Use Model Theory in Nonsmooth Analysis

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Abstract This is a short invitation to using the models of nonstandard set theory in nonsmooth analysis. The techniques of the infinitesimal and Boolean valued versions of analysis are illustrated by the concept of infinitesimal optimality and the operator-valued Farkas Lemma.

1 Introduction

There are significant features distinguishing the modern mathematics whose epoch we count from the turn of the twentieth century.

Mathematics becomes logic. Logic is the calculus of truth and proof. The ideas of mathematical logic has penetrated into many sections of science and technology. Logic organizes and orders our ways of thinking, manumitting us from conservatism in choosing the objects and methods of research. Logic of today is a fine instrument and institution of mathematical freedom. Logic liberates mathematics by model theory. Model theory evaluates and counts truth and proof.

Another crucial circumstance is the universal mathematization of knowledge. Mathematical ideas have trespassed the frontiers of the exact sciences and imbued the humanitarian sphere, primarily, politics, sociology, and economics. Social events are principally volatile and possess a high degree of uncertainty. Economic processes utilize a wide range of the admissible ways of production, organization, and management. The nature of nonunicity in economics transpires: The genuine interests of human beings cannot fail to be contradictory. The unique solution is an oxymoron in any nontrivial problem of economics which refers to the distribution of goods between a few agents. It is not by chance that the social sciences and instances of humanitarian mentality invoke the numerous hypotheses of the best orga-

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nization of production and consumption, the most just and equitable social structure, the codices of rational behavior and moral conduct, etc. The art of decision making has become a science in the twentieth century. The presence of many contradictory conditions and conflicting interests is the main particularity of the social situations under control of today. Management by objectives is an exceptional instance of the stock of rather complicated humanitarian problems of goal agreement which has no candidates for a unique solution.

Nonstandard models of analysis form one of the areas of mathematics that appears in the twentieth century. It still resembles a youngster seeking his ends, means, and opportunities. This article touches a few opportunities that are open up in nonsmooth analysis by model theory. Most attention is paid to the nonstandard concept of infinitesimal optimality and the Boolean valued interpretation of the Farkas Lemma.

2 Infinitesimal Optimality

Let us restrict the discussion of vector optimization to convex problems. Under vector optimization we mean the search of optima under conflicting goals or multiple criteria decision making. Technically this reduces to working in ordered vector space setting.

Assume that X is a vector space, E is an ordered vector space, $E^\bullet := E \cup \{+\infty\}$, $f\colon X \to E^\bullet$ is a convex operator, and $C \subset X$ is a convex set. We define a *vector* (*convex*) *program* to be a pair (C, f) and write it as

$$x \in C, \quad f(x) \to \inf.$$

A vector program is also commonly called a *multiple objective* or *multiple criteria optimization problem*. An operator f is the *objective of the program* and C, the *constraint*. The points $x \in C$ are referred to as *feasible elements*. The above notation of a vector program reflects the fact that we consider the following extremal problem: find a greatest lower bound of the values of f on C. In the case $C = X$ we speak of an *unconstrained problem*.

Constraints in an extremal problem can be posed in different ways, for example, in the form of equation or inequality. Let $g\colon X \to F^\bullet$ be a convex operator, $\Lambda \in L(X,Y)$, and $y \in Y$, where Y is a vector space and F is an ordered vector space. If the constraints C_1 and C_2 have the form

$$\begin{aligned} C_1 &:= \{x \in C : g(x) \leq 0\}, \\ C_2 &:= \{x \in X : g(x) \leq 0,\ \Lambda x = y\}; \end{aligned}$$

then instead of (C_1, f) and (C_2, f) we respectively write (C, g, f) and (Λ, g, f), or more expressively,

$$x \in C,\ \ g(x) \leq 0,\ \ f(x) \to \inf;$$
$$\Lambda x = y,\ \ g(x) \leq 0,\ \ f(x) \to \inf.$$

An element $e := \inf_{x \in C} f(x)$ (if exists) is the *value* of (C, f). It is clear that $e = -f^*(0)$, where $f^* \colon L(X,E) \to \overline{E}$, $f^*(T) = \sup\{Tx - f(x) : x \in C\}$. A feasible element x_0 is an *ideal optimum* or a *solution* if $e = f(x_0)$. Thus, x_0 is an ideal optimum if and only if $f(x_0)$ is the least element of the image $f(C)$, i.e., $f(C) \subset f(x_0) + E^+$.

