PROCEEDINGS Positivity IV - Theory and Applications
Dresden (Germany), 93-106 (2006)

Boolean Valued Analysis and Positivity

A. G. Kusraev and S. S. Kutateladze (Russia)

Abstract. This is an overview of the recent results of interaction of Boolean valued analysis and vector lattice theory.

Key words: Boolean valued analysis, vector lattice, positive operator

MSC 2000: 46 A 40

Boolean valued analysis is a general mathematical method that rests on a special model-theoretic technique. This technique consists generally in comparison between the representations of arbitrary mathematical objects and theorems in two different set-theoretic models whose constructions start with principally distinct Boolean algebras. We usually take as these models the cosiest Cantorian paradise, the von Neumann universe of Zermelo–Fraenkel set theory, and a special universe of Boolean valued "variable" sets trimmed and chosen so that the traditional concepts and facts of mathematics acquire completely unexpected and bizarre interpretations. The use of two models, one of which is *formally* nonstandard, is a family feature of *nonstandard analysis*. For this reason, Boolean valued analysis means an instance of nonstandard analysis in common parlance. By the way, the term *Boolean valued analysis* was minted by G. Takeuti.

Proliferation of Boolean valued models is due to P. Cohen's final breakthrough in Hilbert's Problem Number One. His method of forcing was rather intricate and the inevitable attempts at simplification gave rise to the Boolean valued models by D. Scott, R. Solovay, and P. Vopěnka.

Professor M. Weber had invited us to the Positivity Conference at the end of 2004 when we were completing our book "Introduction to Boolean Valued Analysis." The book was recently published in Russian and so this article is a kind of presentation.

Another recent event of relevance to this article is grievous. Saunders Mac Lane, a cofather of category theory, passed away in San Francisco on April 14, 2005. The power of mathematics rests heavily on the trick of socializing the objects and problems under consideration. The understanding of the social medium of set-theoretic models belongs to category theory.

Topos theory provides a profusion of categories of which classical set theory is an ordinary member. Mathematics has thus acquired infinitely many new degrees of freedom. All these achievements rest on category theory.

"There remains to us, then, the pursuit of truth, by way of proof, the concatenation of those ideas which fit, and the beauty which results when they do fit." So wrote Saunders Mac Lane, a great genius, creator, master, and servant of mathematics. We reverently dedicate this article to the memory of this eternal and tragicomical mathematical Knight of the Sorrowful Figure.

1 Boolean Requisites

We start with recalling some auxiliary facts about the construction and treatment of Boolean valued models.

1.1. Let $\mathbb{B}$ be a complete Boolean algebra. Given an ordinal α, put

$$\mathbb{V}_\alpha^{(\mathbb{B})} := \{x : x \text{ is a function } \wedge\ (\exists\beta)(\beta < \alpha\ \wedge \operatorname{dom}(x) \subset \mathbb{V}_\beta^{(\mathbb{B})}\ \wedge\ \operatorname{im}(x) \subset \mathbb{B})\}.$$

After this recursive definition the *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ or, in other words, the *class of* $\mathbb{B}$*-sets* is introduced by

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha\in\mathrm{On}} \mathbb{V}_\alpha^{(\mathbb{B})},$$

with On standing for the class of all ordinals.

In case of the two element Boolean algebra $2 := \{\mathbb{0}, \mathbb{1}\}$ this procedure yields a version of the classical *von Neumann universe* $\mathbb{V}$ (see 2.1 (2)).

Let φ be an arbitrary formula of ZFC, Zermelo–Fraenkel set theory with choice. The *Boolean truth value* $[\![\varphi]\!] \in \mathbb{B}$ is introduced by induction on the length of a formula φ by naturally interpreting the propositional connectives and quantifiers in the Boolean algebra $\mathbb{B}$ and taking into consideration the way in which this formula is built up from atomic formulas. The Boolean truth values of the *atomic formulas* $x \in y$ and $x = y$, with $x, y \in \mathbb{V}^{(\mathbb{B})}$, are defined by means of the following recursion schema:

$$[\![x \in y]\!] = \bigvee_{t\in\operatorname{dom}(y)} y(t) \wedge [\![t = x]\!],$$

$$[\![x = y]\!] = \bigvee_{t\in\operatorname{dom}(x)} x(t) \Rightarrow [\![t \in y]\!] \wedge \bigvee_{t\in\operatorname{dom}(y)} y(t) \Rightarrow [\![t \in x]\!].$$

The sign $\Rightarrow$ symbolizes the implication in $\mathbb{B}$; i.e., $a \Rightarrow b := a^* \vee b$ where a^* is as usual the *complement* of a.

The universe $\mathbb{V}^{(\mathbb{B})}$ with the Boolean truth value of a formula is a model of set theory in the sense that the following statement is fulfilled.

1.2. Transfer Principle. *For every theorem φ of* ZFC*, we have $[\![\varphi]\!] = \mathbb{1}$; i.e., φ is true inside $\mathbb{V}^{(\mathbb{B})}$.*

Enter into the next agreement: If x is an element of $\mathbb{V}^{(\mathbb{B})}$ and $\varphi(\cdot)$ is a formula of ZFC, then the phrase "x satisfies φ inside $\mathbb{V}^{(\mathbb{B})}$" or, briefly, "$\varphi(x)$ is true inside $\mathbb{V}^{(\mathbb{B})}$" means that $[\![\varphi(x)]\!] = \mathbb{1}$. This is sometimes written as $\mathbb{V}^{(\mathbb{B})} \models \varphi(x)$.

Given $x \in \mathbb{V}^{(\mathbb{B})}$ and $b \in \mathbb{B}$, define the function $bx : z \mapsto bx(z) \quad (z \in \operatorname{dom}(x))$. Here we presume that $b\varnothing := \varnothing$ for all $b \in \mathbb{B}$.

1.3. Mixing Principle. *Let $(b_\xi)_{\xi\in\Xi}$ be a partition of unity in $\mathbb{B}$, i.e. $\sup_{\xi\in\Xi} b_\xi = \sup\mathbb{B} = \mathbb{1}$ and $\xi \neq \eta \to b_\xi \wedge b_\eta = \mathbb{0}$. To each family $(x_\xi)_{\xi\in\Xi}$ in $\mathbb{V}^{(\mathbb{B})}$ there exists a unique element x in the separated universe such that $[\![x = x_\xi]\!] \geq b_\xi \quad (\xi \in \Xi)$.*

This element is called the *mixing* of $(x_\xi)_{\xi\in\Xi}$ by $(b_\xi)_{\xi\in\Xi}$ and is denoted by $\sum_{\xi\in\Xi} b_\xi x_\xi$.

