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CREDENDA OF NONSTANDARD ANALYSIS

S. S. Kutateladze

Abstract. The principal set-theoretic credos of nonstandard analysis are presented. A "naive" justification of the infinitesimal techniques and an overview of the corresponding formal apparatus are provided. The axioms of Nelson's internal set theory are discussed as well as those of the external set theories by Hrbaček and Kawai.

The beginning of the sixties was marked with an outstanding achievement of A. Robinson, the creation of nonstandard analysis. For a long time nonstandard analysis was considered to be a sophisticated and even exotic logical technique appropriate mostly for justifying the method of actual infinities. It was also assumed that this technique has limited applicability and in any case could never lead to a serious reexamination of the general mathematical ideas. In the late seventies, after the publication of the Internal Set Theory by E. Nelson [74] (and somewhat later—of the External Set Theories by K. Hrbaček [62] and T. Kawai [67]) the views of the place and role of nonstandard analysis in mathematics were radically enriched and exchanged. In the light of the new conceptions the nonstandard elements may now be treated as intrinsic parts of all current mathematical objects rather than some "imaginary, surd and ideal entities" added to the traditional sets for the sake of formal convenience. The view becomes prevailing that every set is composed of standard and nonstandard elements. Alongside, the standard sets form a dense grid in the world of all objects studied in mathematics. It was also discovered that the objects usually encountered in nonstandard mathematical analysis (the monads of filters, the standard parts of numbers and vectors, the operator shadows, etc.) form "Cantorian" sets that cannot be found in any of the canonical pictures delineated by the conventional formal set theories. The *von Neumann universe does not exhaust the world of classical mathematics*—this is obviously the first and foremost implication of the new theories.

An important advantage of the new approaches lies in their axiomatic character which makes it possible to pursue them without preliminarily mastering the techniques of ultraproducts, Boolean-valued models or similar devices. The axioms are simple to use and clearly motivated within the framework of the "naive" set theory so typical of and attractive to mathematical analysis. At the same time these axioms considerably expand the world of mathematical objects, open up opportunities

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for developing new formal methods, allow us to substantially reduce the dangerous gaps between the notions, methodology and levels of rigor current in mathematics and in the applications of mathematics to the natural and social sciences.

In 1947 K. Gödel remarked: "There might exist axioms so abundant in the verifiable consequences, shedding so much light upon the whole discipline and furnishing such powerful methods for solving given problems (and even solving them, as far as that is possible, in a constructivistic way), that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well established theory" [58, p. 521].

The purpose of this survey is to make the new paths to nonstandard analysis more accessible. In order to achieve this we aim at two specific goals. The first is to provide an account of qualitative notions of the standard and nonstandard objects as well as of the principles of nonstandard analysis on a "naive" level quite sufficient for effective application of the technique with no need for appealing to logical formalisms. The second goal is to supply the reader with a concise but complete reference to axiomatic foundation of nonstandard analysis in the Cantorian framework, viz. to the theories of E. Nelson, K. Hrbaček and T. Kawai. (Concerning an alternative approach to nonstandard analysis see the works of P. Vopenka and his school [9].)

These two tasks determined the plan of this paper. The first few sections contain some historical data as well as qualitative motivations for the principles of nonstandard analysis and a discussion of their most elementary consequences for differential and integral calculuses, all forming a "naive" foundation for the infinitesimal techniques. The formal details of the corresponding nonstandard set-theoretic machinery are presented in the concluding sections. "Mathematical analysis is not a completed theory as some people imagine it to be, with stable principles from which we only have to derive new corollaries. [...] Mathematical analysis is not different from any other science, with its ideas flowing circularly rather than rectilinearly so that sometimes we return to the ideas seemingly old but always newly elucidated [30, p. 389].

The list of references contains, as a rule, only the sources quoted directly in the text as well as the papers containing extensive bibliographies on the areas. For example, a comprehensive synopsis of the specific research involving the modern infinitesimal techniques can be found in [14, 18, 41, 52, 53, 55, 56, 59–61, 64, 68, 69, 72, 73, 75, 76, 78–80].

The principal idea of this paper was conceived in the fall of 1984 under the influence of the author's discussions of nonstandard analysis with Academicians A. D. Alexandrov, Ya. B. Zel′dovich, S. S. Kutateladze, M. A. Styrikovich, and Yu. G. Reshetnyak (then a Corresponding Member of the Soviet Academy of Sciences). The exposition is based on a transcript [19] of the course that I gave at the University of Novosibirsk. I am deeply grateful to all of my colleagues whose interest and critical remarks helped to improve the paper.

AN EXCURSUS INTO THE HISTORY OF MATHEMATICAL ANALYSIS

1.1. G. W. Leibniz and I. Newton. The differential and integral calculuses have an ancient name "infinitesimal analysis." Such was the title of the first textbook on mathematical analysis published in 1696. The textbook was written by H. de l'Höpital as a result of his contacts with I. Bernoulli, one of the outstanding disciples of G. Leibniz.

"Of all the theoretic feats of knowledge none could be considered as high a triumph of the human spirit as the invention of infinitesimal analysis in the second half of the 17th century. Were it possible to find somewhere a pure and sole achievement of the human spirit, this it would be"—this is F. Engels's appraisal of the new theory [50, p. 582].

The history of the creation of mathematical analysis, the work and relationships of its founders have been studied in detail and even scrutinized (cf. [68, 80]). Here it will be sufficient to turn to the G. W. Leibniz and I. Newton own accounts of their perception of infinitesimals.

G. W. Leibniz's paper in the Leipzig journal *Acta Eruditorum* of 1684 was the first publication on differential calculus. Here Leibniz gave the following definition of differential.

Considering a curve YY and a tangent at a fixed point Y on the curve which corresponds to a coordinate X on the axis AX and denoting by D the intersection point of the tangent and axis, Leibniz wrote: "Now some straight line selected arbitrarily is called dx while another one whose ratio to dx is the same as of ... y ... to XD will be called ... dy or difference (*differentia*) ... of y ...." The essential details of the picture accompanying this text are reproduced here (see Fig. 1).

Therefore, according to G. W. Leibniz for a function $x \mapsto y(x)$ at a point x for an arbitrary dx we have $dy := (YX/XD)dx$, i.e., the differential is defined as the corresponding linear mapping! It was surely clear to G. W. Leibniz that, in order to justify this algorithm of differential calculus (the name that he used for differentiation rules), one should explicate the notion of tangent, he wrote: "... to find the tangent means to draw a line that connects two points on the curve at the infinitely small distance or the continual side of a polygon with infinitely many angles which for us takes the place of the curve." In other words, G. W. Leibniz had based his calculus on the structure of a curve "in the small."

At that time there were two accepted views of the status of infinitely small quantities. The first which was closer to Leibniz treated an infinitesimal as smaller than every "definable" quantity. This conception was accompanied with such images as actually existing "indivisible" elements of which quantities and figures are composed. For I. Newton, the other founder of analysis, the notion of *infinite smallness* was associated primarily with "*vanishing*" *quantities* [38, 44]. The famous "method of prime and ultimate ratios" is formulated in his classical treatise "Principia Mathematicae" as follows: "The quantities and the ratios of quantities which in any finite time converge continuously to equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal" [44, p. 101]. In the process of developing the ideas that are now firmly associated with the theory of limits, I. Newton demonstrated foresight and wisdom in evaluation of the competing views. He wrote: "... to institute an analysis after this manner in finite quantities and investigate the prime or ultimate ratios of these finite quantities when in their nascent or evanescent state is consonant to the geometry of the ancients, and I was willing to show that in the method of fluxions there is no necessity of introducing figures infinitely small into geometry. Yet the analysis may be performed in any kind of figure whether finite or infinitely small, which are imagined similar to the evanescent figures, as likewise in the figures, which, by the method of indivisibles, used to be reckoned as infinitely small provided you proceed with due caution" [38, p. 169].

The views of G. W. Leibniz were just as flexible and dialectical. In his well-known letter to Varignon [44] he stressed that "it is unnecessary to make mathematical analysis depend on metaphysical controversies" and pointed out the unity of the existing perceptions of the objects of the new formalism: "... if any opponent tries to contradict this propositions it follows from our calculus that the error will be less than any possible assignable error since it is in our power to make this incomparably small magnitude small enough for this purpose, in as much as we can always take a magnitude as small as we wish. Perhaps this is what you mean, Sir, when you speak on the inexhaustible and the rigorous demonstration of the infinitesimal calculus which we use. ... So it can also be said that infinitesimals are grounded so that everything in geometry and even in nature takes place as if they were perfect realities. Witness not only our geometrical analysis of transcendental curves but also by my law of continuity in virtue of which we may consider rest as infinitely small motion (i.e., as equivalent of a particular instance of its own contradictory), coincidence as infinitely small distance, equality—as the limit of inequalities, etc."

1.2. K. Marx on the mystical differential calculus. Unfortunately, the requirement that the new techniques possess a sound base that was characteristic for the quoted papers of Leibniz and Newton was not assimilated by their followers who helped to create an additional mystical aura over the already nontrivial and abstract ideas. It will be sufficient to note that the mentioned textbook by l'Höpital declares: "The infinitely small part whereby a variable quantity is continually increased or decreased is called the differential of that quantity." Clearly this is a large step backwards in comparison with the original definition given by G. Leibniz. It is not accidental that on getting familiar with analysis of the 17th century K. Marx called it the "mystical differential calculus." In some interpretations of Marx's memoirs [35] his critique of actual infinities is overdramatized. We will quote verbatim one of the relevant fragments in order to clarify the details.

"Thus there is no choice but to imagine that the increments of the variable h are infinitely small and possess an *independent existence*, e.g. in symbols $\dot{x}$, $\dot{y}$, etc. or dx, dy [etc.]. But the infinitely small as well as infinitely large quantities are quantities too (infinitely [small] only means indefinitely small); thus these dy, dx, etc. or $\dot{x}$, $\dot{y}$, [etc.] participate in the computations just as ordinary algebraic quantities, and in the equation

$$(y+k)-y \quad \text{or} \quad k = 2x\ dx + dx\ dx$$

the term $dx\ dx$ is as valid as $2x\ dx$. But the most remarkable is the reasoning with the help of which this term is deleted exactly because the notion of infinitely small is relative; $dx\ dx$ is deleted because it is infinitely small compared to dx, and thus to$2x\ dx$ and $2x\dot{x}$. Or: if in

$$\dot{y} = \dot{u}z + \dot{z}u + \dot{u}\dot{z}$$

[the addend] $\dot{u}\dot{z}$ is deleted due to its infinite smallness with respect to $\dot{u}z$ or $\dot{z}u$, this could be mathematically justified by recalling that we see $\dot{u}z + \dot{z}u$ as having an approximate value that is imagined to be infinitely close to the exact one. Similar maneuvers can be found in the ordinary algebra. But in this case we witness a still greater miracle: by using this method we can obtain the values of the derivative function at x that are not approximate but exact (or at least symbolically correct, as above), like in the example $\dot{y} = 2x\dot{x} + \dot{x}\dot{x}$. After deleting $\dot{x}\dot{x}$ one obtains

$$\dot{y} = 2x\dot{x},$$

$$\frac{\dot{y}}{\dot{x}} = 2x$$

which is the correct first derivative of x^2, as [the theorem on] the binomial has already proved.

