ideas and methods. The research school created by Aleksandrov assumed worldwide leadership in the field of geometry in the large. As he predicted already in 1948 in a discussion of differential geometry textbooks, problems concerned with the global structure of differential-geometric objects came to the forefront in the whole of modern differential geometry.

Fundamental results in the theory of convex bodies are due to Aleksandrov. In developing Minkowski's classical results, he established new inequalities for mixed volumes of convex bodies. As a byproduct, he also discovered similar algebraic inequalities which forty years later found a totally unexpected use in solving the famous van der Waerden problem on estimating the permanent, formulated as far back as 1926. His inequalities for mixed volumes have now found interesting generalisations and applications also in algebraic geometry and the theory of non-linear elliptic equations, and the concept of mixed volumes has made its way even into the theory of random processes.

At the same time, Aleksandrov introduced techniques of measure theory and functional analysis into the theory of convex bodies. He proposed the idea of considering the function space generated by the support functions and of endowing this space with special measures, 'surface functions' and the 'curvature functions' related to them. He proved theorems on uniqueness up to translations of a convex body with a given curvature function, which included as extreme special cases the known theorems of Christoffel and Minkowski. In doing this, Aleksandrov defined generalised differential equations in measures and the corresponding generalised solutions.

His achievements in the theory of convex polytopes, obtained in the middle of the last century, are impressive even today for the strength and completeness of the results and the beauty of the methods used. He proposed general methods of proving existence and uniqueness theorems for convex polytopes and surfaces satisfying various conditions, and on this basis he obtained a great number of specific results. Most remarkable is his solution of Weyl's problem, posed as far back as 1918. Weyl's problem was to prove that each two-dimensional Riemannian manifold of positive curvature that is homeomorphic to a sphere, is isometric to a closed convex surface in three-dimensional Euclidean space. The solution found by Aleksandrov gave an answer in a much more general situation than originally required.

Weyl's method of solution (which he had not carried through to the end) was based on a reduction of the problem to a certain problem for differential equations. In contrast, the methods employed by Aleksandrov are purely geometrical.

He considered first the analogue of Weyl's problem for polytopes. In this case it becomes an existence problem for a convex polytope with a given unfolding satisfying certain simple necessary conditions (the polygons of the unfolding when glued together must form a manifold homeomorphic to a sphere, and the sum of the angles at each vertex of the unfolding must not exceed 2π). An intrinsic metric arises on the surface of the convex polytope, in which the distance between two points is taken to be the infimum of the lengths of all curves that join these points.

A metric on an arbitrary abstractly given unfolding is introduced in the same way. Cutting the surface of a convex polytope into polygons in an arbitrary manner, we obtain different unfoldings of it, all isometric to each other. Thus, Weyl's problem turns into a finite-dimensional problem for polytopes. There are two sets: the set M_n of convex polytopes with n vertices, and the set Q_n of unfoldings that have n vertices and satisfy the above conditions. Here two isometric unfoldings are regarded as one and the same unfolding. Each set is endowed with a natural topology that turns M_n and Q_n into manifolds of dimension $3n - 6$. Moreover, M_n and Q_n can even be viewed as differentiable manifolds. By assigning to each convex polytope P its unfolding S we obtain a map $\varphi : M_n \to Q_n$.

The problem is to prove that $\varphi(M_n) = Q_n$. It suffices to prove the following assertions: (A) $\varphi(M_n)$ is an open subset in Q_n; (B) every connected component of Q_n contains an element of $\varphi(M_n)$; (C) $\varphi(M_n)$ is closed in Q_n.

Assertion (C) is relatively easy to prove. It means that if an unfolding $S_0 \in Q_n$ is the limit of unfoldings S_m, $m = 1, 2, \ldots$, each of which is realised as the surface of a convex polytope, then S_0 can be realised in the same sense. The main difficulty lies in proving (A). Aleksandrov indicated two different proofs of it. One relies on Brouwer's theorem on invariance of domain. First it is established that φ is continuous and injective. The continuity is obvious. The injectivity follows from the fact that if the surfaces of two convex polytopes are isometric, then they can be superimposed by a certain motion. (The last assertion, also proved by Aleksandrov, is a generalisation of the classical Cauchy theorem which states that two convex polytopes equally composed of identical respective faces are congruent.) Therefore, φ is a homeomorphism. Brouwer's theorem now implies that $\varphi(M_n)$ is an open subset in Q_n. The other proof of (A), also indicated by Aleksandrov, uses the fact that φ is differentiable and its Jacobian is everywhere non-zero. The latter property of φ is geometrically none other than a theorem on rigidity of convex polytopes. The proof of (B), as well as the proof of the fact that Q_n is a $(3n - 6)$-dimensional manifold, is a separate highly technical task.

