Boolean Valued Analysis: Background and Results

A. G. Kusraev and S. S. Kutateladze

Abstract The paper provides a brief overview of the origins, methods and results of Boolean valued analysis. Boolean valued representations of some mathematical structures and mappings are given in tabular form. A list of some problems arising outside the theory of Boolean valued models, but solved using the Boolean valued approach, is given. The relationship between the Kantorovich's heuristic principle and the Boolean valued transfer principle is also discussed.

Keywords Vector lattice · Kantorovich's principle · Gordon's theorem · Boolean valued analysis · Boolean valued representation

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1 Introduction

In 1977, Eugene Gordon, a young teacher of Lobachevsky Nizhny Novgorod State University, published the short note [13] which begins with the words:

> This article establishes that the set whose elements are the objects representing reals in a Boolean valued model of set theory $\mathbb{V}^{(\mathbb{B})}$, can be endowed with the structure of a vector

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space and an order relation so that it becomes an extended K-space[1] with base[2] $\mathbb{B}$. It is shown that in some cases this fact can be used to generalize the theorems about reals to extended K-spaces.

His note has become the bridge between various areas of mathematics which helps, in particular, to solve numerous problems of functional analysis in "semiordered vector spaces" [36] by using the techniques of Boolean valued models of set theory [6].

In the same year, at the Symposium on Applications of Sheaf Theory to Logic, Algebra, and Analysis (Durham, July 9–11, 1977), Gaisi Takeuti, a renowned expert in proof theory, observed that if $\mathbb{B}$ is a complete Boolean algebra of orthogonal projections in a Hilbert space H, then the set whose elements represent reals in the Boolean valued model $\mathbb{V}^{(\mathbb{B})}$ can be identified with the vector lattice of selfadjoint operators in H whose spectral resolutions take values in $\mathbb{B}$; see [93].

These two events marked the birth of a new section of functional analysis, which Takeuti designated by the term *Boolean valued analysis*. The history and achievements of Boolean valued analysis are reflected in [56–58].

It should be mentioned that in 1969 Dana Scott foresaw that the new nonstandard models must be of mathematical interest beyond the independence proof, but he was unable to give a really good evidence of this; see [87]. In fact Takeuti found a narrow path whereas Gordon paved a turnpike to the core of mathematics, which justifies the vision of Scott.

Boolean valued analysis signifies the technique of studying the properties of an arbitrary mathematical object by comparison between its representations in two different Boolean valued models of set theory. As the models, we usually take the *von Neumann universe* $\mathbb{V}$ (the mundane embodiment of the classical Cantorian paradise) and the *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ (a specially-trimmed universe whose construction utilizes a complete Boolean algebra $\mathbb{B}$). The principal difference between $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$ is the way of verification of statements: There is a natural way of assigning to each statement ϕ about $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$ the *Boolean truth-value* $[\![\phi(x_1, \dots, x_n)]\!] \in \mathbb{B}$. The sentence $\phi(x_1, \dots, x_n)$ is called true in $\mathbb{V}^{(\mathbb{B})}$ if $[\![\phi(x_1, \dots, x_n)]\!] = \mathbb{1}$. All theorems of Zermelo–Fraenkel set theory with the axiom of choice are true in $\mathbb{V}^{(\mathbb{B})}$ for every complete Boolean algebra $\mathbb{B}$. There is a smooth and powerful mathematical technique for revealing interplay between the interpretations of one and the same fact in the two models $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$. The relevant *ascending-and-descending machinery* rests on the functors of *canonical embedding* $X \mapsto X^{\wedge}$, *descent* $X \mapsto X{\downarrow}$, and *ascent* $X \mapsto X{\uparrow}$ acting between $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$, see [56, 57]. Everywhere below $\mathbb{B}$ is a complete Boolean algebra and $\mathbb{V}^{(\mathbb{B})}$ the corresponding Boolean valued model of set theory; see [6, 99]. A *partition of unity* in $\mathbb{B}$ is a family $(b_\xi)_{\xi\in\Xi} \subset \mathbb{B}$ such that $\bigvee_{\xi\in\Xi} b_\xi = \mathbb{1}$ and $b_\xi \wedge b_\eta = \mathbb{0}$

[1] A *K-space* or a *Kantorovich space* is a Dedekind complete vector lattice. An *extended K-space* is a universally complete vector lattice, cp. [4] and [104].

[2] The *base* of a vector lattice is the inclusion ordered set of all of its bands (that forms a complete Boolean algebra) [36, 104].

whenever $\xi \neq \eta$. The unexplained terms of vector lattice theory can be found in [4, 70, 71, 85, 104].

