Dido's Problem and Beyond

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Prologue

- David Mumford, one of the most beautiful mathematical minds of today, remarked once that he honestly carried out some ghastly but wholly straightforward calculations while checking something. “It took me several hours to do every bit and as I was no wiser at the end... I shall omit details here.” Narrating this episode, another outstanding mathematician, Yuri Manin, concluded: “The moral: a good proof is one which makes us wiser.”
- The following slight abstraction of this thesis transpires: *In science we appraise and appreciate that which makes us wiser.* The notions of a good theory open up new possibilities of solving particular problems. Rewarding is the problem whose solution paves way to new fruitful concepts and methods.

The Dido Problem

- The Dido problem is ususally acknowleged as the start of the theory of extremal problems. Dido was a mythical Phoenician Princess. Virgil told about the escape of Dido from her treacherous brother in the first chapter of *The Aeneid*. Dido had to decide about the choice of a tract of land near the future city of Carthage, while satisfying the famous constraint of selecting "a space of ground, which (Byrsa call'd, from the bull's hide) they first inclos'd."
- By the legend, Phoenicians cut the oxhide into thin strips and enclosed a large expanse. Now it is customary to think that the decision by Dido was reduced to the isoperimetric problem of finding a figure of greatest area among those surrounded by a curve whose length is given. It is not excluded that Dido and her subjects solved the practical versions of the problem when the tower was to be located at the sea coast and part of the boundary coastline of the tract was somehow prescribed in advance.

Rembrandt's "Dido Divides the Oxhide" (mid-1600s)

Minkowski Duality

- Let $\overline{E} := E \cup \{+\infty\} \cup \{-\infty\}$. Assume that $H \subset E$ is a (convex) cone in E, and so $-\infty$ lies beyond H. A subset U of H is *convex relative to* H or *H-convex* provided that U is the *H-support set* $U_p^H := \{h \in H : h \le p\}$ of some element p of $\overline{E}$.
- Alongside the H-convex sets we consider the so-called H-convex elements. An element $p \in \overline{E}$ is *H-convex* provided that $p = \sup U_p^H$. The H-convex elements comprise the cone which is denoted by $\mathscr{C}(H, \overline{E})$. We may omit the references to H when H is clear from the context.
- Convex elements and sets are "glued together" by the *Minkowski duality* $\varphi : p \mapsto U_p^H$. This duality enables us to study convex elements and sets simultaneously.

The Space of Convex Bodies

- The Minkowski duality makes $\mathscr{V}_N$ into a cone in the space $C(S_{N-1})$ of continuous functions on the Euclidean unit sphere S_{N-1}, the boundary of the unit ball $\mathfrak{z}_N$. This yields the so-called *Minkowski structure* on $\mathscr{V}_N$. Addition of the support functions of convex figures amounts to taking their algebraic sum, also called the *Minkowski addition*. It is worth observing that the *linear span* $[\mathscr{V}_N]$ of $\mathscr{V}_N$ is dense in $C(S_{N-1})$, bears a natural structure of a vector lattice and is usually referred to as the *space of convex sets*.
- The study of this space stems from the pioneering breakthrough of Alexandrov in 1937 and the further insights of Radström, Hörmander, and Pinsker.

Linear Inequalities over Convex Surfaces

RESHETNYAK (1954):

- A measure μ *linearly majorizes* or *dominates* a measure ν on S_{N-1} provided that to each decomposition of S_{N-1} into finitely many disjoint Borel sets $U_1, \dots, U_m$ there are measures $\mu_1, \dots, \mu_m$ with sum μ such that every difference $\mu_k - \nu|_{U_k}$ annihilates all restrictions to S_{N-1} of linear functionals over $\mathbb{R}^N$. In symbols, $\mu \gg_{\mathbb{R}^N} \nu$.
- For all sublinear p on $\mathbb{R}^N$ we have

$$\int_{S_{N-1}} p d\mu \geq \int_{S_{N-1}} p d\nu$$

if $\mu \gg_{\mathbb{R}^N} \nu$.