We can immediately see from the definitions that x_0 is a solution of the unconstrained problem $f(x) \to \inf$ if and only if the zero operator belongs to the *subdifferential* $\partial f(x_0)$, i.e.,

$$f(x_0) = \inf_{x \in X} f(x) \ \ \leftrightarrow \ \ 0 \in \partial f(x_0).$$

The difference between *local* and *global* optima is not essential for us, since we will consider only the problems of minimizing convex operators on convex sets. Indeed, let x_0 be an ideal local optimum for the program (C, f) in the following (very weak) sense: there exists a set $U \subset X$ such that $0 \in \operatorname{core} U$ and

$$f(x_0) = \inf\{f(x) : x \in C \cap (x_0 + U)\}.$$

Given an arbitrary $h \in C$, choose $0 < \varepsilon < 1$ so as to have $\varepsilon(h - x_0) \in U$. Then $z \in C \cap (x_0 + U)$ for $z := x_0 + \varepsilon(h - x_0) = (1 - \varepsilon)x_0 + \varepsilon h$, whence $f(x_0) \leq f(z)$. Hence, $f(x_0) \leq (1 - \varepsilon) f(x_0) + \varepsilon f(h)$ or $f(x_0) \leq f(h)$.

Considering simple examples, we can check that ideal optimum is extremely rare. This circumstance impels us to introduce various concepts of optimality suitable for these or those classes. Among them is *approximate optimality* which is useful even in a scalar situation (i.e., in problems with a scalar objective function).

Fix a positive element $\varepsilon \in E$. A feasible point x_0 is an *ε-solution* of (C, f) if $f(x_0) \leq e + \varepsilon$, where e is the value of the program. Thus, x_0 is an ε-solution of the program (C, f) if and only if $x_0 \in C$ and $f(x_0) - \varepsilon$ is a lower bound of the image $f(C)$, or which is the same, $f(C) + \varepsilon \subset f(x_0) + E^+$. It is obvious that a point x_0 is an ε-solution of the unconstrained problem $f(x) \to \inf$ if and only if zero belongs to $\partial_\varepsilon f(x_0)$, the *$\varepsilon$-subdifferential* of f at x_0:

$$f(x_0) \leq \inf_{x \in X} f(x) + \varepsilon \ \ \leftrightarrow \ \ 0 \in \partial_\varepsilon f(x_0).$$

We call a set $\mathscr{A} \subset C$ a *generalized ε-solution* of the program (C, f) whenever $\inf_{x \in \mathscr{A}} f(x) \leq e + \varepsilon$, where, as above, e is the value of the program. If $\varepsilon = 0$, then we speak simply of a *generalized solution*. Of course, a generalized ε-solution always exists (for instance, $\mathscr{A} = C$); but we however try to choose it as least as possible. An inclusion-minimal generalized ε-solution $\mathscr{A} = \{x_0\}$ is an ideal ε-optimum.

Theorem 1. *Each generalized ε-solution is an ε-solution of some vector convex program.*

PROOF. To demonstrate, we need recalling the concept of canonical operator.

Consider a Dedekind complete vector lattice E and an arbitrary nonempty set $\mathscr{A}$. Denote by $l_\infty(\mathscr{A}, E)$ the set of all order bounded mappings from $\mathscr{A}$ into E; i.e.,

$f \in l_\infty(\mathscr{A}, E)$ if and only if $f\colon \mathscr{A} \to E$ and $\{f(\alpha) : \alpha \in \mathscr{A}\}$ is order bounded in E. It is easy to verify that $l_\infty(\mathscr{A}, E)$, endowed with the coordinatewise algebraic operations and order, is a Dedekind complete vector lattice. The operator $\varepsilon_{\mathscr{A},E}$ acting from $l_\infty(\mathscr{A}, E)$ into E by the rule

$$\varepsilon_{\mathscr{A},E} : f \mapsto \sup\{f(\alpha) : \alpha \in \mathscr{A}\} \quad (f \in l_\infty(\mathscr{A}, E))$$

is the *canonical sublinear operator* relative to $\mathscr{A}$ and E. We write $\varepsilon_{\mathscr{A}}$ instead of $\varepsilon_{\mathscr{A},E}$, when E is clear from the context.

