general remarks concerning differentials can be found. This state of affairs seems to confirm the old prejudice that there exists an abyss between algebra and analysis which cannot be bridged.

To remedy this evil it is absolutely necessary to show that the concept of number can be generalized in such a way as to include the concept of a differential by employing only the concepts upon which algebra is based. The apparatus of concepts of rational semantics permits the complete solution of this problem.

Instead of speaking of functions of a real variable which approaches 0 at the point 0, it is possible to speak of arbitrary sequences of real numbers. Sequences which invariably take on the same value at points sufficiently far from 1, i.e. sequences which beginning with a certain term are equal, can be assembled in separate classes which are called sequential numbers.[1]

Nevertheless in conformity with a remark of Herzberg, sequences will be dealt with directly.

The following definitions are posited :

1. If $E$ is a real number, E is an 0-order number.

2. If $E$ is an $n$-order number, for any value of a natural variable $I$, then the sequence $\bar{I}\, E$ of the values of $E$ is an $(n+1)$-order number.

If $a$, $a_i$, $a_{ik}$ and $a_{ikl}$ are functions of the natural variables $i$, $k$ and $l$, the following are numbers :

$$\bar{\imath}\, a,\ \bar{\imath}\, a_i,\ \bar{k}\,\bar{\imath}\, a_i,\ \bar{\imath}\,\bar{k}\, a_i,\ \bar{l}\,\bar{k}\, a_i,\ \bar{l}\,\bar{k}\, a_{lk},\ \bar{l}\,\bar{k}\,\bar{\imath}\, a_{lki} \ldots$$

The following are likewise numbers :

| Order | Number | The corresponding sequence |
|---|---|---|
| 1 | $\bar{\imath}\, 0$ | $0,\ 0,\ 0, \ldots$ |
| 1 | $\bar{\imath}\, i$ | $1,\ 2,\ 3, \ldots$ |
| 2 | $\bar{\imath}\,\bar{k}\, 0$ | $\bar{k}\, 0,\ \bar{k}\, 0,\ \bar{k}\, 0$ |
| 2 | $\bar{\imath}\,\bar{k}\, k$ | $\bar{k}\, k,\ \bar{k}\, k,\ \bar{k}\, k$ |
| 2 | $\bar{\imath}\,\bar{k}\, i$ | $\bar{k}\, 1,\ \bar{k}\, 2,\ \bar{k}\, 3$ |
| 2 | $\bar{\imath}\,\bar{k}\, \frac{i}{\bar{k}}$ | $\bar{k}\, \frac{1}{\bar{k}},\ \bar{k}\, \frac{2}{\bar{k}},\ \bar{k}\, \frac{3}{\bar{k}}$ |

Numbers which contain an infinity of members all of which are equal to 0 will be called null numbers.

Convention : In these constructions of sequential numbers

[1] Cf. *U. H. M.*, p. 463 ff.



the number 0 will be taken for any expression $\frac{a}{0}$ unless there is an infinity of such expressions. In the latter case the corresponding pattern will not be used at all.

For example $\bar{\imath}\,\frac{1}{i-1}$ is the sequence $0,\ 1,\ \frac{1}{2} \ldots$

$\bar{\imath}\,\frac{1}{\sin\frac{\pi}{2}\, i}$ is a meaningless expression.

The $n$th member of the sequence $X$ will be denoted by $[n]\, X$. For example :

$$[n]\quad \bar{\imath}\, a = a$$
$$[n]\quad \bar{\imath}\, a_i = a_n$$
$$[n]\quad \bar{k}\,\bar{\imath}\, a_{ik} = \bar{k}\, a_{nk}$$

Note that the symbol $[N]\, E$ should not be employed if $E$ contains $N$.

Small latin letters will be employed to denote the real numbers and large latin letters to denote sequences.

## THE DEFINITION OF EQUALITY

I. If $[n]\, X = a$ for almost all values of $n$, i.e. except for a finite number of values of $n$,

$$X = a$$

II. If $[n]\, X = [n]\, Y$ for all values of $n$ then

$$X = Y$$

Examples :

$$a = \bar{\imath}_1\, a = \bar{\imath}_2\,\bar{\imath}_1\, a = \bar{\imath}_3\,\bar{\imath}_2\,\bar{\imath}_1\, a \ldots$$

$$\bar{n}\,\bar{m}\,\frac{m-1}{m(m-1)} = \bar{n}\,\bar{m}\,\frac{1}{m}$$

## DEFINITION OF GREATER AND LESS

I. If $[n]\, A > a$ for almost all values of $n$, then :

$$A > a$$

II. If $a > [n]\, X$ for almost all values of $n$, then:

$$a > X$$

III. If $[n]\, X > [n]\, Y$ for almost all values of $n$, then :

$$X > Y$$

## DEFINITION OF FUNDAMENTAL OPERATIONS

If $X' = X$ and $Y' = Y$, where $X'$ and $Y'$ are of the same order, then

$$X + Y = \bar{n}\,(\,[n]\,X' + [n]\,Y'\,)$$
$$X - Y = \bar{n}\,(\,[n]\,X' - [n]\,Y'\,)$$
$$X \cdot Y = \bar{n}\,(\,[n]\,X' \cdot [n]\,Y'\,)$$
$$\frac{X}{Y} = \bar{n}\,\frac{[n]\,X'}{[n]\,Y'}$$

where $Y$ is not a null number.