Choquet Order

LOOMIS (1962):

- A measure μ *affinely majorizes* or *dominates* a measure ν, both given on a compact convex subset Q of a locally convex space X, provided that to each decomposition of ν into finitely many summands $\nu_1, \dots, \nu_m$ there are measures $\mu_1, \dots, \mu_m$ whose sum is μ and for which every difference $\mu_k - \nu_k$ annihilates all restrictions to Q of affine functionals over X. In symbols, $\mu \gg_{\mathrm{Aff}(Q)} \nu$.
- Cartier, Fell, and Meyer proved in 1964 that

$$\int_Q f d\mu \geq \int_Q f d\nu$$

for each continuous convex function f on Q if and only if $\mu \gg_{\mathrm{Aff}(Q)} \nu$. An analogous necessity part for linear majorization was published in 1970.

Decomposition Theorem

KUTATELADZE (1974):
Assume that $H_1, \dots, H_N$ are cones in a Riesz space X, while f and g are positive functionals on X.

- *The inequality*

$$f(h_1 \vee \dots \vee h_N) \geq g(h_1 \vee \dots \vee h_N)$$

holds for all $h_k \in H_k$ $(k := 1, \dots, N)$ *if and only if to each decomposition of* g *into a sum of* N *positive terms* $g = g_1 + \dots + g_N$ *there is a decomposition of* f *into a sum of* N *positive terms* $f = f_1 + \dots + f_N$ *such that*

$$f_k(h_k) \geq g_k(h_k) \quad (h_k \in H_k;\ k := 1, \dots, N).$$

Alexandrov Measures

- The celebrated *Alexandrov Theorem* proves the unique existence of a translate of a convex body given its surface area function. Each surface area function is an *Alexandrov measure*. So we call a positive measure on the unit sphere which is supported by no great hypersphere and which annihilates singletons.
- Each Alexandrov measure is a translation-invariant additive functional over the cone $\mathscr{V}_N$. The cone of positive translation-invariant measures in the dual $C'(S_{N-1})$ of $C(S_{N-1})$ is denoted by $\mathscr{A}_N$.

Blaschke's Sum

- Given $\mathfrak{x}, \mathfrak{y} \in \mathscr{V}_N$, the record $\mathfrak{x} =_{\mathbb{R}^N} \mathfrak{y}$ means that $\mathfrak{x}$ and $\mathfrak{y}$ are equal up to translation or, in other words, are translates of one another. So, $=_{\mathbb{R}^N}$ is the associate equivalence of the preorder $\geq_{\mathbb{R}^N}$ on $\mathscr{V}_N$ of the possibility of inserting one figure into the other by translation.
- The sum of the surface area measures of $\mathfrak{x}$ and $\mathfrak{y}$ generates the unique class $\mathfrak{x}\#\mathfrak{y}$ of translates which is referred to as the *Blaschke sum* of $\mathfrak{x}$ and $\mathfrak{y}$.

The Natural Duality

- Let $C(S_{N-1})/\mathbb{R}^N$ stand for the factor space of $C(S_{N-1})$ by the subspace of all restrictions of linear functionals on $\mathbb{R}^N$ to S_{N-1}. Let $[\mathscr{A}_N]$ be the space $\mathscr{A}_N - \mathscr{A}_N$ of translation-invariant measures, in fact, the linear span of the set of Alexandrov measures.
- $C(S_{N-1})/\mathbb{R}^N$ and $[\mathscr{A}_N]$ are made dual by the canonical bilinear form

$$\langle f, \mu\rangle = \frac{1}{N}\int_{S_{N-1}} fd\mu$$

$$(f \in C(S_{N-1})/\mathbb{R}^N,\ \mu \in [\mathscr{A}_N]).$$

- For $\mathfrak{x} \in \mathscr{V}_N/\mathbb{R}^N$ and $\mathfrak{y} \in \mathscr{A}_N$, the quantity $\langle \mathfrak{x}, \mathfrak{y}\rangle$ coincides with the *mixed volume* $V_1(\mathfrak{y}, \mathfrak{x})$.