1.4. Maximum Principle. *If φ is a formula of* ZFC *then there is a $\mathbb{B}$-valued set x_0 satisfying $[\![(\exists x)\varphi(x)]\!] = [\![\varphi(x_0)]\!]$.*

2 The Escher Rules

Boolean valued analysis consists primarily in comparison of the instances of a mathematical object or idea in two Boolean valued models. This is impossible to achieve without some dialog between the universes $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$. In other words, we need a smooth mathematical toolkit for revealing interplay between the interpretations of one and the same fact in the two models $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$. The relevant *ascending-and-descending technique* rests on the functors of canonical embedding, descent, and ascent.

2.1. We start with the canonical embedding of the von Neumann universe $\mathbb{V}$.

Given $x \in \mathbb{V}$, we denote by $x^{\wedge}$ the *standard name* of x in $\mathbb{V}^{(\mathbb{B})}$; i.e., the element defined by the following recursion schema: $\varnothing^{\wedge} := \varnothing$, $\operatorname{dom}(x^{\wedge}) := \{y^{\wedge} : y \in x\}$, $\operatorname{im}(x^{\wedge}) := \{\mathbb{1}\}$. Observe some properties of the mapping $x \mapsto x^{\wedge}$ we need in the sequel.

(1) For an arbitrary $x \in \mathbb{V}$ and a formula φ of ZFC we have

$$[\![(\exists y \in x^{\wedge})\, \varphi(y)]\!] = \bigvee_{z \in x} [\![\varphi(z^{\wedge})]\!],$$
$$[\![(\forall y \in x^{\wedge})\, \varphi(y)]\!] = \bigwedge_{z \in x} [\![\varphi(z^{\wedge})]\!].$$

(2) If x and y are elements of $\mathbb{V}$ then, by transfinite induction, we establish $x \in y \leftrightarrow \mathbb{V}^{(\mathbb{B})} \models x^{\wedge} \in y^{\wedge}$, $\quad x = y \leftrightarrow \mathbb{V}^{(\mathbb{B})} \models x^{\wedge} = y^{\wedge}$. In other words, the standard name can be considered as an embedding of $\mathbb{V}$ into $\mathbb{V}^{(\mathbb{B})}$. Moreover, it is beyond a doubt that the standard name sends $\mathbb{V}$ onto $\mathbb{V}^{(2)}$, which fact is demonstrated by the next proposition:

(3) The following holds: $(\forall u \in \mathbb{V}^{(2)})\, (\exists! x \in \mathbb{V})\, \mathbb{V}^{(\mathbb{B})} \models u = x^{\wedge}$.

A formula is called *bounded* or *restricted* if each bound variable in it is restricted by a bounded quantifier; i.e., a quantifier ranging over a particular set. The latter means that each bound variable x is restricted by a quantifier of the form $(\forall x \in y)$ or $(\exists x \in y)$ for some y.

2.2. Restricted Transfer Principle. *For each bounded formula* φ *of* ZFC *and every collection* $x_1, \dots, x_n \in \mathbb{V}$ *the following holds:* $\varphi(x_1, \dots, x_n) \leftrightarrow \mathbb{V}^{(\mathbb{B})} \models \varphi(x_1^{\wedge}, \dots, x_n^{\wedge})$. Henceforth, working in the separated universe $\overline{\mathbb{V}}^{(\mathbb{B})}$, we agree to preserve the symbol $x^{\wedge}$ for the distinguished element of the class corresponding to x.

Observe for example that the restricted transfer principle yields:

$$\text{“}\Phi \text{ is a correspondence from } x \text{ to } y\text{”}$$
$$\leftrightarrow \mathbb{V}^{(\mathbb{B})} \models \text{“}\Phi^{\wedge} \text{ is a correspondence from } x^{\wedge} \text{ to } y^{\wedge}\text{”};$$
$$\text{“}f : x \to y\text{”} \leftrightarrow \mathbb{V}^{(\mathbb{B})} \models \text{“}f^{\wedge} : x^{\wedge} \to y^{\wedge}\text{”}$$

(moreover, $f(a)^{\wedge} = f^{\wedge}(a^{\wedge})$ for all $a \in x$). Thus, the standard name can be considered as a covariant functor of the category of sets (or correspondences) inside $\mathbb{V}$ to an appropriate subcategory of $\mathbb{V}^{(2)}$ in the separated universe $\mathbb{V}^{(\mathbb{B})}$.

2.3. A set X is *finite* if X coincides with the image of a function on a finite ordinal. In symbols, this is expressed as $\operatorname{fin}(X)$; hence,

$$\operatorname{fin}(X) := (\exists\, n)(\exists\, f)(n \in \omega \wedge f \text{ is a function} \wedge\ \operatorname{dom}(f) = n \wedge \operatorname{im}(f) = X)$$

(as usual $\omega := \{0, 1, 2, \dots\}$). Obviously, the above formula is not bounded. Nevertheless there is a simple transformation rule for the class of finite sets under the canonical embedding. Denote by $\mathscr{P}_{\mathrm{fin}}(X)$ the class of all finite subsets of X; i.e., $\mathscr{P}_{\mathrm{fin}}(X) := \{Y \in \mathscr{P}(X) \ : \mathrm{fin}(Y)\}$. For an arbitrary set X the following holds: $\mathbb{V}^{(\mathbb{B})} \models \mathscr{P}_{\mathrm{fin}}(X)^{\wedge} = \mathscr{P}_{\mathrm{fin}}(X^{\wedge})$.

2.4. Given an arbitrary element x of the (separated) Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$, we define the *descent* $x{\downarrow}$ of x as $x{\downarrow} := \{y \in \mathbb{V}^{(\mathbb{B})} : [\![y \in x]\!] = \mathbb{1}\}$. We list the simplest properties of descending:

(1) The class $x{\downarrow}$ is a set, i.e., $x{\downarrow} \in \mathbb{V}$ for all $x \in \mathbb{V}^{(\mathbb{B})}$. If $[\![x \neq \varnothing]\!] = \mathbb{1}$ then $x{\downarrow}$ is a nonempty set.

(2) Let $z \in \mathbb{V}^{(\mathbb{B})}$ and $[\![z \neq \varnothing]\!] = \mathbb{1}$. Then for every formula φ of ZFC we have

$$[\![(\forall x \in z)\,\varphi(x)]\!] = \bigwedge_{x \in z{\downarrow}} [\![\varphi(x)]\!],$$

$$[\![(\exists x \in z)\,\varphi(x)]\!] = \bigvee_{x \in z{\downarrow}} [\![\varphi(x)]\!].$$

Moreover, there exists $x_0 \in z{\downarrow}$ such that $[\![\varphi(x_0)]\!] = [\![(\exists x \in z)\,\varphi(x)]\!]$.

