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CALCULI WITH THE GREATEST NUMBER

V. L. Rvachev, A. N. Shevchenko, and T. I. Sheiko

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The existence of calculi (differential, integral) with the greatest number $\mathfrak{M}$ has been proved in [1-4]. There have been repeated attempts in the past to declare some numbers as the greatest for all cases in life. Among such greatest numbers we can mention 10^{49} (colod), 10^{100} (googol), and others. A declaration that "such and such number is to be regarded as the greatest" is of course not very helpful in mathematics. Below we propose logical approaches to the construction of calculi with the greatest number which are algebraically isomorphic to the classical calculus, and consider some applications of these calculi.

1. THE IDEA OF CONSTRUCTING CALCULI WITH THE GREATEST NUMBER

The classical calculus is based on the set of natural numbers $N = \{0, 1, 2, 3, \dots\}$. This is reflected in Kronecker's well-known aphorism: "The natural numbers are from God, all the rest is the handiwork of man." Kronecker is also credited with another, seemingly equivalent statement: "Zero and one are from God, all the rest is the handiwork of man," However, the first proposition asserts, without saying it directly, that not only zero and one but also addition is from God, while the second proposition leaves it to man to choose the addition operation. We will show below that the construction of the classical calculus uses only the group properties of ordinary addition $x + y$:

1) $x + y \equiv y + x$ – commutativity;
2) $x + (y + y) \equiv (x + y) + z \equiv x + y + z$ – associativity;
3) $\forall x \exists \bar{x}$, $x + \bar{x} \equiv 0$ – existence of the inverse operation;
4) $x + 0 \equiv x$ – existence of the zero element.

This means that if the ordinary addition is replaced with any other two-place operation having the same group properties, we can construct a new calculus which is logically equivalent to the classical calculus.

By tracing on the axiomatic level how the addition operation is introduced in mathematics, we obtain the axiom of arithmetic:

– for any a and b, $a < b$, there is a natural number n such that $a + a + \dots + a$ (n addends) is greater than b.

If the addition operation in this axiom is the ordinary addition, it is called the Archimedean axiom, because in this case it is an exact analog of the Archimedean axiom in Euclidean geometry. The latter is stated as an "obvious" empirical fact for segments on the straight line: given two segments, we can always mark the smaller of the two along the straight line a sufficient number of times so that the result is greater than the larger segment.

Accepting the algebraic and the geometrical versions of the Archimedean axiom, we actually accept the possibility of existence in the real world of arbitrarily large values of physical quantities, as well as the existence of arbitrarily large distances. This conclusion is not undisputable, however, because mathematical axioms must be based on empirical evidence. Thus, for instance, astronomers have established that in the entire Universe there are no distances greater than some $R_0 = 2.5 \cdot 10^{25}$ m [5, 6]. Talking about empirical evidence on which mathematical axioms must rely, we can quote Avinash [7] who stated that physical theories must be constructed in accordance with the principle of finiteness and certainty of all physical quantities in the Universe. For instance, the total mass of matter and the total charge in the Universe are finite. Hence it follows that we should try to track down the sources of the notion of infinite values and eliminate them. The Archimedean axiom is such an initial notion.

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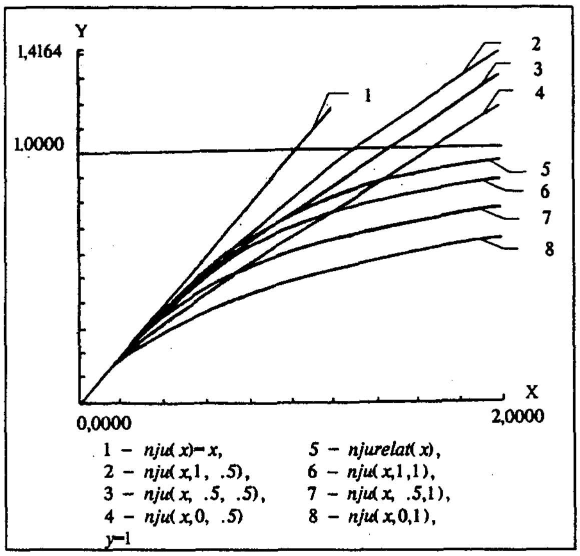

Fig. 1

Let us recall the main stages in the construction of the classical calculus from the triple of "atomic axioms" $\{0, 1, x + y\}$ and the operation of passage to the limit. First the set of natural numbers is constructed. Subtraction is defined as the inverse of addition, and negative numbers are introduced. This leads to the set of integers. Multiplication is defined as group addition, and division as the inverse of multiplication. This leads to rational numbers, and applying the operation of passage to the limit, we augment the set of rational numbers to the set of real numbers, which preserves the algebraic properties of the previously introduced operations (the field properties). Further extension of the field of real numbers leads to the field of complex numbers. Introducing the concept of a function, we can introduce the concepts of derivative, series, integral, differential and other equations. After all, their definition does not require any other concepts and operations in addition to those introduced previously. Elementary and special functions can be defined as solutions of differential equations, sums of power series, etc. All the constructions listed above use only properties 1)-4) of the addition operation.

Now suppose that the triple of classical "atomic axioms" is replaced with the triple $\{0, 1, X +_\nu Y\}$, where $X +_\nu Y$ is a two-place operation which also has the algebraic properties 1)-4). Then all the constructs of the classical calculus remain unchanged. All the logical constructions remain because they only use the algebraic properties of operations, and not the numerical values that the operations produce. We thus obtain a new calculus which is algebraically isomorphic to the classical calculus.

In what follows, we observe the following rule: if a special symbol is introduced for some expression in the classical calculus (for instance, the symbol $\sin x$ for the series $x - x^3/3! + x^5/5! - \ldots$, the symbol d/dx for the derivative, $\int f(x)dx$, etc.), then in the non-Archimedean calculus these expressions are subscripted with the symbol ν used in the "starting" addition operation $x +_\nu y$, i.e., $\sin_\nu x$;... d_ν/dx; $\int_\nu f_\nu(x)dx$, etc.

2. CONSTRUCTION OF RELATIVISTIC CALCULUS

The operation

$$x \overset{\circ}{+} y = \frac{x + y}{1 + \alpha^2 xy}, \qquad \alpha = \mathfrak{M}^{-1}, \tag{1}$$

has been proposed in [1-4, 7]. For $\mathfrak{M} = c$, c, where c is the velocity of light, this is the formula for addition of collinear velocities in special relativity theory. We will show that the operation $x \overset{\circ}{+} y$ satisfies the algebraic properties 1)-4). Commutativity is obvious. Let us check associativity:

$$(x \overset{\circ}{+} y) \overset{\circ}{+} z \equiv \frac{x+y}{1+\alpha^2 xy} \overset{\circ}{+} z = \frac{\frac{x+y}{1+\alpha^2 xy} + z}{1+\alpha^2 (x+y)z(1+\alpha^2 xy)^{-1}} =$$

$$= \frac{x+y+z+\alpha^2 xyz}{1+\alpha^2 xy + \alpha^2 xz + \alpha^2 yz} = x \overset{\circ}{+} (y \overset{\circ}{+} z) .$$

Also

$$x \overset{\circ}{+} 0 = \frac{x+0}{1+\alpha^2 x \cdot 0} = x .$$

The calculus based on the operation $x \overset{\circ}{+} y$ is called relativistic calculus [1-3]. Let us trace how the other operations and operators of relativistic calculus have been constructed in [1, 2].