Cones of Feasible Directions

- Given a cone K in a vector space X in duality with another vector space Y, the *dual* of K is

$$K^* := \{y \in Y \mid (\forall x \in K)\ \langle x, y\rangle \geq 0\}.$$

- To a convex subset U of X and $\bar{x} \in U$ there corresponds

$$U_{\bar{x}} := \mathrm{Fd}(U, \bar{x}) := \{h \in X \mid (\exists \alpha \geq 0)\ \bar{x} + \alpha h \in U\},$$

the *cone of feasible directions* of U at $\bar{x}$.

- Let $\bar{\mathfrak{x}} \in \mathscr{A}_N$. Then the dual $\mathscr{A}^*_{N,\bar{\mathfrak{x}}}$ of the cone of feasible directions of $\mathscr{A}_N n$ at $\bar{\mathfrak{x}}$ may be represented as follows

$$\mathscr{A}^*_{N,\bar{\mathfrak{x}}} = \{f \in \mathscr{A}^*_N \mid \langle \bar{\mathfrak{x}}, f\rangle = 0\}.$$

Dual Cones in Spaces of Surfaces

- Let $\mathfrak{x}$ and $\mathfrak{y}$ be convex figures. Then
  (1) $\mu(\mathfrak{x}) - \mu(\mathfrak{y}) \in \mathscr{V}_N^* \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$;
  (2) If $\mathfrak{x} \geq_{\mathbb{R}^N} \mathfrak{y}$ then $\mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$;
  (3) $\mathfrak{x} \geq_{\mathbb{R}^2} \mathfrak{y} \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^2} \mu(\mathfrak{y})$;
  (4) If $\mathfrak{y} - \bar{\mathfrak{x}} \in \mathscr{A}_{N,\bar{\mathfrak{x}}}^*$ then $\mathfrak{y} =_{\mathbb{R}^N} \bar{\mathfrak{x}}$;
  (5) If $\mu(\mathfrak{y}) - \mu(\bar{\mathfrak{x}}) \in \mathscr{V}_{N,\bar{\mathfrak{x}}}^*$ then $\mathfrak{y} =_{\mathbb{R}^N} \bar{\mathfrak{x}}$.
- It stands to reason to avoid discriminating between a convex figure, the respective coset of translates in $\mathscr{V}_N/\mathbb{R}^N$, and the corresponding measure in $\mathscr{A}_N$.

Comparison Between the Structures

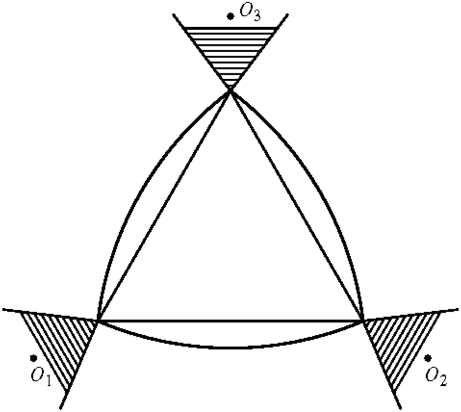
Object of Parametrization	Minkowski's Structure	Blaschke's Structure
cone of sets	$\mathcal{V}_N/\mathbb{R}^N$	$\mathcal{A}_N$
dual cone	$\mathcal{V}_N^*$	$\mathcal{A}_N^*$
positive cone	$\mathcal{A}_N^*$	$\mathcal{A}_N$
typical linear functional	$V_1(\mathfrak{z}_N, \cdot)$ (width)	$V_1(\cdot, \mathfrak{z}_N)$ (area)
concave functional (power of volume)	$V^{1/N}(\cdot)$	$V^{(N-1)/N}(\cdot)$
simplest convex program	isoperimetric problem	Urysohn's problem
operator-type constraint	inclusion of figures	inequalities on "curvatures"
Lagrange's multiplier	surface	function
differential of volume at a point $\bar{\mathfrak{x}}$ is proportional to	$V_1(\bar{\mathfrak{x}}, \cdot)$	$V_1(\cdot, \bar{\mathfrak{x}})$