Let $\Delta_{\mathscr{A}} := \Delta_{\mathscr{A},E}$ be the embedding of E into $l_\infty(\mathscr{A}, E)$ which assigns the constant mapping $\alpha \mapsto e$ $(\alpha \in \mathscr{A})$ to every element $e \in E$ so that $(\Delta_{\mathscr{A}} e)(\alpha) = e$ for all $\alpha \in \mathscr{A}$.

Turning back to demonstration, consider the operator $\mathscr{F}\colon X^{\mathscr{A}} \to E^{\mathscr{A}} \cup \{+\infty\}$ acting for $\chi \in X^{\mathscr{A}}$ by the rule $\mathscr{F}(\chi)\colon \alpha \in \mathscr{A} \mapsto f(\chi(\alpha))$ if $\operatorname{im}\chi \subset \operatorname{dom} f$, and $\mathscr{F}(\chi) = +\infty$ otherwise. Let $\chi_0 \in X^{\mathscr{A}}$ and $\chi_0(\alpha) = \alpha$ $(\alpha \in \mathscr{A})$, and suppose without loss of generality that $\mathscr{F}(\chi_0) \in l_\infty(\mathscr{A}, E)$.

Now take $\mu \in \partial\varepsilon_{\mathscr{A}}(-\mathscr{F}(\chi_0))$, where $\varepsilon_{\mathscr{A}}\colon l_\infty(\mathscr{A}, E) \to E$ is the canonical operator. By [1, 2.1.4(3)], we have

$$\mu \geq 0,\ \mu \circ \Delta_{\mathscr{A},E} = I_E,$$
$$\mu \circ \mathscr{F}(\chi_0) = -\varepsilon_{\mathscr{A}}(-\mathscr{F}(\chi_0)) = \inf_{\alpha \in \mathscr{A}} f(\alpha).$$

If $\mathscr{A}$ is a generalized ε-solution then

$$\mu \circ \mathscr{F}(\chi) \geq -\varepsilon_{\mathscr{A}}(-\mathscr{F}(\chi)) = \inf_{\alpha \in \mathscr{A}} f(\alpha) \geq \inf_{x \in C} f(x)$$
$$\geq \inf_{\alpha \in \mathscr{A}} f(\alpha) - \varepsilon = \mu(\mathscr{F}(\chi_0)) - \varepsilon$$

for $\chi \in C^{\mathscr{A}}$. Consequently, χ_0 is an ε-solution of the program

$$\chi \in C^{\mathscr{A}},\ \ \mathscr{F}(\chi) \to \inf.$$

Conversely, if χ_0 is an ε-solution of the last problem then

$$\mu \circ \mathscr{F}(\chi_0) \leq \mu \circ \mathscr{F} \circ \Delta_{\mathscr{A},X}(x) + \varepsilon = \mu \circ \Delta_{\mathscr{A},E} \circ f(x) + \varepsilon = f(x) + \varepsilon$$

for every $x \in C$. Thus, the following relations hold:

$$\inf_{\alpha \in \mathscr{A}} f(\alpha) = \mu \circ \mathscr{F}(\chi_0) \leq \inf_{x \in C} f(x) + \varepsilon,$$

i.e., $\mathscr{A}$ is a generalized ε-solution of the program (C, f). The proof of the theorem is complete.

From what was said above we can conclude, in particular, that $\mathscr{A} \subset X$ is a generalized ε-solution of the unconstrained problem $f(x) \to \inf$ if and only if the following system of equations is compatible:

$$\mu \in L^+(l_\infty(\mathscr{A}, E), E), \quad \mu \circ \Delta_{\mathscr{A},E} = I_E;$$

$$\mu \circ \mathscr{F}(\chi_0) = \inf_{\alpha \in \mathscr{A}} f(\alpha), \quad 0 \in \partial_\varepsilon(\mu \circ \mathscr{F})\,(\chi_0).$$

The above concepts of optimality are connected with the infimum of the objective function on the set of feasible elements, i.e., with the value of the program. The notion of minimal element leads to a principally different concept of optimality.