Solving the equation $x = y \overset{\circ}{+} z$, we obtain the operation $x \overset{\circ}{-} y$:

$$x \overset{\circ}{-} y = \frac{x-y}{1-\alpha^2 xy} . \tag{2}$$

Properties 1)-4) can be augmented with the following properties:

5) $x \overset{\circ}{-} y \equiv x \overset{\circ}{+} (-y)$;

6) $x \overset{\circ}{-} x \equiv 0$;

7) $0 \overset{\circ}{-} x \equiv -x$ (i.e., $\overset{\circ}{-} x \equiv -x$).

In classical arithmetic, addition and subtraction are linked by the distributive law

$$(x+y)z \equiv xz + yz . \tag{3}$$

We stipulate that the identity (3) is satisfied for the addition $x \overset{\circ}{+} y$ defined by formula (1) and for the new multiplication $x \circ y$, which given its origin from (1) is called relativistic multiplication. Following (3), we obtain

$$\frac{x+y}{1+\alpha^2 xy} \circ z = x \circ z \overset{\circ}{+} y \circ z,$$

or

$$\frac{x+y}{1+\alpha^2 xy} \circ z = \frac{(x \circ y) + (y \circ z)}{1+\alpha^2 (x \circ z)(y \circ z)} . \tag{4}$$

Differentiating (4) with respect to x and y, we obtain

$$\left[\frac{x+y}{1+\alpha^2 xy} \circ z\right]'_{x \overset{\circ}{+} y} \left(\frac{1-\alpha^2 y^2}{(1+\alpha^2 xy)^2}\right) = \frac{(x \circ z)'_x \, [1-\alpha^2 (y \circ z)^2]}{[1+\alpha^2 (x \circ z)(y \circ z)]^2} ,$$

$$\left[\frac{x+y}{1+\alpha^2 xy} \circ z\right]'_{x \overset{\circ}{+} y} \left(\frac{1-\alpha^2 x^2}{(1+\alpha^2 xy)^2}\right) = \frac{(y \circ z)'_y \, [1-\alpha^2 (x \circ z)^2]}{[1+\alpha^2 (x \circ z)(y \circ z)]^2} .$$

Dividing the first equality by the second, we have

$$\frac{1-\alpha^2 y^2}{1-\alpha^2 x^2} = \frac{(x \circ z)'_x \, [1-\alpha^2 (y \circ z)^2]}{(y \circ z)'_y \, [1-\alpha^2 (x \circ z)^2]} . \tag{5}$$

The variables are separated to give

$$\frac{(x \circ z)'_x \, [1-\alpha^2 x^2]}{1-\alpha^2 (x \circ z)^2} = \frac{(y \circ z)'_y \, [1-\alpha^2 y^2]}{1-\alpha^2 (y \circ z)^2} = C(z), \tag{6}$$

where $C(z)$ is any function. Thus, for $(x \circ z)$ we obtain the differential equation

$$\frac{(x \circ z)'_x \, [1 - \alpha^2 x^2]}{1 - \alpha^2 (x \circ z)^2} = C(z), \tag{7}$$

with z a parameter. Denoting $u = x \circ z$, we obtain

$$\int \frac{du}{1 - \alpha^2 u^2} = C(z) \int \frac{dx}{1 - \alpha^2 x^2} . \tag{8}$$

Hence

$$\frac{1}{2\alpha} \ln \left| \frac{1 + \alpha u}{1 - \alpha u} \right| = \frac{C(z)}{2\alpha} \ln \left| \frac{1 + \alpha x}{1 - \alpha x} \right| + C_1(z) . \tag{9}$$

Setting $0 \circ z = u(0, z) = 0$, we find that $C_1 = 0$, and the condition $1 \circ z = u(1, z) = z$ gives

$$\frac{1}{2\alpha} \ln \left| \frac{1 + \alpha z}{1 - \alpha z} \right| = \frac{C(z)}{2\alpha} \ln \left| \frac{1 + \alpha}{1 - \alpha} \right| . \tag{10}$$

Hence

$$C(z) = \gamma \ln \left| \frac{1 + \alpha z}{1 - \alpha z} \right| , \tag{11}$$

where

$$\gamma = \ln^{-1} \left| \frac{1 + \alpha}{1 - \alpha} \right| . \tag{12}$$

Substituting (11) in (9), solving (9) for u, and also replacing z with y, we obtain a formula for relativistic multiplication

$$x \circ y \equiv \frac{1}{\alpha} \left\{ \frac{\exp\left(\gamma \ln \left| \frac{1 + \alpha x}{1 - \alpha x} \right| \ln \left| \frac{1 + \alpha y}{1 - \alpha y} \right| \right) - 1}{\exp\left(\gamma \ln \left| \frac{1 + \alpha x}{1 - \alpha x} \right| \ln \left| \frac{1 + \alpha y}{1 - \alpha y} \right| \right) + 1} \right\} . \tag{13}$$

We see from (13) that $x \circ y \equiv y \circ x$, i.e., the operation $x \circ y$ is commutative, and since it is defined from identity (3), it also satisfies the distributive law. However, the above arguments do not imply associativity of the multiplication $x \circ y$, and we need to check the associativity condition for $x \circ y$.

Denote

$$M(s) = \frac{1}{\alpha} \left(\frac{s - 1}{s + 1} \right), \quad N(s) = \frac{1 + \alpha s}{1 - \alpha s} . \tag{14}$$

It is easily seen that these functions are mutually inverse:

$$M(N(s)) = s, \quad N(M(s)) = s . \tag{15}$$

Using (14), we rewrite formula (13) in the form

$$x \circ y \equiv M[\exp(\gamma \ln |N(x)| \ln |N(y)|)] . \tag{16}$$

Then

$$(x \circ y) \circ z = M\{\exp[\gamma \ln|N(x \circ y)\ln|N(z)|\,]\} =$$
$$= M\{\exp[\gamma \ln|N(M[\exp(\gamma \ln|N(x)|\ \ln|N(y)|)\,])\ |\ \ln|N(z)|\,]\} =$$
$$= M\{\exp[\gamma^2 \ln|N(x)|\ \ln|N(y)|\ \ln|N(z)|\,]\} = x \circ (y \circ z)\,. \tag{17}$$

This proves associativity.