The External Urysohn Problem

- Among the convex figures, circumscribing $\mathfrak{x}_0$ and having integral breadth fixed, find a convex body of greatest volume.
- *A feasible convex body $\bar{\mathfrak{x}}$ is a solution to the external Urysohn problem if and only if there are a positive measure μ and a positive real $\bar{\alpha} \in \mathbb{R}_+$ satisfying*
  (1) $\bar{\alpha}\mu(\mathfrak{z}_N) \gg_{\mathbb{R}^N} \mu(\bar{\mathfrak{x}}) + \mu$;
  (2) $V(\bar{\mathfrak{x}}) + \frac{1}{N}\int_{S_{N-1}} \bar{\mathfrak{x}} d\mu = \bar{\alpha} V_1(\mathfrak{z}_N, \bar{\mathfrak{x}})$;
  (3) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ *for all z in the support of μ.*

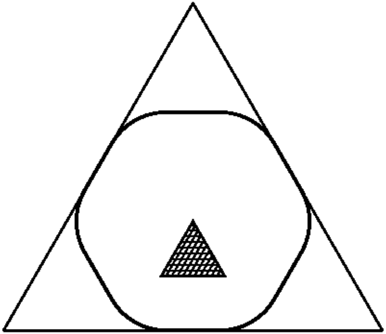
Solutions

- If $\mathfrak{x}_0 = \mathfrak{z}_{N-1}$ then $\bar{\mathfrak{x}}$ is a *spherical lens* and μ is the restriction of the surface area function of the ball of radius $\bar{\alpha}^{1/(N-1)}$ to the complement of the support of the lens to S_{N-1}.
- If $\mathfrak{x}_0$ is an equilateral triangle then the solution $\bar{\mathfrak{x}}$ looks as follows:



Symmetric Solutions

- This is the general solution of the internal Urysohn problem inside a triangle in the class of centrally symmetric convex figures:



Current Hyperplanes

- Find two convex figures $\bar{\mathfrak{x}}$ and $\bar{\mathfrak{y}}$ lying in a given convex body $\mathfrak{x}_o$, separated by a hyperplane with the unit outer normal z_0, and having the greatest total volume of $\bar{\mathfrak{x}}$ and $\bar{\mathfrak{y}}$ given the sum of their integral breadths.
- *A feasible pair of convex bodies $\bar{\mathfrak{x}}$ and $\bar{\mathfrak{y}}$ solves the internal Urysohn problem with a current hyperplane if and only if there are convex figures $\mathfrak{x}$ and $\mathfrak{y}$ and positive reals $\bar{\alpha}$ and $\bar{\beta}$ satisfying*
  (1) $\bar{\mathfrak{x}} = \mathfrak{x} \# \bar{\alpha}\mathfrak{z}_N$;
  (2) $\bar{\mathfrak{y}} = \mathfrak{y} \# \bar{\alpha}\mathfrak{z}_N$;
  (3) $\mu(\mathfrak{x}) \geq \bar{\beta}\varepsilon_{z_0}$, $\mu(\mathfrak{y}) \geq \bar{\beta}\varepsilon_{-z_0}$;
  (4) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ *for all* $z \in \operatorname{supp}(\mathfrak{x}) \setminus \{z_0\}$;
  (5) $\bar{\mathfrak{y}}(z) = \mathfrak{x}_0(z)$ *for all* $z \in \operatorname{supp}(\mathfrak{x}) \setminus \{-z_0\}$, *with* $\operatorname{supp}(\mathfrak{x})$ *standing for the support of* $\mathfrak{x}$, *i.e. the support of the surface area measure* $\mu(\mathfrak{x})$ *of* $\mathfrak{x}$.

Pareto Optimality

- Consider a bunch of economic agents each of which intends to maximize his own income. The *Pareto efficiency principle* asserts that as an effective agreement of the conflicting goals it is reasonable to take any state in which nobody can increase his income in any way other than diminishing the income of at least one of the other fellow members.
- Formally speaking, this implies the search of the maximal elements of the set comprising the tuples of incomes of the agents at every state; i.e., some vectors of a finite-dimensional arithmetic space endowed with the coordinatewise order. Clearly, the concept of Pareto optimality was already abstracted to arbitrary ordered vector spaces.