2 Kantorovich's Heuristic Principle

Definition 1 A *vector lattice* or a *Riesz space* is a real vector space X equipped with a partial order $\leq$ for which the *join* $x \vee y$ and the *meet* $x \wedge y$ exist for all $x, y \in X$, and such that the *positive cone* $X_+ := \{x \in X : 0 \leq x\}$ is closed under addition and multiplication by positive reals and for any $x, y \in X$ the relations $x \leq y$ and $0 \leq y - x$ are equivalent. A *band* in a vector lattice X is the *disjoint complement* Y' of any set $Y \subset X$ where $Y' := \{x \in X : (\forall y \in Y)\, |x| \wedge |y| = 0\}$. Let $\mathbb{P}(X)$ stand for the complete Boolean algebra of all band projections in X.

Definition 2 A subset $U \subset X$ is *order bounded* if U lies in an *order interval* $[a, b] := \{x \in X : a \leq x \leq b\}$ for some $a, b \in X$. A vector lattice X is *Dedekind complete* (respectively, *laterally complete*) if each nonempty order bounded set (respectively, each nonempty set of pairwise disjoint positive vectors) U in X has a least upper bound $\sup(U) \in X$. The vector lattice that is laterally complete and Dedekind complete simultaneously is referred to as *universally complete*.

Definition 3 An *f-algebra* is a vector lattice X equipped with a distributive multiplication such that if $x, y \in X_+$ then $xy \in X_+$, and if $x \wedge y = 0$ then $(ax) \wedge y = (xa) \wedge y = 0$ for all $a \in X_+$. An f-algebra is *semiprime* provided that $xy = 0$ implies $x \perp y$ for all x and y. A *complex vector lattice* $X_{\mathbb{C}}$ is the complexification $X_{\mathbb{C}} := X \oplus iX$ (with i standing for the imaginary unity) of a real vector lattice X.

Leonid Kantorovich was among the first who studied operators in ordered vector spaces. He distinguished an important instance of ordered vector spaces, a Dedekind complete vector lattice, often called a *Kantorovich space* or a *K-space*. This notion appeared in Kantorovich's first fundamental article [35] on this topic where he wrote:

> In this note, I define a new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in such a space) as linear functionals.

Here Kantorovich stated an important methodological principle, the *heuristic transfer principle* for K-spaces, claiming that the elements of a K-space can be considered as generalized reals. Essentially, this principle turned out to be one of those profound ideas that, playing an active and leading role in the formation of a new branch of analysis, led eventually to a deep and elegant theory of K-space rich in various applications. At the very beginning of the development of the theory, attempts were made at formalizing the above heuristic argument. In this direction, there appeared the so-called *identity preservation theorems* which claimed that if some proposition involving finitely many relations is proven for the reals then an analogous fact remains valid automatically for the elements of every K-space (see [36, 71, 104]). The depth and universality of Kantorovich's principle were demonstrated within Boolean valued analysis. See more about the Kantorovich's universal heuristics and innate integrity of his methodology in [67]. The contemporary forms of above mentioned relation preservation theorems, basing on Boolean valued models, may be found in Gordon [15, 18, 21] and Jech [30].

3 Boolean Valued Reals

Boolean valued analysis stems from the fact that each internal field of reals of a Boolean valued model descends into a universally complete vector lattice. Thus, a remarkable opportunity opens up to expand and enrich the mathematical knowledge by translating information about the reals to the language of other branches of functional analysis.

According to the principles of Boolean valued set theory there exists an internal field of reals $\mathscr{R}$ in a model $\mathbb{V}^{(\mathbb{B})}$ which is unique up to isomorphism. In other words, there exists $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ for which $[\![\, \mathscr{R}$ is a field of reals $]\!] = \mathbb{1}$. Moreover, if $[\![\, \mathscr{R}'$ is a field of reals $]\!] = \mathbb{1}$ for some $\mathscr{R}' \in \mathbb{V}^{(\mathbb{B})}$ then $[\![$ the ordered fields $\mathscr{R}$ and $\mathscr{R}'$ are isomorphic $]\!] = \mathbb{1}$.

By the same reasons there exists an internal field of complex numbers $\mathscr{C} \in \mathbb{V}^{(\mathbb{B})}$ which is unique up to isomorphism. Moreover, $\mathbb{V}^{(\mathbb{B})} \models \mathscr{C} = \mathscr{R} \oplus i\mathscr{R}$. We call $\mathscr{R}$ and $\mathscr{C}$ the *internal reals* and *internal complexes* in $\mathbb{V}^{(\mathbb{B})}$.