But this miracle is not a miracle at all. On the contrary, it would be a miracle if the deletion of $\dot{x}\dot{x}$ *would not lead to the correct result. Indeed, we are omitting the computation error* that is an unavoidable corollary of the method that allows to introduce an indefinite increment of the variable, for example, h, immediately as the differential dx or $\dot{x}$, as an operational symbol and thus in a similarly direct manner obtain the differential calculus as a separate computational technique that is different from the ordinary algebra" [35, pp. 151–153].

1.3. L. Euler. In the history of mathematical analysis the eighteenth century is rightfully called the Euler century [5, 22, 23]. Everybody who at least briefly thumbs through Euler's textbooks [46–48] will be impressed by his virtuoso technique and the depth of understanding the subject. We should mention here his systematic approach to the study of mathematical problems which was characteristic of him, in his research he utilized every method that has been developed by then. His effective use of the infinitesimal concepts, and above all, of the actual infinitely large and infinitely small numbers, should be especially emphasized. L. Euler gave a detailed account of his methods which were called the "calculus of zeros." As we intend to show later (3.3, 4.12, 4.14), the common opinion that L. Euler gave an "incorrect foundation" for analysis is erroneous.

1.4. G. Berkeley. The ideas of analysis in their general form are tightly woven into the general atmosphere of the 18th century. The ideas of infinitely small or infinitely large quantities can be found in the books of that time; e.g., "Travels of Lemuel Gulliver" (G. Swift, 1726)—Liliputia and Brobdingneg, or the famous "Micromegas 1752" (F. Voltaire). It is worthy to note that A. Robinson included a quote from Micromegas ([8, p. 91]) in the epigraph to [76].

In 1734 G. Berkeley published his famous pamphlet "The Analyst or a Discourse addressed to an infidel mathematician (in fact, to the astronomer E. Galley—S. K.), wherein it is examined whether the object, principles and inferences of the modern analysis are more distinctly conceived, or more evidently deduced, than religious mysteries and points of faith" [3, pp. 396–422]. Despite the antihuman implication of G. Berkeley's works they demonstrate finesse of observation as well as murderous precision of expression. "... An error can give birth to a truth although not to a science"—this is the leitmotif of his critique of calculus.

V. I. Lenin revealed G. Berkeley's poisonous plans in the following words: "Let us then assume that the external world, the nature is a 'combination of sensations' induced in our mind by the deity. Admit this, refuse to seek the 'basis' for these sensations outside conscience, outside man—and I shall admit (within the frame of my idealistic theory of cognition) modern natural sciences complete with the importance and truth of their conclusions. This frame is exactly what I need for my conclusions in favor of 'the world and religion.' This is Berkeley's position" [26, p. 22]. G. Berkeley's challenge could not, of course, be left unanswered by the best brains of the 18th century.

1.5. J. D'Alembert. A turning-point in the history of the formation of the basic notions of analysis is related to the ideas and activity of J. D'Alembert. He was one of the organizers and principal authors of the immortal masterpiece "The Encyclopedia, or the Lexicon of Sciences, Arts and Trades," and he wrote in the entry for "Differential": "Newton never considered the differential calculus to be the calculus of infinitesimals, he saw it as the method of prime and ultimate ratios" [44, p. 157]. J. D'Alembert became the first mathematician who claimed the possession of the proof that the infinitesimals exist neither in nature nor in the mind of geometers. J. D'Alembert's position reflected in "The Encyclopedia" may most primarily be blamed for the fact that by the end of the 18th century an infinitely small quantity was usually perceived as a vanishing variable.

1.6. B. Bolzano, A. Cauchy, and K. Weierstrass. The 19th century became the time when analysis had been acquiring a firm foundation in the theory of limits. An important contribution to the process was made by B. Bolzano, A. Cauchy, and K. Weierstrass. The influence of these scholars is reflected in every course in differential and integral calculuses. The new canon of rigor proposed by B. Bolzano [65]; the definition by A. Cauchy of an infinitesimal quantity as a vanishing variable; finally, the ε-δ-technique of K. Weierstrass are part and parcel of mathematics as well as the modern culture as a whole.

1.7. N. N. Luzin. The beginning of the 20th century was marked with growing mistrust of the concept of infinitesimal. The trend became stronger as mathematics was rebuilt on the set-theoretic foundation which won the key position in the thirties.

In the first edition of the Soviet Encyclopedia N. N. Luzin wrote: "As far as a constant nonzero infinitely small quantity is concerned, contemporary mathematical analysis, while not rejecting the formal possibility of defining the idea of a constant infinitely small quantity (e.g., as a corresponding segment of "non-Archimedean geometry), considers the idea utterly fruitless since it appears as impossible to introduce the notion into calculus" [30, pp. 293–294]. N. N. Luzin's attitude to infinitesimals is worth special treatment as an important evidence of a dramatic history usually characteristic of genuinely great ideas. N. N. Luzin had a rare ability of deep penetration into the core of most sophisticated mathematical problems as well as the exceptional gift of foresight [23, 28]. Moreover, the idea of infinitely small quantities was very appealing to him psychologically. He stressed: "... our consciousness could never be successfully purged of them. There must be some deeply concealed reasons not fully understood yet which make our mind susceptible to taking them seriously" [30, p. 396]. In another paper N. N. Luzin remarked with profound sorrow: "When the mind begins its acquaintance with analysis, in other words, in its springtime, it begins with the infinitesimals that could be called the "elemental" quantities. But gradually, step by step, as it accumulates knowledge, theories, as it is satiated with abstractions and gets tired, the mind begins to forget the early aspirations and smile down at their "childishness." In short, when the autumn comes, the mind allows one to be convinced that the only true foundation is constructible by means of limits" [7]. N. N. Luzin vigorously defended the last view in his textbook "Differential calculus," pointing out that "no constant quantity can be infinitely small, as well as no number, nor matter how small it is. Hence it would be much more correct to use the term "infinitely decreasing" instead of "infinitely small" since the former expresses the idea of variability in a more explicit manner" [29, p. 61].

1.8. A. Robinson. The seventh (posthumous) edition of the just-mentioned book by N. N. Luzin was printed in the same year of 1961 when A. Robinson published his "Nonstandard Analysis" containing a modern foundation for the method of actual infinitesimals (cf. [39, 76, 77]). A. Robinson's work was based on the local theorem of A. I. Mal′tsev. It is this theorem that he characterized as a "result of fundamental importance for our theory" [76, p. 13], directly citing A. I. Mal′tsev's paper [33] which was published in 1936. The discovery of A. Robinson clarifies the ideas of the founders of differential and integral calculuses and provides a new confirmation of the dialectical nature of the development of mathematics.

THE NOTION OF SET IN NONSTANDARD ANALYSIS

2.1. Modern textbooks on mathematical analysis are based on the notion of set. Nonstandard analysis or, more precisely, *nonstandard mathematical analysis is a part of mathematical analysis*. Therefore, it obviously assumes the usual view of sets (cf., for example, [13]). In other words, nonstandard analysis accepts as sets exactly those collections that are treated as such in the classical theory. Note that the converse statement is also true: nonstandard analysis does not accept as sets those and only those collections that are not accepted as sets in the "standard" mathematics. At the same time *nonstandard analysis uses a refined view of sets*, i.e., *it is built within a nonstandard set-theoretic framework*.

2.2. The naive set theory is based on the classical perceptions of G. Cantor: set is "any many which can be thought of as one, that is, every totality of definite elements which can be united to a whole through a law," or a set is "every collection into a whole of definite distinct objects of our perception or our thought" [15, p. 173]. It is now common knowledge that these conceptions are too general. This fact is circumvented via more precise formulation of the distinctions between sets and nonsets. For example, the unacceptable, or "too large" collections of sets are termed "classes," assuming that a class should not necessarily be a set. In other words, in the process of formalization of the notions of naive set theory we define in a more complete and careful fashion the procedures that enable us to introduce a "Cantorian" set into mathematical usage. All the sets accepted as such in mathematics possess equal rights. Of course, this does not mean that all sets are equal or have no distinctions. They only that they are of the same type and have the same status—they are the "elements of the class of all sets."

2.3. The decisively new principal assumption underlying the nonstandard set theory is extremely simple: *sets can be different*: *standard or nonstandard*. Therefore it is more correct to speak the theory of standard and nonstandard sets rather than of the nonstandard set theory. The intuitive interpretation of the phrase "set A is standard" is that A is clearly and unambiguously described and has thus become an "artefact" of the human cognitive activity. This notion of "standardness" draws a separating line between the objects determined via explicit mathematical constructions (for example, existence and uniqueness theorems),—they are the *standard sets*, and the objects that arise in the course of study in an implicit way, the *nonstandard sets*.

The numbers π, e, $\sin 81$ are defined directly and unambiguously; the sets of natural and real numbers are described in a concrete and clear fashion. These objects are standard. On the other hand, an arbitrary "abstract" real number is defined implicitly as an element of the already-introduced set of all real numbers. This procedure of defining new objects is very common: a vector is an element of a vector space, a filter is a set of subsets of the given set possessing certain properties, etc. Thus there are standard and nonstandard real numbers, standard and nonstandard vectors or filters, and, generally speaking, standard and nonstandard sets.

Consider the example of all grains of sand on Earth. Archimedes in his classical treatise *Psammiths, the Sand-Reckoner* said that: "... the numbers named by me and given in the work which I sent to Zeuxippes some exceed not only the number of the grains of sand equal in magnitude to the earth filled up in the way described but also that of a mass equal in magnitude to the *universe*" [2, p. 358].

2.4. Of course, this view of the difference between the standard and nonstandard sets is of an auxiliary importance if all we want to learn is how to work with them.

This is similar to the situation in geometry where the distinctive and intuitive notions of spatial forms help us in developing the skills of using the axioms that eventually present the rigorous definitions of points, lines, planes, etc. Following A. D. Alexandrov, we should note that "all axioms themselves need a meaningful foundation, they are only a summary of other knowledge and a starting point for the logical construction of a theory" [1, p. 51]. Therefore, *before formal introduction of the axioms of nonstandard set theory we are to discuss them on a qualitative level.*
2.5. As we already know, nonstandard set theory begins with the primary observation that sets may be viewed as different, standard or nonstandard. Furthermore, we assume the following postulates (more precisely, some variants of the following postulates).

2.6. The transfer principle. *Each ordinary statement proving existence of a set at the same time defines some standard set.*

In other words, the existence and uniqueness theorems of classical mathematics are considered as direct and explicit definitions of mathematical objects. An equivalent formulation of this principle which also explains its name is as follows: *in order to prove a statement for all sets we should only prove it for all standard sets.* The transfer principle has an intuitive foundation in the obvious fact that we usually make statements on arbitrary sets while dealing only with already-described or standard sets.

2.7. The idealization principle. *Each infinite set contains a nonstandard element.*

This statement agrees obviously with the conventional understanding of infinity. The idealization principle will often appear in stronger forms that reflect the concept of the infinite profusion of ideal objects. For example, sometimes all standard sets are assumed to be elements of a certain finite set. The number of elements of this "universal" set is enormous and, which is more important, unrealizable, thus—nonstandard. It is not surprising that the universal set itself is also nonstandard.