The solution of Weyl's problem in the general case is obtained from the Aleksandrov theorem for polytopes via approximation of Riemannian metrics by polytopes and passage to the limit.

Weyl's scheme of the proof was implemented by Lewy in 1938 by means of analytic function theory; however, both Weyl and Lewy considered the realisation problem only for an analytic Riemannian metric. Aleksandrov achieved much more: he gave up not only analyticity but even smoothness of the metric. In the language now used in differential equations, we can say that he introduced and developed in

this inherently non-linear problem the theory of its generalised solutions, and he did it at a time when this approach was still in the process of being accepted in the theory of partial differential equations proper, and was used only in problems of the calculus of variations.

Aleksandrov obtained non-trivial generalisations of his results on Weyl's problem to the case of Lobachevsky space and spherical space. Later, important progress was made in this area by Pogorelov. He established theorems on the relationship between the degree of smoothness of a convex surface and that of its intrinsic metric and found a generalisation of Aleksandrov's theorem which concerns immersion of a Riemannian metric into a Riemannian space of curvature bounded above.

Aleksandrov's results on Weyl's problem gave rise to numerous studies on the theory of bending of convex surfaces, among which we mention first of all his own work and also that of S. P. Olovyanishnikov and Pogorelov, and it stimulated other approaches to the theory of bending in the work of N. V. Efimov, I. N. Vekua, and their students. The cutting and gluing method created by Aleksandrov on the basis of his existence theorems dramatically changed the whole theory of bending. His work on Weyl's problem was the origin of a new branch in modern geometry, which can be characterised as the theory of irregular Riemannian spaces. Aleksandr Danilovich Aleksandrov was the creator of this branch and the author of its most important results.

His solution of Weyl's problem relies on approximation of a Riemannian metric of positive curvature by polyhedral metrics of positive curvature. A natural question arises: in general, what are the metrics that admit such approximations? Aleksandrov gave a complete answer to this question. He introduced the notion of a two-dimensional manifold with metric of positive curvature and analysed properties of these manifolds in detail.

His numerous results on this subject are combined in his monograph *Intrinsic geometry of convex surfaces*, published in 1948. Notions such as a shortest path, the angle between two curves, and the area of a set are defined for two-dimensional manifolds with metric of positive curvature. Moreover, a certain non-negative completely additive set function called the (integral) curvature is defined for these manifolds.

In the special case where the manifold is Riemannian (of class C^2) this set function coincides with the area integral of the Gaussian curvature. In general, the curvature is not necessarily an absolutely continuous function with respect to the area, and may even be concentrated at isolated points and on lines. For example, the curvature of the surface of a right circular cone is concentrated on the set that consists of its apex and the base circle.

Among other results from the geometry of manifolds of positive curvature we mention the following remarkable theorem due to Aleksandrov.

Suppose that a triangle on a convex surface is formed by shortest paths joining three points X, Y, Z. We construct a plane triangle $X'Y'Z'$ with the same side lengths. It turns out that the separate angles at the vertices of this plane triangle are less than or equal to the corresponding angles of the original triangle on the convex surface.

This fact was previously unknown, even in the case of two-dimensional Riemannian spaces of positive curvature. An extension of this theorem (usually called

Aleksandrov's comparison theorem in the literature) to Riemannian spaces of positive curvature and arbitrary dimension, due to V. A. Toponogov, has played an important role and contributed to the progress in recent years in the analysis of the global structure of these spaces. These results have served as a model and a stimulus for a number of comparison theorems obtained in modern Riemannian geometry in the large.

After the theory of two-dimensional manifolds of positive curvature was constructed, a new problem was to consider manifolds with curvature a completely additive set function of arbitrary sign. The theory of such manifolds, called two-dimensional manifolds of bounded curvature, was developed mainly by Aleksandrov as far back as the early 1950s. A complete presentation of it appeared in 1962 in the monograph *Two-dimensional manifolds of bounded curvature* (jointly with Zalgaller).

Aleksandrov proposed two different definitions of a two-dimensional manifold of positive curvature. One is axiomatic; the other is constructive and based on approximation of manifolds of bounded curvature by polyhedra. Aleksandrov proved that they are equivalent. We give here only the second definition.

Let M be a two-dimensional manifold equipped with a metric ρ which is intrinsic, that is, for any two points $X, Y \in M$, the value of $\rho(X,Y)$ equals the infimum of the lengths of rectifiable curves joining these points. For each domain $G \subset M$, there is a naturally defined metric ρ_G, where $\rho_G(X,Y)$ equals the infimum of the lengths of rectifiable curves in G joining X and Y. The metric ρ_G is called the induced metric of the domain G.