The fundamental result of Boolean valued analysis is *Gordon's Theorem* [13] which reads as follows: *Each universally complete vector lattice is an interpretation of the reals in an appropriate Boolean valued model.* Formally:

Gordon Theorem *Let $\mathbb{B}$ be a complete Boolean algebra, $\mathscr{R}$ be a field of reals within $\mathbb{V}^{(\mathbb{B})}$. Endow $\mathbf{R} := \mathscr{R}{\downarrow}$ with the descended operations and order. Then*

(1) *The algebraic structure $\mathbf{R}$ is a universally complete vector lattice.*
(2) *The field $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ can be chosen so that $[\![\, \mathbb{R}^{\wedge}$ is a dense subfield of $\mathscr{R} \,]\!] = \mathbb{1}$.*
(3) *There is a Boolean isomorphism χ from $\mathbb{B}$ onto $\mathbb{P}(\mathbf{R})$ such that*

$$\chi(b)x = \chi(b)y \Longleftrightarrow b \le [\![\, x = y \,]\!],$$
$$\chi(b)x \le \chi(b)y \Longleftrightarrow b \le [\![\, x \le y \,]\!]$$
$$(x, y \in \mathbf{R};\ b \in \mathbb{B}).$$

For a detailed proof of the Gordon Theorem, see [45, 56, 58]. Observe also some additional properties of Boolean valued reals, namely multiplicative structure and complexification:

Corollary 1 *The universally complete vector lattice $\mathscr{R}{\downarrow}$ with the descended multiplication is a semiprime f-algebra with the ring unity $\mathbb{1} := 1^{\wedge}$. Moreover, for every $b \in \mathbb{B}$ the band projection $\chi(b) \in \mathbb{P}(\mathbf{R})$ acts as multiplication by $\chi(b)\mathbb{1}$.*

Corollary 2 *Let $\mathscr{C}$ be the field of complex numbers within $\mathbb{V}^{(\mathbb{B})}$. Then the algebraic system $\mathscr{C}{\downarrow}$ is a universally complete complex f-algebra. Moreover, $\mathscr{C}{\downarrow}$ is the complexification of the universally complete real f-algebra $\mathscr{R}{\downarrow}$; i.e., $\mathscr{C}{\downarrow} = \mathscr{R}{\downarrow} \oplus i\mathscr{R}{\downarrow}$.*

Example 1 Assume that a measure space (Ω, Σ, μ) is semi-finite; i.e., if $A \in \Sigma$ and $\mu(A) = \infty$ then there exists $B \in \Sigma$ with $B \subset A$ and $0 < \mu(B) < \infty$. The vector lattice $L^0(\mu) := L^0(\Omega, \Sigma, \mu)$ (of cosets) of μ-measurable functions on Ω is universally complete if and only if (Ω, Σ, μ) is *localizable* ($\equiv$ *Maharam*). In this event $L^p(\Omega, \Sigma, \mu)$ is Dedekind complete; see [11, 241G]. Observe that $\mathbb{P}(L^0(\Omega, \Sigma, \mu)) \simeq \Sigma/\mu^{-1}(0)$.

Example 2 Given a complete Boolean algebra $\mathbb{B}$ of orthogonal projections in a Hilbert space H, denote by $\langle\mathbb{B}\rangle$ the space of all selfadjoint operators on H whose spectral resolutions are in $\mathbb{B}$; i.e., $A \in \langle\mathbb{B}\rangle$ if and only if $A = \int_{\mathbb{R}} \lambda\, dE_\lambda$ and $E_\lambda \in \mathbb{B}$ for all $\lambda \in \mathbb{R}$. Define the partial order in $\langle\mathbb{B}\rangle$ by putting $A \geq B$ whenever $\langle Ax, x\rangle \geq \langle Bx, x\rangle$ holds for all $x \in \mathscr{D}(A) \cap \mathscr{D}(B)$, where $\mathscr{D}(A) \subset H$ stands for the domain of A. Then $\langle\mathbb{B}\rangle$ is a universally complete vector lattice and $\mathbb{P}(\langle\mathbb{B}\rangle) \simeq \mathbb{B}$.

Example 3 Let $\Lambda = \mathscr{R}{\Downarrow}$ stands for the bounded part of the universally complete vector lattice $\mathscr{R}{\downarrow}$, that is, $\Lambda := \{x \in \mathscr{R}{\downarrow} : |x| \leq C^{\wedge}$ for some $C \in \mathbb{R}\}$. Then Λ is a Dedekind complete vector lattice and $\bar{\Lambda} := \Lambda \oplus i\Lambda$ is a complex Dedekind complete vector lattice. Moreover, Λ can be endowed with a norm $\|x\|_\infty := \inf\{\alpha > 0 : |x| \leq \alpha\mathbb{1}\}$.

