

Leibnizian, Robinsonian, and Boolean Valued Monads

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Agenda

- This is an overview of the present-day versions of monadology with some applications to vector lattices and linear inequalities.
- The notion of monad is central to external set theory. Justifying the use of infinitesimals and the technique of descending and ascending in vector lattice theory requires adaptation of monadology for the implementation of filters in Boolean valued universes.
- The two approaches are available now. One is to apply monadology to the descents of objects. The other consists in applying the standard monadology inside the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ over a complete Boolean algebra \mathbb{B} , while ascending and descending by the Escher rules.¹
- These approaches are sketched and illustrated by tests for order convergence and rules for fragmenting and projecting positive operators in vector lattices. Also, Lagrange's principle is shortly addressed in polyhedral environment with inexact data.

¹Cp. [1]

The Origins of Monadology

- The concept of monad stems from Ancient Greece. Monadology as a philosophical doctrine is a creation of Leibniz.²
- The general theory of the monads of filters was proposed by Luxemburg³ within Robinson's nonstandard analysis.⁴

²Cp. [2] and [3].

³Cp. [4].

⁴Cp. [5].

Basics of Monadology

- Let \mathcal{F} be a standard filter; ${}^\circ\mathcal{F}$, the standard core of \mathcal{F} ; and ${}^a\mathcal{F} := \mathcal{F} \setminus {}^\circ\mathcal{F}$, the external set of *remote* elements of \mathcal{F} . Note that

$$\mu(\mathcal{F}) := \bigcap {}^\circ\mathcal{F} = \bigcup {}^a\mathcal{F}$$

is the *monad* of \mathcal{F} . Also, $\mathcal{F} = {}^*\text{fil}(\{\mu(\mathcal{F})\})$; i.e., \mathcal{F} is the standardization of the collection $\text{fil}(\mu(\mathcal{F}))$ of all supersets of $\mu(\mathcal{F})$.

- Let \mathcal{A} be a filter on $X \times Y$, and let \mathcal{B} be a filter on $Y \times Z$. Put $\mathcal{B} \circ \mathcal{A} := \text{fil}\{B \circ A \mid A \in \mathcal{A}, B \in \mathcal{B}\}$, where we may assume all $B \circ A$ nonempty. Then

$$\mu(\mathcal{B} \circ \mathcal{A}) = \mu(\mathcal{B}) \circ \mu(\mathcal{A}).$$

The Granted Horizon Principle

- Let X and Y be standard sets. Assume further that \mathcal{F} and \mathcal{G} are standard filters on X and Y respectively satisfying $\mu(\mathcal{F}) \cap {}^\circ X \neq \emptyset$. Distinguish a remote set F in ${}^a\mathcal{F}$.
- Given a standard correspondence $f \subset X \times Y$ meeting \mathcal{F} , the following are equivalent:
 - (1) $f(\mu(\mathcal{F}) - F) \subset \mu(\mathcal{G})$;
 - (2) $(\forall F' \in {}^a\mathcal{F}) f(F' - F) \subset \mu(\mathcal{G})$;
 - (3) $f(\mu(\mathcal{F})) \subset \mu(\mathcal{G})$.

Boolean Valued Universe

- Let \mathbb{B} be a complete Boolean algebra. Given an ordinal α , put

$$V_{\alpha}^{(\mathbb{B})} := \{x \mid (\exists \beta \in \alpha) x : \text{dom}(x) \rightarrow \mathbb{B}, \text{dom}(x) \subset V_{\beta}^{(\mathbb{B})}\}.$$

- The *Boolean valued universe* $V^{(\mathbb{B})}$ is

$$V^{(\mathbb{B})} := \bigcup_{\alpha \in \text{On}} V_{\alpha}^{(\mathbb{B})},$$

with On the class of all ordinals.

- The truth value $\llbracket \varphi \rrbracket \in \mathbb{B}$ is assigned to each formula φ of ZFC relativized to $V^{(\mathbb{B})}$.

Functional Realization

- Let Q be the Stone space of a complete Boolean algebra \mathbb{B} . Denote by \mathfrak{U} the (separated) Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$. Given $q \in Q$, put $u \sim_q v \leftrightarrow q \in \llbracket u = v \rrbracket$. Consider the bundle

$$V^Q := \{(q, \sim_q(u)) \mid q \in Q, u \in \mathfrak{U}\}$$

and denote $(q, \sim_q(u))$ by $\hat{u}(q)$. So $\hat{u} : q \mapsto \hat{u}(q)$ is a section of V^Q for every $u \in \mathfrak{U}$. Note that to each $x \in V^Q$ there are $u \in \mathfrak{U}$ and $q \in Q$ satisfying $\hat{u}(q) = x$. Moreover, $\hat{u}(q) = \hat{v}(q)$ if and only if $q \in \llbracket u = v \rrbracket$.

Functional Realization

- Make each fiber V^q of V^Q into an algebraic system of signature $\{\in\}$ by letting $V^q \models x \in y \leftrightarrow q \in \llbracket u \in v \rrbracket$, where $u, v \in \mathfrak{U}$ are such that $\hat{u}(q) = x$ and $\hat{v}(q) = y$.
- The class $\{\hat{u}(A) \mid u \in \mathfrak{U}\}$, with A a clopen subset of Q , is a base for some topology on V^Q . Thus V^Q as a continuous bundle called a *continuous polyverse*. By a *continuous section* of V^Q we mean a section that is a continuous function. Denote by \mathfrak{C} the class of all continuous sections of V^Q .
- The mapping $u \mapsto \hat{u}$ is a bijection between \mathfrak{U} and \mathfrak{C} , yielding a convenient functional realization of the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$. This universal construction belongs to Gutman and Losenkov.⁵

⁵Cp. [6].