Note that caution *should be used with caution* exercised with these two postulates (like with anything else, although). By transfer a standard set is unambiguously defined by its standard elements in an environment consisting of standard sets only. But this set is not generally reducible to the collection of its standard elements. There can exist other, nonstandard, sets that contain all standard elements of the original set and no other standard elements. One should also use prudently the notion of *statement*. The transfer principle is valid for the mathematical propositions that do not appeal to the new property of sets—to be or not to be standard. Otherwise we would be able to state that all sets are standard (since all standard sets are standard), which contradicts the idealization principle. Therefore, a sentence saying that a set is standard is not an "ordinary statement."

2.8. The standardization principle. *Each standard set and each property define a new standard set, the subset of the original set such that all its standard elements possess the given property.*

Let A be the standard set and φ, a property. The standardization principle claims that there exists a standard set, usually denoted ${}^*\{x \in A\colon \varphi(x)\}$, such that

$$y \in {}^*\{x \in A : \varphi(x)\} \Leftrightarrow y \in \{x \in A : \varphi(x)\} \Leftrightarrow y \in A \;\&\; \varphi(y)$$

for each standard y. The set ${}^*\{x \in A : \varphi(x)\}$ is often referred to as *standardization* and the parameters used in its definition are often omitted. An intuitive foundation for the standardization principle is as follows: if we have explicit descriptions of mathematical objects, we can also work with the new sets constructed from the given objects by using some procedures. *Standardization supplements the conventional manner of indicating subsets by selecting elements with an a priori given property.* When reflecting on the standardization principle it is useful to observe that it says nothing of the nonstandard elements of the new set. This is not accidental: such elements need not possess the property. It should also be observes that care must be exercised in applying the standardization principle. An attempt to arbitrarily standardize the universal set that contains all standard sets leads to a contradiction immediate.

2.9. These postulates are placed as the ground for the axiomatic presentations of nonstandard set theory. We shall treat them in more detail in Sections 5–7. Right now we will discuss the properties of the elementary objects of mathematical analysis within the above-presented "naive" framework.

SIMPLE PROPERTIES OF STANDARD AND NONSTANDARD REAL NUMBERS

3.1. For a set A we write $a \in {}^\circ A$ instead of the proposition "a is a standard element of A."

3.2. *The following statements are true:*

(1) *the induction principle is valid for the standard natural numbers, i.e., if A is a set such that $1 \in A$ and for $n \in {}^\circ\mathbb{N}$ it is true that $n \in A \Rightarrow n+1 \in A$, then A contains all standard natural numbers; symbolically, ${}^\circ\mathbb{N} \subset A$;*

(2) *all elements of a standard finite set are standard;*

(3) *if all elements of a set are standard then the set is standard and finite;*

(4) *a natural number N is nonstandard if and only if N is greater than each standard natural number;*

(5) *for an infinite standard A the symbol ${}^\circ A$ does not denote a set.*

◁ Let us prove (1) and (5) since other statements are easier to verify.

(1) The following standard set can be formed on using the standardization principle: $B := {}^*\{n \in N : n \notin A\}$. Suppose that $B \neq \varnothing$. Then B contains (by the transfer principle) the smallest standard element m. By assumption $m \neq 1$ (since $1 \in A$). Besides, $m \notin A$; therefore, $m-1 \notin A$.

Since $m-1 \in {}^\circ\mathbb{N}$, we have $m-1 \in B$, and the false inequality $m-1 = m$ appears.

Therefore, $B = \varnothing$; i.e., $(\forall n \in {}^\circ\mathbb{N})\ n \in A$.

(5) Suppose that ${}^\circ A$ is a set. Then it follows from (3) that ${}^\circ A$ is finite and standard; therefore, $A = {}^\circ A$ by the transfer principle and A is finite, contradicting the assumption. ▷

3.3. Due to 3.2 (4) the nonstandard natural numbers are called *infinitely large* or simply infinite. As L. Euler pointed out, "... an infinite number and a number greater than each definable number are synonymous" [47] (cf. [6]). The infinitude of a number N is denoted by the symbol $N \approx \infty$.

3.4. *The following facts are true:*

(1) $(N \approx \infty\ M \approx \infty) \Rightarrow (N+M \approx \infty,\ NM \approx \infty)$*;*

(2) $(N \approx \infty,\ n \in {}^\circ\mathbb{N}) \Rightarrow (N+n \approx \infty,\ N-n \approx \infty,\ nN \approx \infty)$*;* $(N \approx \infty,\ M \geq N) \Rightarrow M \approx \infty$*;*

(3) *"... if 1/0 denotes an infinite number, then since 2/0 is of course only doubled 1/0* (by definition!—**S. K.**) *then clearly every number, even an infinite number, can become two or several times greater" (L. Euler [35, p. 620]).*

3.5. Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ be the extended real line. The element $t \in \overline{\mathbb{R}}$ is called *finite* or *limited* (denoted $t \in {}^{\approx}\mathbb{R}$) if there exists a standard $n \in {}^{\circ}\mathbb{N}$ such that $|t| \le n$. For $t \notin {}^{\approx}\mathbb{R}$ such that $t > 0$ we use the notation $t \approx +\infty$; $t \approx -\infty$ is interpreted in a similar manner. The following convention is often used for the infinite numbers: $t \approx +\infty \Leftrightarrow t \in \mu(+\infty)$, or in a verbal form: the "element t belongs to the monad of $+\infty$." The number $t \in \mathbb{R}$ is called (*actually*) *infinitely small*, or simply *infinitesimal*, if $|t|$ is less than any strictly positive standard number, in other words, if $|t| \le 1/n$ for every $n \in {}^{\circ}\mathbb{N}$. In this case we say that t belongs to the *monad* of zero (denoted as $t \approx 0$ or $t \in \mu(\mathbb{R})$. (The symbol $\mu(\mathbb{R})$ is used interchangeably with $\mu(0)$, stressing the obvious connection with the unique separated vector topology on $\mathbb{R}$.) The term monad [[$\mu o \nu \alpha \zeta$]] has a long history and is traditionally translated in the classical texts as "unit," which is not quite correct. By the definition of Euclid, a monad "is [that] through which a magnitude is considered as a whole" [37, p. 9].

3.6. *The following assertions hold:*

(1) $(s \approx 0, t \approx 0) \Rightarrow s + t \approx 0$*;*

(2) $(s \in {}^{\approx}\mathbb{R}, t \approx 0) \Rightarrow st \approx 0$*;*

(3) $z \approx 0 \ \Leftrightarrow\ 1/z \approx +\infty$ *(for* $z > 0$*)*

"... if z becomes a quantity that is less than any quantity that can be given, i.e., infinitely small, then the ratio $1/z$ *should become greater than any quantity that can be given, i.e., infinitely large" (L. Euler [47, p. 93]);*

(4) $(t \approx 0$ *and* t *is standard)* $\Rightarrow t = 0$.

$\triangleleft$ (1) Suppose that $n \in {}^{\circ}\mathbb{N}$. Then obviously $|s\ | \le 1/2n$ and $|t| \le 1/2n$. Hence $|s + t| \le |s| + |t| \le 1/2n + 1/2n = 1/n$, i.e., $s + t$ is infinitely small.

The other statements are verified just as easily. $\triangleright$

3.7. *The monad* $\mu(\mathbb{R})$ *is not a set.*

$\triangleleft$ If the statement is false then $\mu(\mathbb{R})$ is a subset of $\mathbb{R}$. For any $t > 0, t \in {}^{\circ}\mathbb{R}$ we have $t > \mu(\mathbb{R})$. Therefore, $t = s := \sup \mu(\mathbb{R})$. The number s is of course infinitely small and positive, besides, $2s \le s$. The latter is impossible. $\triangleright$

3.8. If $s - t \approx 0$ for $s,\ t \in \mathbb{R}$, it is denoted as $s \approx t$ and we say that s and t are *infinitely close*. The founders of mathematical analysis if necessary did not distinguish between the numbers such that their difference is infinitesimal. L. Euler expressed this in the following words: "... an infinitesimal quantity is exactly zero" [47, p. 91].

3.9. A useful observation: the relation of infinite closeness of numbers cannot be called a subset of $\mathbb{R} \times \mathbb{R}$. Indeed, otherwise the image of the zero element under the relation, i.e. the monad $\mu(\mathbb{R})$ would also be a set. Note also that the monad $\mu(\mathbb{R})$ is "indivisible," i.e., for each standard n it is true that $1/n\mu(\mathbb{R}) = \mu(\mathbb{R})$. While reflecting on the role of the monad $\mu(\mathbb{R})$ in the construction of integers we can turn to the Euclid definition: "Number is a set that is produced by monads" (cf. [37, p. 9]). Similarly, the whole "nonstandard" extended real line $\overline{\mathbb{R}}$ and even its finite part ${}^{\approx}\mathbb{R}$ are sets of monads. A more rigorous formulation of this statement is based on the following fundamental fact.

3.10. *To each finite number there is a unique standard number infinitely close to it.*

$\triangleleft$ By the standardization principle for a given $t \in {}^{\approx}\mathbb{R}$ we can construct the standard set $A := {}^{*}\{a \in \mathbb{R} : a \le t\}$. Clearly $A \ne \varnothing$ and $A \le n$ where the standard number $n \in {}^{\circ}\mathbb{N}$ is such that $-n \le t \le n$. By the transfer principle we can conclude that $A \le n$. Due to completeness of $\mathbb{R}$ it is true that $s := \sup A \in \mathbb{R}$. Obviously s

is standard. Let us show that $s \approx t$. Otherwise for a standard $\varepsilon > 0$ we would get $|s - t| > \varepsilon$. If $s \geq t$ then $s \geq t + \varepsilon$, i.e., $s \geq a + \varepsilon$ for any standard $a \in A$. But in that case $s \geq s + \varepsilon$ which is not true. The other possibility $s < t$ leads to an equally quick contradiction: we would have $t > s + \varepsilon$ and again $s \geq s + \varepsilon$. ▷

3.11. A standard number that is infinitely close to a finite number $t \in {}^{\approx}\mathbb{R}$ is called the *standard part of* t and denoted $\mathrm{st}(t)$, or $^{\circ}t$. It is assumed that $^{\circ}t := \pm\infty$, if $t \approx \pm\infty$.

3.12. Thus the extended line of reals in nonstandard analysis should be thought of in connection with the following schema (Fig. 2). When we point out a (finite) number t at the axis, we draw a blob t, the monad $\mu(t) := t + \mu(\mathbb{R})$, the "indivisible faithful image of $^{\circ}t$." If we examine a neighborhood of t under a powerful microscope, we shall see a cloud with fuzzy edges which represents the image of $\mu(t)$. If we use more powerful lenses, the visual portion of the "point-monad" will appear in more detail with part of it outside of the field of observation. But we shall still be dealing with the same standard real number that is in a way described by this process of "studying the microstructure of the physical line."