If μ is a Maharam measure and $\mathbb{B}$ in the Gordon Theorem is the algebra of all μ-measurable sets modulo μ-negligible sets, then $\mathscr{R}{\downarrow}$ is lattice isomorphic to $L^0(\mu)$; see Example 1. If $\mathbb{B}$ is a complete Boolean algebra of projections in a Hilbert space H then $\mathscr{R}{\downarrow}$ is isomorphic to $\langle\mathbb{B}\rangle$; see Example 2. The two indicated particular cases of Gordon's Theorem were intensively and fruitfully exploited by Takeuti [92–95]. The object $\mathscr{R}{\downarrow}$ for general Boolean algebras was also studied by Jech [30, 31], and [32] who in fact rediscovered Gordon's Theorem. The difference is that in [30] a (complex) universally complete vector lattice with unit is defined by another system of axioms and is referred to as a complete *Stone algebra*. By selecting special $\mathbb{B}$'s, it is possible to obtain some properties of $\mathscr{R}$.

Remark 1 In 1963 P. Cohen discovered his *method of 'forcing'* and also proved the independence of the Continuum Hypothesis. A comprehensive presentation of the Cohen forcing method gave rise to the *Boolean valued models of set theory*, which were first introduced by D. Scott and R. Solovay (see Scott [87]) and P. Vopěnka [103]. A systematic account of the theory of Boolean valued models and its applications to independence proofs can be found in [6, 33, 91, 99].

Remark 2 Gordon came to his theorem, while trying to solve the Solovay's famous problem. Assuming the consistency with ZFC of the existence of inaccessible cardinal, R. Solovay established the following result: *The statement "Every subset of* $\mathbb{R}$ *is Lebesgue measurable" is consistent with* ZF+DC (*Dependent choice*), see [90]. The Solovay's problem asks whether or not this result remains true without assumption of consistency of existence of inaccessible cardinal? Gordon failed to solve this problem but proved the following weaker assertion: *The statement "The Lebesgue measure on* $\mathbb{R}$ *can be extended to a* σ*-additive invariant measure on the* σ*-algebra of sets definable by a countable sequence of ordinals" is consistent with* ZFC,[3] see [13, Theorem 7] and [16]. In order to prove this result he needed to examine a Boolean algebra $\mathscr{B}$ with a measure $\mu : \mathscr{B} \to \mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$ and identify the descent $\mu{\downarrow} : \mathscr{B}{\downarrow} \to \mathscr{R}{\downarrow}$ of μ in $\mathbb{V}$. Thus, he discovered that the algebraic structure of $\mathscr{R}{\downarrow}$ is a well-known object, and it is K-space, which he learned from the book [101].

Remark 3 Many delicate properties of the objects inside $\mathbb{V}^{(\mathbb{B})}$ depend essentially on the structure of the initial Boolean algebra $\mathbb{B}$. The diversity of opportunities together with a great stock of information on particular Boolean algebras ranks Boolean valued models among the most powerful tools of foundational studies, see [6, 33, 99]. Here it is worth mentioning two deep independence results in analysis: The sentences SH[4] (Souslin's Hypothesis) and NDH[5] (No Discontinuous Homomorphisms) are independent of ZFC, see [29, 91] and [10], respectively.

4 Boolean Valued Representation of Structures

Every Boolean valued universe has the collection of mathematical objects in full supply. Available in plenty are all sets with extra structure: groups, rings, algebras, normed spaces, operators etc. Applying the descent functor to these internal algebraic systems of a Boolean valued model, we distinguish some bizarre entities or recognize old acquaintances, which leads to revealing the new facts of their life and structure.

[3] Earlier G. Saks [88] without assumption of existence of inaccessible cardinal proved that the statement "The Lebesgue measure on $\mathbb{R}$ can be extended to the σ-additive invariant measure defined on all subsets of $\mathbb{R}$" is consistent with ZF + DC.

[4] H: Every order complete order dense linearly ordered set having no first or last element is order isomorphic to the ordered set of reals $\mathbb{R}$, provided that every collection of mutually disjoint non-empty open intervals in it is countable.

[5] NDH: For each compact space X, each homomorphism from $C(X, \mathbb{C})$ into a Banach algebra is continuous.

It thus stands to reason to raise the following question: What structures significant for mathematical practice are obtainable by the Boolean values interpretation of the most typical algebraic systems? The answer is given in terms of Boolean sets.