Here it is convenient to assume that E is a *preordered vector space*, i.e., the cone of positive elements is not necessarily sharp. Thereby the subspace $E_0 := E^+ \cap (-E^+)$, generally speaking, does not reduce to the zero element alone. Given $u \in E_0$, we denote

$$[u] := \{v \in E : u \le v,\ \ v \le u\}.$$

The record $u \sim v$ means that $[u] = [v]$.

A feasible point x_0 is *Pareto ε-optimal* in the program (C, f) if $f(x_0)$ is a minimal element of $f(C)+\varepsilon$, i.e., if $(f(x_0)-E^+)\cap(f(C)+\varepsilon) \subset [f(x_0)]$. In detail, the Pareto ε-optimality of a point x_0 means that $x_0 \in C$ and for every point $x \in C$ the inequality $f(x_0) \ge f(x)+\varepsilon$ implies $f(x_0) \sim f(x)+\varepsilon$. If $\varepsilon = 0$, then we simply speak of the Pareto optimality. Studying the Pareto optimality, we often use the *scalarization method*, i.e., the reduction of the program under consideration to a scalar extremal problem with a single objective. Scalarization proceeds in different ways. We will consider one of the routes.

Suppose that the preorder $\le$ on E is defined as follows:

$$u \le v \quad \leftrightarrow \quad (\forall l \in \partial q)\ lu \le lv,$$

where $q\colon E \to \mathbb{R}$ is a sublinear functional. This is equivalent to the fact that the cone E^+ has the form $E^+ := \{u \in E : (\forall l \in \partial q)\ lu \ge 0\}$. Then a feasible point x_0 is Pareto ε-optimal in the program (C, f) if and only if for every $x \in C$ either $f(x_0) \sim f(x)+\varepsilon$, or there exists a functional $l \in \partial q$ for which $l(f(x_0)) < l(f(x)+\varepsilon)$. In particular, a Pareto ε-optimal point $x_0 \in C$ satisfies

$$\inf_{x \in C} q(f(x) - f(x_0) + \varepsilon) \ge 0.$$

The converse is not true, since the last inequality is equivalent to a weaker concept of optimality. Say that a point $x_0 \in C$ is *Pareto weakly ε-optimal* if for every $x \in C$ there exists a functional $l \in \partial q$ such that $l(f(x)-f(x_0)+\varepsilon) \ge 0$, i.e., if for any $x \in C$ the system of strict inequalities $l(f(x_0)) < l(f(x)+\varepsilon)$ $(l \in \partial q)$ is not compatible. As we can see, Pareto weak ε-optimality is equivalent to the fact that $q(f(x) - f(x_0)+\varepsilon) \ge 0$ for all $x \in C$ and this concept is not trivial only in the case $0 \notin \partial q$.

The role of ε-subdifferentials is revealed, in particular, by the fact that for a sufficiently small ε an ε-solution can be considered as a competitor for a "practical optimum," "practically exact" solution to the initial problem.

The rules for calculating ε-subdifferentials yield a formal apparatus for calculating the limits of exactness for a solution to the extremal problem but do not agree completely with the practical methods of optimization in which simplified rules for "neglecting infinitesimals" are employed.

Let us illustrate this by example. Recall that some cones K_1 and K_2 in a topological vector space X are *in general position* provided that

(1) the algebraic span of K_1 and K_2 is some subspace $X_0 \subset X$; i.e., $X_0 = K_1 - K_2 = K_2 - K_1$;

(2) the subspace X_0 is complemented; i.e., there exists a continuous projection $P\colon X \to X$ such that $P(X) = X_0$;

(3) K_1 and K_2 constitute a nonoblate pair in X_0.

Finally, observe that the two nonempty convex sets C_1 and C_2 are *in general position* if so are their Hörmander transforms $H(C_1)$ and $H(C_2)$.