(3) Let Φ be a correspondence from X to Y in $\mathbb{V}^{(\mathbb{B})}$. Thus, Φ, X, and Y are elements of $\mathbb{V}^{(\mathbb{B})}$ and, moreover, $[\![\Phi \subset X \times Y]\!] = \mathbb{1}$. There is a unique correspondence $\Phi{\downarrow}$ from $X{\downarrow}$ to $Y{\downarrow}$ such that $\Phi{\downarrow}(A{\downarrow}) = \Phi(A){\downarrow}$ for every nonempty subset A of X inside $\mathbb{V}^{(\mathbb{B})}$. The correspondence $\Phi{\downarrow}$ from $X{\downarrow}$ to $Y{\downarrow}$ of the above proposition is called the *descent* of the correspondence Φ from X to Y inside $\mathbb{V}^{(\mathbb{B})}$.

(4) The descent of the composite of correspondences inside $\mathbb{V}^{(\mathbb{B})}$ is the composite of their descents: $(\Psi \circ \Phi){\downarrow} = \Psi{\downarrow} \circ \Phi{\downarrow}$.

(5) If Φ is a correspondence inside $\mathbb{V}^{(\mathbb{B})}$ then $(\Phi^{-1}){\downarrow} = (\Phi{\downarrow})^{-1}$.

(6) Let Id_X be the identity mapping inside $\mathbb{V}^{(\mathbb{B})}$ of a set $X \in \mathbb{V}^{(\mathbb{B})}$. Then $(\mathrm{Id}_X){\downarrow} = \mathrm{Id}_{X{\downarrow}}$.

(7) Suppose that $X, Y, f \in \mathbb{V}^{(\mathbb{B})}$ are such that $[\![f : X \to Y]\!] = \mathbb{1}$, i.e., f is a mapping from X to Y inside $\mathbb{V}^{(\mathbb{B})}$. Then $f{\downarrow}$ is a unique mapping from $X{\downarrow}$ to $Y{\downarrow}$ satisfying $[\![f{\downarrow}(x) = f(x)]\!] = \mathbb{1}$ for all $x \in X{\downarrow}$.

By virtue of (1)–(7), we can consider the descent operation as a functor from the category of $\mathbb{B}$-valued sets and mappings (correspondences) to the category of the usual sets and mappings (correspondences) (i.e., in the sense of $\mathbb{V}$).

(8) Given $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$, denote by $(x_1, \dots, x_n)^{\mathbb{B}}$ the corresponding ordered n-tuple inside $\mathbb{V}^{(\mathbb{B})}$. Assume that P is an n-ary relation on X inside $\mathbb{V}^{(\mathbb{B})}$; i.e., $X, P \in \mathbb{V}^{(\mathbb{B})}$ and $[\![P \subset X^{n^{\wedge}}]\!] = \mathbb{1}$, where $n \in \omega$. Then there exists an n-ary relation P' on $X{\downarrow}$ such that $(x_1, \dots, x_n) \in P' \leftrightarrow [\![(x_1, \dots, x_n)^{\mathbb{B}} \in P]\!] = \mathbb{1}$. Slightly abusing notation, we denote the relation P' by the same symbol $P{\downarrow}$ and call it the *descent* of P.

2.5. Let $x \in \mathbb{V}$ and $x \subset \mathbb{V}^{(\mathbb{B})}$; i.e., let x be some set composed of $\mathbb{B}$-valued sets or, in other words, $x \in \mathscr{P}(\mathbb{V}^{(\mathbb{B})})$. Put $\varnothing{\uparrow} := \varnothing$ and $\mathrm{dom}(x{\uparrow}) := x$, $\quad \mathrm{im}(x{\uparrow}) := \{\mathbb{1}\}$ if $x \neq \varnothing$. The element $x{\uparrow}$ (of the separated universe $\mathbb{V}^{(\mathbb{B})}$, i.e., the distinguished representative of the class $\{y \in \mathbb{V}^{(\mathbb{B})} : [\![y = x{\uparrow}]\!] = \mathbb{1}\}$) is called the *ascent* of x.

(1) For all $x \in \mathscr{P}(\mathbb{V}^{(\mathbb{B})})$ and every formula φ we have the following:

$$[\![(\forall z \in x{\uparrow})\,\varphi(z)]\!] = \bigwedge_{y\in x}[\![\varphi(y)]\!],$$
$$[\![(\exists z \in x{\uparrow})\,\varphi(z)]\!] = \bigvee_{y\in x}[\![\varphi(y)]\!].$$

Introducing the ascent of a correspondence $\Phi \subset X \times Y$, we have to bear in mind a possible distinction between the domain of departure X and the domain $\mathrm{dom}(\Phi) := \{x \in X : \Phi(x) \neq \varnothing\}$. This circumstance is immaterial for the sequel; therefore, speaking of ascents, we always imply total correspondences; i.e., $\mathrm{dom}(\Phi) = X$.

(2) Let $X, Y, \Phi \in \mathbb{V}^{(\mathbb{B})}$, and let Φ be a correspondence from X to Y. There exists a unique correspondence $\Phi{\uparrow}$ from $X{\uparrow}$ to $Y{\uparrow}$ inside $\mathbb{V}^{(\mathbb{B})}$ such that $\Phi{\uparrow}(A{\uparrow}) = \Phi(A){\uparrow}$ is valid for every subset A of $\mathrm{dom}(\Phi)$ if and only if Φ is *extensional*; i.e., satisfies the condition $y_1 \in \Phi(x_1) \to [\![x_1 = x_2]\!] \leq \bigvee_{y_2\in\Phi(x_2)}[\![y_1 = y_2]\!]$ for $x_1, x_2 \in \mathrm{dom}(\Phi)$. In this event, $\Phi{\uparrow} = \Phi'{\uparrow}$, where $\Phi' := \{(x,y)^{\mathbb{B}} : (x,y) \in \Phi\}$. The element $\Phi{\uparrow}$ is called the *ascent* of the initial correspondence Φ.

(3) The composite of extensional correspondences is extensional. Moreover, the ascent of a composite is equal to the composite of the ascents inside $\mathbb{V}^{(\mathbb{B})}$: On assuming that $\mathrm{dom}(\Psi) \supset \mathrm{im}(\Phi)$ we have $\mathbb{V}^{(\mathbb{B})} \models (\Psi \circ \Phi){\uparrow} = \Psi{\uparrow} \circ \Phi{\uparrow}$.

Note that if Φ and Φ^{-1} are extensional then $(\Phi{\uparrow})^{-1} = (\Phi^{-1}){\uparrow}$. However, in general, the extensionality of Φ in no way guarantees the extensionality of Φ^{-1}.

(4) It is worth mentioning that if an extensional correspondence f is a function from X to Y then the ascent $f{\uparrow}$ of f is a function from $X{\uparrow}$ to $Y{\uparrow}$. Moreover, the extensionality property can be stated as follows: $[\![x_1 = x_2]\!] \leq [\![f(x_1) = f(x_2)]\!]$ for all $x_1, x_2 \in X$.