3.13. *The following statements are true:*

(1) *for* $s, t \in {}^{\circ}\mathbb{R}$ *it is true that*

$$^{\circ}(s + t) = {}^{\circ}s + {}^{\circ}t, {}^{\circ}(st) = ({}^{\circ}s)({}^{\circ}t),$$

$$s \leq t \Rightarrow {}^{\circ}s \leq {}^{\circ}t;$$

(2) *The law of transition from a standard number to its standard part is not a set (and, in particular, a function).*

◁ Prove, for example, (2). If the law $t \mapsto {}^{\circ}t$ were a set then the monad $\mu(\mathbb{R})$ would be a set too since $t \in \mu(\mathbb{R} \Leftrightarrow {}^{\circ}t = 0$. What is now left is to apply 3.7. ▷

DISCUSSION OF THE INITIAL NOTIONS OF ANALYSIS ON THE REAL LINE

4.1. The nonstandard criteria for limits. *For a standard sequence* (a_n) *and a standard number* $a \in \overline{\mathbb{R}}$ *the following assertions hold:*

(1) *the number* a *is a partial limit of* (a_n) *if and only if* $a = st(a_N)$ *for an infinitely large* N.

(2) *the number* a *is the limit of* (a_n) *if and only if* a_N *is infinitely close to* A *for all infinitely large* N, *i.e.* $a = \lim a_n \Leftrightarrow (\forall N \approx +\infty)\ a_N \approx a$.

◁ Let us verify, for example, (2). Suppose first that $a_n \to a$ and $a \in \mathbb{R}$. By assumption for each positive number $\varepsilon > 0$ and some $n \in \mathbb{N}$ we have $|a_N - a| \leq \varepsilon$ whenever $N \in \mathbb{N}$ and $N = n$. Therefore, it follows from the transfer principle that for any standard $\varepsilon > 0$ there exists a standard n with the same property. Each infinitely large N is larger than n; i.e., $a_N \approx a$.

Assume now that for $N \approx +\infty$ a is known to belong to $\mu(a)$. Suppose that $a = -\infty$ and $n \in {}^{\circ}\mathbb{N}$. From 3.4 (2) and the definition of monad we conclude that if $N \geq M$, where M is an infinite number, then $a_N \leq -n$. Thus we have proved "something" for any standard n, namely $(\exists M)(\forall N \geq M) a_N \leq -n$. By transfer this "something" is valid for arbitrary $n \in \mathbb{N}$, i.e., $a_n \to -\infty$. ▷

4.2. Criteria 4.1 possess the following merits. We have seen that the partial limits of a standard sequence are the definable numbers corresponding to all infinite indexes. In other words a *partial limits is the observable value of an infinitely far member of the sequence.* This statement has a clear intuitive content and differs drastically from the conventional definition (an instructive discussion of the latter by N. N. Luzin can be found in [31, pp. 98–99]).

Criterion 4.1 (2) remarkably grasps the dynamic nature of the limit which is superbly described by R. Courant [20, pp. 66–67] who asserted however that it does not accept an exact mathematical formulation. At the same time, the nonstandard criterion for limit is applicable to standard sequences only (if $a_n := N/n$, where $N \approx +\infty$, then $a_n \to 0$ and $a_N = 1$). In other words 4.1 complements the classical notions of limit but does not reject them. Furthermore, by pointing out all convergent standard sequences we at the same time automatically define the standard set of all convergent sequences by the standardization principle. Thus, the *traditional ε-N-constructions and the nonstandard formula are closely related*.

It is useful to stress that in specific applications (e.g., in physics) we encounter "real," explicitly defined, i.e., standard sequences. Moreover, in such situations the "infinite" has a clear (physical) meaning—the corresponding scales and bounds are indicated directly. Keeping in mind additionally that the existence problems are solved in practice by turning to meaningful arguments (if there is no physical velocity then we need not look for it), one encounters the problem of recognizing the limit that is known to exist. Nonstandard analysis offers a simple prescription: "take an entry in your sequence with an (arbitrary) infinite index; the number defined by this element is the sought limit." This also makes clearer the foundations of the infinitesimal methods employed by the forefathers of differential and integral calculuses who sought answers to questions on exact numerical values of standard quantities: the areas of specific figures, the equations of tangents to curves, the integrals of explicitly given analytic expressions, etc.

4.3. An important new contribution of nonstandard analysis is the notion of the limit of a *finite sequence* $a[N] := (a_1, \dots, a_N)$, where N is an infinitely large natural number. The following definition is based on an intuitive idea that agrees nicely with the practical procedures for obtaining numerical characteristics of unobservable discrete collections: the thermodynamic parameters of a fluid or gas, the estimates for the population demand, etc.

4.4. A number a is called the *microlimit* of a sequence $a[N]$ if $a_M \approx a$ for all infinite M less than N. Let us demonstrate how this notion relates to the conventional notion of convergence.

4.5. *Let (a_n) be a standard (countable) sequence, $N \approx +\infty$ and $a \in {}^{\approx}\mathbb{R}$. Then the following statements are equivalent:*

(1) *${}^\circ a$ is the microlimit of $a[N]$;*

(2) *the sequence (a_n) converges to ${}^\circ a$.*

$\lhd$ The implication (2) $\Rightarrow$ (1) is contained in 4.1 (2). In order to prove (1) $\Rightarrow$ (2) choose an arbitrary standard $\varepsilon > 0$ and consider the set $A := \{m \in \mathbb{N} : (\forall n)(m \leq n \leq N) \Rightarrow \mid a_n - {}^\circ a \mid \leq \varepsilon \}$. The set A is nonempty since $N \in A$. Therefore A contains the least element m. If $m \approx +\infty$ then $m - 1 \approx +\infty$ and by assumption $m - 1 \in A$, with m thus being standard. Besides, if $n \geq m$ and n is standard then $n \leq N$ and $\mid a_n - {}^\circ a \mid \leq \varepsilon$. Thus, $(\forall \varepsilon \in {}^\circ\mathbb{R},\ \varepsilon > 0)(\exists m \in {}^\circ\mathbb{N}) n \geq m \Rightarrow |a_n - {}^\circ a \mid \leq \varepsilon$, and we can conclude from the transfer principle that (a_n) converges to ${}^\circ a$. $\rhd$

4.6. The nonstandard criterion for continuity. *Let f be a standard real-valued function and x, a standard point in its standard domain. The following statements are equivalent:*

(1) *f is continuous at x;*

(2) *f maps the points that are infinitely close to x into points infinitely close to $f\ (x)$; i.e., f is microcontinuous in x.*

4.7. In discussing this nonstandard criterion the argument of 4.2 can be repeated. Following R. Courant one can also note that "as in the case of the limit of a sequence, the definition by Cauchy is based, so to speak, on reversing the intuitively acceptable order in which the variables are considered" [20, p. 73]. Nonstandard analysis liberates us from the unpleasant necessity of reversing quantifiers for all available (standard) functions and points. At the same time the complete ε-δ-definition can be restored only implicitly from microcontinuity at a point by standardization. The following statements will help us to achieve a deeper understanding of microcontinuity (cf. [11, 53, 75]).

4.8. *A standard function f is microcontinuous at every (possibly nonstandard) point if and only if f is uniformly continuous.*

4.9. *A standard set consists of functions microcontinuous at every point if and only if the set is (uniformly) equicontinuous.*

4.10. Let y be a standard function defined in a neighborhood of a standard point x and differentiable at this point. Furthermore, let dx be an arbitrary nonzero infinitely small number. Following G. W. Leibniz, we denote by dy the differential of the function y at the point x, which is applied to the element dx.

4.11. *The following relationships hold:*

$$dy \approx 0, \quad dy \approx y(x+dx) - y(x),$$

$$\frac{dy}{dx} \approx \frac{y(x+dx)-y(x)}{dx}.$$

4.12. The nonstandard relationships of 4.11 supplement the following remark of L. Euler: "As for the differential calculus, I have already noted that the problem of finding differentials should be understood in a relative rather than absolute sense; this means that if y is a function of x, we need to find not the differential itself but its ratio to the differential dx. Indeed, since all differentials are equal to zero, $dy = 0$ for any function y of the quantity x; thus we cannot hope to find more in the absolute sense. The correct formulation of the question is as follows: x receives an infinitely small, i.e., vanishing increment dx; we should find the ratio of the corresponding increment of the function y to dx. Although both increments $= 0$, there is a certain relation between them that is found in differential calculus" [48, p. 9].

Note that L. Euler uses the symbol $=$ where we write $\approx$ (cf. 3.8), and is looking for the derivative that he assumes to exist while working with specific differentiable functions. Under these conditions it is perfectly legitimate to use an arbitrarily chosen infinitesimal dx. Repeating F. Engels's maxim, one could say: "…dx is infinitely small but active and produces everything" [50, p. 580].

4.13. The nonstandard representation of the Riemann integral. *Let $f : [a,b] \to \mathbb{R}$ be a standard continuous function and $a = x_1 < x_2 < \dots < x_N < x_{N+1} = b$, a partition of $[a,\ b], \xi_k \in [x_k,\ x_{k+1}]$, and $x_k \approx x_{k+1}$ for $k := 1, \dots, N$. Then*

$$\int\limits_a^b f(x)dx = {}^\circ\left(\sum_{k=1}^{N} f(\xi_k)(x_{k+1} - x_k) \right).$$

◁ Observe that N is infinitely large and use the definition of integral and the nonstandard criteria 4.1 (2) and 4.8. ▷

4.14. Proposition 4.13 provides a formal base for the understanding of integration as a specific variant of the conventional summation. Indeed, it turns out that *in order to find the integral of a standard continuous function one should compute the exact value* (= the standard part) *of just one finite sum* of an infinitely large number of infinitely small summands. We recall the definition of integral (with "variable upper limit") given by L. Euler:

"Integration is usually defined in the following manner: one says that it is the summation of all values of the differential expression $X\ dx$ when x assumes all values differing by the increment dx, beginning from a given value up to x; the increment dx should be assumed infinitely small .... It is clear that the proposed method at least enables us to obtain integration from summation with arbitrary precision; the exact value can be computed only if one assumes that the increments are infinitely small, i.e., zeros" [48, p. 163].

Note that the technique described in 4.13 is not generally applicable to arbitrary nonstandard functions. In other words, in this case we discover again that the *nonstandard theories of the objects of mathematical analysis extend and refine their classical analogs rather than expel them.*

4.15. Due to the listed reasons it can be said that *nonstandard analysis is a direct descendant from infinitesimal calculus.* This is why the term "infinitesimal analysis" has been gaining acceptance. It is worth to note that the *concept of infinitely large or infinitely small numbers has never disappeared from the tool-kit of natural sciences but only was absent from mathematics during the last thirty years.* Therefore we need not dwell on applications and importance of nonstandard analysis in more detail.

NELSON'S INTERNAL SET THEORY

5.1. After having discussed on a "naive" level the distinctions between the standard or definable and nonstandard or implicit ways of introducing objects we are now able to fill the concepts that we propose under the founders of mathematical analysis with an intuitively clear meaning and so to acquire a deeper understanding of their mode of reasoning. At the same time we encounter serious problems even in the simplest situations. First of all, it is still not clear how to distinguish between standard and nonstandard sets; therefore, we must be aware of the danger of incorrect application of the principles of nonstandard analysis. We are increasingly disturbed by the appearance of objects formed in a manner seemingly acceptable although these objects might not be considered as ordinary sets. These are various monads, collections of standard elements, the objects $\approx, {}^{\approx}\mathbb{R}$, etc. Even more annoying is the fact that the "mathematical law" $x \mapsto \mathrm{st}(x)$ acting from $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$ is not a function. Incidentally, the notion of function has formed long before the set-theoretic approach was taken. As early as in 1774 L. Euler wrote: "When certain quantities depend on other quantities in such a manner that after changing the latter they themselves are changed—the former quantities are called functions of the latter. The scope of this name is very wide; it encompasses all procedures determining how one quantity is determined from certain others. Thus ... all quantities that depend on x in any way, i.e., are determined by the x, are called functions of it" [47, p. 38]. This dynamic concept of transforming one object into another is not adequately expressed in the presently prevailing stationary view of a function as a set. The last view is a "formal set-theoretic *model* for the intuitive idea of function—the model which reflects only one aspect of this idea but not its general meaning" [10, p. 32]. The reader recalls that for $s, t \in [0, 1]$ we have: ${}^{\circ}(s+t) = {}^{\circ}s + {}^{\circ}t, {}^{\circ}0 = 0$, ${}^{\circ}1 = 1$ and also $\mathrm{st}(x) = 0$ in an interval $[0, h]$, where h

is a positive number (an arbitrary actual infinitely small number). The existence of such a "numerical function" is an indication of the presence of antinomies. All these circumstances call for immediate and explicit refinement of the concepts and constructs used as well as for description of their foundation.