Descending and Ascending

- Given φ , a formula of ZFC, and $y \in \mathbb{V}^{\mathbb{B}}$; put $A_\varphi := A_{\varphi(\cdot, y)} := \{x \mid \varphi(x, y)\}$.
- The *descent* $A_\varphi \downarrow$ of a class A_φ is

$$A_\varphi \downarrow := \{t \mid t \in \mathbb{V}^{(\mathbb{B})}, \llbracket \varphi(t, y) \rrbracket = \mathbb{1}\}.$$

- If $t \in A_\varphi \downarrow$, then it is said that t *satisfies* $\varphi(\cdot, y)$ *inside* $\mathbb{V}^{(\mathbb{B})}$.
- The *descent* $x \downarrow$ of $x \in \mathbb{V}^{(\mathbb{B})}$ is defined as

$$x \downarrow := \{t \mid t \in \mathbb{V}^{(\mathbb{B})}, \llbracket t \in x \rrbracket = \mathbb{1}\},$$

i.e. $x \downarrow = A_{\in x \downarrow}$. The class $x \downarrow$ is a set.

- If x is a nonempty set inside $\mathbb{V}^{(\mathbb{B})}$ then

$$(\exists z \in x \downarrow) \llbracket (\exists t \in x) \varphi(t) \rrbracket = \llbracket \varphi(z) \rrbracket.$$

- The *ascent* functor acts in the opposite direction.

The Reals Within

- There is an object \mathcal{R} inside $\mathbb{V}^{(\mathbb{B})}$ modeling \mathbb{R} ; i.e., $\llbracket \mathcal{R} \text{ is the reals} \rrbracket = \mathbb{1}$.
- Let $\mathcal{R}\downarrow$ be the descent of the carrier $|\mathcal{R}|$ of the algebraic system

$$\mathcal{R} := (|\mathcal{R}|, +, \cdot, 0, 1, \leq)$$

inside $\mathbb{V}^{(\mathbb{B})}$.

- Implement the descent of the structures on $|\mathcal{R}|$ to $\mathcal{R}\downarrow$ as follows:

$$x + y = z \leftrightarrow \llbracket x + y = z \rrbracket = \mathbb{1};$$

$$xy = z \leftrightarrow \llbracket xy = z \rrbracket = \mathbb{1};$$

$$x \leq y \leftrightarrow \llbracket x \leq y \rrbracket = \mathbb{1};$$

$$\lambda x = y \leftrightarrow \llbracket \lambda \wedge x = y \rrbracket = \mathbb{1} \quad (x, y, z \in \mathcal{R}\downarrow, \lambda \in \mathbb{R}).$$

- **Gordon Theorem.**⁶ $\mathcal{R}\downarrow$ with the descended structures is a universally complete vector lattice with base $\mathbb{B}(\mathcal{R}\downarrow)$ isomorphic to \mathbb{B} .

⁶Cp. [1, p. 349].

Filters within $\mathbb{V}^{(\mathbb{B})}$

- Let \mathcal{G} be a filterbase on X , with $X \in \mathcal{P}(\mathbb{V}^{(\mathbb{B})})$. Put

$$\mathcal{G}' := \{F \in \mathcal{P}(X\uparrow)\downarrow \mid (\exists G \in \mathcal{G}) [F \supset G\uparrow] = \mathbb{1}\};$$
$$\mathcal{G}'' := \{G\uparrow \mid G \in \mathcal{G}\}.$$

Then $\mathcal{G}'\uparrow$ and $\mathcal{G}''\uparrow$ are bases of the same filter \mathcal{G}^\uparrow on $X\uparrow$ inside $\mathbb{V}^{(\mathbb{B})}$ —the *ascent* of \mathcal{G} . If $\text{fil}(\mathcal{G})$ is the set of all mixings of nonempty families of elements of \mathcal{G} and \mathcal{G} consists of cyclic sets; then $\text{fil}(\mathcal{G})$ is a filterbase on X and $\mathcal{G}^\uparrow = \text{fil}(\mathcal{G})^\uparrow$.

- If \mathcal{F} is a filter on X inside $\mathbb{V}^{(\mathbb{B})}$ then put $\mathcal{F}^\downarrow := \text{fil}(\{F\downarrow \mid F \in \mathcal{F}\downarrow\})$. The filter \mathcal{F}^\downarrow is the *descent* of \mathcal{F} . A filterbase \mathcal{G} on $X\downarrow$ is *extensional* provided that $\text{fil}(\mathcal{G}) = \mathcal{F}$ for some filter \mathcal{F} on X .
- The descent of an ultrafilter on X is a *proultrafilter* on $X\downarrow$. A filter with a base of cyclic sets is *cyclic*. Proultrafilters are maximal cyclic filters.