A curve in M is called a shortest path if its length equals the distance between its endpoints. For any two sufficiently close points there is a shortest path between them. A manifold M equipped with an intrinsic metric ρ is called locally flat if every point X of it has a neighbourhood U that is isometric (in the metric ρ) to a disc $x^2+y^2<\delta^2$ in the usual Euclidean plane. A manifold M is called a polyhedron if it has a finite subset $H = \{A_1, \ldots, A_k\}$ such that the set $M \setminus H$ is locally flat. The points $A_1, \ldots, A_k$ are called the vertices of the polyhedron. A metric ρ on a two-dimensional manifold M is called polyhedral if it is intrinsic and the manifold M endowed with this metric is a polyhedron. Associated to each vertex $A \in M$ is a number $\theta(A)$, called the total angle at the vertex. It is defined as follows.

Some neighbourhood of A can be subdivided by shortest paths issuing from A into finitely many domains, each isometric (in the induced metric) to a plane triangle. The number $\theta(A)$ is equal to the sum of the angles of these plane triangles at the point A. (It is easily proved that the sum does not depend on the choice of a neighbourhood and its subdivision.) We always have $\theta(A) > 0$. If $\omega(A) \equiv 2\pi - \theta(A) = 0$, then some neighbourhood of the point A is isometric to a disc on the plane, so that $\omega(A)$ can be viewed as a measure of non-Euclideanness of the polyhedron near A. Accordingly, $\omega(A)$ is called the curvature at the vertex A.

Let $\omega(E)$ denote the sum of the curvatures at all vertices of M that belong to a set $E \subset M$, and $|\omega|(E)$ the sum of the absolute values of these curvatures. Then $\omega(E)$ is called the curvature, and $|\omega|(E)$ the absolute curvature of the set E. A two-dimensional manifold M with intrinsic metric ρ is called a two-dimensional manifold of bounded curvature if for every point $A \in M$ there is a neighbourhood

U and a sequence of polyhedral metrics ρ_n, $n = 1, 2, \ldots$, in U that converges uniformly to ρ and is such that the sequence $|\omega_n|(U)$, $n = 1, 2, \ldots$, is bounded (ω_n is the curvature in the metric ρ_n).

A two-dimensional Riemannian manifold whose metric is defined by the linear element $E\,du^2 + 2F\,du\,dv + G\,dv^2$, where the functions E, F, and G satisfy smoothness conditions that are needed to define the Gaussian curvature at a point (it suffices to assume that $E, F, G \in C^2$), is a special case of a two-dimensional manifold of bounded curvature. Another special case is given by manifolds with polyhedral metrics.

Fundamental notions of classical two-dimensional Riemannian geometry such as the length of a curve, the curvature of a curve, a geodesic, the area of a set, and the curvature of a manifold have analogues in the general case of two-dimensional manifolds of bounded curvature. (There, instead of the curvature of a curve at its points, one considers the turn of a curve, a quantity equal in the regular case to the integral of the curvature with respect to the arclength of the curve, and instead of the curvature of the manifold itself at a point, a set function that is an analogue of the integral of the curvature over a set.) Numerous specific results in the theory of two-dimensional manifolds of bounded curvature are due to Aleksandrov, many of them new even for two-dimensional Riemannian manifolds. He developed techniques that enable us easily to find our way in this theory. Among them are set functions (curvatures of sets and one-way turns of parts of curves) and comparison theorems. Another equally effective apparatus is the generalised isothermal line element introduced for such spaces by Yu. G. Reshetnyak, a student of Aleksandrov. A surprising new area for applications of manifolds of bounded curvature appeared in the theory of meromorphic functions.

Thus, the class of two-dimensional Riemannian manifolds received a compactification that facilitates its investigation, and at the same time preserves the structure of a manifold and the boundedness of the integral curvature. This enabled Aleksandrov and his students to give exhaustive solutions to a great number of extremal problems in the theory of surfaces. In the regular case many of them simply had no solution, because the extremum was realised on objects outside the regular class. An example is the problem, solved by Aleksandrov, of finding the surface of greatest area among surfaces of a given perimeter that are homeomorphic to a disc and have positive part $\omega^+(S)$ of the curvature (that is, the upper variation of the set function ω) less than or equal to a given number $\eta > 0$. In the case $\eta \geq 2\pi$ the problem has no solution. In the case $\eta < 2\pi$ the solution is the lateral surface of a right circular cone whose angle at the apex equals $2\pi - \eta$. (If the surface is cut along a generatrix of the cone, then it can be unfolded into the plane as a sector of a disc with angle $2\pi - \eta$.)