5.2. We have already noted that nonstandard analysis is grounded on the set-theoretic principles. In other words, the *naive nonstandard set theory can be based on the same foundation as the current Cantorian theory* [15], or, to be precise, the axiomatic set theories that "approximate it from below" (cf. [16, 18, 21, 34, 45, 71]).

Note that analysis is the "science of the infinite" (according to G. W. Leibniz) or the "mathematics of the infinite" (according to F. Engels) and is thus firmly tied to set theory (via the notion of infinity). Nevertheless, we should not forget that the classical works of G. Cantor appeared 200 years after differential calculus was discovered. *The most essential parts of the modern mathematics rest on set theory.* To be more precise, now there is a set-theoretic foundation laid under its "living quarters." Only time will show what is to happen. Instantly we are to state that the construction of the mathematical edifice continues, causing an endless quest for new ideas and severe clashes of opinions (cf. [4, 9, 17, 40, 57, 66, 70]). It should be noted here that only when one realizes the unfeasibility of a final and "absolute" foundation of nonstandard analysis it should he or she launches into studying the implementation of the project.

5.3. Axiomatic theories carefully describe correct procedures for set formation. Qualitatively speaking, they describe worlds, or *universes*, of sets that are expected to adequately reflect our intuitive perception of the "Cantor paradise," the universe of naive set theory.

5.4. The Zermelo-Fraenkel theory is mostly often used in analysis (cf. [16, 18, 43, 45, 71]). We shall now briefly discuss some of its notions accentuating the necessary details. The discussion will be continued on the same current and common level of rigor (which is inevitable), in particular, we shall use the assignment operator definor, $:=$ for introducing abbreviations and skip the accompanying subtler points.

5.5. The *alphabet of Zermelo-Fraenkel theory* (abbreviated as ZF or ZFC) consists of: symbols of variables; letters; the parentheses; the propositional connectives $\&, \neg, \Rightarrow, \Leftrightarrow, \wedge$; the quantifiers $\forall, \exists$, the sign of equality $=$ and the special binary predicate of "containment" $\in$. The domain of values for the variables of ZF is thought of as the *world of sets*. In other words, sets are the only objects in the universe of ZF. The relation $\in(x, y)$ is usually written as $x \in y$ and pronounced "*x is an element of y.* "

5.6. The *formulas of* ZF are defined in a usual manner, i.e., as the finite texts formed from the atomic formulae $x = y$ and $x \in y$ (where x and y are ZF variables) by using parentheses, quantifiers and connectives. The terms *free and bound variables* (or, equivalently, the scope of a quantifier) retain their usual meaning. Hereafter if we want to stress that the variables $x_1, \dots, x_n$ (and only these variables) are free in the formula φ, we shall write $\varphi = \varphi(x_1, \dots, x_n)$ or just $\varphi(x_1, \dots, x_n)$.

5.7. Let $\varphi = \varphi(x)$ be a formula of ZF. We write $y \in \{x : \varphi(x)\}$ instead of $\varphi(y)$, i.e., $y \in \{x : \varphi(x)\} := \varphi(y)$. In this case we say that y possesses the *property* φ or *y lies in the class* $\{x : \varphi(x)\}$. In this sense the concepts of property, formula and class are equivalent in ZF. Note that the *classification symbols* {and} do not belong to the alphabet of ZF. The following abbreviations are often used within ZF:

$$V := \{x : \ x = x\} \text{—\textit{the class of all sets}};$$

$$\{x : \ \varphi(x)\} \in V := (\exists z)(\forall y)\varphi(y) \ \Leftrightarrow \ y \in z;$$

$$\varnothing := \{x : \ x \neq x\} \text{—}\textit{the empty class};$$

$$x \subset y := (\forall z) z \in x \Rightarrow z \in y := x \textit{ is a subset of } y;$$

$$\mathcal{P}(x) := \{z : \ z \subset x\} \text{—}\textit{the class of all subsets of } x;$$

$$f : x \to y := f \text{—}\textit{a function from } x \textit{ into } y.$$

5.8. The axioms of ZF include the usual logical principles of a first-order prepositional calculus with equality and the following special postulates:

(1) the *axiom of extensionality*—

$$(\forall x)(\forall y)(x \subset y \ \& \ y \subset x) \Rightarrow x = y;$$

(2) the *axiom of union*—

$$(\forall x) \cup x \in V;$$

(3) the *axiom of powerset*—

$$(\forall x) \mathcal{P}(x) \in V;$$

(4) the *axiom schema of replacement*—

$$(\forall x)(((\forall y)(\forall z)(\forall w) \varphi(y, z) \in \varphi(y, \ w) \Rightarrow z = w) \Rightarrow \{v \colon (\exists \ y \in x) \varphi(y, v)\} \in V)$$

for every formula φ;

(5) the *axiom of foundation*—

$$(\forall x \neq \varnothing)(\exists y \in x) y \cap x = \varnothing;$$

(6) the *axiom of infinity*—

$$(\exists \omega)(\varnothing \in \omega \ \& \ (\forall x \in \omega) x \cup \{x\} \in \omega);$$

(7) the *axiom of choice*—

$$(\forall F)(\forall x)(\forall y)$$

$$((x \neq \varnothing) \ \& \ F : x \to \mathcal{P}(y)) \Rightarrow ((\exists f)(f : x \to y \ \& \ (\forall z \in x) f(z) \in F(z)))).$$

5.9. On the basis of these axioms we can form a precise understanding of the class of all sets V as of the "*von Neumann universe*." The initial object in the construction is the empty set. An elementary step in the introduction of the new sets is to form the union of powersets of the existing sets. The transfinite repetition of such steps provides the *world of sets* V, i.e.,

$$V = \bigcup_{\alpha \in \mathrm{Od}} V_\alpha,$$

$$V_\alpha := \{x : (\exists \beta \in \alpha) x \in \mathcal{P}(V_\beta)\},$$

where Od is the class of all ordinals. We can view an arbitrary class as an object external to V, a collection of elements of the von Neumann universe that possess a fixed set-theoretic property defined by a ZF formula. Therefore a class consisting of some elements of a set is (by the axiom of replacement) a set too. There exists an extension of ZF, the *Gödel-Bernays theory*, which provides formal means for working with classes. It should be stressed that any set in this theory is introduced as an element of a class. Therefore, *neither* ZF *nor the Gödel-Bernays theory contain monads as objects.*

5.10. This discussion of the properties of standard and nonstandard sets showed that the von Neumann universe can accommodate all infinitely small numbers but not the whole collection of them. Therefore, *neither the world of Zermelo-Fraenkel sets, nor the world of Gödel-Bernays classes do exhaust the universe of "naive" sets.* In other words each of these theories (intended for description of the objects of classical mathematics) defines a proper part of the "Cantorian paradise." In order to stress this important fact the nonstandard set theory calls the elements of the von Neumann universe *internal sets.* Thus an internal set is synonymous to a ZF set. A convenient formal foundation for nonstandard analysis is offered by *Internal Set Theory*—IST, developed by E. Nelson in [74].

5.11. The *alphabet of* IST consists of the alphabet of ZF with one additional symbol for the unary predicate St expressing the property "to be a standard set." In other words, the texts of IST can contain fragments like $\mathrm{St}(x)$, or more verbosely "x is a standard set." Thus, the domain of the variables of IST is the von Neumann universe separated into standard and nonstandard sets.

5.12. The *formulas of* IST are defined via the usual procedure. The atomic formulae may include the texts like $\mathrm{St}(x)$ where x is a variable. Each formula of ZF is also a formula of IST; in symbols: $\varphi \in (\mathrm{ZF}) \Rightarrow \varphi \in (\mathrm{IST})$. The formulae of ZF are called *internal*, while the formulae of IST that are not formulae of ZF—*external* (or strictly external). This distinction leads naturally to the notions of *internal and external classes.* So if φ is an external formula of IST then the text $\varphi(y)$ is described as: "y is an element of the external class $\{x : \varphi(x)\}$." The external classes of elements of an internal set are called *external sets* or external subsets of the set. Note that an internal class composed of elements of an internal set is an internal set too.

IST uses some additional conventions together with the conventional abbreviations of ZF. Here follow some of them:

$$V^{\mathrm{St}} := \{x : \mathrm{St}(x)\} \text{—}\textit{the class of standard sets};$$

$$x \in V^{\mathrm{St}} := x \textit{ is standard} := (\exists y)\mathrm{St}(y) \ \&\ y = x;$$

$$(\forall^{\mathrm{St}} x)\varphi := (\forall x)(x \textit{ is standard } \Rightarrow \varphi);$$

$$(\exists^{\mathrm{St}} x)\varphi := (\exists x)(x \textit{ is standard } \& \ \varphi);$$

$$(\forall^{\mathrm{Stfin}} x)\varphi := (\forall^{\mathrm{St}} x)(x \textit{ is finite } \Rightarrow \varphi);$$

$$(\exists^{\mathrm{Stfin}} x)\varphi := (\exists^{\mathrm{St}} x)(x \textit{ is finite } \& \ \varphi);$$

5.13. The axioms of IST contain the postulates of ZF plus the three new schemata that are called the principles of nonstandard set theory;

(1) the transfer principle—

$$(\forall^{\mathrm{St}} x_1) \dots (\forall^{\mathrm{St}} x_n)((\forall^{\mathrm{St}} x)\varphi(x, x_1, \dots, x_n) \Rightarrow (\forall x)\varphi(x, x_1, \dots, x_n))$$

for each internal formula φ;

(2) the idealization principle—

$$(\forall x_1) \dots (\forall x_n)((\forall^{\mathrm{Stfin}} x \subset z)(\forall y \in z)\varphi(x, x_1, \dots, x_n) \Leftrightarrow$$

$$\Leftrightarrow (\exists x)(\forall^{\mathrm{St}} y)\varphi(x, x_1, \dots, x_n)),$$

(3) the standardization principle—

$$(\forall x_1)\dots(\forall x_n)((\forall^{\mathrm{St}} x)(\exists^{\mathrm{St}} y)(\forall^{\mathrm{St}} z) z \in y \Leftrightarrow\ z \in x \ \&\ \varphi(z, x_1, \dots, x_n))$$

for each formula φ.

It should be noted that when applying these principles of nonstandard analysis one is required to strictly follow our convention of listing all free variables in a formula (cf. 5.7).