1. A *Boolean set* or, more precisely, a $\mathbb{B}$-*set* is by definition a pair (X, d), where $X \in \mathbb{V}$, $X \neq \varnothing$, and d is a mapping from $X \times X$ to $\mathbb{B}$ satisfying for all $x, y, z \in X$ the conditions: (1) $d(x, y) = \mathbb{O}$ if and only if $x = y$; (2) $d(x, y) = d(y, x)$; (3) $d(x, y) \leq d(x, z) \vee d(z, y)$. Each nonempty subset $\varnothing \neq X \subset \mathbb{V}^{(\mathbb{B})}$ provides an example of a $\mathbb{B}$-set on assuming that $d(x, y) := [\![x \neq y]\!] = [\![x = y]\!]^*$ for all $x,\ y \in X$. Another example arises if we furnish a nonempty set X with the "discrete $\mathbb{B}$-metric" d; i. e., on letting $d(x, y) = \mathbb{1}$ in case $x \neq y$ and $d(x, y) = \mathbb{O}$ in case $x = y$.

2. For every $\mathbb{B}$-set (X, d) there are a member $\mathscr{X}$ of $\mathbb{V}^{(\mathbb{B})}$ and an injection $\imath : X \to X' := \mathscr{X}\!\downarrow$ such that $d(x, y) = [\![\imath(x) \neq \imath(y)]\!]$ for all $x, y \in X$ and every $x' \in X'$ admits the representation $x' = \mathrm{mix}_{\xi\in\Xi}(b_\xi \imath(x_\xi))$, with $(x_\xi)_{\xi\in\Xi} \subset X$ and $(b_\xi)_{\xi\in\Xi}$ a partition of unity in $\mathbb{B}$. The element $\mathscr{X}$ of $\mathbb{V}^{(\mathbb{B})}$ is said to be the *Boolean valued representation* of the $\mathbb{B}$-set X. If X is a discrete $\mathbb{B}$-set then $\mathscr{X} = X^\wedge$ and $\imath(x) = x^\wedge$ for all $x \in X$. If $X \subset \mathbb{V}^{(\mathbb{B})}$ then $\imath\!\uparrow$ is an injection from $X\!\uparrow$ to $\mathscr{X}$ within $\mathbb{V}^{(\mathbb{B})}$. Say that X is $\mathbb{B}$-*complete* (or $\mathbb{B}$-*cyclic*), whenever $\imath(X) = X'$.

3. A mapping f from a $\mathbb{B}$-set (X, d) to a $\mathbb{B}$-set (X', d') is *contractive* provided that $d'(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$. Assume that X and Y are some $\mathbb{B}$-sets. Assume further that $\mathscr{X}$ and $\mathscr{Y}$ are the Boolean valued representations of X and Y, while $\imath : X \to \mathscr{X}\!\downarrow$ and $\jmath : Y \to \mathscr{Y}\!\downarrow$ are the corresponding injections. If $f : X \to Y$ is a contractive mapping then there is a unique member g of $\mathbb{V}^{(\mathbb{B})}$ such that $[\![g : \mathscr{X} \to \mathscr{Y}]\!] = \mathbb{1}$ and $f = \jmath^{-1} \circ g\!\downarrow \circ\, \imath$.

4. In case a $\mathbb{B}$-set X has some a priori structure we may try to furnish the Boolean valued representation of X with an analogous structure, so as to apply the technique of ascending and descending to the study of the original structure of X. Consequently, the above questions may be treated as instances of the unique problem of searching a well-qualified Boolean valued representation of a $\mathbb{B}$-set with some additional structure, *algebraic* $\mathbb{B}$-*systems*.

5. Thus an algebraic $\mathbb{B}$-system refers to a $\mathbb{B}$-set endowed with a few contractive operations and $\mathbb{B}$-predicates, the latter meaning $\mathbb{B}$-valued contractive mappings. The Boolean valued representation of an algebraic $\mathbb{B}$-system appears to be a conventional two valued algebraic system of the same type. This means that an appropriate $\mathbb{B}$-completion of each algebraic $\mathbb{B}$-system coincides with the descent of some two valued algebraic system.

6. The following table shows Boolean valued representations of some structures. Of course, all these representation results are applied to the study of their properties by means of Boolean valued analysis. For details, we refer to the sources indicated in the third column of the table (Table 1).