Cyclic Filters and Monads

- Fix a standard complete Boolean algebra B and think of $\mathbb{V}^{(B)}$ to be composed of internal sets. If A is external then the *cyclic hull* $\text{fil}(A)$ of A consists of $x \in \mathbb{V}^{(B)}$ admitting an internal family $(a_\xi)_{\xi \in \Xi}$ of elements of A and an internal partition $(b_\xi)_{\xi \in \Xi}$ of unity in \mathbb{B} such that x is the mixing of $(a_\xi)_{\xi \in \Xi}$ by $(b_\xi)_{\xi \in \Xi}$; i.e., $b_\xi x = b_\xi a_\xi$ for $\xi \in \Xi$ or, equivalently, $x = \text{fil}_{\xi \in \Xi}(b_\xi a_\xi)$.
- Given a filter \mathcal{F} on $X \downarrow$, let

$$\mathcal{F} \uparrow \downarrow := \text{fil}(\{F \uparrow \downarrow \mid F \in \mathcal{F}\}).$$

Then $\text{fil}(\mu(\mathcal{F})) = \mu(\mathcal{F} \uparrow \downarrow)$ and $\mathcal{F} \uparrow \downarrow$ is the greatest cyclic filter coarser than \mathcal{F} .

- The monad of \mathcal{F} is called *cyclic* if $\mu(\mathcal{F}) = \text{fil}(\mu(\mathcal{F}))$. Unfortunately, the cyclicity of a monad is not completely responsible for extensionality of a filter.

Monad Hulls

- The *cyclic monad hull* $\mu_c(U)$ of an external set U is defined as follows:

$$x \in \mu_c(U) \leftrightarrow (\forall^{\text{st}} V = V \uparrow \downarrow) V \supset U \rightarrow x \in \mu(V).$$

If $\mathbb{B} = \mathbb{2}$, then $\mu_c(U)$ is the monad of the standardization of the external filter of supersets of U , i.e. the (*discrete*) *monad hull* $\mu_d(U)$.

- *The cyclic monad hull of a set is the cyclic hull of its monad hull*

$$\mu_c(U) = \text{fil}(\mu_d(U)).$$

Essential Points

- A special role is played by the *essential points* of $X\downarrow$ constituting the external set eX . By definition, an essential point of eX belongs to the monad of some proultrafilter on $X\downarrow$. The collection eX of all essential points of X is usually external.
- $x \in {}^eX$ if and only if x can be separated by a standard set from every standard cyclic set not containing x .
- If there is an essential point in the monad of an ultrafilter \mathcal{F} then $\mu(\mathcal{F}) \subset {}^eX$; moreover, $\mathcal{F}\uparrow\downarrow$ is a proultrafilter.
- A filter \mathcal{F} is *extensional* if and only if $\mu(\mathcal{F}) = \mu_c({}^e\mu(\mathcal{F}))$.
A standard set A is *cyclic* if and only if A is the cyclic monad hull of eA .

Test for the Mixing of Filters

- Let $(\mathcal{F}_\xi)_{\xi \in \Xi}$ be a standard family of extensional filters, and let $(b_\xi)_{\xi \in \Xi}$ be a standard partition of unity. The filter \mathcal{F} is the mixing of $(\mathcal{F}_\xi)_{\xi \in \Xi}$ by $(b_\xi)_{\xi \in \Xi}$ if and only if

$$(\forall^{\text{St}} \xi \in \Xi) b_\xi \mu(\mathcal{F}) = b_\xi \mu(\mathcal{F}_\xi).$$

Properties of Essential Points

- **(1)** The image of an essential point under an extensional mapping is an essential point of the image;
- **(2)** Let E be a standard set, and let X be a standard element of $\mathbb{V}^{(\mathbb{B})}$. Consider the product X^{E^\wedge} inside $\mathbb{V}^{(\mathbb{B})}$, where E^\wedge is the standard name of E in $\mathbb{V}^{(\mathbb{B})}$. If x is an essential point of $X^{E^\wedge} \downarrow$ then for every standard $e \in E$ the point $x \downarrow (e)$ is essential in $X \downarrow$;
- **(3)** Let \mathcal{F} be a cyclic filter in $X \downarrow$, and let ${}^e\mu(\mathcal{F}) := \mu(\mathcal{F}) \cap {}^eX$ be the set of essential points of its monad. Then ${}^e\mu(\mathcal{F}) = {}^e\mu(\mathcal{F}^{\uparrow\downarrow})$.

Procompactness

- Let (X, \mathcal{U}) be a uniform space inside $\mathbb{V}^{(\mathbb{B})}$. The descent $(X\downarrow, \mathcal{U}\downarrow)$ is *procompact* or *cyclically compact* if (X, \mathcal{U}) is compact inside $\mathbb{V}^{(\mathbb{B})}$. A similar sense resides in the notion of *pro-total-boundedness* and so on.
- *Every essential point of $X\downarrow$ is nearstandard, i.e., infinitesimally close to a standard point, if and only if $X\downarrow$ is procompact.*
- Existence of many procompact but not compact spaces provides a lot of examples of inessential points.

Test for Proprecompactness

- *A standard space is the descent of a totally bounded uniform space if and only if its every essential point is prenearstandard, i.e. belongs to the monad of a Cauchy filter.*

Vector Lattice Environment

- Let Y to be a universally complete vector lattice. By Gordon's Theorem, $Y = \mathcal{R} \downarrow$ of the reals \mathcal{R} inside $\mathbb{V}^{(\mathbb{B})}$ over the base \mathbb{B} of Y .
- Denote by \mathcal{E} the filter of order units in Y , i.e. the set $\mathcal{E} := \{\varepsilon \in Y_+ \mid \llbracket \varepsilon = 0 \rrbracket = 0\}$.
- Put $x \approx y \leftrightarrow (\forall^{\text{st}} \varepsilon \in \mathcal{E}) (|x - y| < \varepsilon)$. Given $a, b \in Y$, write $a < b$ if $\llbracket a < b \rrbracket = 1$; in other words, $a > b \leftrightarrow a - b \in \mathcal{E}$. Thus, there is some deviation from the understanding of the theory of ordered vector spaces. Clearly, this is done in order to adhere to the principles of introducing notations while descending and ascending.
- Let $\approx Y$ be the *nearstandard part* of Y . Given $y \in \approx Y$, denote by ${}^\circ y$ (or by $\text{st}(y)$) the *standard part* of y , i.e. the unique standard element infinitely close to y .