5.14. The Powell theorem [74]. *The theory* IST *is conservative over* ZF*.*

5.15. This assertion means that the internal theorems of IST are theorems of Zermelo-Fraenkel theory. In other words, when we are proving "standard" theorems on sets in the von Neumann universe, the formalism of IST can be used with the same degree of reliability as that we used to in ZF. At the same time one should not forget that the *ultimate foundation of* ZF *is the absence of revealed contradictions and the existence of meaningful justifications.* You can find more on the relationship between ZF and nonstandard analysis in [61].

5.16. While reflecting on the meaning of the formal axioms of IST one immediately notices that the idealization principle looks slightly cumbersome. Therefore, now we shall verify that 5.13 (2) guarantees existence for nonstandard elements.

5.17. *There is a finite internal set that containing every standard set among its elements.*

$\triangleleft$ Let $\varphi := (x$ is finite & $y \in x)$. Then $\varphi \in$ (ZF). For each standard finite z we can find an x such that for all $y \in z$ we shall have $\varphi(x, y)$ (e.g., z itself). Now we only have to apply 5.13 (2). $\triangleright$

5.18. A useful observation: standard finite sets are exactly those sets consisting only of standard elements. This was proved in 3.2 but now we shall give another instructive proof based on 5.13 (2).

5.19. *For an internal set A*

$$A = {}^\circ A \ \Leftrightarrow\ (A \textit{ is standard } \& \ A \textit{ is finite}).$$

$\triangleleft$ Let $\varphi := (x \in A \ \&\ x \neq y)$. Then it follows from 5.13 (2) that

$$(\forall^{\mathrm{Stfin}} z)(\exists x)(\forall y \in z)\varphi(x, y, A) \Leftrightarrow (\exists x)(\forall^{\mathrm{St}} y)(x \in A \ \&\ x \neq y) \Leftrightarrow$$

$$\Leftrightarrow (\exists x \in A) x \text{ is nonstandard } \ \Leftrightarrow A \setminus {}^\circ A \neq \varnothing.$$

In other words,

$$A = {}^\circ A \Leftrightarrow (\exists^{\mathrm{St}} z)(\forall x)(\exists y \in z) x \notin A \wedge x = y \Leftrightarrow$$

$$\Leftrightarrow (\exists^{\mathrm{St}} z)(\forall x \notin A)(\exists y \in z) x = y \Leftrightarrow (\exists^{\mathrm{St}} z) A \subset z. \ \triangleright$$

5.20. *Let X, Y be standard sets and $\varphi = \varphi(x, y, z)$—a formula of IST. Then the following rule for introduction of standard functions is correct:*

$$(\forall^{\mathrm{St}} x)(\exists^{\mathrm{St}} y) x \in X \Rightarrow y \in Y \ \& \ \varphi(x, y, z) \Leftrightarrow$$

$$\Leftrightarrow (\exists^{\mathrm{St}} y(\cdot))(\forall^{\mathrm{St}} x)(y(\cdot) \text{ is a function from } X \text{ to } Y \ \&$$
$$\& \ x \in X \Rightarrow \varphi(x, y(x), z)).$$

◁ Consider the standardization $\overline{F}(x) := {}^*\{y \in Y : \varphi(x, y, z)\}$. We can use 5.13 (3) again to form a standard set

$$F := {}^*\{(x, A) \in X \times \mathcal{P}(Y) : \overline{F}(x) = A\}.$$

By assumption we have $(\forall^{\mathrm{St}} x \in X)\overline{F}(x) \neq \varnothing$. At the same time $F(x) = \overline{F}(x)$ by the definition of F. Therefore,

$$(\forall^{\mathrm{St}} x \in A) F(x) \neq = \varnothing \Rightarrow (\forall x \in X) F(x) \neq \varnothing$$

by 5.13 (1). Now we conclude with the help of the axiom of choice:

$$(\exists y(\cdot))(y(\cdot) \text{ is a function from } X \text{ to } Y) \ \& \ (\forall x \in X) y(x) \in F(x).$$

Using 5.13 (1) again we can deduce that there exists a standard function $y(\cdot)$ with domain in X and values in Y such that $y(x) \in F(x)$ for all $x \in X$. ▷

5.21. These rules allow us to translate many (but not all of course) concepts and statements of nonstandard analysis into equivalent definitions and theorems free from the notion of "standardness." In other words, the formulae of IST expressing "something unusual" about standard objects can be translated into equivalent formulae of ZF which are conventional mathematical expressions. The procedure, yielding this result, is called the *Nelson algorithm* or the *reduction algorithm.* The essence of the "decoding" algorithm is in using standard functions, idealization and transposition of quantifiers for reducting an expression to the form more suitable for transfer. The translation is ultimately equivalent to *elimination of the external notion of standardness.* In every case when the relations of 5.13 or 5.20 are used we should ensure the validity of their application.

5.22. The Nelson algorithm consists of the following steps:

(1) a theorem of nonstandard analysis is rewritten as a formula of IST, i.e. all abbreviations are decoded;

(2) the formula of IST is reduced to the prenex normal form

$$(Q_1 x_1) \dots (Q_n x_n) \varphi(x_1, \dots, x_n),$$

where $\varphi \in (\mathrm{ZF})$ and $Q_k \in \{\forall, \exists, \forall^{\mathrm{St}}, \exists^{\mathrm{St}}\}$ for $k = 1, \dots, n$;

(3) if Q_n is an "internal" quantifier, i.e. either $\forall$ or $\exists$, then we assign $\varphi := (Q_n x_n)\varphi(x_1, \dots, x_n)$ and return to the step (2);

(4) if Q_n is an "external" quantifier, i.e., $\forall^{\mathrm{St}}$ or $\exists^{\mathrm{St}}$, then the quantifier prefix is searched from left to right until the first internal quantifier is found;

(5) if no internal quantifiers were found in step (4) then the quantifier Q_n is replaced by the corresponding internal quantifier (according to 5.13 (1)) and we return to step (2) (i.e., we delete all superscripts St) in right-to-left order);

(6) let Q_m be the first internal quantifier. Suppose that $Q_m + 1$ is an external quantifier of the same kind as Q_m. Transpose the quantifiers and go back to (2);

(7) if all quantifiers $Q_{m+1}, \dots, Q_n$ are of the same type, apply 5.13 (2) and go back to step (2);

(8) if the type of quantifiers changes, i.e., Q_{p+1} and Q_m are of the same type, and the type of all quantifiers $Q_{m+1}, \dots, Q_p$ is of the other type, then use 5.20 and start again with (2).

5.23. Note that an assertion is expressible in different forms, some of which can be absolutely incomprehensible. Hence *in practical applications of the Nelson algorithm one should look for possibilities of accelerating procedures of "extracting the external quantifiers."* In particular, sometimes it is not necessary to complete Step 5.22 (2), i.e., to reduce the formula to the prenex normal form. For example, decoding 3.6 (1) yields

$$(\forall s \in \mathbb{R})(\forall t \in \mathbb{R}) s \approx 0 \mathbin{\&} t \approx 0 \Rightarrow s + t \approx 0 \ \Leftrightarrow$$

$$\Leftrightarrow (\forall^{\mathrm{St}} \varepsilon > 0)(\forall s \in \mathbb{R})(\forall t \in \mathbb{R}) s \approx 0 \mathbin{\&} t \approx 0 \Rightarrow |s + t| \le \varepsilon \Leftrightarrow \dots$$

$$\dots \Leftrightarrow \ (\forall \varepsilon > 0)(\exists \delta > 0)(\forall s \in \mathbb{R})(\forall t \in \mathbb{R}) |s| \le \delta \mathbin{\&} |t| \le \delta \Rightarrow |\ s + t\ | \le \varepsilon,$$

i.e., as we might expect, we arrive at the usual ε-δ-definition of the continuity of addition at the origin.

EXTERNAL SET THEORIES

6.1. Credenda of nonstandard analysis can be adequately expressed in the language of the formal machinery of Nelson's internal set theory. The Powell theorem enables us to view IST as a technique for studying the von Neumann universe. At the same time the existence of external objects completely undermines a common belief that Zermelo-Fraenkel theory provides the sufficient freedom of operation from the viewpoint of naive set theory. Within the framework of IST we cannot, for example, even ask the following question: "Is it possible to select some numbers so that each element of $\mathbb{R}$ could be represented as their linear combination with standard coefficients (since $\mathbb{R}$ can obviously be considered as a vector space over $^\circ\mathbb{R}$)?" The number of these illegitimate although mathematically meaningful questions is so large that the need to extend the limits of IST becomes a must. A practical solution of the problem of returning to the "Cantorian paradise" is, in particular, in designing a formalism that would allow us to use the conventional means for working with sets that are external to the von Neumann universe. We shall now review the axiomatic approaches to study of external sets. The first variant of the corresponding formalism was proposed by K. Hrbaček [62, 63]. A similar version, the theory NST, was constructed by T. Kawai [67]. These nonstandard set theories demonstrate that from the Philistine pragmatic point of view the *world of external sets is at least as good as the universe of naive sets*, since it admits the classic set-theoretic operations including selection of subsets with the help of properties (the axioms of comprehension) and well-ordering of arbitrary sets (the axiom of choice). At the same time the external sets include the complete collection of standard and nonstandard internal sets satisfying some variants of the transfer, idealization and standardization principles close to their intuitive formulations. More strictly, we can say that internal sets are included in the world of external sets by definition.

As regards the actual demands of modern mathematical analysis (standard as well as nonstandard) both theories, EXT and NST provide almost equivalent opportunities that are quite sufficient for safe by founding the usage of common analytic constructs. It is of course necessary to study carefully the details of the axiomatic theories in order to avoid the illusions accompanying the euphoria of universal permissiveness. Thus, it is worth noting that the *world of external sets does not coincide win to the von Neumann universe* (the absence of the axiom of foundation is essential). Besides, the exact expressions of the principles of nonstandard analysis in EXT are different from their analogs in IST. Therefore, although EXT is a conservative extension of ZF, it does not contain all of Nelson's theory IST. This gap was bridged by T. Kawai, whose theory NST enriched the formal techniques of IST while providing, along with EXT and IST, new reliable means for studying ZF.

6.2. The *alphabet of the formal theory* EXT consists of the alphabet of IST and some new symbol for the unary predicate Int expressing the property of "being an internal set." In other words, we accept for consideration the texts containing $\mathrm{Int}(x)$, or, in a more verbose form, "x is an internal set." Intuitively: the range of the variables of EXT is the universe of all external sets $V^{\mathrm{Ext}} := \{x : x = x\}$ which includes the world of standard sets $V^{\mathrm{St}} := \{x \in V^{\mathrm{Ext}} : \mathrm{St}(x)\}$ as well as the world of internal sets $V^{\mathrm{Int}} := \{x \in V^{Ext} : \mathrm{Int}(x)\}$.

6.3. The *conventions of* EXT are similar to those of ZF and IST. In particular, we shall of course continue using the "classifiers"—the curly brackets—in EXT (cf. 5.7) and the traditional notation for the operations on classes of external sets. Using the previous patterns, we shall write for a formula φ of EXT (in notation: $\varphi \in (\mathrm{EXT})$):

$$(\forall^{\mathrm{St}} x)\varphi := (\forall x)\mathrm{St}(x) \Rightarrow \varphi; (\exists^{\mathrm{Int}} x)\varphi := (\exists x)\mathrm{Int}(x) \ \& \ \varphi.$$

Similar rules, easily understood from context, will be hereupon used without explanations. Further, we shall need the following concepts.