2.6. Given a set $X \subset \mathbb{V}^{(\mathbb{B})}$, we denote by the symbol $\mathrm{mix}(X)$ the set of all mixings of the form $\mathrm{mix}(b_\xi x_\xi)$, where $(x_\xi) \subset X$ and (b_ξ) is an arbitrary partition of unity. The following propositions are referred to as the *arrow cancellation rules* or *ascending-and-descending rules*. There are many good reasons to call them simply the *Escher rules*.

(1) Let X and X' be subsets of $\mathbb{V}^{(\mathbb{B})}$ and let $f : X \to X'$ be an extensional mapping. Suppose that $Y, Y', g \in \mathbb{V}^{(\mathbb{B})}$ are such that $[\![\, Y \neq \varnothing]\!] = [\![\, g : Y \to Y']\!] = \mathbb{1}$. Then $X{\uparrow}{\downarrow} = \mathrm{mix}(X), Y{\downarrow}{\uparrow} = Y, f{\uparrow}{\downarrow} = f$, and $g{\downarrow}{\uparrow} = g$.

(2) From 2.3 (8) we easily infer the useful relation: $\mathscr{P}_{\mathrm{fin}}(X{\uparrow}) = \{\theta{\uparrow} : \theta \in \mathscr{P}_{\mathrm{fin}}(X)\}{\uparrow}$.

Suppose that $X \in \mathbb{V}$, $X \neq \varnothing$; i.e., X is a nonempty set. Let the letter ι denote the standard name embedding $x \mapsto x^{\wedge}$ $(x \in X)$. Then $\iota(X){\uparrow} = X^{\wedge}$ and $X = \iota^{-1}(X^{\wedge}{\downarrow})$. Using the above relations, we may extend the descent and ascent operations to the case in which Φ is a correspondence from X to $Y{\downarrow}$ and $[\![\Psi$ is a correspondence from $X^{\wedge}$ to $Y]\!] = \mathbb{1}$, where $Y \in \mathbb{V}^{(\mathbb{B})}$. Namely, we put $\Phi{\Uparrow} := (\Phi \circ \iota){\uparrow}$ and $\Psi{\Downarrow} := \Psi{\downarrow} \circ \iota$. In this case, $\Phi{\Uparrow}$ is called the *modified ascent* of Φ and $\Psi{\Downarrow}$ is called the *modified descent* of Ψ. (If the context excludes ambiguity then we briefly speak of ascents and descents using simple arrows.) It is easy to see that $\Psi{\Uparrow}$ is a unique correspondence inside $\mathbb{V}^{(\mathbb{B})}$ satisfying the relation $[\![\Phi{\Uparrow}(x^{\wedge}) = \Phi(x){\uparrow}]\!] = \mathbb{1}\quad (x \in X)$. Similarly, $\Psi{\Downarrow}$ is a unique correspondence from X to $Y{\downarrow}$ satisfying the equality $\Psi{\Downarrow}(x) = \Psi(x^{\wedge}){\downarrow}\quad (x \in X)$. If $\Phi := f$ and $\Psi := g$ are functions then these relations take the form $[\![f{\Uparrow}(x^{\wedge}) = f(x)]\!] = \mathbb{1}$ and $g{\Downarrow}(x) = g(x^{\wedge})$ for all $x \in X$.

2.7. Various function spaces reside in functional analysis, and so the problem is natural of replacing an abstract Boolean valued system by some function-space analog, a model whose elements are functions and in which the basic logical operations are calculated "pointwise." An example of such a model is given by the class $\mathbb{V}^Q$ of all functions defined on a fixed nonempty set Q and acting into $\mathbb{V}$. The truth values on $\mathbb{V}^Q$ are various subsets of Q: The truth value $[\![\varphi(u_1,\dots,u_n)]\!]$ of $\varphi(t_1,\dots,t_n)$ at functions $u_1,\dots,u_n \in \mathbb{V}^Q$ is calculated as follows:

$$[\![\varphi(u_1,\dots,u_n)]\!] = \big\{q \in Q:\ \varphi\big(u_1(q),\dots,u_n(q)\big)\big\}.$$

A. G. Gutman and G. A. Losenkov solved the above problem by the concept of continuous polyverse which is a continuous bundle of models of set theory. It is shown that the class of continuous sections of a continuous polyverse is a Boolean valued system satisfying all basic principles of Boolean valued analysis and, conversely, each Boolean valued algebraic system can be represented as the class of sections of a suitable continuous polyverse. More details are collected in [15, Chapter 6].

2.8. Every Boolean valued universe has the collection of mathematical objects in full supply: available in plenty are all sets with extra structure: groups, rings, algebras, normed spaces, etc.

An *abstract Boolean set* or *set with* $\mathbb{B}$-*structure* is a pair (X,d), where $X \in \mathbb{V}$, $X \neq \varnothing$, and d is a mapping from $X \times X$ to $\mathbb{B}$ such that $d(x,y) = \mathbb{0} \leftrightarrow x = y$; $d(x,y) = d(y,x)$; $d(x,y) \leq d(x,z) \vee d(z,y)$ all $x,y,z \in X$.

To obtain an easy example of an abstract $\mathbb{B}$-set, given $\varnothing \neq X \subset \mathbb{V}^{(\mathbb{B})}$ put

$$d(x,y) := [\![x \neq y]\!] = \neg[\![x = y]\!]$$

for $x,\ y \in X$.

Another easy example is a nonempty X with the *discrete* $\mathbb{B}$-*metric* d; i.e., $d(x,y) = \mathbb{1}$ if $x \neq y$ and $d(x,y) = \mathbb{0}$ if $x = y$.

Let (X,d) be some abstract $\mathbb{B}$-set. There exist an element $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ and an injection $\iota : X \to X' := \mathscr{X}{\downarrow}$ such that $d(x,y) = [\![\iota x \neq \iota y]\!]$ for all $x,y \in X$ and every element $x' \in X'$ admits the representation $x' = \mathrm{mix}_{\xi\in\Xi}(b_\xi \iota x_\xi)$, where $(x_\xi)_{\xi\in\Xi} \subset X$ and $(b_\xi)_{\xi\in\Xi}$ is a partition of unity in $\mathbb{B}$. The element $\mathscr{X} \in \mathbb{V}^{(\mathbb{B})}$ is referred to as the *Boolean valued realization* of X.

If X is a discrete abstract $\mathbb{B}$-set then $\mathscr{X} = X^\wedge$ and $\iota x = x^\wedge$ for all $x \in X$. If $X \subset \mathbb{V}^{(\mathbb{B})}$ then $\iota{\uparrow}$ is an injection from $X{\uparrow}$ to $\mathscr{X}$ (inside $\mathbb{V}^{(\mathbb{B})}$). A mapping f from a $\mathbb{B}$-set (X,d) to a $\mathbb{B}$-set (X',d') is said to be *contractive* if $d(x,y) \geq d'(f(x),f(y))$ for all $x,y \in X$.