The proof of this theorem can be outlined as follows. It suffices to consider the case when the manifold is a polyhedron. The polyhedron S, with a given perimeter and $\omega^+(S) \leq \eta < 2\pi$, is successively transformed so that its area grows and its curvature, in the end, is concentrated at a single point. Each separate step of the transformation consists in making a cut and gluing a certain polyhedron into the cut.

Similar techniques are useful in other problems of the geometry of manifolds of bounded curvature. Together, they constitute Aleksandrov's cutting and gluing method.

Extensive research on two-dimensional manifolds of bounded curvature has been done by other authors, mainly students of Aleksandrov. In particular, questions on the theory of manifolds of bounded curvature have been studied by Yu. F. Borisov, Yu. D. Burago, V. A. Zalgaller, Yu. G. Reshetnyak, V. V. Strel'tsov, and others. One of the problems in this theory was to distinguish classes of two-dimensional surfaces, determined by certain natural conditions, which in their intrinsic geometry would be two-dimensional manifolds of bounded curvature. Important results in this direction were obtained by Aleksandrov, who proved that if a surface is given by an equation $z = f(x, y)$ with f a difference of two convex functions, then this surface is a two-dimensional manifold of bounded curvature. (Other classes of surfaces with this property were indicated by Pogorelov, Burago, and others.) It should be noted that there are many unsolved problems in the study of the extrinsic geometry of irregular surfaces with metric of bounded curvature in the sense of Aleksandrov, and on the whole this area of investigation is far from complete. (This circle of problems gave rise to an interesting new direction in the theory of immersed manifolds, developed by S. Z. Shefel'.)

The theory of manifolds of bounded curvature constructed by Aleksandrov is two-dimensional. The task of constructing a multidimensional analogue of it seems rather difficult. The most significant progress in this direction is due to Aleksandrov. A particular case of two-dimensional manifolds of bounded curvature is formed by manifolds of curvature bounded above or below by a number K_0. (In the regular case these are Riemannian manifolds whose Gaussian curvature $K(X)$ is either at most K_0 at every point X or at least K_0 at all points X.) Aleksandrov showed that these manifolds can be described by a system of axioms in which the two-dimensionality is not used. This makes it possible to introduce a general notion of a metric space with curvature bounded from one side and with topology satisfying rather weak (from the viewpoint of differential geometers) conditions. Such a space is not even necessarily a manifold. Aleksandrov studied in detail spaces whose curvature is less than or equal to a $K_0 < \infty$.

These studies were continued and extended by other Siberian geometers, students and followers of Aleksandrov. In particular, they solved the problem of an axiomatic construction of classical Riemannian geometry. Namely, I. G. Nikolaev and V. N. Berestovskii proved the following assertion. A space with intrinsic metric that is an n-dimensional manifold of bounded curvature in the sense of Aleksandrov is a Riemannian space smooth enough to satisfy the classical theory of curvature.

In differential geometry and the theory of convex bodies there are well-known uniqueness theorems that establish the equality (in some sense or another) of geometric objects that satisfy certain additional conditions. Results of this kind were obtained in their time by Cauchy, Liouville, and other prominent mathematicians.

Uniqueness theorems, together with existence theorems, occupy an important place in Aleksandrov's creative work. They are the subject of a cycle of his research carried out in 1956–1966. The main tool of this research was theorems on the solutions to elliptic differential equations combined with various arguments of a

geometrical nature. To give an idea of this work of Aleksandrov we present one of his theorems.

Theorem A. *Let S and S_0 be closed convex analytic surfaces, and $k_1 \geq k_2$, $k_{01} \geq k_{02}$ their principal curvatures at points $x \in S$, $x_0 \in S_0$ with parallel normals. Let $f(\xi, \eta, \bar{n})$ be a function of scalar parameters ξ, η and a unit vector $\bar{n}$ such that $f(\xi, \eta, \bar{n}) > f(\xi', \eta', \bar{n})$ for $\xi > \xi'$ and $\eta > \eta'$. Suppose that $f(k_1, k_2, \bar{n}) = f(k_{01}, k_{02}, \bar{n})$ for every $x \in S$, where $\bar{n}$ is the normal at $\bar{x}$. Then the surfaces S and S_0 can be transformed into each other by a parallel translation.*

Theorem A in this formulation was proved by Aleksandrov in 1938. It was natural to ask whether the analyticity requirement can be replaced by a weaker condition. Aleksandrov obtained also an analogue of Theorem A for convex polytopes. His proof is based on an idea similar to that used for proving Cauchy's theorem on congruence of polytopes.