The formula for n-ary multiplication has the form

$$\prod_{i=1}^{n}{}_{\circ}\, x_i = x_1 \circ \ldots \circ x_n = M[\exp(\gamma^{n-1} \ln|N(x_1)| \ldots \ln|N(x_n)|)]\,. \tag{18}$$

Solving the equation $z \circ y = x$, we obtain the relativistic division operation

$$x \,\mathring{\circ}\!\!:\, y \equiv M[\exp(\gamma^{-1} \ln|N(x)| \ln^{-1}|N(y)|)] \tag{19}$$

or, using (14), we reduce (19) to the form

$$x \,\mathring{:}\, y = \frac{1}{\alpha}\left\{\frac{\exp\left(\frac{1}{\gamma}\ln\left|\frac{1+\alpha x}{1-\alpha x}\right| \ln^{-1}\left|\frac{1+\alpha y}{1-\alpha y}\right|\right) - 1}{\exp\left(\frac{1}{\gamma}\ln\left|\frac{1+\alpha x}{1-\alpha x}\right| \ln^{-1}\left|\frac{1+\alpha y}{1-\alpha y}\right|\right) + 1}\right\}. \tag{20}$$

Properties 1)-7) are augmented with the following properties:

8) $x \circ y \equiv y \circ x$;

9) $(x \circ y) \circ z \equiv x \circ (y \circ z) \equiv x \circ y \circ z$;

10) $x \circ 1 \equiv x$;

11) $x \circ 0 \equiv 0$;

12) $x \circ (1 \,\mathring{:}\, x) \equiv 1$;

13) $(x \circ y) \,\mathring{:}\, y \equiv x$;

14) $(x \,\mathring{:}\, y) \circ y \equiv x$;

15) $(x \,\mathring{:}\, y) \,\mathring{:}\, z \equiv x \,\mathring{:}\, (y \circ z)$;

16) $(x \circ y) \,\mathring{:}\, z \equiv (x \,\mathring{:}\, z) \circ y$;

17) $(x \overset{\circ}{+} y) \circ z \equiv (x \circ z) \overset{\circ}{+} (y \circ z)$;

18) $(x \overset{\circ}{-} y) \circ z \equiv (x \circ z) \overset{\circ}{-} (y \circ z)$;

19) $|x \circ y| < \mathfrak{M}\ \forall x, y \in \mathfrak{X} = (-\mathfrak{M}, \mathfrak{M})$;

20) $|x \,\mathring{:}\, y| \leqslant \mathfrak{M}\ \forall x, y \in \mathfrak{X} = (-\mathfrak{M}, \mathfrak{M})$;

21) $\lim_{y \to \pm 0} (x \,\mathring{:}\, y) = \pm\mathfrak{M} \operatorname{sign} x \quad (x \neq 0)$;

22) $\lim_{y \to \mathfrak{M}} (x \circ y) = \mathfrak{M} \operatorname{sign} x$;

23) $(x \circ z) \,\mathring{:}\, (y \circ z) \equiv x \,\mathring{:}\, y$.

Properties 1)-23) are written with some redundancy (it is sufficient to write some of these properties, treating the rest as consequences). We see from 1)-23) that the relativistic operations

$$\mathring{\Pi} = \{x \overset{\circ}{+} y,\ x \overset{\circ}{-} y,\ x \circ y,\ x \,\mathring{:}\, y\} \tag{21}$$

have the same properties as the ordinary arithmetic operations

$$\Pi = \{x + y,\ x - y,\ xy,\ x/y\}\,. \tag{22}$$

We thus find that the operations $\mathring{\Pi}$ define a field on the set $\mathfrak{X} = (-\mathfrak{M}, \mathfrak{M})$. Since the fields of real and complex numbers and the quaternion field are the only connected locally compact associative bodies, we expect that either the new algebraic system is not a field due to some deficiencies (violation of connectedness or local compactness) or it is isomorphic to the field of real numbers.

Let us construct the set of relativistic natural numbers. Properties 1)-23) of relativistic operations include only two numbers (0 and 1) that are smaller than $\mathfrak{M}$. Applying the addition operation $x \dot{+} y$ and denoting $1 \dot{+} 1 = \dot{2}$, we obtain

$$\overset{\circ}{2} = 1 \overset{\circ}{+} 1 = \frac{2}{1+\alpha^2} < 2.$$

Similarly, $\dot{2} \dot{+} 1 = 1 \dot{+} 1 \dot{+} 1 = \dot{2} < 3$, and so on.

We thus obtain the set of relativistic natural numbers

$$\overset{\circ}{\mathbb{N}} = \{0, 1, \overset{\circ}{2}, \overset{\circ}{3}, \ldots, \overset{\circ}{n}, \ldots\}, \tag{23}$$

which is a sequence that converges to $\mathfrak{M}$ in the sense of the ordinary metric $\rho(x, y) = |x - y|$. The application of the other relativistic operations (in conjunction with passage to the limit) produces all other real (and then complex) relativistic constants [1, 2]. The numbers entering the set Nplay the role of positive integers in relativistic arithmetic. We can easily obtain a formula for the relativistic image of the integer n:

$$n_\circ = \overset{\circ}{n} = \frac{1 - \left(\frac{1-\alpha}{1+\alpha}\right)^n}{\alpha\left[1 + \left(\frac{1-\alpha}{1+\alpha}\right)^n\right]} = M\{[N(1)]^n\}. \tag{24}$$

Using (20), we construct the relativistic image of the rational number m/n as

$$\overset{\circ}{m} \overset{\circ}{:} \overset{\circ}{n} = M\{[N(1)]^m\} \overset{\circ}{:} M\{[N(1)]^n\} =$$

$$= M\{\exp\frac{1}{\gamma}\ [\ln N(M\{[N(1)]^m\}) \ln^{-1} N(M\{[N(1)]^n\})]\} =$$

$$= M\{\exp[\frac{m}{\gamma} \ln N(1)/n \ln N(1)]\} = M\left\{\exp\frac{1}{\gamma}\left(\frac{m}{n}\right)\right\}.$$

We will show later that the relativistic image of any real number can be constructed using formula (24).

We now apply the previous results to construct a relativistic metric space. The distance between the numbers $\mathfrak{M}$ and $\mathfrak{M}$, measured in the ordinary Euclidean metric is $2\mathfrak{M}$. Following the isomorphism principle, we construct the "inner" (for $\dot{\Pi}$) metric $\dot{\rho}(x, y) = |x \dot{-} y|$, which corresponds to the Euclidean metric $\rho(x, y) = |x - y|$ on the real line $\mathbb{R}$. It is easy to prove that $\dot{\rho}(x, y)$ satisfies the metric space axioms. The only nontrivial axiom is the triangle axiom.

