Contents

Foreword to the English Translation

This is a translation of the book that opens the series "Nonstandard Methods of Analysis" in print by the Sobolev Institute Press at Novosibirsk.

Nonstandard methods of analysis consist generally in comparative study of two interpretations of a mathematical claim or construction given as a formal symbolic expression by means of two different set-theoretic models: one, a "standard" model and the other, a "nonstandard" model. The second half of the twentieth century is a period of significant progress in these methods and their rapid development in a few directions.

The first of the latter appears often under the name coined by its inventor, A. Robinson. This memorable but slightly presumptuous and defiant term, *nonstandard analysis*, often swaps places with the term *Robinson's* or *classical nonstandard analysis*. The characteristic feature of Robinson's nonstandard analysis is a frequent usage of many controversial concepts appealing to the actual infinitely small and infinitely large quantities that have happily resided in natural sciences from ancient times but were strictly forbidden in modern mathematics for many decades. The present-day achievements revive the forgotten term *infinitesimal analysis* which expressively reminds us of the heroic bygones of Calculus.

Infinitesimal analysis expands rapidly, bringing about radical reconsideration of the general conceptual system of mathematics. The principal reasons for this progress are twofold. Firstly, infinitesimal analysis provides us with a novel understanding for the method of indivisibles rooted deeply in the mathematical classics. Secondly, it synthesizes both classical approaches to differential and integral calculuses which belong to the noble inventors of the latter. Infinitesimal analysis finds newer and newest applications and merges into every section of contemporary mathematics. Sweeping changes are on the march in nonsmooth analysis, measure theory, probability, the qualitative theory of differential equations, and mathematical economics.

The second direction, *Boolean valued analysis* distinguishes itself by ample usage of such terms as the technique of ascending and descending, cyclic envelopes and mixings, B-sets and representation of objects in $\mathbf{V}^{(B)}$. Boolean valued analysis originated with the famous works by P. J. Cohen on the continuum hypothesis. Progress in this direction has evoked radically new ideas and results in many sections of functional analysis. Among them we list Kantorovich space theory, the theory of von Neumann algebras, convex analysis, and the theory of vector measures.

The book [135], printed by the Siberian Division of the Nauka Publishers in 1990 and translated into English by Kluwer Academic Publishers in 1994, gave a first unified treatment of the two disciplines forming the core of the present-day nonstandard methods of analysis.

The reader's interest as well as successful research into the field assigns a task of updating the book and surveying the state of the art. Implementation of the task has shown soon that it is impossible to compile new topics and results in a single book. Therefore, the Sobolev Institute Press decided to launch the series "Nonstandard Methods of Analysis" which will consist of the monographs devoted to various aspects of this direction of mathematical research.

The present book opens the series and treats Boolean valued analysis. The formal technique of the discipline is expounded in detail. The book also pays much attention to studying the classical objects of functional analysis, namely, Banach spaces and algebras by means of Boolean valued models.

This edition was typeset using $\mathcal{AMS}$-TEX, the American Mathematical Society's TEX macro package.

As the editor of the series, I am deeply grateful to Kluwer Academic Publishers for cooperation and support of the new project.

S. Kutateladze

Preface

As the title implies, the present book treats *Boolean valued analysis*. This term signifies a technique for studying the properties of an arbitrary mathematical object by means of comparison between its representations in two different set-theoretic models whose construction utilizes principally distinct Boolean algebras. We usually take as these models the classical Cantorian paradise in the shape of the von Neumann universe and a specially-trimmed Boolean valued universe in which the conventional set-theoretic concepts and propositions acquire bizarre interpretations. Usage of two models for studying a single object is a family feature of the so-called *nonstandard methods of analysis*. For this reason, Boolean valued analysis means an instance of nonstandard analysis in common parlance.

Proliferation of Boolean valued analysis stems from the celebrated achievement of P. J. Cohen who proved in the beginning of the sixties that the negation of the continuum hypothesis, CH, is consistent with the axioms of Zermelo–Fraenkel set theory, ZFC. This result by P. J. Cohen, alongside the consistency of CH with ZFC established earlier by K. Gödel, proves that CH is independent of the conventional axioms of ZFC.