Another natural question: is there an analogue of Theorem A for surfaces in n-dimensional space for $n > 3$? This and similar questions were investigated by Aleksandrov in 1956–1966. Specifically, Pogorelov first showed that for Theorem A the analyticity requirement may be relaxed to fourth-order differentiability. The function f is assumed to be of class C^1, with $\frac{\partial f}{\partial \xi}\frac{\partial f}{\partial \eta} > 0$ everywhere in the domain of definition.

In 1956 Aleksandrov proved that (under the same assumptions on f) the analyticity requirement can be replaced with second-order differentiability. He also showed that if S and S_0 are analytic surfaces homeomorphic to a sphere, then the condition that S be convex can be removed (this is established in a paper of 1966). He proved a great number of theorems similar in formulation to Theorem A on convex surfaces in n-dimensional Euclidean space for an arbitrary $n \geq 3$ and on surfaces in general Riemannian spaces and spaces of constant curvature. The form of these theorems is as follows: the points of two surfaces are put into a correspondence in some way or another; if the principal curvatures of the surfaces at corresponding points are in a certain relation, then the surfaces coincide. (A word-for-word translation of Theorem A to the multidimensional case seems impossible, although Aleksandrov did obtain a partial generalisation.)

As an application of the uniqueness theorems, Aleksandrov deduced general theorems on a characteristic property of an $(n-1)$-dimensional sphere. Namely, if on a surface S that bounds a body in E^n we have $\Phi(k_1, \dots, k_{n-1}) = \text{const}$, where $k_1 \geq k_2 \geq \cdots \geq k_{n-1}$ are the principal curvatures at a point of the surface, and the function Φ is such that the derivatives $\partial\Phi/\partial k_i$ are continuous and have the same sign for all $k_1, \dots, k_{n-1}$, then S is a sphere. In particular, a closed surface in three-dimensional space that has constant mean curvature and no self-intersections is a sphere. (In the language of physics this means that no soap bubble can be different from a sphere.) Related to uniqueness questions are problems on estimating the change of an object under small changes of characteristics that uniquely determine the object. New methods and results were established by Aleksandrov in this area also. The difficult problem of Cohn-Vossen on estimating the change of the shape of a closed convex surface under a small change of its intrinsic metric was solved by Aleksandrov's student Yu. A. Volkov.

Aleksandrov created a new branch in the theory of elliptic differential equations—the geometric theory of elliptic equations.

We give a brief outline of the results of Aleksandrov's work in differential equations in the period from 1956 to 1965. First of all, these are theorems on the existence of generalised solutions to the first boundary value problem for Monge–Ampère type equations, namely, equations of the form

$$f(\nabla z, z, x)\,\mathrm{Det}\left(\frac{\partial^2 z}{\partial x_i \partial x_j}(x)\right) = h(x), \tag{1}$$

where f and h are non-negative functions. The solution is sought in the class of convex functions. This is a natural requirement, since (1) is elliptic only for such functions.

The equation (1) makes it possible to construct for each convex function z two set functions, $\omega_f(M, z)$ and $\nu(M)$. Here

$$\nu(M) = \int_M h(x)\,dx,$$

so ν is determined by the function h. In the regular case (namely, in the case $z \in C^2$),

$$\omega_f(M, z) = \int_M f(\nabla z(x), z(x), x)\,\mathrm{Det}\left(\frac{\partial^2 z}{\partial x_i \partial x_j}(x)\right) dx.$$

In the general case the function $\omega_f(M, z)$ is defined using the notion of a normal map, which is introduced as follows. Let $z = z(x)$ be a convex function defined in a closed convex domain $\Omega \subset \mathbb{R}^n$. A vector $\zeta(x) = (\zeta_1, \ldots, \zeta_n)$ is called a generalised gradient of z at a point x_0 if $z = \langle \zeta, x - x_0 \rangle + z(x_0)$ is a supporting hyperplane for the hypersurface $S = \{(x, z) \mid z = z(x)\}$. If z is differentiable at x_0, then the generalised gradient obviously coincides with the usual one. If we associate with each point $x \in \Omega$ all vectors that are generalised gradients of the function z at this point, we obtain a map φ, in general multi-valued, of the domain Ω into $\mathbb{R}^n$, which is what is called the normal map. Let $E = \varphi(\Omega)$. For each point $\zeta \in E$ there is a point $(x, z) \in S$ such that ζ is a generalised gradient at x. We set $x = x(\zeta)$, $z = z(\zeta)$. The function $\omega_f(M, z)$ is defined by the equality

$$\omega_f(M, z) = \int_{\varphi(M)} f(\zeta, z(\zeta), x(\zeta))\,d\zeta.$$

Aleksandrov considered the following problem: find a convex function z with given values on the boundary $\partial\Omega$ and such that the set function $\omega_f(M, z)$ coincides with a specified set function $\nu(M)$. If the function obtained is of class C^2, then it will obviously be a solution to (1). He established the existence of a generalised solution to this problem under the condition that f and the prescribed boundary values of the unknown solution satisfy certain natural restrictions. We omit the details and refer the reader to his paper in *Vestnik Leningrad. Univ.* **1958**, no. 1. Pogorelov later proved that Aleksandrov's generalised solutions are smooth if $f \equiv 1$ and if $z\big|_{\partial\Omega}$ and h are sufficiently smooth positive functions.