Order Convergence

- For a standard filter \mathcal{F} in Y and a standard $z \in Y$, the following are true:
 - **(1)** $\inf_{F \in \mathcal{F}} \sup F \leq z \leftrightarrow (\forall y \in {}^e\mu(\mathcal{F}\uparrow\downarrow))^\circ y \leq z$;
 - **(2)** $\sup_{F \in \mathcal{F}} \inf F \geq z \leftrightarrow (\forall y \in {}^e\mu(\mathcal{F}\uparrow\downarrow))^\circ y \geq z$;
 - **(3)** $\inf_{F \in \mathcal{F}} \sup F \geq z \leftrightarrow (\exists y \in {}^e\mu(\mathcal{F}\uparrow\downarrow))^\circ y \geq z$;
 - **(4)** $\sup_{F \in \mathcal{F}} \inf F \leq z \leftrightarrow (\exists y \in {}^e\mu(\mathcal{F}\uparrow\downarrow))^\circ y \leq z$;
 - **(5)** $\mathcal{F} \xrightarrow{(o)} z \leftrightarrow (\forall y \in {}^e\mu(\mathcal{F}\uparrow\downarrow))y \approx z \leftrightarrow (\forall y \in \mu(\mathcal{F}\uparrow\downarrow))y \approx z$.

Infinitesimal Modeling in $\mathbb{V}(\mathbb{B})$

- Let us follow the classical approach of Robinson inside $\mathbb{V}(\mathbb{B})$. In other words, the classical and internal universes and the corresponding $*$ -map (Robinson's standardization) are understood to be members of $\mathbb{V}(\mathbb{B})$. Moreover, the nonstandard world is supposed to be properly saturated.

Descent Standardization

- The descent of the $*$ -map is referred to as *descent standardization*. Alongside the term “descent standardization” we also use the expressions like “ B -standardization,” “prostandardization,” etc. Furthermore, Denote the Robinson standardization of a B -set A by the symbol $*A$.
- The *descent standardization* of a set A with B -structure, i.e. a subset of $\mathbb{V}^{(B)}$, is defined as $(*(A\uparrow))\downarrow$ and is denoted by $*A$ (it is meant here that $A\uparrow$ is an element of the standard universe located inside $\mathbb{V}^{(B)}$).
- Thus, $*a \in *A \leftrightarrow a \in A\uparrow\downarrow$. The *descent standardization* $*\Phi$ of an *extensional correspondence* Φ is also defined in a natural way.
- Considering the descent standardizations of the standard names of elements of the von Neumann universe \mathbb{V} , use the abbreviations $*x := *(x^\wedge)$ and $*x := (*x)\downarrow$ for $x \in \mathbb{V}$. The rules of placing and omitting asterisks (by default) in descent standardization are also assumed as liberal as those for the Robinson $*$ -map.

Transfer

- Let $\varphi = \varphi(x, y)$ be a formula of ZFC without any free variables other than x and y . Then

$$(\exists x \in {}^*F) \llbracket \varphi(x, {}^*z) \rrbracket = \mathbb{1} \leftrightarrow (\exists x \in F \downarrow) \llbracket \varphi(x, z) \rrbracket = \mathbb{1};$$

$$(\forall x \in {}^*F) \llbracket \varphi(x, {}^*z) \rrbracket = \mathbb{1} \leftrightarrow (\forall x \in F \downarrow) \llbracket \varphi(x, z) \rrbracket = \mathbb{1}$$

for a nonempty element F in $\mathbb{V}^{(\mathbb{B})}$ and for every z .

Idealization

- Let $X \uparrow$ and Y be classical elements of $\mathbb{V}^{(\mathbb{B})}$, and let $\varphi = \varphi(x, y, z)$ be a formula of ZFC. Then

$$(\forall^{\text{fin}} A \subset X) (\exists y \in {}_* Y) (\forall x \in A) \llbracket \varphi({}_* x, y, z) \rrbracket = \mathbb{1}$$

$$\leftrightarrow (\exists y \in {}_* Y) (\forall x \in X) \llbracket \varphi({}_* x, y, z) \rrbracket = \mathbb{1}$$

for an internal element z in $\mathbb{V}^{(\mathbb{B})}$.