Table 1 Structures

Algebraic structure with order, norm, etc.	Boolean valued representation	Author [·], year
Complete Boolean algebra with a complete subalgebra	Complete Boolean algebra	Solovay and Tennenbaum [91]
Amalgated free product of Boolean algebras over $\mathbb{B}$	Free product of Boolean algebras	Can be extracted from [91]
Universally complete Kantorovich space	Field of reals	Gordon [13]
Boolean extension of a uniform space	Complete uniform space	Gordon and Lyubetskiĭ [22–24]
Rationally complete semiprime abelian ring	Arbitrary field	Gordon [19]
Complete ring of fractions of a semiprime abelian ring	The field of fractions of an integral domain	Gordon [19]
Unital separated injective module	Vector space	Gordon [20]
Continuous geometry[a]	Irreducible CG[b]	Nishimura [73]
Von Neumann algebra	Von Neumann factor	Ozawa [78], Takeuti [96]
Kaplansky–Hilbert module	Hilbert space	Takeuti [96], Ozawa [79, 80]
$\mathbb{B}$-complete C^*-algebra	C^*-algebra	Takeuti [97]
Type I AW^*-algebra	W^*-algebra $\mathrm{End}(H)$ for a Hilbert space H	Ozawa [80]
AW^*-module	Hilbert space	Ozawa [80]
Embeddable AW^*-algebra	Von Neumann algebra	Ozawa [81]
Banach–Kantorovich space	Banach space	Kusraev [41]
Operator caps and faces	Caps and faces of sets of functionals	Kutateladze [64, 65]
$\mathbb{B}$-simple groups and $\mathbb{B}$-simple rings	Simple groups and Simple rings	Takeuti [98]
$\mathbb{B}$-complete Banach space	Banach space	Kusraev [41, 42], Ozawa [84]
$\mathbb{B}$-compactification (or cyclic compactification)	Stone-Čech compactification	Abasov and Kusraev [1]
$\mathbb{B}$-Dedekind domain[b]	Dedekind domain[b]	Nishimura [75]
$\mathbb{B}$-complete Lie algebra over a Stone algebra	Lie algebra	Nishimura [76]
AL^*-algebra[c]	L^*-algebra[c]	Nishimura [77]

$\mathbb{B}$-complete JB-algebra	JB-algebra	Kusraev [43]
$\mathbb{B}$-complete $\mathbb{B}$-dual JB-algebra	Dual JB-algebra	Kusraev [43]
Injective Banach lattice	AL-space (L_1 space)	Kusraev [50, 54][e]
Kaplansky–Hilbert lattice[d]	Hilbert lattice	Kusraev [51]
Ordered preduals to injective Banach lattices	L^1-preduals	Kusraev, Kutateladze [59]

[a]A *continuous geometry* (= CG) is a complete complemented modular lattice L satisfying the axioms: $\sup_{\alpha\in\mathrm{A}}(x_\alpha \wedge z) = (\sup_{\alpha\in\mathrm{A}} x_\alpha) \wedge z$ and $\inf_{\alpha\in\mathrm{A}}(y_\alpha \vee z) = (\inf_{\alpha\in\mathrm{A}} y_\alpha) \vee z$ for all $z \in L$, increasing family $(x_\alpha)_{\alpha\in\mathrm{A}}$, and decreasing family $(y_\alpha)_{\alpha\in\mathrm{A}}$ in L. A continuous geometry with a trivial center is called *irreducible, Neuman [102]*

[b]A $\mathbb{B}$-*Dedekind domain* is a $\mathbb{B}$-integral domain that is $\mathbb{B}$-hereditary. A $\mathbb{B}$-*integral domain* is a $\mathbb{B}$-complete ring R in which every $\mathbb{B}$-ideal of R is $\mathbb{B}$-projective and for all $a, b \in R$ with $ab = 0$ there exist $e, f \in \mathbb{B}$ such that $ef = 0$, $e + f = 1$, $ea = 0$, and $fb = 0$; see [75, p. 69]. A Dedekind domain is an integral domain in which every ideal is projective or, equivalently, each nonzero ideal is a product of prime ideals [7, Chap. 7, § 2]

[c]An AL^*-algebra is an AW^*-module $\mathscr{L}$ over a commutative von Neumann algebra A endowed with an A-bilinear operation $[\cdot,\cdot] : \mathscr{L} \times \mathscr{L} \to \mathscr{L}$ and a unary $*$-operation $(\cdot)^* : \mathscr{L} \to \mathscr{L}$ such that for all $u, v, w \in \mathscr{L}$ we have: (1) $[u, u] = 0$; (2) $[[u, v], w] + [[v, w]u] + [[w, u]v] = 0$; (3) $\langle [u, v], w\rangle = \langle v, [u^*, w]\rangle$; see [77, p. 245]. An L^*-algebra is a complex Lie algebra $\mathscr{L}$ that is simultaneously a Hilbert space endowed with a $*$-operation satisfying $\langle [u, v], w\rangle = \langle v, [u^*, w]\rangle$ for all $u, v, w \in \mathscr{L}$; see [86]