Theorem 2.[1] *Let $f_1 : X \times Y \to E^\bullet$ and $f_2 : Y \times Z \to E^\bullet$ be convex operators and $\delta, \varepsilon \in E^+$. Suppose that the convolution*

$$f_2 \vartriangle f_1 = \inf\{f_1(x,y) + f_2(y,z) \mid y \in Y\}$$

is δ-exact at some point (x,y,z); i.e., $\delta + (f_2 \vartriangle f_1)(x,y) = f_1(x,y) + f_2(y,z)$. If, moreover, the convex sets $\operatorname{epi}(f_1, Z)$ and $\operatorname{epi}(X, f_2)$ are in general position, then

$$\partial_\varepsilon (f_2 \vartriangle f_1)(x,y) = \bigcup_{\substack{\varepsilon_1 \ge 0,\ \varepsilon_2 \ge 0, \\ \varepsilon_1 + \varepsilon_2 = \varepsilon + \delta}} \partial_{\varepsilon_2} f_2(y,z) \circ \partial_{\varepsilon_1} f_1(x,y).$$

In practice ε is viewed as an actual infinitesimal, which is happily formalized within infinitesimal analysis by A. Robinson and his followers. Model theory suggests the concept of infinitesimal solution within Nelson's theory of internal sets.

Distinguish some downward-filtered subset $\mathscr{E}$ of E that is composed of positive elements. Assuming E and $\mathscr{E}$ standard, define the *monad* $\mu(\mathscr{E})$ of $\mathscr{E}$ as $\mu(\mathscr{E}) := \bigcap\{[0,\varepsilon] \mid \varepsilon \in {}^\circ\mathscr{E}\}$. The members of $\mu(\mathscr{E})$ are *positive infinitesimals* with respect to $\mathscr{E}$. As usual, ${}^\circ\mathscr{E}$ denotes the external set of all standard members of $\mathscr{E}$, the *standard part* of $\mathscr{E}$.

Assume that the monad $\mu(\mathscr{E})$ is an external cone over ${}^\circ\mathbb{R}$ and, moreover, $\mu(\mathscr{E}) \cap {}^\circ E = \{0\}$. In application, $\mathscr{E}$ is usually the filter of order-units of E. The relation of *infinite proximity* or *infinite closeness* between the members of E is introduced as follows:

$$e_1 \approx e_2 \ \leftrightarrow \ |e_1 - e_2| \in \mu(\mathscr{E}).$$

Now

$$Df(x_0) := \bigcap_{\varepsilon \in {}^\circ\mathscr{E}} \partial_\varepsilon f(x_0) = \bigcup_{\varepsilon \in \mu(\mathscr{E})} \partial_\varepsilon f(x_0),$$

which is the *infinitesimal subdifferential* of f at x_0. The elements of $Df(x_0)$ are *infinitesimal subgradients* of f at x_0.

Assume that there exists a limited value $e := \inf_{x \in C} f(x)$ of the program (C, f). A feasible point x_0 is called an *infinitesimal solution* if $f(x_0) \approx e$, i.e., if $f(x_0) \le$

[1] Cp. [1, Th. 4.2.8].

$f(x)+\varepsilon$ for every $x \in C$ and every standard $\varepsilon \in \mathscr{E}$. Using the definition of infinitesimal subdifferential, we see that a point $x_0 \in X$ is an infinitesimal solution of the unconstrained problem $f(x) \to \inf$ if and only if $0 \in Df(x_0)$.

Consider some *Slater regular program*

$$\Lambda x = \Lambda \bar{x}, \quad g(x) \leq 0, \quad f(x) \to \inf;$$

i.e., first, $\Lambda \in L(X, \mathscr{X})$ is a linear operator with values in some vector space $\mathscr{X}$, the mappings $f\colon X \to E^{\bullet}$ and $g\colon X \to F^{\bullet}$ are convex operators (for the sake of convenience we assume that $\operatorname{dom}(f) = \operatorname{dom}(g) = X$); second, F is an Archimedean ordered vector space, E is a standard Dedekind complete vector lattice of bounded elements; and, at last, the element $-g(x)$ with some feasible point x is a strong order unit in F.