If x and y have different signs, then

$$|x| + |z| \geqslant \left|\frac{x - z}{1 - \alpha^2 xz}\right| = \frac{|x| + |z|}{1 + \alpha^2 |x|\,|z|}.$$

If x and z have the same sign, then

$$\left|\frac{x-z}{1-\alpha^2 xz}\right| = |x| + |z| - \frac{|z|(1-\alpha^2 x^2) + |z|(1-\alpha^2|x|\,|z|)}{1-\alpha^2|x|\,|z|} \leqslant$$

$$\leqslant |x| + |z| \quad \forall x, z \in \mathfrak{X}.$$

We thus have the inequality

$$|x| + |z| \geqslant \left|\frac{x - z}{1 - \alpha^2 xz}\right| = |x \overset{\circ}{-} z|.$$

Setting $x = x \dot{-} y$, $z = z \dot{-} y$, we obtain

$$|x \mathring{-} y| + |z \mathring{-} y| \geq |(x \mathring{-} y) \mathring{-} (z \mathring{-} y)| = |x \mathring{-} z|.$$

Q.E.D.

From (24) we see that both in the metric ρ and in the metric $\mathring{\rho}$ the distance from the number $\dot{n}$ to the origin 0 is the same, because $|\dot{n} - 0| = \dot{n} \dot{-} 0| = |\dot{n}|$. However, if we take the distance between neighboring numbers $(n + k)^{\cdot}$ and $\dot{n}$, then in the Euclidean metric we have

$$\rho((n + k)_{\circ}, \mathring{n}) = \left| \frac{(1+\alpha)^{n+k} - (1-\alpha)^{n+k}}{\alpha[(1+\alpha)^{n+k} + (1-\alpha)^{n+k}]} - \frac{(1+\alpha)^{n} - (1-\alpha)^{n}}{\alpha[(1+\alpha)^{n} + (1-\alpha)^{n}]} \right| =$$
$$= \frac{2(1+\alpha)^{n}(1-\alpha)^{n}[(1+\alpha)^{k} - (1-\alpha)^{k}]}{\alpha[(1+\alpha)^{n+k} + (1-\alpha)^{n+k}][(1+\alpha)^{n} + (1-\alpha)^{n}]} =$$
$$= \frac{2\left[1 - \left(\frac{1-\alpha}{1+\alpha}\right)^{k}\right]\left(\frac{1-\alpha}{1+\alpha}\right)^{n}}{\alpha\left[1 + \left(\frac{1-\alpha}{1+\alpha}\right)^{n+k}\right]\left[1 + \left(\frac{1-\alpha}{1+\alpha}\right)^{n}\right]}, \tag{25}$$

and in the metric $\mathring{\rho}(x, y)$

$$\mathring{\rho}((n + k)_{\circ}, \mathring{n}) = |(n + k)_{\circ} \mathring{-} \mathring{n})| = |\mathring{n} \mathring{+} \mathring{k} \mathring{-} \mathring{n})| = \mathring{k} \tag{26}$$

or, by (24),

$$\mathring{\rho}((n + k)_{\circ}, \mathring{n}) = \frac{1 - \left(\frac{1-\alpha}{1+\alpha}\right)^{k}}{\alpha\left[1 + \left(\frac{1-\alpha}{1+\alpha}\right)^{k}\right]}. \tag{27}$$

It follows from (25) and (27) that as $n \to \infty$ the Euclidean distance between the points $(n + k)_{\circ}$ and $\dot{n}$ tends to zero, while the relativistic distance between these points is independent of n and as $k \to \infty$ tends to $1/\alpha = \mathfrak{M}$. Thus, from our (Euclidean) point of view the natural numbers in $\dot{\mathbb{N}}$display condensation as we approach the greatest number $\mathfrak{M}$. Taking the "inner" point of view of the relativistic set $\dot{\mathbb{N}}$and using the metric $\mathring{\rho}$, we see that the distance between points corresponding to pairs of equally spaced natural numbers is the same. If we partition the interval $[\dot{n}, (n + k)_{\circ}]$ into k segments, $[(n = s)_{\circ}, (n + s + 1)_{\circ}]$, $s = 0, 1, \ldots, k - 1$, measure their lengths (in the metric $\mathring{\rho}$), and add them up (in the usual sense), then we obtain that the length of each segment is $|(n + s + 1)_{\circ} \dot{-} (n + s)_{\circ}| = \dot{1} = 1$.

The sum of lengths of these segments is $k > \mathring{k}$. The relativistic number axis in the space with its "natural" metric thus cannot be considered as a straight line, because on a straight line the length of any segment is equal to the sum of the lengths of its constituents. We will therefore view the space $\dot{\mathbb{R}}$ as a curve in some (probably infinite-dimensional) space on which natural numbers from $\dot{\mathbb{R}}$ are marked with equal spacing and it has a constant curvature, so that three neighboring points generate a triangle with two unit sides and the third side equal to $\mathring{2} < 2$:

$$\mathring{2} = \frac{(1+\alpha)^{2} - (1-\alpha)^{2}}{\alpha[(1+\alpha)^{2} + (1-\alpha)^{2}]} = \frac{2}{1+\alpha} < 2.$$

If we draw a circle through these three points and assume that the parameter α is sufficiently small (i.e., $\mathfrak{M}$ is sufficiently large), then the radius of this circle is approximately equal to the radius of curvature of the curve. For this radius we easily obtain the formula

$$r = \frac{1}{\sqrt{(1+\alpha^2)^2 - 1}} = \frac{1}{\alpha\sqrt{2+\alpha^2}} \approx \frac{1}{\alpha\sqrt{2}} = \frac{\mathfrak{M}}{\sqrt{2}},$$

and the curvature of the space $\dot{\mathbb{R}}$ for sufficiently large $\mathfrak{M}$ equals $\alpha\sqrt{2}$.

A metric space with this metric on the set of points with coordinates bounded in absolute value by the greatest number $\mathfrak{M}$ agrees with Nostradamus's statement, "The center of the Universe is everywhere, its boundary is nowhere."

3. GENERAL APPROACH TO THE CONSTRUCTION OF NON-ARCHIMEDEAN CALCULI

Consider the problem of choosing the set of non-Archimedean axioms. Let $\mathring{x} \in (-\mathfrak{M}, \mathfrak{M})$. In formula (16) set $x = \mathring{x}$, $y = 1$. As a result, noting that $\mathring{x} \circ 1 = \mathring{x}$, we obtain

$$\mathring{x} \circ 1 = M[\exp(\gamma \ln|N(\mathring{x})| \ln|N(1)|)] \equiv \mathring{x}. \tag{28}$$

Since $\ln|N(1)| = \ln|1+\alpha|/|1-\alpha| = \gamma^{-1}$, formula (28) can be replaced with two formulas:

$$M[\exp\frac{x}{\gamma}] = \mathring{x}, \quad x = \gamma \ln|N(\mathring{x})|. \tag{29}$$

Denote

$$\mathring{\nu}(x) = M[\exp(x\gamma^{-1})], \quad \mathring{\tau}(x) = \gamma \ln|N(x)|. \tag{30}$$