[d]A *Kaplansky–Hilbert lattice* over Λ is a real Banach lattice X such that $X \oplus iX$ is a Kaplansky–Hilbert module over $\bar{\Lambda}$ and $\|x\| := \left\|\sqrt{\langle x, x\rangle}\right\|_\infty$ for all $x \in X$, see Example 3. A Kaplansky–Hilbert lattice over $\Lambda = \mathbb{R}$ is called a *Hilbert lattice,* see [71]. The norm $\|x + iy\| := \sqrt{\|\langle x, x\rangle + \langle y, y\rangle\|_\infty}$ is given incorrectly in [51]

[e]Some related results can be found in [51, 59, 60]

5 Boolean Valued Representation of Operators

1. Let X be a normed space and let E be a vector lattice. Say that a linear operator $T : X \to E$ has an *abstract norm* or is *dominated* if the image $T(B_X)$ of the unit ball B_X of X is order bounded in E. Assume now that X is a multinormed space and E has an order unit $\mathbb{1}$. An operator T is called *piecewise bounded* if there is a partition of unity (π_α) in $\mathbb{P}(E)$ and a family of continuous seminorms (p_α) such that $|\pi_\alpha Tx| \le \mathbb{1}p_\alpha(x)$ for all α and $x \in X$

2. An operator $T : E \to F$ between two vector lattices is said to be *interval preserving* whenever T is a positive operator and $T[0, x] = [0, Tx]$ holds for each $x \in E_+$. A Maharam operator is an order continuous interval preserving operator. An operator $T : E \to E$ on vector lattice is said to be *band preserving* if $x \perp y$ implies $Tx \perp y$ for all $x, y \in E$ or, equivalently, whenever T keeps all bands of E invariant, i. e., $T(B) \subset B$ holds for each band B of E.

3. Consider a $\mathbb{B}$-complete Banach space Y. Denote by $\mathrm{Prt}_\sigma(\mathbb{B})$ the set of all countable partitions of unity in $\mathbb{B}$. Say that a *sequence* $(y_n)_{n\in\mathbb{N}}$ $\mathbb{B}$-*approximates* $y \in Y$ if, for each $k \in \mathbb{N}$, we have $\inf\{\sup_{n\geqslant k} \|\pi_n(y_n - y)\| : (\pi_n)_{n\ge k} \in \mathrm{Prt}_\sigma(\mathbb{B})\} = 0$. Call a set $K \subset Y$ $\mathbb{B}$-*compact* if K is $\mathbb{B}$-complete and every sequence $(y_n)_{n\in\mathbb{N}} \subset K$ $\mathbb{B}$-approximates some $y \in K$. An operator from a normed space X to Y is called $\mathbb{B}$-*compact* or *cyclically compact* if the image of every norm bounded subset of X lies in some $\mathbb{B}$-compact subset of Y.

4. Suppose E is a Banach lattice. A linear operator $T : E \to Y$ is *cone* $\mathbb{B}$*-summing* if and only if there exists a positive constant C such that for every finite collection $x_1, \dots x_n \in E$ there is a countable partition of unity $(\pi_k)_{k\in\mathbb{N}}$ in $\mathbb{B}$ such that the inequality

$$\sup_{k\in\mathbb{N}} \sum_{i=1}^{n} \|\pi_k T x_i\| \le C \left\| \sum_{i=1}^{n} |x_i| \right\|$$

holds, see [50]. Observe that if $\mathbb{B} = \{0, I_Y\}$ then a cone $\mathbb{B}$-summing operator is a *cone absolutely summing operator*; cp. [85, Ch. 4].

5. Let $\mathbb{P} = \mathbb{R}$ or $\mathbb{P} = \mathbb{C}$. Given an algebra A over the field $\mathbb{P}$, we call a $\mathbb{P}$-linear operator $D : A \to A$ a *derivation* provided that $D(uv) = D(u)v + uD(v)$ for all $u, v \in A$. It can be easily seen that an order bounded derivation of a universally complete f-algebra is trivial (Table 2).