For example

$$\bar{\imath}\,i + a = \bar{n}\,(\,[n]\,\bar{\imath}\,i + [n]\,\bar{\imath}\,a\,) = \bar{n}\,[n + a]$$
$$\bar{\imath}\,i \cdot \bar{\imath}\,(i - 1) = \bar{n}\,n\,(n - 1)$$

## DEFINITION OF POSITIVE AND NEGATIVE NUMBERS

Positive numbers are greater than 0 and negative numbers less than 0.

## DEFINITION OF DIFFERENTIALS

Positive numbers which are less than any positive $(n - 1)$-order number are $n$-order differentials.

The $n$-order differentials are denoted by $d_n\,x,\ d_n\,y,\ d_n\,z \ldots$

For example $\bar{\imath}\,\frac{1}{i}$ is a 1-order differential

$\bar{k}\,\bar{\imath}\,\frac{1}{i}$ is a 2-order differential

$\bar{l}\,\bar{k}\,\bar{\imath}\,\frac{1}{i}$ is a 3-order differential

## DEFINITION OF A LIMIT-VALUE OF A SEQUENTIAL NUMBER

The $n$-order number $G$ is an $n$-order limit-value of $A$ if for the differential $d_{n+1}x$

$$-\,d_{n+1}x \leqslant A - G \leqslant d_{n+1}x$$

Then we write:

$$\lim\,(n)\,A = G$$

It can be proved that there are never two different $n$-order limit-values of a given number.



We have, e.g. $\lim\,(0)\,d_n\,x = 0$

$$\lim\,(n + p)\,d_n\,x = d_n\,x$$
$$\lim\,(0)\,a = a$$
$$\lim\,(0)\,\bar{\imath}\,\frac{(x + \frac{1}{i})^2 - x}{\frac{1}{i}} = 2\,x$$

## DEFINITION OF $n$-ORDER FUNCTIONS

The expression $f(X)$ is an $n$-order function of $X$ in the domain $\Omega$, if for any $n$-order value of $\Omega$, $f(X)$ is a uniquely determined $n$-order number.

If the recursive pattern

$$f(X) = n\,f\,([n]\,X)$$

is employed, the following $n$-order elementary functions are obtained:

$$-\,X = \bar{n}\,(-\,[n]\,X)$$
$$|X| = \bar{n}\,|[n]\,X|$$
$$e^x = \bar{n}\,e^{\,[n]\,x}$$
$$\text{(E)}\qquad \log X = \bar{n}\,\log\,[n]\,X$$
$$\sin X = \bar{n}\,\sin\,[n]\,X$$
$$\arcsin X = \bar{n}\,\arcsin\,[n]\,X$$

Also:

$$E(X) = \bar{n}\,E\,([n]\,X),$$

where $E(x)$ is the next integer not greater than $x$.

We have the following definitions for series

$$\sum_{k=1}^{E(X)} f(k, Z) = \bar{n} \sum_{k=1}^{E(\,[n]\,X)} [n]\,f(k, Z)$$

## LIMITS

The following definitions are posited for $n$-order functions.

$$\lim_{Z \leftarrow X} f(X) = \lim\,(n)\,f(Z + d_{n+1}x)$$
$$\lim_{X \rightarrow Z} f(X) = \lim\,(n)\,f(Z - d_{n+1}x)$$

$$\lim_{X \doteq Z} f(X) = \lim_{Z \leftarrow X} f(X) = \lim_{X \rightarrow Z} f(X)$$

$$\lim_{X \doteq \infty} f(X) = \lim (n) f\left(\frac{1}{d_{n+1}x}\right)$$

$$\lim_{X \doteq -\infty} f(X) = \lim (n) f\left(\frac{-1}{d_{n+1}x}\right)$$

for any $d_{n+1}x$.

The derivative and the definite integral are then defined as follows :

$$f^1(X) = \lim_{Z \doteq X} \frac{f(Z) - f(X)}{Z - X}$$

$$\int_A^B f(X)\, dX = \lim_{X \doteq \infty} \sum_{k=1}^{(n)E(B-A)(X)} f\left(A + \frac{k}{X}\right)\frac{1}{X}$$

Thus the definite integral is the limit of an infinite series of infinitely narrow rectangles.

## MULTIPLE SEQUENTIAL NUMBERS

A simple analysis of multiple series and multiple integrals cannot be given in terms of simple sequential numbers. For this purpose *multiple-sequential numbers* which are double, triple, etc., sequences are employed.

## THE FULFILLED LINEAR CONTINUUM OF 1-ORDER NUMBERS

We will begin with the real 1-order numbers $\bar{\imath}\, a$, and $\bar{\imath}\, i$ ; the fundamental operations and the elementary functions $(E)$ will be assumed, where the domain of $X$ in $\sin X$ is limited to the real numbers.