The genuine value of the great step forward by P. J. Cohen could be understood better in connection with the serious difficulty explicated by J. Shepherdson and absent from the case settled by K. Gödel. The crux of J. Shepherdson's observation lies in the impossibility of proving the consistency of (ZFC) + ($\neg$ CH) by means of any standard models of set theory. Strictly speaking, we can never find a subclass of the von Neumann universe which models (ZFC) + ($\neg$ CH) provided that we use the available interpretation of membership. P. J. Cohen succeeded in inventing a new powerful method for constructing noninner, *nonstandard*, models of ZFC. He coined the term *forcing*. The technique by P. J. Cohen invokes the axiom of existence of a standard transitive model of ZFC in company with the forcible and forceful transformation of the latter into an immanently nonstandard model by the method of forcing. His tricks fall in an outright contradiction with the routine mathematical intuition stemming "from our belief into a natural nearly physical model of the mathematical world" as P. J. Cohen phrased this himself [30].

Miraculously, the difficulties in comprehension of P. J. Cohen's results gained a perfect formulation long before they sprang into life. This was done in the famous talk "Real Function Theory: State of the Art" by N. N. Luzin at the All-Russia Congress of Mathematicians in 1927. Then N. N. Luzin said: "The first idea that might leap to mind is that the determination of the cardinality of the continuum is a matter of a free axiom like the parallel postulate of geometry. However, when we vary the parallel postulate, keeping intact the rest of the axioms of Euclidean geometry, we in fact change the precise meanings of the words we write or utter, that is, 'point,' 'straight line,' etc. What words are to change their meanings if we attempt at making the cardinality of the continuum movable along the scale of alephs, while constantly proving consistency of this movement? The cardinality of the continuum, if only we imagine the latter as a set of points, is some unique entity that must reside in the scale of alephs at the place which the cardinality of the continuum belongs to; no matter whether the determination of this place is difficult or even 'impossible for us, the human beings' as J. Hadamard might comment" [159, pp. 11–12].

P. S. Novikov expressed a very typical attitude to the problem: "...it might be (and it is actually so in my opinion) that the result by Cohen conveys a purely negative message and reveals the termination of the development of 'naive' set theory in the spirit of Cantor" [192, p. 209].

Intention to obviate obstacles to mastering the technique and results by P. J. Cohen led D. Scott and R. Solovay to constructing the so-called Boolean valued models of ZFC which are not only visually attractive from the standpoint of classical mathematicians but also are fully capable of establishing consistency and independence theorems. P. Vopěnka constructed analogous models in the same period of the early sixties.

The above implies that the Boolean valued models, achieving the same ends as P. J. Cohen's forcing, must be nonstandard in some sense and possess some new features that distinguish them from the standard models.

Qualitatively speaking, the *notion of Boolean valued model involves a new conception of modeling* which might be referred to as *modeling by correspondence* or *long-distance modeling.* We explain the particularities of this conception as compared with the routine approach. Encountering two classical models of a single theory, we usually seek for a bijection between the universes of the models. If this bijection exists then we translate predicates and operations from one model to the other and speak about isomorphism between the models. Consequently, this conception of isomorphism implies a direct contact of the models which consists in witnessing to bijection of the universes of discourse.

Imagine that we are physically unable to compare the models pointwise. Happily, we take an opportunity to exchange information with the owner of the other model by using some means of communication, e.g., by having long-distance calls. While communicating, we easily learn that our interlocutor uses his model to operate on some objects that are the namesakes of ours, i.e., sets, membership, etc. Since we are interested in ZFC, we ask the interlocutor whether or not the axioms of ZFC are satisfied in his model. Manipulating the model, he returns a positive answer. After checking that he uses the same inference rules as we do, we cannot help but acknowledge his model to be a model of the theory we are all investigating. It is worth noting that this conclusion still leaves unknown for us the objects that make up his universe and the procedures he uses to distinguish between true and false propositions about these objects.†

All in all, the *new conception of modeling implies not only refusal from identification of the universes of discourse but also admission of various procedures for verification of propositions.*