In the 1950s Aleksandrov developed a method for finding upper and lower estimates for functions that satisfy second-order elliptic equations or inequalities, but do not have the classical smoothness (do not have second-order derivatives at each point but merely belong to the space $W_n^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$). We present only one such estimate, far from the most general, but one which led to much progress in the study of quasi-linear problems and even of a certain class of extremely non-linear elliptic problems. It has the form

$$\max_{x\in\Omega} z(x) \le \max_{x\in\partial\Omega} z(x) + C_1 \operatorname{diam}\Omega\, e^{C_2|b|_{n,\Omega}} |Lz(x)_-|_{n,\Omega}. \tag{2}$$

Here

$$Lz(x) = \sum_{i,j=1}^n a_{ij}(x) z_{x_i x_j}(x) + \sum_{i=1}^n b_i(x) z_{x_i}(x);$$

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge 0$$

for each $\xi \in \mathbb{R}^n$; C_1 and C_2 are constants depending only on n; Ω is an arbitrary bounded domain in $\mathbb{R}^n$; and $z \in W_n^2(\Omega)$. The seminorm $|\cdot|_{n,\Omega}$ is computed by the rule

$$|\nu|_{n,\Omega} = \left(\int_\Omega |\nu(x)|^n \left(\operatorname{Det}(a_{ij}(x))\right)^{-1} dx\right)^{1/n},$$

and $\nu_-(x) = \max\{0, -\nu(x)\}$. The inequality (2) is remarkable in many respects (including the character of dependence on Ω), and a purely analytical proof of it seems highly improbable.

We illustrate the main idea of Aleksandrov's method of proof of (2) by a simple example. Let $z(x)$ be a solution to the equation

$$\sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 z}{\partial x_i \partial x_j}(x) = f(x) \tag{3}$$

in a domain G in the space $\mathbb{R}^n$, where the functions $a_{ij}(x)$ are such that the eigenvalues of the quadratic form

$$\sum_{i,j=1}^n a_{ij}\xi_i\xi_j$$

are contained in an interval $[\lambda_1, \lambda_2]$, where $0 < \lambda_1 \le \lambda_2 < \infty$ for all x. Suppose that G is convex, $z(x) = 0$ on the boundary, and it is required to estimate $\min z(x)$. Let $\Gamma_z = \{(x,y) \in \mathbb{R}^{n+1} : x \in G,\ y \ge z(x)\}$ be the epigraph of z. Further, let V_z be the convex hull of Γ_z. The set V_z is bounded below by a surface $y = \widetilde{z}(x)$. Here $z(x) \ge \widetilde{z}(x)$ for all $x \in G$, and the function $\widetilde{z}(x)$ is convex. Suppose that $z(x)$ attains its minimum at a point $x_0 \in G$. We construct the convex cone K in $\mathbb{R}^{n+1}$ formed by intervals that join the point $(x_0, z(x_0))$ to boundary points of G. If the absolute value of $z(x_0)$ is large, then the cone K is stretched and its supporting spherical image is large. On the other hand, it is clear that the supporting image

of K is contained in the supporting image of the surface $z = \widetilde{z}(x)$. However, the latter image cannot be too large, for the following reason. In evaluating the supporting image of the surface $z = \widetilde{z}(x)$ it suffices to take into account only the points where $\widetilde{z}(x) = z(x)$. They are the convexity points for the function $z(x)$; therefore, the quadratic form $\sum_{i,j=1}^{n} z_{ij}\xi_i\xi_j$, where $z_{ij} = \frac{\partial^2 z}{\partial x_i \partial x_j}(x)$, is non-negative at these points. Consequently, at the points where $\widetilde{z}(x) = z(x)$, we have

$$\sum_{i,j=1}^{n} a_{ij} z_{ij} \geq \lambda_1 \sum_{i=1}^{n} z_{ii} \geq n\lambda_1 \left(\operatorname{Det}(z_{ij})\right)^{1/n}. \tag{4}$$

From (4) it follows that the area of the supporting image of the surface $z = \widetilde{z}(x)$ does not exceed

$$\frac{1}{(n\lambda_1)^n} \int_G (f(x))^n \, dx. \tag{5}$$

Thus, we see that the cone K cannot be arbitrarily long, since the area of its normal image is less than or equal to (5). It is not difficult to obtain an explicit bound for the height of the cone. This will give an estimate for

$$|z(x_0)| = |\min_{x \in G} z(x)|.$$

To justify all our conclusions, it is sufficient that the function z belong to the class $W_n^2(G)$, that is, have generalised second derivatives that are integrable to the power n.