Although these functions are also defined for complex values x, for real numbers they are mappings of the form

$$\mathring{\nu}(x) : (-\infty, \infty) \to (-\mathfrak{M}, \mathfrak{M}), \quad \mathring{\tau}(x) : (-\mathfrak{M}, \mathfrak{M}) \to (-\infty, \infty), \tag{31}$$

and formula (30) implies that

$$\mathring{\nu}[\mathring{\tau}(x)] \equiv x, \tag{32}$$

i.e., $\mathring{\nu}(x)$ and $\mathring{\tau}(x)$ are mutually inverse functions. Thus, $\mathring{\nu}(x)$ maps all the numbers of the line $(-\infty, \infty)$ to the domain $(-\mathfrak{M}, \mathfrak{M})$ For instance, the integer n goes into the number $\mathring{n}$:

$$\mathring{\nu}(n) = M[\exp\frac{n}{\gamma}] = M\left[\exp\ln\left|\frac{1+\alpha}{1-\alpha}\right|^n\right] = M\{[N(1)]^n\} = \mathring{n}, \tag{33}$$

which agrees with (24). From formulas (31), (32) we obtain a general rule for transforming ordinary arithmetic numbers into their relativistic images:

$$\mathring{\nu}(C) = \mathring{C}. \tag{34}$$

The inverse transformation has the form

$$\mathring{\tau}(\mathring{C}) = C. \tag{35}$$

Using the functions $\mathring{\nu}(x)$ and $\mathring{\tau}(x)$, we obtain the following formulas for the operations in the field $\dot{\Pi}$:

$$x \mathbin{\mathring{\pm}} y = \mathring{\nu}[\mathring{\tau}(x) \pm \mathring{\tau}(y)], \quad x \circ y = \mathring{\nu}[\mathring{\tau}(x)\mathring{\tau}(y)], \quad x \mathbin{\mathring{:}} y = \mathring{\nu}[\mathring{\tau}(x)/\mathring{\tau}(y)]. \tag{36}$$

The representation of relativistic operations in the form (36) essentially simplifies the verification of properties 1)-23). All these properties hold if the functions $\mathring{\nu}(x)$ and $\mathring{\tau}(x)$ related by condition (32) are mappings of the form (31) and for $x = 0$ and $x = 1$ take the form

$$\mathring{\nu}(0) = \mathring{\tau}(0) = 0, \quad \mathring{\nu}(1) = \mathring{\tau}(1) = 1. \tag{37}$$

The specific form of (30) used to represent the functions $\mathring{\nu}(x)$ and $\mathring{\tau}(x)$ is irrelevant, i.e., the formulas (30) are only one of the possible starting variants for the construction of formulas of the form (36) and thus the corresponding non-Archimedean calculus. Instead of the function $\mathring{\nu}(x)$ (and respectively $\mathring{\tau}(x)$) we can axiomatically define a function of a single argument $\nu(x)$ that satisfies the following conditions:

$$\nu[\tau(x)] \equiv \tau[\nu(x)] \equiv x, \qquad \nu(0) = \tau(0) = 0, \quad \nu(1) = \tau(1) = 1, \tag{38}$$

$$\nu(x) : (-\infty, \infty) \to (-\mathfrak{M}, \mathfrak{M}), \qquad \tau(x) : (-\mathfrak{M}, \mathfrak{M}) \to (-\infty, \infty). \tag{39}$$

The corresponding arithmetic field

$$\Pi_\nu = \{x \pm_\nu y, x \circ_\nu y, x \,\mathring{\circ}_\nu\, y = x/_\nu y\}, \tag{40}$$

where $\pm_\nu$, $\circ_\nu$, $\mathring{\circ}_\nu = /_\nu$ are the operation symbols, is defined by the formulas

$$x \pm_\nu y = \nu[\tau(x) \pm \tau(y)], \quad x \pm_\nu y = \nu[\tau(x)\,\tau(y)], \quad x \mathring{\underset{\nu}{\circ}} y = \nu[\tau(x) / \tau(y)], \tag{41}$$

which preserve the algebraic properties of the arithmetic operations under the mapping (39). This means that instead of starting our construction with the relativistic addition $x \dot{+} y$, we could axiomatically define the function $\mathring{\nu}(x)$ by formula (30). The manipulations would be essentially simplified, but this approach would hide the motivation.

By formulas (39), the operations of the field $\mathring{\Pi}$ are mappings of the form $(-\mathfrak{M}, \mathfrak{M}) \times (-\mathfrak{M}, \mathfrak{M}) \to (-\mathfrak{M}, \mathfrak{M})$. If we set $\mathfrak{M} = \infty$, and choose the function $\nu(x)$ in the form $\nu(x) \equiv x$, then $\tau(x) \equiv x$ and the field (40) becomes an ordinary arithmetic field.

Thus, to construct a non-Archimedean calculus, we can follow a simpler technique instead of axiomatically defining addition, i.e., a function of two variables with the properties of an additive Abelian group. We can axiomatically define a monotone increasing function of a single argument $\nu(x)$ that satisfies the properties (38), (39). This automatically ensures the arithmetic field properties. In other words, formula (40) defines a bundle of fields Π_ν in which every field is defined by specifying the function $\nu(x)$. With $\nu(x) \equiv \mathring{\nu}(x)$, we obtain the relativistic field, and with $\nu(x) \equiv x$ the ordinary field.

To stress the origin of the non-Archimedean calculus from the function $\nu(x)$ (we call this function the calculus axiom), we denote it by $\Xi(-\mathfrak{M}, \mathfrak{M}, \nu(x))$. In particular, $\Xi(-\mathfrak{M}, \mathfrak{M}, \mathring{\nu}(x))$ is the relativistic calculus and $\Xi(-\infty, \infty, x)$ the classical calculus.

4. ALGEBRAIC ISOMORPHISM PRINCIPLE

The above discussion leads to the following proposition: if some identity

$$\varphi(x_1, \ldots, x_n, C_1, \ldots, C_m) \equiv \psi(x_1, \ldots, x_n, C_1, \ldots, C_m), \tag{42}$$

where φ and ψ are formulas obtained by ordinary arithmetic operations on the arguments $x_1, \ldots, x_n$ and the constants $C_1, \ldots, C_m$, holds in the ordinary arithmetic field Π, then by replacing the arithmetic operations of the field Π with the operations of the field $\mathring{\Pi}$ and simultaneously transforming the constants C_i into the constants $C_{i\nu} = \nu(C_i)$ we obtain the identity

$$\varphi_\nu(x_1, \ldots, x_n, C_{1\nu}, \ldots, C_{m\nu}) \equiv \psi_\nu(x_1, \ldots, x_n, C_{1\nu}, \ldots, C_{m\nu}) . \tag{43}$$

For instance, the identities

$$(x-y)^3 \equiv x^3 - 3x^2y + 3xy^2 - y^3, \quad (4xy^3) / 2y \equiv 2xy^2$$

correspond in the relativistic calculus $\Xi\,(-\,\mathfrak{M},\, \mathfrak{M},\, \mathring{\nu}(x))$ to the identities

$$(x \mathbin{\mathring{-}} y)^{\mathring{3}} \equiv x^{\mathring{3}} \mathbin{\mathring{-}} \mathring{3} \circ x^{\mathring{2}} \circ y \mathbin{\mathring{+}} \mathring{3} \circ x \circ y^{\mathring{2}} \mathbin{\mathring{-}} y^{\mathring{3}}, \quad (\mathring{4} \circ x \circ y^{\mathring{3}}) \mathbin{\mathring{:}} (\mathring{2} \circ y) \equiv \mathring{2} \circ x \circ y^{\mathring{2}},$$

where $y^{\dot{2}} = y^1 \circ y^1 = y^{1+1}$, $y^{\dot{3}} = y^{\dot{2}} \circ y^{\dot{2}+1}$.

