BOOLEAN VALUED ANALYSIS AND POSITIVITY

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Boolean valued analysis is a technique of studying properties of an arbitrary mathematical object by means of comparison between its representations in two different set-theoretic models whose construction utilizes principally distinct Boolean algebras. Boolean valued analysis is usually viewed as an instance of nonstandard analysis since it uses models with variable sets. The latter are characteristic of topos theory, a section of category theory dealing with the social neighborhood of the classical category of sets.

Saunders Mac Lane, a cofather of category theory, passed away on April 14, 2005 in San Francisco and we devote this talk to the memory of this outstanding scholar and personality who linked us with D. Hilbert since P. Bernays was an advisor of Mac Lane's thesis approved in Göttingen in 1934.

It happened incidentally that our new book "Invitation to Boolean Valued Analysis" was printed just at the begining of May. This talk is also a kind of presentation.

Various function spaces reside in functional analysis, and so the intention is natural of replacing an abstract Boolean valued system by some function analog, a model whose elements are functions and in which the basic logical operations are calculated "pointwise." An example of such a model is given by the class $\mathbb{V}^Q$ of all functions defined on a fixed nonempty set Q and acting into the class $\mathbb{V}$ of all sets. Truth values in the model $\mathbb{V}^Q$ are various subsets of Q and, in addition, the truth value $[\![\varphi(u_1,\dots,u_n)]\!]$ of $\varphi(t_1,\dots,t_n)$ at functions $u_1,\dots,u_n\in\mathbb{V}^Q$ is calculated as follows:

$$[\![\varphi(u_1,\dots,u_n)]\!]=\big\{q\in Q:\ \varphi\big(u_1(q),\dots,u_n(q)\big)\big\}.$$

A polyverse by A. G. Gutman and G. A. Losenkov is a continuous bundle of models of set theory. The class of continuous sections of a continuous polyverse brings about a system satisfying all principles of Boolean valued analysis and, conversely, every Boolean valued algebraic system can be represented as the class of sections of a suitable continuous polyverse.

Every Boolean valued universe has the collection of mathematical objects in full supply: available in plenty are all sets with extra structure: groups, rings, algebras, normed spaces, etc. Applying the descent functor to the established algebraic systems in a Boolean valued model, we distinguish bizarre entities or recognize old acquaintances, which leads to revealing the new facts of their life and structure.

This technique of research, known as *direct Boolean valued interpretation*, allows us to produce new theorems or, to be more exact, to extend the semantical content of the available theorems by means of slavish translation. The information we so acquire might fail to be vital, valuable, or intriguing, in which case the direct Boolean valued interpretation turns out to be a leisurely game.

It thus stands to reason to raise the following questions: What structures significant for mathematical practice are obtainable by the Boolean valued interpretation

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of the most common algebraic systems? What transfer principles hold true in this process? Clearly, the answers should imply specific objects whose particular features enable us to deal with their Boolean valued representation which, if understood duly, is impossible to implement for arbitrary algebraic systems.

We show that an abstract B-set U embeds in the Boolean valued universe $\mathbb{V}^{(B)}$ so that the Boolean distance between the members of U becomes the Boolean truth-value of the negation of their equality. The corresponding element of $\mathbb{V}^{(B)}$ is, by definition, the *Boolean valued representation* of U. In case the B-set U has some a priori structure we may try to furnish the Boolean valued representation of U with an analogous structure, intending to apply the technique of ascending and descending to studying the original structure of U. Consequently, the above questions may be treated as instances of the unique problem of searching a well-qualified Boolean valued representation of a B-set furnished with some additional structure.

We analyzes the problem for the main objects of general algebra. Located at the epicenter of exposition, the notion of an algebraic B-system refers to a nonempty B-set endowed with a few contractive operations and B-predicates, the latter meaning B-valued contractive mappings.

The Boolean valued representation of an algebraic B-system appears to be a conventional two-valued algebraic system of the same type. This means that an appropriate completion of each algebraic B-system coincides with the descent of some two-valued algebraic system inside $\mathbb{V}^{(B)}$. On the other hand, each two-valued algebraic system may be transformed into an algebraic B-system on distinguishing a complete Boolean algebra of congruences of the original system. In this event, the task is in order of finding the formulas holding true in direct or reverse transition from a B-system to a two-valued system. In other words, we have to seek here some versions of the transfer principle or the identity preservation principle of long standing in some branches of mathematics.

We illustrate the general facts of Boolean valued analysis with particular algebraic systems in which complete Boolean algebras of congruences are connected with the relations of order and disjointness. We restrict exposition mainly to the descents of the systems under study and demonstrate the opportunities that are opened up by Boolean valued analysis.

Many delicate mathematical properties of the members of $\mathbb{V}^{(B)}$ depends essentially on the structure of B. We show here how the choice of a Boolean algebra affects the specific properties of cardinals (and not only cardinals) in the corresponding Boolean valued universe. The canonical embedding of the von Neumann universe $\mathbb{V}$ to $\mathbb{V}^{(B)}$ sends ordinals to Boolean valued ordinals, preserving the order on ordinals. The same happens to cardinals provided that B enjoys the countable chain condition. However, the choice of B is available such that the canonical embedding "glue together" infinite cardinals; i.e., the standard names of two distinct infinite cardinals may have the same cardinality in an appropriate Boolean valued model. There are various mathematical constructions distorted under the canonical embedding. We discuss a few of them but focus exposition on the classical Gödel–Cohen solution of the continuum problem.