It is impossible to describe in a single article all the new and valuable results in this cycle of Aleksandrov's papers. Many of them are still waiting for a user, and without any doubt will bear fruit. An example is the inequality (2), which led to progress in the study of non-linear elliptic equations (O. A. Ladyzhenskaya, N. V. Krylov, M. V. Safonov, N. N. Ural'tseva, and others). Analogous inequalities for parabolic operators, proved by Krylov, Ural'tseva, and A. I. Nazarov, were an important step in the study of quasi-linear parabolic equations.

In the 1970s Aleksandrov's research interests concerned mainly geometrical questions on the foundations of relativity theory. This research began in his work (jointly with V. A. Ovchinnikova) as early as 1953. Aleksandrov turned to relativity theory regularly in various periods of his life. His ideas in this area were continued and developed in the work of his students: Yu. F. Borisov, A. K. Guts, A. V. Kuz'minykh, A. V. Levichev, P. I. Pimenov, and A. V. Shaidenko.

Geometrically, space-time (that is, the totality of all events in the physical universe) can be viewed as a four-dimensional affine space equipped with an order relation. If x and y are two points of this space, then $x \prec y$ means that the event x precedes the event y, or in other words, x can affect y. For each point x we have the set K_x of all events following x. In Newtonian mechanics K_x is a half-space. In the mechanics of relativity theory K_x is a right circular cone with apex at x, and the cones K_x corresponding to different points x are obtained from one another by parallel translation. A one-to-one transformation of the space $\mathbb{R}^4$ that preserves the order relation of the special relativity theory is Lorentzian.

In physics, this is proved under smoothness conditions for the transformation. The work of Aleksandrov and V. A. Ovchinnikova showed that in fact no smoothness conditions are needed.

Aleksandrov introduced a general notion of kinematics. In his sense, kinematics is a topological space endowed with an order relation which properly agrees with the topology. The problem consists in describing minimal conditions (axioms) under which a given kinematics is the kinematics of special relativity theory.

He also made a large contribution to the theory of functions of a real variable. This is connected with his goal of studying irregular geometric images and extending to them certain principal concepts of differential geometry.

One of his results in the theory of functions of a real variable is the classical theorem that a convex function of n variables is twice differentiable almost everywhere. However, his most significant achievement in this area is his work on the abstract theory of set functions. An analysis of various completely additive set functions that arise naturally in the theory of convex bodies was his stimulus for studying general questions of measure theory in the most abstract form.

The main results of Aleksandrov in this area include, first of all, a theorem on the general form of a linear functional on the space $C(X)$ of bounded continuous functions defined on a normal topological space X. He considered slightly more general spaces than the ones traditionally accepted in general topology. According to the Riesz theorem, each continuous linear functional on $C([a,b])$ is represented by a Stieltjes integral. A. A. Markov proved that if X is a compact topological space, then each linear functional on $C(X)$ is represented by an integral with respect to a completely additive set function. However, if X is not compact, then the Markov theorem does not hold. Aleksandrov proved that if the condition of complete additivity is replaced by the condition of regularity (which is equivalent to the former in the case of compact spaces), then the theorem on the representability of each linear functional on $C(X)$ in the form of an integral with respect to an additive set function remains true in the general case. His second important achievement in the theory of set functions is his theory of weak convergence for sequences of such functions. The results of this series of Aleksandrov's work made up the content of his DSc dissertation. They are widely used in probability theory and functional analysis.

Aleksandrov's work in mathematics, as deep, original, and valuable as it was, does not exhaust his legacy. Philosophical questions of mathematics and theoretical physics were always in the circle of his interests. His thoughts on the essence of mathematics, gained over more than 20 years of experience, were summed up in his paper "Mathematics and dialectics" (*Siberian Mathematical Journal*, **1970**, no. 2). It is not by chance that professors of the humanities have often recommended that students majoring in the exact sciences read Aleksandrov's papers on general questions of science.

He also wrote profound papers on philosophical problems of relativity theory and quantum mechanics. His papers in philosophy and his talks discuss a very broad variety of questions of life.