5. EXTENSION OF THE ALGEBRAIC ISOMORPHISM PRINCIPLE TO THE ENTIRE CALCULUS $\Xi\,(-\,\mathfrak{M},\, \mathfrak{M},\, \nu(x))$.

To ensure that the algebraic isomorphism principle is valid also in the higher orders of calculi, we must replace the operations of the field Π with the operations of the field Π_ν, without changing anything in the classical mathematical constructs. Indeed, only the operations, and not their numerical values, are relevant for the corresponding constructions. For instance, guided by this idea, the non-Archimedean derivative is defined as the limit

$$\frac{d_\nu}{dx}[f(x)] = \lim_{h \to 0} \{[f(x +_\nu h) -_\nu f(x)] \mathbin{\underset{\nu}{\mathring{:}}} h\} . \tag{44}$$

If using (41) we express the non-Archimedean operations in (44) in terms of ordinary operations and take the limit, then we obtain the formula

$$\frac{d_\nu}{dx}[f(x)] = \nu\,[f'(x)\,\frac{\tau'(f(x))}{\tau'(x)}] . \tag{45}$$

In particular, for the relativistic derivative we obtain the formula

$$\frac{\mathring{d}}{dx} f(x) = M\left\{\exp\left(\frac{f'(x)(1-\alpha^2x^2)}{\gamma(1-\alpha^2f^2(x))}\right)\right\} . \tag{46}$$

Non-Archimedean series, integrals, differential equations, etc., are defined similarly.

The algebraic isomorphism principle implies (formal) applicability of the ordinary rules of differentiation and integration and the properties of elementary and special functions. For instance, for the relativistic derivative of the product and the compound function we respectively have

$$\frac{\mathring{d}}{dx}(u \circ v) = \frac{\mathring{d}u}{dx} \circ v \mathbin{\mathring{+}} \frac{\mathring{d}v}{dx} \circ u , \qquad \frac{d_\nu^\circ}{dx}\{f[\varphi(x)]\} = f'^{\circ}_{\nu}\,[\varphi(x)] \mathbin{\underset{\nu}{\mathring{\circ}}} \varphi'^{\circ}_{\nu}(x) .$$

Let us consider the construction of ν-images of functions and constants. The ν-image of the function $\varphi(x_1, \ldots, x_n)$ is the function

$$\varphi_\nu = \nu[\varphi[\tau(x_1), \ldots, \tau(x_n)]] . \tag{47}$$

An alternative notation is $\varphi_\nu = \mathrm{rim}_\nu\, \varphi$ [2]. In particular, if $a = \mathrm{const}$, then we obtain $\mathrm{rim}_\nu\, a = a_\nu = \nu(a)$, and if $\mathfrak{X} = \{a_i\}$ is a set, then $\mathfrak{X}_\nu = \mathrm{rim}_\nu\, \mathfrak{X} = \{\mathrm{rim}_\nu a_i\}$.

Let

$$H = \{\varphi_i(x) : \mathfrak{X}^{n_i} \to \mathfrak{X}^{m_i},\ i \in \mathfrak{I}_0\},$$

where $\mathfrak{I}_0$ is an index set, $\mathfrak{M}(H)$ is the set of all H-realizable functions [15] (i.e., compound functions constructed by H), and

$$H_\nu = \{\mathrm{rim}_\nu \varphi_i(x) : \mathfrak{X}_\nu^{n_i} \to \mathfrak{X}_\nu^{m_i},\ i \in \mathfrak{I}_0\}.$$

Then, as shown in [2], we have the following theorem.

THEOREM 1. If $f \in \mathfrak{M}(H)$ and $f_1 \in \mathfrak{M}(H_\nu)$ are superpositions of the same structure, then $\mathrm{rim}_\nu f = f_1$.

For instance,

$$\overset{\circ}{\mathrm{rim}}_\nu(\sin(2x^2 + 3)) = \sin_\circ(\overset{\circ}{2} \circ x^{\overset{\circ}{2}} \overset{\circ}{+} \overset{\circ}{3}),$$

$$\overset{\circ}{\nu}[\sin(2\overset{\circ}{\tau}{}^2(x) + 3)] = \overset{\circ}{\nu}[\sin[\overset{\circ}{\tau}(\overset{\circ}{\nu}(2\overset{\circ}{\tau}{}^2(x) + 3))]] =$$

$$= \overset{\circ}{\nu}[\sin \overset{\circ}{\tau}[\overset{\circ}{\nu}[\overset{\circ}{\tau}(\overset{\circ}{\nu}(2\overset{\circ}{\tau}{}^2(x))) + \overset{\circ\circ}{\tau\nu}(3)]]] = \sin_\circ[\overset{\circ}{\nu}(2\tau^2(x)) \overset{\circ}{+} \overset{\circ}{3}] =$$

$$= \sin_\circ[\overset{\circ}{\nu}[\overset{\circ}{\tau}(\overset{\circ}{\nu}(2))\overset{\circ}{\tau}(\overset{\circ}{\nu}(\overset{\circ}{\tau}{}^2(x))) \overset{\circ}{+} \overset{\circ}{3}] =$$

$$= \sin_\circ[\overset{\circ}{2} \circ \overset{\circ}{\nu}(\overset{\circ\circ}{\tau\nu}\tau(x)\overset{\circ\circ\circ}{\tau\nu\tau}(x)) \overset{\circ}{+} \overset{\circ}{3}] = \sin(\overset{\circ}{2} \circ x^{\overset{\circ}{2}} \overset{\circ}{+} \overset{\circ}{3}).$$

This theorem extends the algebraic isomorphism principle to all formulas represented in terms of unknown (elementary or nonelementary) functions if their ν-images are defined by (47). For instance, $\exp_\nu x = \nu(\exp \tau(x))$, $\sin_\nu x = \nu(\sin \tau(x))$, etc. The following theorem is also easily proved.