We see that an abstract $\mathbb{B}$-set X embeds in the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ so that the Boolean distance between the members of X becomes the Boolean truth value of the negation of their equality. The corresponding element of $\mathbb{V}^{(\mathbb{B})}$ is, by definition, the *Boolean valued representation* of X.

In case a $\mathbb{B}$-set X has some a priori structure we may try to furnish the Boolean valued representation of X with an analogous structure, so as to apply the technique of ascending and descending to the study of the original structure of X. Consequently, the above questions may be treated as instances of the unique problem of searching a well-qualified Boolean valued representation of a $\mathbb{B}$-set with some additional structure.

We call these objects *algebraic* $\mathbb{B}$-*systems*. Located at the epicenter of exposition, the notion of an algebraic $\mathbb{B}$-system refers to a nonempty $\mathbb{B}$-set endowed with a few contractive operations and $\mathbb{B}$-predicates, the latter meaning $\mathbb{B}$-valued contractive mappings.

The Boolean valued representation of an algebraic $\mathbb{B}$-system appears to be a conventional two valued algebraic system of the same type. This means that an appropriate completion of each algebraic $\mathbb{B}$-system coincides with the descent of some two valued algebraic system inside $\mathbb{V}^{(\mathbb{B})}$.

On the other hand, each two valued algebraic system may be transformed into an algebraic $\mathbb{B}$-system on distinguishing a complete Boolean algebra of congruences of the original system. In this event, the task is in order of finding the formulas holding true in direct or reverse transition from a $\mathbb{B}$-system to a two valued system. In other words, we have to seek here for some versions of the transfer or identity preservation principle of long standing in some branches of mathematics.

3 Boolean Valued Numbers

Boolean valued analysis stems from the fact that each internal field of reals of a Boolean valued model descends into a universally complete Kantorovich space. Thus, a remarkable opportunity opens up to expand and enrich the treasure-trove of mathematical knowledge by translating information about the reals to the language of other noble families of functional analysis. We will elaborate upon the matter in this section.

3.1. Recall a few definitions. Two elements x and y of a vector lattice E are called *disjoint* (in symbols $x \perp y$) if $|x| \wedge |y| = 0$. A *band* of E is defined as the *disjoint complement* $M^{\perp} := \{x \in E : (\forall y \in M)\, x \perp y\}$ of a nonempty set $M \subset E$.

The inclusion-ordered set $\mathfrak{B}(E)$ of all bands in E is a complete Boolean algebra with the Boolean operations:

$$L \wedge K = L \cap K, \quad L \vee K = (L \cup K)^{\perp\perp}, \quad L^* = L^{\perp} \quad (L, K \in \mathfrak{B}(E)).$$

The Boolean algebra $\mathfrak{B}(E)$ is often referred as to the *base* of E.

A *band projection* in E is a linear idempotent operator in $\pi : E \to E$ satisfying the inequalities $0 \leq \pi x \leq x$ for all $0 \leq x \in E$. The set $\mathfrak{P}(E)$ of all band projections ordered by $\pi \leq \rho \Longleftrightarrow \pi \circ \rho = \pi$ is a Boolean algebra with the Boolean operations:

$$\pi \wedge \rho = \pi \circ \rho, \quad \pi \vee \rho = \pi + \rho - \pi \circ \rho, \quad \pi^* = I_E - \pi \quad (\pi, \rho \in (E)).$$

Let $u \in E_+$ and $e \wedge (u - e) = 0$ for some $0 \leq e \in E$. Then e is a *fragment* or *component* of u. The set $\mathfrak{E}(u)$ of all fragments of u with the order induced by E is a Boolean algebra where the lattice operations are taken from E and the Boolean complement has the form $e^* := u - e$.

3.2. A Dedekind complete vector lattice is also called a *Kantorovich space* or *K-space,* for short. A K-space E is *universally complete* if every family of pairwise disjoint elements of E is order bounded.

(1) Theorem. *Let E be an arbitrary K-space. Then the correspondence $\pi \mapsto \pi(E)$ determines an isomorphism of the Boolean algebras $\mathfrak{P}(E)$ and $\mathfrak{B}(E)$. If there is an order unity $\mathbb{1}$ in E then the mappings $\pi \mapsto \pi\mathbb{1}$ from $\mathfrak{P}(E)$ into $\mathfrak{E}(E)$ and $e \mapsto \{e\}^{\perp\perp}$ from $\mathfrak{E}(E)$ into $\mathfrak{B}(E)$ are isomorphisms of Boolean algebras too.*

(2) Theorem. *Each universally complete K-space E with order unity $\mathbb{1}$ can be uniquely endowed by multiplication so as to make E into a faithful f-algebra and $\mathbb{1}$ into a ring unity. In this f-algebra each band projection $\pi \in \mathfrak{P}(E)$ is the operator of multiplication by $\pi(\mathbb{1})$.*

3.3. By a *field of reals* we mean every algebraic system that satisfies the axioms of an Archimedean ordered field (with distinct zero and unity) and enjoys the axiom of completeness. The same object can be defined as a one-dimensional K-space.

Recall the well-known assertion of ZFC: *There exists a field of reals* $\mathbb{R}$ *that is unique up to isomorphism.*

Successively applying the transfer and maximum principles, we find an element $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ for which $[\![\, \mathscr{R}$ is a field of reals $]\!] = \mathbb{1}$. Moreover, if an arbitrary $\mathscr{R}' \in \mathbb{V}^{(\mathbb{B})}$ satisfies the condition $[\![\mathscr{R}'$ is a field of reals $]\!] = \mathbb{1}$ then $[\![$ the ordered fields $\mathscr{R}$ and $\mathscr{R}'$ are isomorphic $]\!] = \mathbb{1}$. In other words, there exists an internal field of reals $\mathscr{R} \in \mathbb{V}^{(\mathbb{B})}$ which is unique up to isomorphism.

By the same reasons there exists an internal field of complex numbers $\mathscr{C} \in \mathbb{V}^{(\mathbb{B})}$ which is unique up to isomorphism. Moreover, $\mathbb{V}^{(\mathbb{B})} \models \mathscr{C} = \mathscr{R} \oplus i\mathscr{R}$. We call $\mathscr{R}$ and $\mathscr{C}$ the *internal reals* and *internal complexes* in $\mathbb{V}^{(\mathbb{B})}$.