Aleksandrov devoted much time and effort to the training of new specialists. He is known for his scientific generosity not only as a leader in research but also as a direct adviser of graduate students and young researchers. He always inspired

their creativity and stimulated them to scientific exploration. His ideas, presented in his lectures and seminars, written in his work notes, and outlined in personal conversations, provided the basis for many studies carried out by his students.

A responsible person, Aleksandrov could not keep aloof from one of the most important problems of school education reform: the creation of new high school textbooks in geometry. He enlisted the participation of A. L. Verner and an experienced teacher, V. I. Ryzhik, in this work. Together they wrote two trial solid geometry textbooks, and then in 1983 a textbook in geometry for grades 9–10 in special schools and classes with advanced mathematical studies. In 1981 Aleksandrov began to work on a new structure for a course in plane geometry, and he published the course in a series of preprints. Trial textbooks based on this course and written jointly with Verner and Ryzhik were published in 1984–1986 for grades 6–8. The experiment involving the entire cycle of these textbooks ended in a whole series of textbooks for ordinary schools as well as for schools with advanced mathematical studies.

For 12 years, from 1952 until 1964, Aleksandrov was Rector of Leningrad State University. He started in the hard postwar years, and he managed to mobilize the staff remaining in the university, invited good scholars from other places, and helped the growth of young researchers in many ways. As a result of his 12 years as Rector new directions and new schools appeared, and the network of seminars was expanded. The specialists trained during that period are today's leaders, together with young successors.

As Rector, Aleksandrov actively and efficiently supported biologists at the university in their struggle with Lysenko's pseudo-science. Leningrad University began teaching the science of genetics already in the 1950s, whereas in other universities genetics was reestablished only in 1965. This was not easy: it suffices to recall Khrushchev's outcry when Aleksandrov refused to obey the order of the Ministry of Education to re-instate at Leningrad University an infamous obscurantist proponent of 'Michurin' biology. This refusal was qualified as a manifestation of Menshevism. Aleksandrov did not falter, and the individual was not hired by the university. At the same time students expelled from other universities for attempts to study genetics illegally were given an opportunity to continue their education at Leningrad University.

Also associated with the name of Rector Aleksandrov was the formation of directions of research new for that time such as sociology and mathematical economics, which found his active support within the walls of the university in the period of persecution. In October 1990 he was awarded the Order of the Red Banner of Labour for his special contribution to the preservation and development of genetics and natural selection; he was the only mathematician among the other recipients, all biologists. This unusual honour was a result of the high esteem for his noble actions by the majority of scientists in our country.

Aleksandrov enjoyed immense authority among venerable scholars as well as among the young. "He led the university not by the force of order but by moral authority", wrote V. I. Smirnov in an address on the occasion of Aleksandrov's leaving the post of Rector. "Aleksandr Danilovich is the conscience of the faculty", said D. K. Faddeev on the same occasion.

Aleksandrov spent 25 years of his life in Siberia. In 1964, by invitation of M. A. Lavrent'ev, he and his family moved to Novosibirsk. There he found many faithful friends and students. He gave Siberia not only his heart and soul but also his health: he suffered from tick-borne encephalitis. Aleksandrov created a large and ramified school of research. Among his Leningrad students are dozens with PhD and DSc degrees, and also in Novosibirsk new Doctors of the Sciences grew up under his guidance, along with a pleiad of young PhDs in geometry. They continue their creative work in many cities in Siberia.

Aleksandrov posessed a whole scientific world view that enabled him to analyse philosophical and social problems in depth, as well as to respond to the challenges of current circumstances during his entire life. As the basis of the system of his moral beliefs he mentioned kindness or universal humanism, a scientific approach, and responsibility. He was faithful to the ideals of his youth to the last days of his life.

His merits were marked by many awards and distinctions. Among the very last, he valued the first Euler Gold Medal, given to him by the Presidium of the Russian Academy of Sciences in 1992.

Aleksandrov was notable for his indomitable aspiration for the highest results in any activity he undertook, in mathematics as much as in sport (he was an experienced mountaineer), in philosophy as much as in the history of science (he delivered lectures on the history of mathematics at Leningrad University and Novosibirsk University), and in many other areas. His family and friends together with his students and colleagues remember well the characteristic features of his personality: devotion to the truth, constant readiness to fight for it and to support and protect it to the very end.

The scientific ideas of Academician A. D. Aleksandrov will long live in the work of his students and successors. His inimitable charm, the combination of a young spirit and the wisdom of experience, his passionate temperament and subtle intellect, his selflessness and tenderness have become dear memories and a consolation for those who were fortunate enough to be near him.

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