Let $\varphi \in (\mathrm{ZF})$ be a formula of EXT that is also a formula of ZF (i.e., does not contain the symbols St and Int). Let us replace each quantifier Q in the text of φ by Q^{St}. This formula is denoted φ^{St} and called the *standardization of* φ or the *relativization of* φ *on* V^{St}. Similarly the substitute Q^{Int} for Q will yield a formula φ^{Int} that is called the *internalization* of φ or the *relativization of* φ *on* V^{St}. Note that nothing happens to the free variables of φ. Finally, we shall say that the external set A *is of standard size* (symbolically: $A \in V^{size}$, if there exist a standard set a and an external function f such that $(\forall X)(X \in A \Leftrightarrow (\exists^{\mathrm{St}} x \in a) X = f(x))$.

6.4. The special axioms of EXT can be divided into three groups: (i) the rules for formation of external sets, (ii) the axioms describing connections between the universes $V^{\mathrm{St}}, V^{\mathrm{Int}}$ and V^{Ext}, (iii) the transfer, idealization and standardization principles.

6.5. The laws of *Zermelo's set theory* Z are valid in EXT, i.e. the following axioms for construction of external sets are assumed:

(1) the *axiom of extensionality*—

$$(\forall A)(\forall B) A \subset B \ \& \ B \subset A \Leftrightarrow A = B;$$

(2) the *axiom of pairing*—

$$(\forall A)(\forall B)\{A, B\} \in V^{\mathrm{Ext}};$$

(3) the *axiom of union*—

$$(\forall A) \cup A \in V^{\mathrm{Ext}};$$

(4) the *axiom of powerset*—

$$(\forall A)\mathcal{P}(A) \in V^{\mathrm{Ext}};$$

(5) the *axiom schema of comprehension*—

$$(\forall A)(\forall X_1)\dots(\forall X_n)\{X \in A : \varphi(X, X_1, \dots, X_n)\} \in V^{\mathrm{Ext}}.$$

(6) the *axiom of well-ordering*—each external set can be well-ordered.

The last property (also known as the Zermelo theorem) implies the axiom of choice in the conventional multiplicative form or in the form of the Kuratowski-Zorn lemma. Note also that the theory Z usually contains the axiom of infinity which will appear in EXT later.

6.6. The second group of axioms of EXT contains the following statements:

(1) the *modeling principle*— the world of internal sets V^{Int} is the von Neumann universe, i.e. for each axiom φ of the Zermelo-Fraenkel theory its internalization φ^{Int} is an axiom of EXT;

(2) the *axiom of transitivity*—

$$(\forall x \in V^{\mathrm{Int}}) x \subset V^{\mathrm{Int}},$$

i.e. all internal sets are composed of internal elements only;

(3) the *axiom of immersion*—

$$V^{\mathrm{St}} \subset V^{\mathrm{Int}},$$

i.e., all standard sets are internal.

6.7. The third group of axioms of EXT consists of the postulates of nonstandard analysis:

(1) the *transfer principle*—

$$(\forall^{\mathrm{St}} x_1)\dots(\forall^{\mathrm{St}} x_n)\varphi^{\mathrm{St}}(x_1, \dots, x_n) \;\Leftrightarrow\; \varphi^{\mathrm{Int}}(x_1, \dots, x_n)$$

for any formula $\varphi \in$ (ZF).

(2) the *idealization principle*—

$$(\forall^{\mathrm{Int}} x_1)\dots(\forall^{\mathrm{Int}} xn)((\forall A \in V^{size})((\forall^{\mathrm{fin}} z \subset A)(\exists^{\mathrm{Int}} x)$$
$$(\forall y \in z)\varphi^{\mathrm{Int}}(x, y, x_1, \dots, x_n)) \Rightarrow (\exists^{\mathrm{Int}} x)(\forall^{\mathrm{Int}} y \in A)\varphi^{\mathrm{Int}}(x, y, x_1, \dots, x_n))$$

for arbitrary formula $\varphi \in$ (ZF).

(3) the *standardization principle*—

$$(\forall A)(\exists^{\mathrm{St}} {}^*A)(\forall^{\mathrm{St}} x) x \in {}^*A \Leftrightarrow x \in A,$$

in other words, given an external set A, there exists its standardization ${}^*A$.

6.8. The Hrbaček theorem [62]. *The theory* EXT *is conservative over* ZF*, i.e., for a* $\varphi \in$ *(ZF)*

$$\varphi \text{ is a theorem of ZF} \Leftrightarrow \varphi^{\mathrm{Int}} \text{ is a theorem of EXT} \Leftrightarrow \varphi^{\mathrm{St}} \text{ is a theorem of EXT}.$$

6.9. When reflecting on these axioms it is useful to realize that EXT *is not an extension of* IST. In other words, the universe V^{Int} is not a model for the Nelson theory of internal sets, since the idealization and standardization principles are worded differently in EXT and IST. The conditions for standardization in V^{Int} are considerably less restrictive than in IST. Thus, for $\varphi \in$ (IST) and $A \in V^{\mathrm{Int}}$ we can define the set ${}^*\{x \in A : \varphi(x)\}$, since $\{x \in A : \varphi(x)$ is an external subset of A. In IST this is possible only when A is standard—a set that contains all standard elements cannot be standardized in IST. In EXT the collection of all standard elements is not contained in any external (or internal) set.

6.10. *There is no such external set, i.e., an element of V^{Ext}, that contains as elements all standard sets.*

◁ Suppose the converse is true, i.e. $V^{\mathrm{St}} \subset X$ for a $X \in V^{\mathrm{Ext}}$. Applying the axiom of comprehension 6.5 (5) to the formula $\varphi := \mathrm{St}(x)$, we conclude that V^{St} is an external set. The standardization ${}^*V^{\mathrm{St}}$ turns out to be a standard finite set containing every standard set, which is certainly impossible. ▷

6.11. This statement shows that the idealization principle in EXT (relativized on V^{Int}) differs from its analog in IST not only in form but also in essence. At the same time the importance of these differences should not be exaggerated.

6.12. *The following statements are true:*

(1) *the external natural numbers are the same as the standard natural numbers;*

(2) *a finite external set is standard if and only if all its elements are standard;*

(3) *if A is an arbitrary external set then the size of its standard core ${}^\circ A = \{a \in A : \ St(a)\}$ is standard;*

(4) *each infinite internal set has a nonstandard element.*

◁ (1) By the principle of induction on the standard natural numbers (which is obviously true in EXT—cf. 3.2 (1)) for the set of external natural numbers we have ${}^\circ\mathbb{N} \subset \mathbb{N}^{\mathrm{Ext}}$. Moreover, it is clear that ${}^*\varnothing = \varnothing$ and ${}^*1 = {}^*\{\varnothing\} = \{\varnothing\} = 1$. Therefore, by the principle of induction on external natural numbers we have $\mathbb{N}^{\mathrm{Ext}} \subset {}^\circ\mathbb{N}$.

(2) A standard set is internal. Therefore, due to 6.6 (2) we can use 3.2.

(3) Let ${}^*A$ be the standardization of A. Let $f(a) := a$ for $a \in {}^\circ A$. It is obvious that $(\forall X) X \in {}^\circ A \Leftrightarrow (\forall^{\mathrm{St}} x \in {}^*A) f(x) = X$.

(4) Let A denote an internal set. By (3) the size of ${}^\circ A$ is standard. Therefore we can apply 6.7 (2) with $\varphi := x = y \ \& \ x \in A$. For each finite $z \subset {}^\circ A$ it is true, of course, that $(\exists x \in A)(\forall y \in z) x \neq y$, since the set A is infinite. Finally, $(\exists x \in A)(\forall y \in {}^\circ A) x \neq y$. ▷

6.13. In relation to 6.12 and 6.8 it is convenient to distinguish a variant of the theory of internal sets INT that is a conservative extension of ZF so that EXT be conservative over INT. This theory differs from IST in the form of the idealization and standardization principles:

(1)

$$(\forall A)(\forall x_1)\dots(\forall x_n)((\forall^{\mathrm{Stfin}} z \subset A)(\exists x)(\forall y \in z)$$

$$\varphi(x, y, x_1, \dots, x_n) \Leftrightarrow (\exists x)(\forall^{\mathrm{St}} y \in A)\varphi(x, y, x_1, \dots, x_n))$$

for any $\varphi \in$ (ZF);

(2) $(\forall A)(\exists^{\mathrm{St}} {}^*A)(\forall^{\mathrm{St}} x) x \in {}^*A \Leftrightarrow x \in A \ \& \ \varphi(x)$

for arbitrary $\varphi \in$ (INT).

6.14. We now describe the theory NST in a variant that is closest to EXT and IST (T. Kawai has in fact constructed a slightly different system that permits one to consider Gödel-Bernays classes as external sets).

6.15. The *alphabet and conventions of* NST are exactly the same as those of EXT. Furthermore, NST includes all axioms for constructing external sets, all axioms on inter between the worlds of sets and the transfer principle from EXT. The principal differences between NST and EXT lie in the formulations of the standardization and idealization principles and in the following additional postulate.

6.16. The *axiom of acceptability*— $V^{\mathrm{St}} \in V^{\mathrm{Ext}}$, i.e., the world of standard sets of Kawai's theory is an external set. According to the axiom an external set A in NST is said to be of *acceptable size* (written as $A \in V^{acs}$), if there exists an external function F that maps V^{St} onto A. Note that the size of V^{St} is acceptable. The

notation $acfin(A)$ will hereafter mean that there is a bijective external mapping of A onto a standard finite set.
6.17 The *standardization principle* in NST says that:

$$(\forall A)((\exists^{\rm St} X)X \supset A \Rightarrow (\exists^{\rm St}{}^*A)(\forall^{\rm St} x)x \in {}^*A \Leftrightarrow x \in A).$$

In other words, it is possible in NST to standardize external subsets of standard sets (but not arbitrary external sets like in EXT).
6.18. The *idealization principle* in NST means that

$$(\forall^{\rm Int} x_1)\dots(\forall^{\rm Int} x_n)(\forall A \in V^{acs})(((\forall z)$$

$$z \subset A \ \&\ acfin(z) \Rightarrow (\exists^{\rm Int} x)(\forall y \in z)\varphi^{\rm Int}(x,\ y, x_1,\dots,x_n)) \Rightarrow$$

$$\Rightarrow (\exists^{\rm Int} x)(\forall^{\rm Int} y \in A)\varphi^{\rm Int}(x,\ y,\ x_1,\dots,x_n)) \Rightarrow$$

for any formula $\varphi \in$ (ZF).