Theorem 3.[2] *A feasible point x_0 is an infinitesimal solution of a Slater regular program if and only if the following system of conditions is compatible:*

$$\beta \in L^{+}(F,E), \quad \gamma \in L(\mathscr{X},E), \quad \beta g(x_0) \approx 0,$$
$$0 \in Df(x_0) + D(\beta \circ g)(x_0) + \gamma \circ \Lambda.$$

The models of infinitesimal analysis reside in many places (cp. [2] and the reference within). The complete details of use of infinitesimal analysis in vector optimization are collected in [3].

3 The Boolean Valued Farkas Lemma

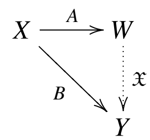
Boolean valued models were invented for simplifying the Cohen method of forcing. We will demonstrate how the technique of these models may be applied to simultaneous linear inequalities with operators. Recall that the Farkas Lemma plays a key role in linear programming and the relevant areas of optimization.

Assume that X is a real vector space, Y is a *Kantorovich space*, i.e., a Dedekind complete vector lattice. Let $\mathbb{B} := \mathbb{B}(Y)$ be the *base* of Y, i.e., the complete Boolean algebra of positive projections in Y; and let $m(Y)$ be the *universal completion* of Y. Let $L(X,Y)$ denote the space of linear operators from X to Y. In case X is furnished with some Y-seminorm on X, by $L^{(m)}(X,Y)$ we mean the *space of dominated operators* from X to Y. As usual,

$$\{T \leq 0\} := \{x \in X \mid Tx \leq 0\}, \qquad \ker(T) = T^{-1}(0) \qquad \text{for} \quad T : X \to Y.$$

[2] Cp. [3, Sect. 5.7].

Kantorovich's Theorem. *Find $\mathfrak{X}$ satisfying*



(1) $(\exists \mathfrak{X})\ \mathfrak{X}A = B \ \ \leftrightarrow \ \ \ker(A) \subset \ker(B)$.
(2) *If W is ordered by W_+ and $A(X) - W_+ = W_+ - A(X) = W$ then*[3]

$$(\exists \mathfrak{X} \geq 0)\ \mathfrak{X}A = B \ \ \leftrightarrow \ \ \{A \leq 0\} \subset \{B \leq 0\}.$$

Alternative Theorem. *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that B and $A_1, \dots, A_N$ belong to $L^{(m)}(X,Y)$.*
Then one and only one of the following holds:
(1) *There are $x \in X$ and $b, b' \in \mathbb{B}$ such that $b' \leq b$ and*

$$b'Bx > 0,\ bA_1x \leq 0,\ \dots,\ bA_Nx \leq 0.$$

(2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))$ such that*

$$B = \sum_{k=1}^{N} \alpha_k A_k.$$

Farkas Lemma. *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume given some dominated operators $A_1, \dots, A_N, B \in L^{(m)}(X,Y)$ and elements $u_1, \dots, u_N, v \in Y$. Assume further that the simultaneous inhomogeneous operator inequalities $A_1x \leq u_1, \dots, A_Nx \leq u_N$ are consistent. Then the following are equivalent:*
(1) *For all $b \in \mathbb{B}$ the inhomogeneous operator inequality $bBx \leq bv$ is a consequence of the simultaneous inhomogeneous operator inequalities $bA_1x \leq bu_1, \dots, bA_Nx \leq bu_N$, i.e.,*

$$\{bB \leq bv\} \supset \{bA_1 \leq bu_1\} \cap \dots \cap \{bA_N \leq bu_N\}.$$

(2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))$ satisfying*

$$B = \sum_{k=1}^{N} \alpha_k A_k; \quad v \geq \sum_{k=1}^{N} \alpha_k u_k.$$

These theorems are obtained by using Boolean valued models of set theory. The latter were invented for simplifying Cohen's final solution of the problem of the

[3] Cp. [1, p. 51].

cardinality of the continuum within ZFC. The honor of creation of these models belong to Scott, Solovay, and Vopěnka.[4]

Takeuti coined the term "Boolean valued analysis" for applications of the models to analysis. Scott wrote in 1969:[5] "We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is, do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good argument."