Table 2 Operators

Operator, representation homomorphism, etc.	Boolean valued representation	Author [·], year
Unitary representation of an LCA group	Character of an LCA group	Takeuti [93]
Ordinary differential operator with parameters in $\langle\mathbb{B}\rangle$[a]	Ordinary differential operator	Takeuti [93]
$\langle\mathbb{B}\rangle$-valued Fourier transform on LCA groups	Fourier transform on LCA groups	Takeuti [94]
Linear operator with abstract norm	Norm bounded linear functional	Gordon [14, 17]
Conditional expectation	Lebesgue integral	Gordon [17]
$\mathbb{B}$-Compact operator	Compact operator	Kusraev [39]
Maharam operator	Order continuous positive functional	Kusraev [40]
Piecewise bounded linear operator	Continuous linear functional	Sikorskiĭ [89]
Differential polynomial on $\mathscr{D}'(\mathbb{R}^n, \mathbf{C})$ or $\mathscr{S}'(\mathbb{R}^n, \mathbf{C})$ with coefficients in $\mathbf{C}$[b]	Constant coefficients differential polynomial on $\mathscr{D}'(\mathbb{R}^n)$, $\mathscr{S}'(\mathbb{R}^n)$	Sikorskiĭ [88, 89]
Unitary representation of a locally compact group	Irreducible unitary representation	Nishimura [74]
Band preserving operator	$\mathbb{R}$^-linear function on Boolean valued reals	Kusraev [46]
Derivation on a universally complete f-algebra over $\mathbb{C}$	Derivation on the complex plane	Kusraev [47]
Cone $\mathbb{B}$-summing operator	Cone absolutely summing operator	Kusraev [49]
Weighted conditional expectation type operator	Weighted conditional expectation operator	Kusraev, Kutateladze [58]

[a]See Example 2 in § 3
[b]$\mathscr{D}'(\mathbb{R}^n, \mathbf{C})$ (resp. $\mathscr{S}'(\mathbb{R}^n, \mathbf{C})$) is the space of all piecewise bounded operators from $\mathscr{D}(\mathbb{R}^n)$ (resp. $\mathscr{S}(\mathbb{R}^n)$ to $\mathbf{C}$), where $\mathbf{C} := \mathscr{C}{\downarrow} = \mathbf{R} \oplus i\mathbf{R}$ is a complex universally complete vector lattice, see Corollary 2

6 Problems and Solutions

Boolean valued analysis sheds new light on some old problems and generates a large number of new ones. We now give a small list of problems that arose independently of the theory of Boolean valued models, but which were solved by means of Boolean valued analysis. Details as well as many other aspects of Boolean valued analysis may be found in the books [10, 21, 56–58, 92] and the survey papers [23, 34, 61] (Table 3).

Table 3 Problems and solutions

Problem	Stems from	Reduced to (by means of BA)	Solved
Intrinsic characterization of subdifferentials	Kutateladze [63]	Weakly compact convex sets of functionals	Kusraev and Kutateladze [55]
General disintegration in Kantorovich spaces	Ioffe, Levin [28]; Neumann [72]	Hahn–Banach and Radon–Nikodým theorems	Kusraev [40]
Kaplansky Problem: Homogeneity of a type I AW^*-algebra	Kaplansky [38]	Homogeneity of $\mathrm{End}(H)$ with H a Hilbert space	Ozawa [80]
The trace problem for finite AW^*-algebra	Kaplansky [37]	The trace problem for a W^*-factor	Ozawa [82, 83]
Wickstead problem: Order boundedness of all band preserving operators	Wickstead [105]	Solvability of Cauchy type functional equations	Gutman [26] and Kusraev [47]
Maharam extension of a positive operator	Luxemburg and Schep [69]	Daniel extension of an elementary integral	Akilov, Kolesnikov, and Kusraev [2, 3]
Goodearl problem 18 in [12]	Goodearl [12]	Theorem 12.16 in [12]	Chupin [9]
$\mathbb{B}$-Atomic decomposition of vector measures (into a sum of spectral measures)	Hoffman-Jørgenson [27]	Hammer–Sobczyc decomposition theorem	Kusraev and Malyugin [62]
Classification of AJW-algebras[a]	Topping [100]	Classification of predual JB-factors (JBW-factors)	Kusraev [52, 53]
Description of operators T with $\|T\|$ a sum of two lattice homomorphisms	Grothendieck [25]	Description of functionals with the same property	Kutateladze [66]
Classification of injective Banach lattices	Cartwright [8] and Lotz [68]	Classification of AL-spaces (L_1 spaces)	Kusraev [52, 53]

[a]An *AJW-algebra* is a JB-algebra with a Jordan counterpart of Baire condition (= annihilators are generated by projections), see [5]. For some related results, see [44, 48]

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