Descending Monads

- Given a filter \mathcal{F} of sets with B -structure, define its the *descent monad* $m(\mathcal{F})$ of \mathcal{F} as $m(\mathcal{F}) := \bigcap_{F \in \mathcal{F}} *F$.
- Let \mathcal{E} be a set of filters, and let $\mathcal{E}^\uparrow := \{\mathcal{F}^\uparrow \mid \mathcal{F} \in \mathcal{E}\}$ be its ascent to $\mathbb{V}^{(\mathbb{B})}$. The following are equivalent:
 - **(1)** the set of cyclic hulls of \mathcal{E} , i.e. $\mathcal{E}^\uparrow\downarrow := \{\mathcal{F}^\uparrow\downarrow \mid \mathcal{F} \in \mathcal{E}\}$, is bounded above;
 - **(2)** the set \mathcal{E}^\uparrow is bounded above inside $\mathbb{V}^{(\mathbb{B})}$;
 - **(3)** $\bigcap \{m(\mathcal{F}) \mid \mathcal{F} \in \mathcal{E}\} \neq \emptyset$.
- Moreover, in this event

$$m(\sup \mathcal{E}^\uparrow\downarrow) = \bigcap \{m(\mathcal{F}) \mid \mathcal{F} \in \mathcal{E}\}; \quad \sup \mathcal{E}^\uparrow = (\sup \mathcal{E})^\uparrow.$$

- It is worth noting that for an infinite set of descent monads, its union, and even the cyclic hull of this union, is not a descent monad in general. The situation here is the same as for ordinary monads.

Nonstandard Tests for a Proultrafilter

- *The following are equivalent:*
- **(1)** \mathcal{U} is a proultrafilter;
- **(2)** \mathcal{U} is an extensional filter with inclusion-minimal descent monad;
- **(3)** $\mathcal{U} = (x)^\downarrow := \text{fil}(\{U \uparrow \downarrow \mid x \in {}_*A\})$ for each point x of the descent monad $m(\mathcal{U})$;
- **(4)** \mathcal{U} is an extensional filter whose descent monad is easily caught by a cyclic set; i.e. either $m(\mathcal{U}) \subset {}_*U$ or $m(\mathcal{U}) \subset {}_*(X \setminus U)$ for every $U = U \uparrow \downarrow$;
- **(5)** \mathcal{U} is a cyclic filter satisfying the condition: for every cyclic U , if ${}_*U \cap m(\mathcal{A}) \neq \emptyset$ then $U \in \mathcal{U}$.

Nonstandard Test for the Mixing of Filters

- Let $(\mathcal{F}_\xi)_{\xi \in \Xi}$ be a family of filters, let $(b_\xi)_{\xi \in \Xi}$ be a partition of unity, and let $\mathcal{F} = \text{fil}_{\xi \in \Xi}(b_\xi \mathcal{F}_\xi^\uparrow)$ be the mixing of \mathcal{F}_ξ^\uparrow by b_ξ . Then

$$m(\mathcal{F}^\downarrow) = \text{fil}_{\xi \in \Xi}(b_\xi m(\mathcal{F}_\xi)).$$

Nonstandard Test for Procompactness

- A point y of the set ${}_*X$ is *descent-nearstandard* or simply *nearstandard* if there is no danger of misunderstanding whenever ${}_*x \approx y$ for some $x \in X\downarrow$; i.e., $(x, y) \in m(\mathcal{U}\downarrow)$, with \mathcal{U} the uniformity on X .
- A set $A\uparrow\downarrow$ is *procompact* if and only if every point of ${}_*A$ is *descent-nearstandard*.

Rules of Descent Standardization

- Let $\varphi = \varphi(x)$ be a formula of ZFC. The truth value of φ is constant on the descent monad of every poultrafilter \mathcal{A} .
- Let $\varphi = \varphi(x, y, z)$ be a formula of ZFC, and let \mathcal{F} and \mathcal{G} be filters of sets with B -structure.

The following quantification rules are valid (for internal y, z in $\mathbb{V}^{(B)}$):

- **(1)** $(\exists x \in m(\mathcal{F})) \llbracket \varphi(x, y, z) \rrbracket = 1$
 $\Leftrightarrow (\forall F \in \mathcal{F}) (\exists x \in {}^*F) \llbracket \varphi(x, y, z) \rrbracket = 1;$
- **(2)** $(\forall x \in m(\mathcal{F})) \llbracket \varphi(x, y, z) \rrbracket = 1$
 $\Leftrightarrow (\exists F \in \mathcal{F}^{\uparrow\downarrow}) (\forall x \in {}^*F) \llbracket \varphi(x, y, z) \rrbracket = 1;$
- **(3)** $(\forall x \in m(\mathcal{F})) (\exists y \in m(\mathcal{G})) \llbracket \varphi(x, y, z) \rrbracket = 1$
 $\Leftrightarrow (\forall G \in \mathcal{G}) (\exists F \in \mathcal{F}^{\uparrow\downarrow}) (\forall x \in {}^*F) (\exists y \in {}^*G) \llbracket \varphi(x, y, z) \rrbracket = 1;$
- **(4)** $(\exists x \in m(\mathcal{F})) (\forall y \in m(\mathcal{G})) \llbracket \varphi(x, y, z) \rrbracket = 1$
 $\Leftrightarrow (\exists G \in \mathcal{G}^{\uparrow\downarrow}) (\forall F \in \mathcal{F}) (\exists x \in {}^*F) (\forall y \in {}^*G) \llbracket \varphi(x, y, z) \rrbracket = 1.$