6.19. The Kawai theorem [67]. NST is conservative over ZF.

6.20. We recall that the *world of internal sets* $V^{\rm Int}$ *in the universe of* NST *with relativized standardization, idealization and transfer principles is a model of* IST. In other words, the techniques that NST offers for working with the external sets arising in IST can be safely utilized for proving the statements of the "standard" mathematics. Note also that the proof of the theorem of Kawai as well as of the theorems of Hrbaček and Powell is based on using suitable analogs of the local Mal′tsev's theorem or, more precisely, on the techniques of ultraproducts and ultralimits [49, 53, 54, 79].
6.21. Let V^E denote the universe of external sets (without specifying one of the two theories NST or EXT). Similarly, we shall use the notation V^I (or V^S) when referring to the world of internal (or standard) sets. Repeating the construction of the von Neumann universe, i.e. iterating the operations of union and taking the collection of all external subsets of the set, we can grow from the empty set the world V^C, the *universe of "classic" sets*. More precisely, one assumes

$$V^C := \bigcup_{\alpha\ \in {\rm Od}\ ^{\rm St}} V^C_\alpha,$$

$$V^C_\alpha := \{x : (\exists^{\rm St}\beta \in \alpha)\ x \in \mathcal{P}(V^C_\beta)\},$$

where Od $^{\rm St}$ is the class of all standard ordinals. Thus the empty set is "classic" and each "classic" set is composed only of "classic" elements. Now using recursion, "walking the floors" of the universe of "classic" sets, we can define the Robinson standardization or $*$-mapping.
6.22. The standard set $*A$ is called the *Robinson standardization* or the $*$-image of a "classic" set A if and only if each standard element of $*A$ is an $*$-image of an element of A. Symbolically: $*\varnothing := \varnothing$, $*A := {}^*\{*a : a \in A\}$.

Note that using the ordinary standardization is undoubtedly legal within EXT. The possibility of using this operation in the construction of Robinson's standardization follows from the construction of V^C. A similar reasoning, the "transfinite induction on rank," shows that the $*$-mapping bijectively identifies the worlds V^C and V^S. Moreover, the *transfer principle* is ensured:

$$(\forall A_1 \in V^C)\dots(\forall A_n \in V^C)\varphi^C(A_1,\dots,A_n) \Leftrightarrow \varphi^S(*A_1,\dots,*A_n)$$

for an arbitrary formula φ of Zermelo-Fraenkel theory (as usual, φ^C and φ^S are the relativizations of φ on V^C and V^S respectively).

THE SET-THEORETIC FOUNDATIONS OF NONSTANDARD ANALYSIS

7.1. The discussion of the previous sections enriched and extended the initial "naive" notions of sets utilized in nonstandard analysis. From the conventional von Neumann universe V we came to the world V^I of the theory of internal sets with reference points, the standard sets forming the class V^S (Fig. 3). Further analysis showed that V^I lies in a new class V^E, the universe of external sets that compose the Zermelo world. In V^E we have selected the universe of "classical" sets V^C, another realization of the world of standard sets, and constructed the corresponding Robinson's $*$-mapping that is a bijection from V^C to V^S. Due to the transfer principles V^C,V^S and V^I might be considered as hypostases of the von Neumann universe (Fig. 4).

7.2. This picture along with other known relations between the worlds V^E, V^I, V^S and V^C leads us to formulating three general set-theoretic approaches to nonstandard analysis. These approaches, or stances, or credos, they are called classic, neoclassic and radical—fix certain notions of the object and method of study. The acceptance of one of the credos determines, for example, the manner of presentation of the mathematical results obtained by use made of nonstandard methods. Thus familiarity with these approaches should be considered as absolute necessity.

7.3. The *classic credo* of nonstandard analysis relates to the techniques used by its founder A. Robinson, and the corresponding formalism is mostly common at present (cf., for example, [11]). In this approach the *principal object of research is the world of classical mathematics* that is identified with the universe of the classical sets V^C. The latter is considered to be the "standard universe" (in practice one usually deals with a sufficiently large part of V^C which contains the specific objects to study, with a so-called "superstructure"). The main technique for the study of the initial "standard universe" is the "nonstandard universe" of internal sets V^I (or a suitable part of it) and the $*$-mapping that glues together the usual "standard" sets and their images in the nonstandard universe. It is useful to note here a specific usage of the words "standard" and "nonstandard." The Robinson standardizations, as elements of the universe V^S, are viewed as "nonstandard" objects. *A "standard" set is an arbitrary representative of the world of "classical" sets V^C, a member of the "standard" universe.* It is said that the $*$-mapping usually adds new "ideal" elements to the set. Here one assumes that $*A = \{*a : a \in A\}$ only if the "classic," or "standard," set A is finite. For example, if we place $\mathbb{R}$ into V^C and study its $*$-image $*\mathbb{R}$, we shall see that $*\mathbb{R}$ plays the role of the field of real numbers in the sense of the universe of internal sets. At the same time $*\mathbb{R}$ is not equal to the set of its standard elements: $^\circ(*\mathbb{R}) = \{*t : t \in \mathbb{R}\}$. Considering that $*\mathbb{R}$ is the "internal set of real numbers $\mathbb{R}$," and $^\circ(*\mathbb{R})$ is its standard core, one sometimes takes the liberty of writing $^\circ\mathbb{R} := \{*t : t \in \mathbb{R}\}$ and even $\mathbb{R} := \{*t : t \in \mathbb{R}\}$. The presence of new elements in $*\mathbb{R}$ is expressed as $*\mathbb{R} \setminus \mathbb{R} \neq \varnothing$, and we talk of the system of "*hyperreal numbers*" $*\mathbb{R}$ that extends the ordinary field of reals $\mathbb{R}$. A similar policy is pursued when considering an arbitrary classical set X; namely, one assumes that $X := \{*x : x \in X\}$ and, therefore, $X \subset *X$. If X is infinite, $*X \setminus X \neq \varnothing$. In other words, the *Robinson standardization adds new elements to all infinite sets.* Furthermore, there is a considerable number of these additional "ideal" numbers since the idealization principle is valid in V^I (this principle in the presented approach is also called the *technique of concurrence or saturation.*

7.4. Let U be an arbitrary correspondence, whereas A and B, sets. Then U is said to be *concurrent* or *directed* (from A to B) if for each nonempty finite subset A_0 of

A there is an element $b \in B$ such that $(a_0, b_0) \in U$ for all $a_0 \in A_0$.

7.5. The concurrence principle. *For a correspondence U directed from A to B there is an element $b \in *B$ such that $(*a,\ b) \ \in \ *U$ for each $a \ \in \ A$.*

7.6. Sometimes one uses the forms of this principle which ensure additional possibilities of introducting new elements and are more adequate to the idealization principle in more precise forms.

The strong concurrence principle. *Let a correspondence U be such that $*U$ is directed from A to $*B$. Then there exists an element $b \in *B$ such that for all $a \in A$ it holds $(*a,\ b) \in *U$.*

7.8. The saturation principle. *Let $A_1 \supset A_2 \supset \dots$ be a decreasing sequence of nonempty internal sets. Then $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$.*

7.9. Note that in the "extended" or "nonstandard" world, in the universe of internal sets V^I, the transfer principle is valid; namely, in terms of the Robinson standardization

$$(\forall x_1 \in V^C)\dots(\forall x_n \in V^C)\varphi^C(x_1,\dots,x_n) \Leftrightarrow \varphi^I(*x_1,\dots,*x_n)$$

for each formula φ of the Zermelo-Fraenkel set theory. This form of the transfer principle is often called the *Leibniz principle.*

7.10. The so-called "*technique of internal sets*" is sometimes specially emphasized in study of the nonstandard universe. By this one usually means the method of proof which is based on the fact that all external sets defined "in a conventional way" are internal. An illustration is readily available:

7.11. *Let A be an infinite set. For each set-theoretic property φ it is false that $\{x : \varphi^I(x)\} = *A \setminus A$.*

$\triangleleft$ Suppose the contrary. Then the class $\{x : \varphi^I(x)\}$ is an internal subset of $*A$. Therefore, A is an internal set. But for an infinite A the external set $*A \setminus A$ is not internal by 6.12. $\triangleright$

7.12. Concluding the discussion, we can say that in confessing the classical credo one works with the two universes, standard and nonstandard. There is a formal possibility of linking the properties of standard and nonstandard objects with the help of the $*$-mapping. At the same time one can freely translate statements about objects of one world into those about their images in the other world; i.e., the Leibniz principle is valid. The nonstandard world is abundant in "ideal" elements; various transfinite constructs are realizable in it because of the concurrence principle. The sets falling beyond the nonstandard universe are called external (this is a peculiarity of the terminology: the internal sets are not considered external in this approach). The technique of internal sets is very useful.

The principal advantage of the classic approach is the availability of the $*$-mapping making it possible to apply the machinery of nonstandard analysis to arbitrary ordinary sets. For example, we can assert that a function $f : [a, b] \to \mathbb{R}$ is uniformly continuous if and only if $*f : *[a, b] \to *\mathbb{R}$ is microcontinuous; i.e., if $*f$ preserves the infinite proximity between the "hyperreal" numbers. The principal complication in absorbing these notions lies in the necessity of imagining the enormous number of the new "ideal" objects inserted into the ordinary sets. Considerable problems are caused by the natural desire to work (at least in the beginning) with two sets of variables that correspond to the two universes. (When we are constructing the internalization φ^I of a formula φ we implicitly assume the existence of such a procedure.) Thus, the *integral attributes of the classical approach, its "bilingual" nature and the Robinson standardization*, determine all of its peculiarities, advantages and disadvantages of the formal apparatus.

7.13. The *neoclassic credo* of nonstandard analysis corresponds to the ideas developed by E. Nelson. In this approach the *principal object of study is the world of mathematics considered as the universe* V^I that lies in the environment of external sets, the elements of V^E. The "classic" sets are not used for analysis separately. The *standard and nonstandard elements are demonstrated in the ordinary objects*, the internal sets composing V^I. Thus, the field of real numbers is $\mathbb{R}$ from the world V^I that is, of course, the same as $*R$ the field of hyperreal numbers—the "ideal" object of the classic theory. The views presented in the Sections 2–4 correspond to the neoclassic credo. Its advantages are determined by the possibility of studying the already-known sets with the goal of finding something new in their structure. The shortcomings of the neoclassic approach are caused by the necessity of implicitly transferring definitions and properties from the standard objects to their internal ones (on use made of standardization). We have encountered the phenomenon in Section 4.

7.14. The *radical credo* of nonstandard analysis assumes that the *object of mathematical research is the universe of external sets* in all completeness and complexity of its intrinsic structure. The classical and neoclassical stances on nonstandard analysis as a technique for study of mathematics (based on the Zermelo-Fraenkel formalism) are declared "parochial" or "shy" and discarded. At a first glance this approach cannot be accepted earnestly and must be dismissed as overextremist; but upon reflection these accusations of the radical credo should be rejected. This is an illusory, superficial "extremism." A widely-accepted view of mathematics as a science of forms and relations considered separately from their content and even the considerably less restrictive classical set-theoretic credo (originated with G. Cantor) certainly contain the "extreme" thoughts of the object of nonstandard analysis. Therefore, the most "radical" views of sets that resulted from laborious studies ultimately merge into the original theory, extending and enriching it. Observe that we started with a "modest" statement that nonstandard analysis operates on exactly the same sets as the rest of mathematics (cf. 2.1). Here it is appropriate to recall the V. I. Lenin observations that refer to the dynamics of cognition:

"*Each shade of thought = a circle on the great circle* (*spiral*) *of the development of human thought*" [27, p. 221].

The "*Human cognition is not* (*respective does not follow*) *a straight line, but a curve that infinitely approximates a series of circles, a spiral*" [27, p. 322].

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