THEOREM 2. We have the equality [2]

$$\frac{d_\nu^n}{dx}\left[\mathrm{rim}_\nu f(x)\right] = \mathrm{rim}_\nu f^{(n)}(x) \quad (n = 1, 2, \ldots). \tag{48}$$

Let D be the system of differentiation rules and $\mathfrak{N}(H, D)$ the extension of the set $\mathfrak{N}(H)$ which includes all the possible products of its elements. (If H is the set of basic elementary functions, arithmetic operations, and constants, then $\mathfrak{N}(H) = \mathfrak{N}(H, D)$. Then by Theorems 1 and 2, $\mathrm{rim}_\nu\, \mathfrak{N}(H, D) = \mathfrak{N}(H_\nu, D_\nu)$, and for any pair of functions (f, f_1), $f \in \mathfrak{N}(H, D)$, $f_1 \in \mathfrak{N}(H_\nu, D_\nu)$, with the same structure we have the equality

$$f_1(x) = \mathrm{rim}_\nu f(x). \tag{49}$$

The sets $\mathfrak{N}(H, D)$ and $\mathfrak{N}(H_\nu, D_\nu)$ are thus algebraically isomorphic. This means that functions from $\mathfrak{N}(H_\nu, D_\nu)$ satisfy the same identities (in form) as functions from $\mathfrak{N}(H, D)$. For instance,

$$\exp_\nu(x +_\nu y) = (\exp_\nu x) \circ_\nu (\exp_\nu y), \quad \sin_\nu(2_\nu \circ_\nu x) = 2_\nu \circ_\nu \sin_\nu x \circ_\nu \cos_\nu x,$$

$$\ln_\circ \exp_\circ x = |x|, \quad (|x|^{p_\nu})^{q_\nu} = |x|^{p_\nu \circ_\nu q_\nu}, \quad \frac{d_\nu}{dx}(\exp_\nu x) = \exp_\nu x,$$

$$\frac{d_\circ}{dx}(\sin_\circ \ln_\circ |\overset{\circ}{2} \circ x| = \cos_\circ \ln |\overset{\circ}{2} \circ x| \circ (1 \overset{\circ}{:} (2 \circ x)) \circ \overset{\circ}{2}$$

and so on.

The transformation rim_ν can be applied to infinite superpositions (series, infinite products, and continuing functions [6]), integrals, and all other constructs containing passage to the limit (for instance, the Newton–Leibniz integral formula holds), i.e., $\mathrm{rim}_\nu \Xi(-\infty, \infty, x) = \Xi(-\mathfrak{M}, \mathfrak{M}, \nu(x))$.

Hence it follows that the transformation rim_ν can be applied in various physical theories and mathematical models containing equations of any type. For instance, the non-Archimedean version of special relativity theory can be obtained by passing in all the formulas of this theory from the operations and operators of the classical calculus $\Xi(-\infty, \infty, x)$ to the corresponding operations and operators of the calculus $\Xi(-\mathfrak{M}, \mathfrak{M}, \nu(x))$.

6. SOME VERSIONS OF NON-ARCHIMEDEAN CALCULI

Alongside the relativistic calculus $\Xi(-\mathfrak{M}, \mathfrak{M}, \mathring{\nu}(x))$, with the axiom $\mathring{\nu}(x)$ (30), we consider a two-parameter family of functions with the parameters ε, δ, $\delta \le 1$:

$$\nu(x, \varepsilon, \delta) = \frac{\mathfrak{M}x\delta + (1-\delta)x^2}{\left(a + \sqrt{x^2 + b^2}\right)\delta + (1-\delta)x}, \tag{50}$$

where $a = \mathfrak{M} - \sqrt{1 + \varepsilon^2(\mathfrak{M}^2 - 1)}$, $b^2 = (\mathfrak{M}^2 - 1)\varepsilon^2$.

If we take the function (50) as the axiom of a non-Archimedean calculus, then for $\delta = 1$ we obtain the non-Archimedean calculi $\Xi(-\mathfrak{M}, \mathfrak{M}, \nu(x, \varepsilon, 1))$ with the greatest number $\mathfrak{M}$, and for $\delta < 1$ calculi of the form $\Xi(-\infty, \infty, \nu(x, \varepsilon, \delta))$. In particular, for $\delta = 0$, $\nu(x, \varepsilon, 0) \equiv x$ we obtain the classical calculus $\Xi(-\infty, \infty, x)$. Figure 1 plots the function $y = \nu(x, \varepsilon, \delta)$ for some values of the parameters ε, δ and the function $y = \mathring{\nu}(x)$.

Changing the values of the parameters ε and δ, we can "deform" the classical calculus while preserving all its formulas (by algebraic isomorphism of structures).

In what follows, alongside the relativistic calculus, we use some specific calculi $\Xi(-\mathfrak{M}, \mathfrak{M}, \nu(x, \varepsilon, 1))$ with the greatest number $\mathfrak{M}$. The corresponding expressions for $\nu(x, \varepsilon, 1)$ and $\tau(x, \varepsilon, 1)$ are defined by the formulas

$$\nu(x, \varepsilon, 1) = \frac{\mathfrak{M}x}{a + \sqrt{x^2 + b^2}},$$
$$\tau(x, \varepsilon, 1) = \frac{x}{\mathfrak{M}^2 - x^2}\left[a\mathfrak{M} + \sqrt{b^2(\mathfrak{M}^2 - x^2) + a^2x^2}\right]. \tag{51}$$

It is easy to see that the functions $\nu(x, \varepsilon, 1)$ and $\tau(x, \varepsilon, 1)$ satisfy conditions (38), (39).

7. THE MAXI CONDITION

We have shown that when the formulas of non-Archimedean calculi with the greatest number $\mathfrak{M}$ are applied to input values not exceeding $\mathfrak{M}$ in absolute value the results never exceed $\mathfrak{M}$ in absolute value. However, if we retain the previous formulas for relativistic operations and operators, then inside these formulas (as a result of intermediate operations) we can obtain values greater than $\mathfrak{M}$. For instance, with $x > 1/2\mathfrak{M}$, $y > 1/2\mathfrak{M}$, the numerator of formula (1) is greater than $\mathfrak{M}$. If (1) is rewritten in the form

$$x \mathring{+} y = \frac{\alpha x + \alpha y}{\alpha(1 + (\alpha x)(\alpha y))}, \tag{52}$$

this can be avoided.