In this way a set of numbers is obtained which is called the elementary continuum of 1-order numbers. Step by step this continuum can be completed by constructing new numbers. With the help of Zermelo's axiom it can be proved that there is a *fulfilled continuum*, i.e. a linear field which cannot be supplemented by a larger linear field.

It should be noted that there is no way of defining a norm of sequential numbers. This conclusion is derived from a



general theorem of the theory of functional operations of Banach,[1] which was proved by Mazur.[2]

## NORMAL 1-ORDER FUNCTIONS

The 1-order function $f(X)$ will be called a *normal function* if

$$f(X) = \bar{n} f_n([n] X)$$

where $f_n(X)$ is a function of a real variable.

The theorems

$$f'(X) = n f_n'([n] X)$$

$$\int_A^B f(X)\, dX = n \int_{[n]A}^{[n]B} f_n(x)\, dx$$

can be proved without any difficulty.

The theorem :

$$\int_0^\infty f(X)\, dX = \bar{n} \int_0^\infty f_n(x)\, dx$$

can also be proved.

## DIRAC'S FUNCTIONS

The 1-order function

$$\frac{1}{\pi} \frac{d_1 a}{X^2 + (d_1 a)^2}$$

denoted by $\delta(X)$ is clearly a normal function. The ordinary calculus of classic analysis can therefore be applied to this function. It will be shown that it is a Dirac's function.

If $X$ is a member of a fulfilled linear continuum, the corresponding value of $\delta(X)$ is also a member of this continuum and $\delta(X)$ has the following properties :

| $X$ | $(X)$ |
|---|---|
| $0$ | $\frac{1}{d_1 a}$ |
| $n\sqrt{d_1 a}$ | $\frac{1}{n^2 + d_1 a}$ |
| $(d_1 a)^{\frac{n-1}{2n}}$ | $d_1 y$ |
| $\infty$ | $0$ |

[1] S. Banach : *Théorie des opérations linéaires*, Warszawa-Lwów, 1932.
[2] S. Mazur : " Sur les anneaux linéaires," *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences*, Paris, Tome 207, 1938, pp. 1025–1027.

The value of $\delta(X)$ at the point 0 is infinite. It decreases to a real number in the infinitely small interval $(0, n\sqrt{d_1 a})$.

In the infinitely small interval $\left(n\sqrt{d_1 a}, (d_1 a)^{\frac{n-1}{2n}}\right)$ it decreases to a differential and it is constantly decreasing along the axis $\left((d_1 a)^{\frac{n-1}{2n}}, \infty\right)$, its limit being 0.

For negative values of $X$ the same values are obtained for $\delta(X)$.

Therefore :

$$\delta'(X) = -\frac{1}{\pi}\frac{2 X d_1 a}{\left(X^2 + (d_1 a)^2\right)^2}$$

and $\delta^{(n)}(X)$ is obtained by the elementary calculus.

By applying this calculus the following equalities may also be obtained :

$$\int_0^A \delta(X)\,dX = \frac{1}{\pi}\arctan\frac{A}{d_1 a}$$

$$\int_0^{\sqrt{d_1 a}} \delta(X)\,dX = \frac{1}{2} - \frac{1}{\pi}\arctan\sqrt{d_1 a}$$

$$\int_0^{\infty} \delta(X)\,dX = \frac{1}{2}$$

$$\int_{-\infty}^{\infty} \delta(X)\,dX = 1$$

## Chapter IX

## PROBLEMS OF THE METHODOLOGY OF THE EXACT SCIENCES

1. The system of rational metamathematics can be regarded as a tool quite similar to a counting machine. The former enables us to obtain results in addition to those obtained by a counting machine. In particular this system can be utilized to obtain expressions which permit the prediction of other expressions at a later stage. While a counting machine permits the prediction of the results of definite experiences, namely of the results of counting, the scope of the experience which can be apprehended by the system of rational metamathematics cannot be established in advance.

The ability to formulate in terms of a pattern experiences which depend upon making spatial measurements was the first advance over a counting machine. In other words the system of rational metamathematics includes the science called geometry.

Geometry is an experimental science. It depends upon the measurement of segments, angles, and areas. The Egyptians conceived it in this way and it has remained essentially the same up to this very day. To-day what is generally regarded as geometry, i.e. what is included in textbooks, is the peculiar mixture of experimental geometry and the geometrical metaphysics which was inherited from the Greeks as Euclid's *Elements*.

Closer consideration of the constructions with which euclidean geometry operates reveals that they are as inaccessible to the imagination and to experience as Cantor's aggregates.[1] The illustrations taken from experience, which are given in textbooks, whether of straight lines, planes, or points differ fundamentally from what is actually meant by these terms. Their only value is that they create the illusion that they explain something. As a result every student is convinced that he knows what a point, straight line, or plane is. He speaks of them with great freedom just as he speaks of grains of sand, the rays of the sun, or the surface of a calm lake. He does not realize that the common properties of both these types of objects are few in comparison with their differences. Once

[1] Cf. 6.5.