3.4. Consider another well-known assertion of ZFC: *If* $\mathbb{P}$ *is an Archimedean ordered field then there is an isomorphic embedding* h *of the field* $\mathbb{P}$ *into* $\mathbb{R}$ *such that the image* $h(\mathbb{P})$ *is a subfield of* $\mathbb{R}$ *containing the subfield of rational numbers. In particular,* $h(\mathbb{P})$ *is dense in* $\mathbb{R}$*.*

Note also that $\varphi(x)$, presenting the conjunction of the axioms of an Archimedean ordered field x, is bounded; therefore, $[\![\, \varphi(\mathbb{R}^\wedge) \,]\!] = \mathbb{1}$, i.e., $[\![\, \mathbb{R}^\wedge$ is an Archimedean ordered field $]\!] = \mathbb{1}$. "Pulling" 3.2 (2) through the transfer principle, we conclude that $[\![\, \mathbb{R}^\wedge$ is isomorphic to a dense subfield of $\mathscr{R} \,]\!] = \mathbb{1}$. We further assume that $\mathbb{R}^\wedge$ is a dense subfield of $\mathscr{R}$ and $\mathbb{C}^\wedge$ is a dense subfield of $\mathscr{C}$. It is easy to note that the elements $0^\wedge$ and $1^\wedge$ are the zero and unity of $\mathscr{R}$.

Observe that the equalities $\mathscr{R} = \mathbb{R}^\wedge$ and $\mathscr{C} = \mathbb{C}^\wedge$ are not valid in general. Indeed, the axiom of completeness for $\mathbb{R}$ is not a bounded formula and so it may thus fail for $\mathbb{R}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$.

3.5. Look now at the descent $\mathscr{R}{\downarrow}$ of the algebraic system $\mathscr{R}$. In other words, consider the descent of the underlying set of the system $\mathscr{R}$ together with descended operations and order. For simplicity, we denote the operations and order in $\mathscr{R}$ and $\mathscr{R}{\downarrow}$ by the same symbols $+$, $\cdot$, and $\leq$. In more detail, we introduce addition, multiplication, and order in $\mathscr{R}{\downarrow}$ by the formulas

$$
\begin{gathered}
z = x + y \leftrightarrow [\![\, z = x + y \,]\!] = \mathbb{1}, \\
z = x \cdot y \leftrightarrow [\![\, z = x \cdot y \,]\!] = \mathbb{1}, \\
x \leq y \leftrightarrow [\![\, x \leq y \,]\!] = \mathbb{1} \quad (x, y, z \in \mathscr{R}{\downarrow}).
\end{gathered}
$$

Also, we may introduce multiplication by the usual reals in $\mathscr{R}{\downarrow}$ by the rule

$$
y = \lambda x \leftrightarrow [\![\, \lambda^\wedge x = y \,]\!] = \mathbb{1} \quad (\lambda \in \mathbb{R},\ x, y \in \mathscr{R}{\downarrow}).
$$

The fundamental result of Boolean valued analysis is Gordon's Theorem which reads as follows: *Each universally complete Kantorovich space is an interpretation of the reals in an appropriate Boolean valued model.* Formally, we have the following

3.6. Gordon Theorem. *Let* $\mathscr{R}$ *be the reals inside* $\mathbb{V}^{(\mathbb{B})}$*. Then* $\mathscr{R}{\downarrow}$*, (with the descended operations and order, is a universally complete* K*-space with order unity* 1*. Moreover, there exists an isomorphism* χ *of* $\mathbb{B}$ *onto* $\mathfrak{P}(\mathscr{R}{\downarrow})$ *such that*

$$
\chi(b)x = \chi(b)y \leftrightarrow b \leq [\![\, x = y \,]\!], \quad \chi(b)x \leq \chi(b)y \leftrightarrow b \leq [\![\, x \leq y \,]\!]
$$

for all $x, y \in \mathscr{R}{\downarrow}$ *and* $b \in \mathbb{B}$*.*

The converse is also true: Each Archimedean vector lattice embeds in a Boolean valued model, becoming a vector sublattice of the reals (viewed as such over some dense subfield of the reals).

3.7. Theorem. *Let E be an Archimedean vector lattice, let $\mathscr{R}$ be the reals inside $\mathbb{V}^{(\mathbb{B})}$, and let $\jmath$ be an isomorphism of $\mathbb{B}$ onto $\mathfrak{B}(E)$. Then there is $\mathscr{E} \in \mathbb{V}^{(\mathbb{B})}$ such that*

(1) *$\mathscr{E}$ is a vector sublattice of $\mathscr{R}$ over $\mathbb{R}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$;*

(2) *$E' := \mathscr{E}{\downarrow}$ is a vector sublattice of $\mathscr{R}{\downarrow}$ invariant under every band projection $\chi(b)$ $(b \in \mathbb{B})$ and such that each set of positive pairwise disjoint sets in it has a supremum;*

(3) *there is an o-continuous lattice isomorphism $\iota : E \to E'$ such that $\iota(E)$ is a coinitial sublattice of $\mathscr{R}{\downarrow}$;*

(4) *for every $b \in \mathbb{B}$ the band projection in $\mathscr{R}{\downarrow}$ onto $\{\iota(\jmath(b))\}^{\perp\perp}$ coincides with $\chi(b)$.*

Note also that $\mathscr{E}$ and $\mathscr{R}$ coincide if and only if E is Dedekind complete. Thus, each theorem about the reals within Zermelo–Fraenkel set theory has an analog in an arbitary Kantorovich space. Translation of theorems is carried out by appropriate general functors of Boolean valued analysis. In particular, the most important structural properties of vector lattices such as the functional representation, spectral theorem, etc. are the ghosts of some properties of the reals in an appropriate Boolean valued model. More details and references are collected in [15].

3.8. The theory of vector lattices with a vast field of applications is thoroughly covered in many monographs (for instance, see [19, 23]). The credit for finding the most important instance among ordered vector spaces, an order complete vector lattice or K-space, is due to L. V. Kantorovich. This notion appeared in Kantorovich's first article on this topic, where he wrote: "In this note, I define a new type of space that I call a semiordered linear space. The introduction of such a space allows us to study linear operations of one abstract class (those with values in such a space) as linear functionals."

Thus the *heuristic transfer principle* was stated for K-spaces which becomes the Ariadna thread of many subsequent studies. The depth and universality of Kantorovich's principle are explained within Boolean valued analysis.

3.9. Applications of Boolean valued models to functional analysis stem from the works by E. I. Gordon and G. Takeuti. If $\mathbb{B}$ in 3.6 is the algebra of μ-measurable sets modulo μ-negligible sets then $\mathscr{R}{\downarrow}$ is isomorphic to the universally complete K-space $L^0(\mu)$ of measurable functions. This fact (for the Lebesgue measure on an interval) was already known to D. Scott and R. Solovay (see [15]). If $\mathbb{B}$ is a complete Boolean algebra of projections in a Hilbert space then $\mathscr{R}{\downarrow}$ is isomorphic to the space of selfadjoint operators $\mathfrak{A}(\mathbb{B})$. These two particular cases of Gordon's Theorem were intensively and fruitfully exploited by G. Takeuti (see the bibliography in [15]). The object $\mathscr{R}{\downarrow}$ for general Boolean algebras was also studied by T. Jech [8]. Theorem 3.7 was obtained by A. G. Kusraev [10]. A close result (in other terms) is presented in T. Jech's article [8] where some Boolean valued interpretation is revealed of the theory of linearly ordered sets. More details can be found in [15].