To construct a Boolean valued model, we start with a complete Boolean algebra B, a cornerstone of a special Boolean valued universe $\mathbf{V}^{(B)}$ consisting of "B-valued sets" that are defined recursively as B-valued functions over available B-valued sets. This $\mathbf{V}^{(B)}$ will serve as a universe of discourse for ZFC. Also, we appoint B as the target of the truth value sending each formula of ZFC to a member of B. More explicitly, to each formula φ of ZFC whose every variable ranges now over $\mathbf{V}^{(B)}$, we put in correspondence some element $[\![\varphi]\!]$ of the parental Boolean algebra B. The quantity $[\![\varphi]\!]$ is the *truth value* of φ. We use truth values for validating formulas of ZFC. In particular, every theorem φ of ZFC acquires the greatest truth value $\mathbf{1}_B$, and we declare φ holding inside the model $\mathbf{V}^{(B)}$.

This construction is elaborated in Chapters 1–3. Application of Boolean valued models to problems of analysis rests ultimately on the procedures of *ascending and descending*, the two natural functors acting between $\mathbf{V}^{(B)}$ and the von Neumann universe $\mathbf{V}$. Preliminaries to the axiomatics of Zermelo–Fraenkel set theory are gathered in the Appendix in order to alleviate the burden of the reader. This Appendix also contains preliminaries to category theory.

In the concluding chapters we demonstrate the main advantages of Boolean valued analysis: tools for transforming function spaces to subsets of the reals; operators, to functionals; vector functions, to numerical mappings, etc. Surely, selection of analytical topics and objects and the respective applications to functional analysis is mainly determined from the personal utility functions of the authors.

We start with thorough examination of the Boolean valued representations of algebraic systems in Chapter 4. The theory of algebraic systems, propounded in the works by A. I. Maltsev and A. Tarski, ranks among the most vital mathematical

† The "*E*, *Eir*, and *Em*" of the celebrated Personal Pronoun Pronouncement seems by far a better choice of pronouns for this paragraph (cf. [228]).

achievements of general import. A profusion of algebraic systems makes information on their Boolean valued representation a must for meaningful application to every section of the present-day mathematics.

Of the same high relevance are the constructions of Chapter 5. Mathematics, in any case mathematics as the Science of Infinity, is inconceivable without the reals. Boolean valued analysis has revealed the particular role of a universally complete Kantorovich space. It turns out that each of these spaces serves as a lawful and impeccable model of the reals. Recall that L. V. Kantorovich was the first who introduced *Dedekind complete* (that is, boundedly order complete) vector lattices as fruitful abstraction of the reals in the thirties. These spaces are also called *K-spaces* or *Kantorovich spaces* in memory of Leonid Vital′evich Kantorovich, a great mathematician and a Nobel Laureate in economics. Considering the new objects, L. V. Kantorovich propounded the *heuristic transfer principle.* Kantorovich's principle claims that the members of a K-space are analogs of real numbers and to each theorem about functionals there corresponds a similar theorem about operators taking values in a K-space. Time enables us to ascribe a clear and rigorous meaning to this heuristic transfer principle. The relevant tools, including the fundamental theorem by E. I. Gordon, comprise the bulk of Chapter 5. Here we also expatiate upon the problem of Boolean valued representation for Banach space, the central object of classical functional analysis. It turns out miraculously that the so-called *lattice normed spaces*, discovered at the cradle of K-space theory, depict the conventional normed spaces.

Chapter 6 deals with the theory of operator algebras. Boolean valued analysis of these algebras is the direction of research originated with the pioneer works by G. Takeuti. Study in this direction is very intensive in the recent decades. Our exposition leans upon the results of Chapter 5 about Boolean valued representation of lattice normed spaces. This approach enables us to treat in a unified fashion various analytical objects such as involutive Banach algebras, Banach modules, Jordan–Banach algebras, algebras of unbounded operators, etc.

Our book is intended to the reader interested in the modern set-theoretic models as applied to functional analysis. We tried to make the book independent to the utmost limits. However, we are fully aware that our attempts at independence were mostly foiled. Clearly, the topic of exposition needs the mathematical ideas and objects plenty above our ability to devour them. We nevertheless hope that the reader will understand our problems and forgive unintentional gaps and inaccuracies.

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