The Case of Standardized Free Variables

- **(1)** $(\exists x \in m(\mathcal{F})) \llbracket \varphi(x, *y, *z) \rrbracket = \mathbb{1}$
 $\Leftrightarrow (\forall F \in \mathcal{F})(\exists x \in F \uparrow \downarrow) \llbracket \varphi(x, y, z) \rrbracket = \mathbb{1};$
- **(2)** $(\forall x \in m(\mathcal{F})) \llbracket \varphi(x, *y, *z) \rrbracket = \mathbb{1}$
 $\Leftrightarrow (\exists F \in \mathcal{F}^{\uparrow \downarrow})(\forall x \in F) \llbracket \varphi(x, y, z) \rrbracket = \mathbb{1};$
- **(3)** $(\forall x \in m(\mathcal{F}))(\exists y \in m(\mathcal{G})) \llbracket \varphi(x, y, *z) \rrbracket = \mathbb{1}$
 $\Leftrightarrow (\forall G \in \mathcal{G})(\exists F \in \mathcal{F}^{\uparrow \downarrow})(\forall x \in F)(\exists y \in G \uparrow \downarrow) \llbracket \varphi(x, y, z) \rrbracket = \mathbb{1};$
- **(4)** $(\exists x \in m(\mathcal{F}))(\forall y \in m(\mathcal{G})) \llbracket \varphi(x, y, *z) \rrbracket = \mathbb{1}$
 $\Leftrightarrow (\exists G \in \mathcal{G}^{\uparrow \downarrow})(\forall F \in \mathcal{F})(\exists x \in F \uparrow \downarrow)(\forall y \in G) \llbracket \varphi(x, y, z) \rrbracket = \mathbb{1}.$

Again in Vector Lattices

- The fact that E is a vector lattice is a formula, say, $\varphi(E, \mathbb{R})$. Hence, recalling the bounded transfer principle, we come to the equality $\llbracket \varphi(E^\wedge, \mathbb{R}^\wedge) \rrbracket = \mathbb{1}$; i.e., E^\wedge is a vector lattice over the ordered field \mathbb{R}^\wedge inside $\mathbb{V}^{(\mathbb{B})}$.
- Let $E^{\wedge\sim}$ be the space of regular \mathbb{R}^\wedge -linear functionals from E^\wedge to \mathcal{R} . It is easy that $E^{\wedge\sim} := L^\sim(E^\wedge, \mathcal{R})$ is a K -space, i.e. a Dedekind complete vector lattice, inside $\mathbb{V}^{(\mathbb{B})}$. Since $E^{\wedge\sim}$ is a K -space, the descent $E^{\wedge\sim}\downarrow$ of $E^{\wedge\sim}$ is a K -space too.
- Turn to the universally complete vector lattice $F := \mathcal{R}\downarrow$. For every operator $T \in L^\sim(E, F)$ the ascent $T\uparrow$ is defined by the equality $\llbracket Tx = T\uparrow(x^\wedge) \rrbracket = \mathbb{1}$ for all $x \in E$. If $\tau \in E^{\wedge\sim}$, then $\llbracket \tau : E^\wedge \rightarrow \mathcal{R} \rrbracket = \mathbb{1}$; hence, the operator $\tau\downarrow : E \rightarrow F$ is available. Moreover, $\tau\downarrow\uparrow = \tau$. On the other hand, $T\uparrow\downarrow = T$.
- For every $T \in L^\sim(E, F)$ the ascent $T\uparrow$ is a regular \mathbb{R}^\wedge -functional on E^\wedge inside $\mathbb{V}^{(\mathbb{B})}$; i.e., $\llbracket T\uparrow \in E^{\wedge\sim} \rrbracket = \mathbb{1}$. The mapping $T \mapsto T\uparrow$ is a linear and lattice isomorphism between $L^\sim(E, F)$ and $E^{\wedge\sim}\downarrow$.

A Few Classes of Operators

- An operator $S \in L^{\sim}(E, F)$ is a *fragment* of $0 \leq T \in L^{\sim}(E, F)$ if $S \wedge (T - S) = 0$. Say that T is *F-discrete* whenever $[0, T] = [0, I_F] \circ T$; i.e., for every $0 \leq S \leq T$ there is an operator $0 \leq \alpha \leq I_F$ satisfying $S = \alpha \circ T$. Let $L_a^{\sim}(E, F)$ be the band of $L^{\sim}(E, F)$ generated by *F-discrete* operators, and write $L_d^{\sim}(E, F) := L_a^{\sim}(E, F)^{\perp}$. The bands $(E^{\wedge\sim})_a$ and $(E^{\wedge\sim})_d$ are introduced similarly.
- The elements of $L_d^{\sim}(E, F)$ are usually referred to as *F-diffuse* operators. The \mathbb{R} -discrete or \mathbb{R} -diffuse operators are *discrete* or *diffuse* functionals.

Applying the Escher Rules

- Consider $S, T \in L^{\sim}(E, F)$ and put $\tau := T\uparrow$, $\sigma := S\uparrow$. The following are true:
 - **(1)** $T \geq 0 \leftrightarrow \llbracket \tau \geq 0 \rrbracket = \mathbb{1}$;
 - **(2)** S is a fragment of $T \leftrightarrow \llbracket \sigma \text{ is a fragment of } \tau \rrbracket = \mathbb{1}$;
 - **(3)** T is F -discrete $\leftrightarrow \llbracket \tau \text{ is discrete} \rrbracket = \mathbb{1}$;
 - **(4)** $T \in L_a^{\sim}(E, F) \leftrightarrow \llbracket \tau \in (E^{\wedge\sim})_a \rrbracket = \mathbb{1}$;
 - **(5)** $T \in L_d^{\sim}(E, F) \leftrightarrow \llbracket \tau \in (E^{\wedge\sim})_d \rrbracket = \mathbb{1}$.
 - **(6)** T is a lattice homomorphism \leftrightarrow
 $\llbracket \tau \text{ is a lattice homomorphism} \rrbracket = \mathbb{1}$.