Vector Isoperimetric Problem

- Given are some convex bodies $\mathfrak{y}_1, \dots, \mathfrak{y}_M$. Find a convex body $\mathfrak{x}$ encompassing a given volume and minimizing each of the mixed volumes $V_1(\mathfrak{x}, \mathfrak{y}_1), \dots, V_1(\mathfrak{x}, \mathfrak{y}_M)$. In symbols,

$$\mathfrak{x} \in \mathscr{A}_N;\ \widehat{p}(\mathfrak{x}) \geq \widehat{p}(\bar{\mathfrak{x}});\ (\langle \mathfrak{y}_1, \mathfrak{x} \rangle, \dots, \langle \mathfrak{y}_M, \mathfrak{x} \rangle) \to \inf.$$

  Clearly, this is a Slater regular convex program in the Blaschke structure.

- *Each Pareto-optimal solution $\bar{\mathfrak{x}}$ of the vector isoperimetric problem has the form*

$$\bar{\mathfrak{x}} = \alpha_1 \mathfrak{y}_1 + \dots + \alpha_m \mathfrak{y}_m,$$

  where $\alpha_1, \dots, \alpha_m$ are positive reals.

The Leidenfrost Problem

- Given the volume of a three-dimensional convex figure, minimize its surface area and vertical breadth.
- By symmetry everything reduces to an analogous plane two-objective problem, whose every Pareto-optimal solution is by 2 a *stadium*, a weighted Minkowski sum of a disk and a horizontal straight line segment.
- *A plane spheroid, a Pareto-optimal solution of the Leidenfrost problem, is the result of rotation of a stadium around the vertical axis through the center of the stadium.*

Internal Urysohn Problem with Flattening

- Given are some convex body $\mathfrak{x}_0 \in \mathscr{V}_N$ and some flattening direction $\bar{z} \in S_{N-1}$. Considering $\mathfrak{x} \subset \mathfrak{x}_0$ of fixed integral breadth, maximize the volume of $\mathfrak{x}$ and minimize the breadth of $\mathfrak{x}$ in the flattening direction:
$\mathfrak{x} \in \mathscr{V}_N;\ \mathfrak{x} \subset \mathfrak{x}_0;\ \langle \mathfrak{x}, \mathfrak{z}_N \rangle \geq \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle;\ (-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \to \inf.$
- *For a feasible convex body $\bar{\mathfrak{x}}$ to be Pareto-optimal in the internal Urysohn problem with the flattening direction $\bar{z}$ it is necessary and sufficient that there be positive reals α, β and a convex figure $\mathfrak{x}$ satisfying*

$$\mu(\bar{\mathfrak{x}}) = \mu(\mathfrak{x}) + \alpha\mu(\mathfrak{z}_N) + \beta(\varepsilon_{\bar{z}} + \varepsilon_{-\bar{z}});$$
$$\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z) \quad (z \in \operatorname{supp}(\mu(\mathfrak{x})).$$

Rotational Symmetry

- Assume that a plane convex figure $\mathfrak{x}_0 \in \mathscr{V}_2$ has the symmetry axis $A_{\bar{z}}$ with generator $\bar{z}$. Assume further that $\mathfrak{x}_{00}$ is the result of rotating $\mathfrak{x}_0$ around the symmetry axis $A_{\bar{z}}$ in $\mathbb{R}^3$.

$$\mathfrak{x} \in \mathscr{V}_3;$$
$$\mathfrak{x} \text{ is a convex body of rotation around } A_{\bar{z}};$$
$$\mathfrak{x} \supset \mathfrak{x}_{00};\ \langle \mathfrak{z}_N, \mathfrak{x} \rangle \geq \langle \mathfrak{z}_N, \bar{\mathfrak{x}} \rangle;$$
$$(-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \to \inf.$$

- *Each Pareto-optimal solution is the result of rotating around the symmetry axis a Pareto-optimal solution of the plane internal Urysohn problem with flattening in the direction of the axis.*

Soap Bubbles

- Little is known about the analogous problems in arbitrary dimensions. An especial place is occupied by the result of Porogelov who demonstrated that the "soap bubble" in a tetrahedron has the form of the result of the rolling of a ball over a solution of the internal Urysohn problem, i.e. the weighted Blaschke sum of a tetrahedron and a ball.