4 Band Preserving Operators

This section deals with the class of band preserving operators. Simplicity of these operators notwithstanding, the question about their order boundedness is far from trivial.

4.1. Recall that a complex K-space is the complexification $G_{\mathbb{C}} := G \oplus iG$ of a real K-space G. A linear operator $T : G_{\mathbb{C}} \to G_{\mathbb{C}}$ is *band preserving* or *contractive* or a *stabilizer* if, for all $f, g \in G_{\mathbb{C}}$, from $f \perp g$ it follows that $Tf \perp g$. Disjointness in $G_{\mathbb{C}}$ is defined just as in G (see 3.1), whereas $|z| := \sup\{\mathrm{Re}(e^{i\theta}z) : 0 \leq \theta \leq \pi\}$ for $z \in G_{\mathbb{C}}$.

(1) Let $\mathrm{End}_N(G_{\mathbb{C}})$ stand for the set of all band preserving linear operators in $G_{\mathbb{C}}$, with $G := \mathscr{R}{\downarrow}$. Clearly, $\mathrm{End}_N(G_{\mathbb{C}})$ is a complex vector space. Moreover, $\mathrm{End}_N(G_{\mathbb{C}})$ becomes a faithful unitary module over the ring $G_{\mathbb{C}}$ if we define gT as $gT : x \mapsto g \cdot Tx$ for all $x \in G$. This follows from the fact that multiplication by a member of $G_{\mathbb{C}}$ is a band preserving operator and the composite of band preserving operators is band preserving too.

(2) Denote by $\mathrm{End}_{\mathbb{C}^{\wedge}}(\mathscr{C})$ the element of $\mathbb{V}^{(\mathbb{B})}$ representing the space of all $\mathbb{C}^{\wedge}$-linear mappings from $\mathscr{C}$ to $\mathscr{C}$. Then $\mathrm{End}_{\mathbb{C}^{\wedge}}(\mathscr{C})$ is a vector space over $\mathbb{C}^{\wedge}$ inside $\mathbb{V}^{(\mathbb{B})}$, and $\mathrm{End}_{\mathbb{C}^{\wedge}}(\mathscr{C}){\downarrow}$ is a faithful unitary module over $G_{\mathbb{C}}$.

4.2. Following [11] it is easy to prove that a linear operator T in the K-space $G_{\mathbb{C}}$ is band preserving if and only if T is extensional. Since each extensional mapping has an ascent, $T \in \mathrm{End}_N(G_{\mathbb{C}})$ has the ascent $\tau := T{\uparrow}$ which is a unique internal functional from $\mathscr{C}$ to $\mathscr{C}$ such that $[\![\tau(x) = Tx]\!] = \mathbb{1}$ $(x \in G_{\mathbb{C}})$. We thus arrive at the following assertion:

The modules $\mathrm{End}_N(G_{\mathbb{C}})$ *and* $\mathrm{End}_{\mathbb{C}^{\wedge}}(\mathscr{C}){\downarrow}$ *are isomorphic by sending each band preserving operator to its ascent.*

By Gordon's Theorem this assertion means that the problem of finding a band preserving operator in G amounts to solving (for $\tau : \mathscr{C} \to \mathscr{C}$) inside $\mathbb{V}^{(\mathbb{B})}$ the *Cauchy functional equation*: $\tau(x + y) = \tau(x) + \tau(y) \quad (x, y \in \mathscr{C})$ under the subsidiary condition $\tau(\lambda x) = \lambda\tau(x) \quad (x \in \mathscr{C}, \lambda \in \mathbb{C}^{\wedge})$.

As another subsidiary condition we may consider the *Leibniz rule* $\tau(xy) = \tau(x)y + x\tau(y)$(in which case τ is called a $\mathbb{C}^{\wedge}$-*derivation*) or multiplicativity $\tau(xy) = \tau(x)\tau(y)$. These situations are addressed in 4.5.

4.3. An element $g \in G^+$ is *locally constant* with respect to $f \in G^+$ if $g = \bigvee_{\xi\in\Xi} \lambda_\xi \pi_\xi f$ for some numeric family $(\lambda_\xi)_{\xi\in\Xi}$ and a family $(\pi_\xi)_{\xi\in\Xi}$ of pairwise disjoint band projections. A universally complete K-space $G_{\mathbb{C}}$ is called *locally one-dimensional* if all elements of G^+ are locally constant with respect to some order unity of G (and hence each of them). Clearly, a K-space $G_{\mathbb{C}}$ is locally one-dimensional if each $g \in G_{\mathbb{C}}$ may be presented as $g = o\text{-}\sum_{\xi\in\Xi} \lambda_\xi \pi_\xi \mathbb{1} f$ with some family $(\lambda_\xi)_{\xi\in\Xi} \subset \mathbb{C}$ and partition of unity $(\pi_\xi)_{\xi\in\Xi} \subset \mathfrak{P}(G)$.

4.4. A σ-complete Boolean algebra $\mathbb{B}$ is called *σ-distributive* if

$$\bigvee_{n\in\mathbb{N}} \bigwedge_{m\in\mathbb{N}} b_{n,m} = \bigwedge_{\varphi\in\mathbb{N}^{\mathbb{N}}} \bigvee_{n\in\mathbb{N}} b_{n,\varphi(n)}.$$

for every double sequence $(b_{n,m})_{n,m\in\mathbb{N}}$ in $\mathbb{B}$. As an example of a σ-distributive Boolean algebra we may take a complete atomic Boolean algebra, i.e., the boolean of a nonempty set. It is worth observing that there are nonatomic σ-distributive complete Boolean algebras (see [12, 5.1.8]).

We now address the problem which is often referred to in the literature as *Wickstead's problem*: Characterize the universally complete vector lattices spaces in which every band preserving linear operator is order bounded.

According to 4.2, Boolean valued analysis reduces Wickstead's problem to that of order boundedness of the endomorphisms of the field $\mathscr{C}$ viewed as a vector space and algebra over the field $\mathbb{C}^{\wedge}$.