Generating Sets of Projections

- Let E stand for a vector lattice and F , for a K -space. A set \mathcal{P} of band projections in $L^\sim(E, F)$ *generates the fragments of* T , $0 \leq T \in L^\sim(E, F)$, provided that $Tx^+ = \sup\{pTx \mid p \in \mathcal{P}\}$ for all $x \in E$. If this happens for all $0 \leq T \in L^\sim(E, F)$, then \mathcal{P} is a *generating set*.
- Put $F := \mathcal{R}\downarrow$ and let p be a band projection in $L^\sim(E, F)$. Then there is a unique element $p\uparrow \in \mathbb{V}^{(\mathbb{B})}$ such that $\llbracket p\uparrow$ is a band projection in $E^{\wedge\sim} \rrbracket = \mathbb{1}$ and $(pT)\uparrow = p\uparrow T\uparrow$ for all $T \in L^\sim(E, F)$.

Scalarizing Fragments

- Consider some set \mathcal{P} of band projections in $L^\sim(E, F)$ and a positive operator $T \in L^\sim(E, F)$. Put $\tau := T^\uparrow$ and $\mathcal{P}^\uparrow := \{p^\uparrow \mid p \in \mathcal{P}\}^\uparrow$. Then $\llbracket \mathcal{P}^\uparrow \text{ is a set of band projections in } E^{\wedge\sim} \rrbracket = \mathbb{1}$ and the following are true:
 - **(1)** \mathcal{P} generates the fragments of $T \leftrightarrow \llbracket \mathcal{P}^\uparrow \text{ generates the fragments of } \tau \rrbracket = \mathbb{1}$;
 - **(2)** \mathcal{P} is a generating set $\leftrightarrow \llbracket \mathcal{P}^\uparrow \text{ is a generating set} \rrbracket = \mathbb{1}$.

- Given a set A in a K -space, denote by A^\vee the result of adjoining to A suprema of every nonempty finite subset of A . Let A^\uparrow stand for the result of adjoining to A suprema of nonempty increasing nets of elements of A . The symbols $A^{\uparrow\downarrow}$ and $A^{\uparrow\downarrow\uparrow}$ are understood naturally.⁷
- Put $\mathcal{P}(f) := \{pf \mid p \in \mathcal{P}\}$ and note that E will for a time being stand for a vector lattice over a dense subfield of \mathbb{R} while \mathcal{P} is a set of band projections in E^\sim . Let $\mathfrak{E}(f)$ be the set of all fragments of f .

⁷Cp. [7].

Generating Scalar Fragments

- The following are equivalent:
- **(1)** $\mathcal{P}(f)^{\vee(\uparrow\downarrow\uparrow)} = \mathfrak{E}(f)$;
- **(2)** \mathcal{P} generates the fragments of f ;
- **(3)** $(\forall x \in {}^\circ E)(\exists p \in \mathcal{P})pf(x) \approx f(x^+)$;
- **(4)** a functional g in $[0, f]$ is a fragment of f if and only if

$$\inf_{p \in \mathcal{P}} (p^\perp g(x) + p(f - g)(x)) = 0$$

for every $0 \leq x \in E$;

- **(5)** $(\forall g \in {}^\circ \mathfrak{E}(f))(\forall x \in {}^\circ E_+)(\exists p \in \mathcal{P})|pf - g|(x) \approx 0$;
- **(6)** $\inf\{|pf - g|(x) \mid p \in \mathcal{P}\} = 0$ for all fragments $g \in \mathfrak{E}(f)$ and $x \geq 0$;
- **(7)** for $x \in E_+$ and $g \in \mathcal{E}(f)$ there is an element $p \in \mathcal{P}(f)^{\vee(\uparrow\downarrow\uparrow)}$, satisfying $|pf - g|(x) = 0$.

Principal Bands in the Scalar Case

- For positive functionals f and g and for a generating set of band projections \mathcal{P} , the following are equivalent:
 - **(1)** $g \in \{f\}^{\perp\perp}$;
 - **(2)** If x is a limited element of E , i.e.
 $x \in {}^{\text{fin}}E := \{x \in E \mid (\exists \bar{x} \in {}^\circ E) |x| \leq \bar{x}\}$, then $pg(x) \approx 0$ whenever $pf(x) \approx 0$ for $p \in \mathcal{P}$;
 - **(3)** $(\forall x \in E_+)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall p \in \mathcal{P}) pf(x) \leq \delta \rightarrow pg(x) \leq \varepsilon$.