In 2009 he added:[6] "At the time, I was disappointed that no one took up my suggestion. And then I was very surprised much later to see the work of Takeuti and his associates. I think the point is that people have to be trained in Functional Analysis in order to understand these models. I think this is also obvious from your book and its references. Alas, I had no students or collaborators with this kind of background, and so I was not able to generate any progress."

Boolean valued models reveal that each mathematical result has many interpretations that are invisible from the start. For instance, all L_p spaces may be considered as subspaces of the reals in a suitable Boolean valued model. Some details may clarify the matter.

Let $\mathbb{B}$ be a complete Boolean algebra. Given an ordinal α, put

$$V_\alpha^{(\mathbb{B})} := \{x \mid (\exists \beta \in \alpha)\ x\colon \operatorname{dom}(x) \to \mathbb{B}\ \&\ \operatorname{dom}(x) \subset V_\beta^{(\mathbb{B})}\}.$$

The *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ is

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(\mathbb{B})},$$

with On the class of all ordinals.

The truth value $[\![\varphi]\!] \in \mathbb{B}$ is assigned to each formula φ of ZFC relativized to $\mathbb{V}^{(\mathbb{B})}$. The phrase "*$x$ satisfies φ inside* $\mathbb{V}^{(\mathbb{B})}$" means that $[\![\varphi(x)]\!] = \mathbb{1}$, with $\mathbb{1}$ the top of $\mathbb{B}$. Application of the so-called "Frege–Russel–Scott trick" makes the Boolean valued universe *separated*: $x = y \leftrightarrow [\![x = y]\!] = \mathbb{1}$ for all $x, y \in \mathbb{V}^{(\mathbb{B})}$.

The *descent* $x{\downarrow}$ of $x \in \mathbb{V}^{(\mathbb{B})}$ is defined as

$$x{\downarrow} := \{t \mid t \in \mathbb{V}^{(\mathbb{B})}\ \&\ [\![t \in x]\!] = \mathbb{1}\}.$$

The class $x{\downarrow}$ is a set. If x is a nonempty set inside $\mathbb{V}^{(\mathbb{B})}$ then

$$(\exists z \in x{\downarrow})\ [\![(\exists t \in x)\ \varphi(t)]\!] = [\![\varphi(z)]\!].$$

The *ascent* functor acts in the opposite direction.

There is an object $\mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$ modeling $\mathbb{R}$, i.e.,

$$[\![\mathscr{R} \text{ is the reals}]\!] = \mathbb{1}.$$

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[4] Cp. [7].

[5] Cp. [5].

[6] Letter of April 29, 2009 to S. S. Kutateladze.

Let $\mathscr{R}{\downarrow}$ be the descent of the carrier $|\mathscr{R}|$ of the algebraic system

$$\mathscr{R} := (|\mathscr{R}|, +, \cdot, 0, 1, \leq)$$

inside $\mathbb{V}^{(\mathbb{B})}$. Implement the descent of the structures on $|\mathscr{R}|$ to $\mathscr{R}{\downarrow}$ as follows:

$$\begin{aligned} x+y=z &\leftrightarrow [\![x+y=z]\!]=\mathbb{1}; \\ xy=z &\leftrightarrow [\![xy=z]\!]=\mathbb{1}; \\ x\leq y &\leftrightarrow [\![x\leq y]\!]=\mathbb{1}; \\ \lambda x=y &\leftrightarrow [\![\lambda^{\wedge}x=y]\!]=\mathbb{1} \quad (x,y,z\in\mathscr{R}{\downarrow},\ \lambda\in\mathbb{R}). \end{aligned}$$

Gordon Theorem.[7] *$\mathscr{R}{\downarrow}$ with the descended structures is a universally complete vector lattice with base $\mathbb{B}(\mathscr{R}{\downarrow})$ isomorphic to $\mathbb{B}$.*

This beautiful result makes it possible to derive the Farkas Lemma for operators by descending the classical version from a suitable Boolean valued universe to vector space environment. The complete details on Boolean valued analysis are collected in [6] and [7]. About application to nonsmooth analysis cp. [8] and [9].

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[7] Cp. [7, p. 349].