The Boolean valued inverse $\mathbb{V}^{(B)}$ associated with a fixed Boolean algebra B is one of the arenas of mathematical events. Indeed, by virtue of the transfer and maximum principles, $\mathbb{V}^{(B)}$ contains numbers and groups as well as the Lebesgue and

Riemann integrals, with the Radon–Nikodým and Hahn–Banach theorems available by virtue of the transfer and maximum principles.

The elementary technique of ascending and descending which we become acquainted with when considering algebraic systems shows each of mathematical objects in $\mathbb{V}^{(B)}$ to be a representation of an analogous classical object with an additional structure determined by B. In particular, this is also true in regard to functional-analytical objects.

Our main topic is Banach spaces in Boolean valued universes. It turns out that these spaces are inseparable from the concepts of the theory of ordered vector spaces and, above all, with the Dedekind complete vector lattices which were introduced by L. V. Kantorovich at the beginning of the 1930s under the name of K-spaces. They are often referred to as *Kantorovich spaces* nowadays.

The fundamental result of Boolean valued analysis in regard to this aspect is Gordon's Theorem which reads as follows: *Each universally complete Kantorovich space is an interpretation of the reals in an appropriate Boolean valued model.* Conversely, each Archimedean vector lattice embeds in a Boolean valued model, becoming a vector sublattice of the reals viewed as such over some dense subfield of the reals.

Moreover, each theorem about the reals within Zermelo–Fraenkel set theory has an analog in the original Kantorovich space. Translation of theorems is carried out by appropriate general operations of Boolean valued analysis.

Boolean valued analysis reveals the structure and properties of a vector space with some norm taking values in a vector lattice. Such a vector space is called a *lattice normed space*. The most important peculiarities of these spaces are connected with decomposability. Use of decomposability allows us in particular to distinguish a complete Boolean algebra of linear projections in a lattice normed space which is isomorphic to the Boolean algebra of band projections of the norm lattice. Most typical in analysis are the lattice normed spaces of continuous or measurable functions.

In much the same way as many structural properties of a Kantorovich space are some properties of the reals in an appropriate Boolean valued model, the basic properties of a lattice normed space presents the Boolean valued interpretations of the relevant properties of normed spaces. The most principal connections are reflected by the three facts:

(1) The internal Banach spaces and external universally complete Banach–Kantorovich spaces are bijective under the procedure of bounded descent from a Boolean valued model.

(2) Each lattice normed space is realizable as a dense subspace of a Banach space viewed a vector space over some field, e.g. the rationals, in an appropriate Boolean valued model.

(3) Each Banach space X is a result of the bounded descent of some Banach space in a Boolean valued model if and only if X includes a complete Boolean algebra of norm one projections which possesses the cyclicity property. In other words, X is a Dedekind complete lattice normed space with a mixed norm.

The theory of Banach algebras is one of the most attractive traditional sections of functional analysis. The possibility of applying Boolean valued analysis to operator algebras rests on the following observation: If the center of an algebra is properly qualified and perfectly located then it becomes a one dimensional subalgebra after immersion in a suitable Boolean valued universe. This might lead to a simpler algebra. On the other hand, the transfer principle implies that the scope of the formal theory of the initial algebra is the same as that of its Boolean valued representation.

Exposition focuses on the analysis of AW^*-algebras and JB-algebras, i.e. Baer C^*-algebras and Jordan–Banach algebras. These algebras are realized in a Boolean valued model as AW^*-factors and JB-factors. The problem of representation of these objects as operator algebras leas to studying Kaplansky–Hilbert modules.

The dimension of a Hilbert space inside a Boolean valued model is a Boolean valued cardinal which is naturally called the Boolean dimension of the Kaplansky–Hilbert module that is the descent of the original Hilbert space. The cardinal shift reveals itself: some isomorphic Kaplansky–Hilbert modules may fail to have all bases of the same cardinality. This implies that a type I AW^*-algebra may generally split in a direct sum of homogeneous subalgebras in many ways. This was conjectured by I. Kaplansky as far back as in 1953.

Leaning on the results about the Boolean valued immersion of Kaplansky–Hilbert modules, we derive some functional representations of these objects. To put it more precisely, we prove that each AW^*-module is unitarily equivalent to the direct sum of some homogeneous AW^*-modules consisting of continuous vector functions ranging in a Hilbert space. An analogous representation holds for an arbitrary type I AW^*-algebra on replacing continuous vector functions with operator valued functions continuous in the strong operator topology.

We call an AW^*-algebra *embeddable* if it is $*$-isomorphic with the double commutant of some type I AW^*-algebra. Each embeddable AW^*-algebra admits a Boolean valued representation, becoming a von Neumann algebra or factor. We give several characterizations for embeddable AW^*-algebras. In particular, we prove that an AW^*-algebra A is embeddable if and only if the center valued normal states of A separate A. We also consider similar problems for the JB-algebras, a kind of real nonassociative analogs of C^*-algebras.