Formulas in which intermediate operations do not produce values greater than $\mathfrak{M}$ are called formulas that satisfy the MAXI condition. For instance, formula (52) satisfies the MAXI condition, while formula (1) does not satisfy this condition. In general, transformation of formulas of the calculus $\Xi(-\mathfrak{M}, \mathfrak{M}, \nu(x))$ to a form that satisfies the MAXI condition is a fairly complicated undertaking. In particular, this is difficult for formulas (13) and (20). Hence it follows that the calculus axiom $\nu(x)$ should be chosen with due regard to this point. We will show that for calculi constructed from the axioms of the one-parameter family (51) the MAXI condition is fairly easy to satisfy. For simplicity, we consider only two extreme cases, when $\varepsilon = 0$ and $\varepsilon = 1$:

$$\nu(x, 0, 1) = \frac{x}{1 - \alpha + |\alpha x|}, \quad \tau(x, 0, 1) = \frac{(1 - \alpha)x}{1 - |\alpha x|}, \tag{53}$$

$$\nu(x, 1, 1) = \frac{x}{\sqrt{1 - \alpha^2 + (\alpha x)^2}}, \quad \tau(x, 1, 1) = \frac{x\sqrt{1 - \alpha^2}}{\sqrt{1 - (\alpha x)^2}}. \tag{54}$$

For axiom (53) the corresponding arithmetic operations have the form

$$x \pm_\nu y = \frac{\alpha x(1 - |\alpha y|) \pm \alpha y(1 - |\alpha x|)}{\alpha[(1 - |\alpha x|)(1 - |\alpha y|) + |\alpha x(1 - |\alpha y|) \pm \alpha y(1 - |\alpha x|)|]},$$

$$x \circ_\nu y = \frac{(1 - \alpha)(\alpha x)(\alpha y)}{\alpha(\alpha - \alpha|\alpha x| - \alpha|\alpha y| + |(\alpha x)(\alpha y)|)}, \tag{55}$$

$$x \underset{\nu}{\overset{\circ}{\circ}} y = \frac{\alpha x(1 - |\alpha y|)\operatorname{sign} y}{(1 - \alpha)|\alpha y| + |\alpha x|(\alpha - |\alpha y|)},$$

and for axiom (54)

$$x \pm_\nu y = \frac{\alpha x\sqrt{1 - (\alpha y)^2} \pm \alpha y\sqrt{1 - (\alpha x)^2}}{\alpha\sqrt{(1 - (\alpha x)^2)(1 - (\alpha y)^2) + (\alpha x\sqrt{1 - (\alpha y)^2} \pm \alpha y\sqrt{1 - (\alpha x)^2})^2}},$$

$$x \circ_\nu y = \frac{\sqrt{1 - \alpha^2}(\alpha x)(\alpha y)}{\alpha\sqrt{\alpha^2 - \alpha^2(\alpha x)^2 - \alpha^2(\alpha y)^2 + (\alpha x)^2(\alpha y)^2}}, \tag{56}$$

$$x \underset{\nu}{\overset{\circ}{\circ}} y = \frac{\alpha x\sqrt{1 - (\alpha y)^2}}{\sqrt{(\alpha y)^2(1 - \alpha^2) + (\alpha x)^2(\alpha^2 - (\alpha y)^2)}}.$$

Since the operations $x \pm_\nu y$, $x \circ_\nu y$, $x \,\dot{:}_\nu\, y$ enter as basic operations all the other operators and elementary functions, the latter can be obtained by replacing $x \pm y$, xy, x/y with $x \pm_\nu y$, $x \circ_\nu y$, $x \,\dot{:}_\nu\, y$.

The proof of the MAXI condition for these operators and functions falls outside the scope of this article.

8. APPLICATIONS

In various areas of science, engineering, and economics we often face a situation when the greatest number $\mathfrak{M}$ (or the numbers $\mathfrak{M}_1$, $\mathfrak{M}_2$) is given. For instance, in banking we can name the sum of all asset values expressed in smallest money units; for velocities, the greatest value is the velocity of light; for temperatures, the least value is $\mathfrak{M}_1 = -273°C$, and so on. We naturally expect that the mathematical apparatus incorporating these constants and dependences in its axioms will be useful for mathematical modeling in the corresponding field. Otherwise, the mathematical axioms may contradict the assumptions and postulates obtained by modeling an object or a phenomenon. For instance, the axioms of the Euclidean geometry were originally viewed as the absolute truth necessary for all mathematicians and physicists [10], but they could not be used to construct the theory of relativity.

One of the unexpected applications of non-Archimedean calculi is the solution of the red shift problem for spectra of distant cosmic objects [5, 11]. The starting point for the analysis is the well-known astronomical fact that in the visible Universe there are no distances exceeding the value $R = 2.576 \cdot 10^{25}$ m. To visualize such a bounded Universe from purely methodological considerations, we describe one of its possible models (without asserting that this is the true model). First, consider the ordinary school globe as a model of the real Earth. It is easy to figure out that the coat of paint on the globe corresponds approximately to a 30-km surface layer of the Earth, which includes all the continents and oceans with their flora and fauna. If we abstract from vertical displacements within this layer, because they are small compared with horizontal displacements of people on the Earth's surface, then we can visualize all life on Earth as existing in a two-dimensional world. The coat of paint on the globe provides the simplest model of this "Universe". Now replace the three-dimensional body (Earth) with a large four-dimensional ball, whose three-dimensional surface is our Universe. Then this Universe is finite, each point may be regarded as the center of the entire Universe, the distance between any two points in this Universe is bounded, and the Universe has no boundary (just as in Nostradamus's statement cited previously: the boundary of the Universe is nowhere, its center is everywhere).

From the existence of an upper bound R_0 on distances it follows that the Archimedean axiom does not hold for distances greater than $R_0/2$. Indeed, if the segment lengths are $a, b > 1/2\ R_0 (a < b)$, then the smaller of the two segments cannot be marked even twice within the physical Universe. Hence it follows that in astronomy the mathematical models of the main physical laws (the Newton–Einstein law, the Maxwell–Lorentz law, the Lorentz transformation, the Minkowski metric) written in the classical calculus may lead to errors for large distances. This is what actually happened in the discussion of the red shift in the spectra of distant cosmic objects. For stationary objects the classical model leads to red shifts that are negligibly small compared to the observed red shifts, and the Doppler effect has been invoked to explain the observed phenomenon by the assumption that the Universe is expanding as a result of the "big bang" which occurred some 15 billion years ago at some point in space. This model has been the foundation for various conjectures of the physical origin of the big bang, the course of subsequent processes, etc. In the end, the expanding universe conjecture came to dominate modern astrophysics [12]. It has been shown in [5, 11, 13], however, that the equations of special relativity theory written in non-Archimedean calculi lead to a red shift in the spectra of stationary objects whose order of magnitude is comparable to the actually observed red shift. The formulas of the relativistic calculus $\Xi(-\mathfrak{M}, \mathfrak{M}, \dot{\nu}(x))$ produce a good fit with the results of thousands of astronomical observations [6, 14, 16], which also support the static universe hypothesis.

A possible application of a calculus with the greatest number is the design of an appropriate computer. The authors have modeled such a computer on an ordinary PC. If $\mathfrak{M}$ is chosen to be less than the machine infinity, then algorithms written in a non-Archimedean calculus with the greatest number will not lead to overflow in calculations. A complex-number version of the non-Archimedean calculus may avoid difficulties with complex values. The question of error buildup remains unclear, especially for operations with algebraic numbers havingt substantially different orders of magnitude. All this requires the development of an appropriate algorithmic and computer support, and also suitable computer experiments and simulations.

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