The External Urysohn Problem with Flattening

- Given are some convex body $\mathfrak{x}_0 \in \mathscr{V}_N$ and flattening direction $\bar{z} \in S_{N-1}$. Considering $x \supset \mathfrak{x}_0$ of fixed integral breadth, maximize volume and minimizing breadth in the flattening direction: $\mathfrak{x} \in \mathscr{V}_N;\ \mathfrak{x} \supset \mathfrak{x}_0;\ \langle \mathfrak{x}, \mathfrak{z}_N \rangle \geq \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle;\ (-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \to \inf.$
- *For a feasible convex body $\bar{\mathfrak{x}}$ to be a Pareto-optimal solution of the external Urysohn problem with flattening it is necessary and sufficient that there be positive reals α, β, and a convex figure $\mathfrak{x}$ satisfying*

$$\mu(\bar{\mathfrak{x}}) + \mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \alpha\mu(\mathfrak{z}_N) + \beta(\varepsilon_{\bar{z}} + \varepsilon_{-\bar{z}});$$
$$V(\bar{\mathfrak{x}}) + V_1(\mathfrak{x}, \bar{\mathfrak{x}}) = \alpha V_1(\mathfrak{z}_N, \bar{\mathfrak{x}}) + 2N\beta b_{\bar{z}}(\bar{\mathfrak{x}});$$
$$\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z) \quad (z \in \operatorname{supp}(\mu(\mathfrak{x})).$$

Optimal Convex Hulls

- Given $\mathfrak{y}_1, \dots, \mathfrak{y}_m$ in $\mathbb{R}^N$, place $\mathfrak{x}_k$ within $\mathfrak{y}_k$, for $k := 1, \dots, m$, maximizing the volume of each of the $\mathfrak{x}_1, \dots, \mathfrak{x}_m$ and minimize the integral breadth of their convex hull:

$$\mathfrak{x}_k \subset \mathfrak{y}_k;\ (-p(\mathfrak{x}_1), \dots, -p(\mathfrak{x}_m), \langle \mathrm{co}\{\mathfrak{x}_1, \dots, \mathfrak{x}_m\}, \mathfrak{z}_N \rangle) \to \inf .$$

- *For some feasible $\bar{\mathfrak{x}}_1, \dots, \bar{\mathfrak{x}}_m$ to have a Pareto-optimal convex hull it is necessary and sufficient that there be $\alpha_1, \dots, \alpha_m \mathbb{R}_+$ not vanishing simultaneously and positive Borel measures $\mu_1, \dots, \mu_m$ and $\nu_1, \dots, \nu_m$ on S_{N-1} such that*

$$\nu_1 + \dots + \nu_m = \mu(\mathfrak{z}_N);$$
$$\bar{\mathfrak{x}}_k(z) = \mathfrak{y}_k(z) \quad (z \in \mathrm{supp}(\mu_k));$$
$$\alpha_k \mu(\bar{\mathfrak{x}}_k) = \mu_k + \nu_k \ (k := 1, \dots, m).$$

Is Dido's Problem Solved?

- From a utilitarian standpoint, the answer is definitely in the affirmative. There is no evidence that Dido experienced any difficulties, showed indecisiveness, and procrastinated the choice of the tract of land. Practically speaking, the situation in which Dido made her decision was not as primitive as it seems at the first glance. The appropriate generality is unavailable in the mathematical model known as the classical isoperimetric problem.
- Dido's problem inspiring our ancestors remains the same intellectual challenge as Kant's starry heavens above and moral law within.