Principal Projections in the Scalar Case

- Let f and g be positive functionals on E , and let x be a positive element of E . The following representations of the band projection b_f onto the band $\{f\}^{\perp\perp}$ are valid:
- **(1)** $b_f g(x) \Rightarrow \inf^* \{pg(x) \mid p^\perp f(x) \approx 0, p \in \mathcal{P}\}$ (the symbol \Rightarrow means that the formula is *exact*, i.e., the equality is attained);
- **(2)** $b_f g(x) = \sup_{\varepsilon > 0} \inf \{pg(x) \mid p^\perp f(x) \leq \varepsilon, p \in \mathcal{P}\}$;
- **(3)** $b_f g(x) \Rightarrow \inf^* \{g(y) \mid f(x - y) \approx 0, 0 \leq y \leq x\}$;
- **(4)** $(\forall \varepsilon > 0) (\exists \delta > 0) (\forall p \in \mathcal{P}) pf(x) < \delta \rightarrow b_f g(x) \leq p^\perp g(x) + \varepsilon$;
 $(\forall \varepsilon > 0) (\forall \delta > 0) (\exists p \in \mathcal{P}) pf(x) < \delta \wedge p^\perp g(x) \leq b_f g(x) + \varepsilon$;
- **(5)** $(\forall \varepsilon > 0) (\exists \delta > 0) (\forall 0 \leq y \leq x) f(x - y) \leq \delta \rightarrow b_f g(x) \leq g(y) + \varepsilon$;
 $(\forall \varepsilon > 0) (\forall \delta > 0) (\exists 0 \leq y \leq x) f(x - y) \leq \delta \wedge g(y) \leq b_f g(x) + \varepsilon$.

Up-Down for Operators

- For a set of band projections \mathcal{P} in $L^\sim(E, F)$ and $0 \leq S \in L^\sim(E, F)$ the following are equivalent:
 - **(1)** $\mathcal{P}(S)^{\vee(\uparrow\downarrow\uparrow)} = \mathfrak{E}(S)$;
 - **(2)** \mathcal{P} generates the fragments of S ;
 - **(3)** an operator $T \in [0, S]$ is a fragment of S if and only if

$$\inf_{p \in \mathcal{P}} (p^\perp T x + p(S - T)x) = 0$$

for all $0 \leq x \in E$;

- **(4)** $(\forall x \in {}^\circ E) (\exists p \in \mathcal{P}\uparrow\downarrow) pSx \approx Sx^+$.

Principal Bands in the Operator Case

- For positive operators S and T and a generating set \mathcal{P} of band projections in $L^{\sim}(E, F)$, the following are equivalent:
 - **(1)** $T \in \{S\}^{\perp\perp}$;
 - **(2)** $(\forall x \in {}^{\text{fin}}E) (\forall p \in \mathcal{P}) (\forall b \in \mathbb{B}) bpSx \approx 0 \rightarrow bpTx \approx 0$;
 - **(3)** $(\forall x \in {}^{\text{fin}}E) (\forall b \in \mathbb{B}) bSx \approx 0 \rightarrow bTx \approx 0$;
 - **(4)** $(\forall x \geq 0) (\forall \varepsilon \in \mathcal{O}) (\exists \delta \in \mathcal{O}) (\forall p \in \mathcal{P}) (\forall b \in \mathbb{B}) bpSx \leq \delta \rightarrow bpTx \leq \varepsilon$;
 - **(5)** $(\forall x \geq 0) (\forall \varepsilon \in \mathcal{O}) (\exists \delta \in \mathcal{O}) (\forall b \in \mathbb{B}) bSx \leq \delta \rightarrow bTx \leq \varepsilon$.

Principal Projections in the Operator Case

- Let E be a vector lattice, and let F be a K -space having the filter of order units \mathcal{E} and the base \mathbb{B} . Suppose that S and T are positive operators in $L^{\sim}(E, F)$ and R is the band projection of T to the band $\{S\}^{\perp\perp}$. For a positive $x \in E$, the following are valid:
- **(1)** $Rx = \sup_{\varepsilon \in \mathcal{E}} \inf \{bTy + b^{\perp}Sx \mid 0 \leq y \leq x, b \in \mathbb{B}, bS(x - y) \leq \varepsilon\}$;
- **(2)** $Rx = \sup_{\varepsilon \in \mathcal{E}} \inf \{(bp)^{\perp}Tx \mid bpSx \leq \varepsilon, p \in \mathcal{P}, b \in \mathbb{B}\}$,
where \mathcal{P} is a generating set of band projections in F .

The Polyhedral Lagrange Principle

- Turn to the revisited Farkas Lemma.⁸
- Let X be a Y -seminormed real vector space, with Y a K -space. Given are some dominated polyhedral sublinear operators P_1, \dots, P_N from X to Y and a dominated sublinear operator $P : X \rightarrow Y$.
- *The finite value of the constrained problem*

$$P_1(x) \leq u_1, \dots, P_N(x) \leq u_N, \quad P(x) \rightarrow \inf$$

is the value of the unconstrained problem for an appropriate Lagrangian without any constraint qualification but polyhedrality.





⁸Cp.[10]–[12].

Interval Operator Inequalities




- Polyhedrality finds applications in inexact data processing.⁹ Let X be a Y -seminormed real space, with Y a K -space. Assume given a dominated polyhedral sublinear operator $P : X \rightarrow Y$, a dominated sublinear operator $Q : X \rightarrow Y$, and $u, v \in Y$. Assume further that $\{P \leq u\} \neq \emptyset$.
- *The following are equivalent:*
- **(1)** *for all $b \in \mathbb{B}$, with \mathbb{B} the base of Y , the sublinear operator inequality $bQ \circ \sim(x) \geq -bv$ is a consequence of the polyhedral sublinear operator inequality $bP(x) \leq bu$, i.e., $\{bP \leq bu\} \subset \{bQ \circ \sim \geq -bv\}$, with $\sim(x) := -x$ for all $x \in X$;*
- **(2)** *there are $A \in \partial(P)$, $B \in \partial(Q)$, and a positive orthomorphism $\alpha \in \text{Orth}(m(Y))$ on the universal completion $m(Y)$ of Y satisfying $B = \alpha A$, $\alpha u \leq v$.*

⁹Cp. [13].

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