4.5. Theorem. *Let $\mathbb{P}$ be an algebraically closed and topologically dense subfield of the field of complexes $\mathbb{C}$. The following are equivalent:*

(1) $\mathbb{P} = \mathbb{C}$*;*

(2) *every $\mathbb{P}$-linear function on $\mathbb{C}$ is order bounded;*

(3) *there are no nontrivial $\mathbb{P}$-derivations on $\mathbb{C}$;*

(4) *each $\mathbb{P}$-linear endomorphism on $\mathbb{C}$ is the zero or identity function;*

(5) *there is no $\mathbb{P}$-linear automorphism on $\mathbb{C}$ other than the identity.*

The equivalence (1) $\leftrightarrow$ (2) is checked by using a Hamel basis of the vector space $\mathbb{C}$ over $\mathbb{P}$. The remaining equivalences rest on replacing a Hamel basis with a transcendence basis (for details see [13]).

Recall that a linear operator $D : G_{\mathbb{C}} \to G_{\mathbb{C}}$ is a $\mathbb{C}$-*derivation* if $D(fg) = D(f)g + fD(g)$ for all $f, g \in G_{\mathbb{C}}$. It can be easily checked that every $\mathbb{C}$-derivation is band preserving.

Interpreting Theorem 4.5 in $\mathbb{V}^{(\mathbb{B})}$, we arrive at

4.6. Theorem. *If $\mathbb{B}$ is a complete Boolean algebra then the following are equivalent:*

(1) $\mathscr{C} = \mathbb{C}^{\wedge}$ *inside* $\mathbb{V}^{(\mathbb{B})}$*;*

(2) *every band preserving linear operator is order bounded in the complex vector lattice $\mathscr{C}{\downarrow}$;*

(3) *there is no nontrivial $\mathbb{C}$-derivation in the complex f-algebra $\mathscr{C}{\downarrow}$;*

(4) *each band preserving endomorphism is a band projection in $\mathscr{C}{\downarrow}$;*

(5) *there is no band preserving automorphism other than the identity in $\mathscr{C}{\downarrow}$.*

(6) *the K-space $\mathscr{R}{\downarrow}$ is locally one-dimensional;*

(7) $\mathbb{B}$ *is σ-distributive.*

4.7. The question was raised by A. W. Wickstead in [22] whether every band preserving linear operator in a universally complete vector lattice is automatically order bounded. The first example of an unbounded band preserving linear operator was suggested by Yu. A. Abramovich, A. I. Veksler, and A. V. Koldunov in [1, 2]. The equivalence (1) $\leftrightarrow$ (6) is trivial, whereas (2) $\leftrightarrow$ (6) combines a result of Yu. A. Abramovich, A. I. Veksler, and A. V. Koldunov [1, Theorem 2.1] and that of P. T. N. McPolin and A. W. Wickstead [21, Theorem 3.2]. The equivalence (6) $\leftrightarrow$ (7) was obtained by A. E. Gutman who also found an example of a purely nonatomic locally one-dimensional Dedekind complete vector lattice (see [7]). The equivalences (1) $\leftrightarrow$ (3) $\leftrightarrow$ (4) $\leftrightarrow$ (5) belong to A. G. Kusraev [13].

5 Boolean Valued Positive Functionals

A linear functional on a vector space is determined up to a scalar from its zero hyperplane. In contrast, a linear operator is recovered from its kernel up to a simple multiplier on a rather special occasion. Fortunately, Boolean valued analysis prompts us that some operator analog of the functional case is valid for each operator with target a Kantorovich space, a Dedekind complete vector lattice. We now proceed along the lines of this rather promising approach.

5.1. Let E be a vector lattice, and let F be a K-space with base a complete Boolean algebra $\mathbb{B}$. By 3.2, we may assume that F is a nonzero space embedded as an order dense ideal in the universally complete Kantorovich space $\mathscr{R}{\downarrow}$ which is the descent of the reals $\mathscr{R}$ inside the separated Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ over $\mathbb{B}$.

An operator T is F-*discrete* if $[0, T] = [0, I_F] \circ T$; i.e., for all $0 \le S \le T$ there is some $0 \le \alpha \le I_F$ satisfying $S = \alpha \circ T$. Let $L_a^{\sim}(E, F)$ be the band in $L^{\sim}(E, F)$ spanned by F-

discrete operators and $L_d^\sim(E,F) := L_a^\sim(E,F)^\perp$. By analogy we define $(E^{\wedge\sim})_a$ and $(E^{\wedge\sim})_d$. The members of $L_d^\sim(E,F)$ are usually called F-*diffuse*.

5.2. As usual, we let $E^\wedge$ stand for the standard name of E in $\mathbb{V}^{(\mathbb{B})}$. Clearly, $E^\wedge$ is a vector lattice over $\mathbb{R}^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$. Denote by $\tau := T\!\uparrow$ the ascent of T to $\mathbb{V}^{(\mathbb{B})}$. Clearly, τ acts from $E^\wedge$ to the ascent $F\!\uparrow = \mathscr{R}$ of F inside the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$. Therefore, $\tau(x^\wedge) = Tx$ inside $\mathbb{V}^{(\mathbb{B})}$ for all $x \in E$, which means in terms of truth values that $[\![\tau : E^\wedge \to \mathscr{R}]\!] = \mathbb{1}$ and $(\forall x \in E)\ [\![\tau(x^\wedge) = Tx]\!] = \mathbb{1}$.

Let $E^{\wedge\sim}$ stand for the space of all order bounded $\mathbb{R}^\wedge$-linear functionals from $E^\wedge$ to $\mathscr{R}$. Clearly, $E^{\wedge\sim} := L^\sim(E^\wedge, \mathscr{R})$ is a K-space inside $\mathbb{V}^{(\mathbb{B})}$. The descent $E^{\wedge\sim}\!\downarrow$ of $E^{\wedge\sim}$ is a K-space. Given $S, T \in L^\sim(E,F)$, put $\tau := T\!\uparrow$ and $\sigma := S\!\uparrow$.

5.3. Theorem. *For each $T \in L^\sim(E,F)$ the ascent $T\!\uparrow$ of T is an order bounded $\mathbb{R}^\wedge$-linear functional on $E^\wedge$ inside $\mathbb{V}^{(\mathbb{B})}$; i.e., $[\![T\!\uparrow \in E^{\wedge\sim}]\!] = \mathbb{1}$. The mapping $T \mapsto T\!\uparrow$ is a lattice isomorphism of $L^\sim(E,F)$ and $E^{\wedge\sim}\!\downarrow$. Moreover, the following hold:*

(1) $T \geq 0 \ \leftrightarrow\ [\![\, \tau \geq 0 \,]\!] = \mathbb{1}$;

(2) *S is a fragment of $T \leftrightarrow [\![\, \sigma$ is a fragment of $\tau \,]\!] = \mathbb{1}$;*

(3) *T is a lattice homomorphism if and only if so is τ inside $\mathbb{V}^{(\mathbb{B})}$;*

(4) *T is F-diffuse $\leftrightarrow [\![\, \tau$ is diffuse $]\!] = \mathbb{1}$;*

(5) $T \in L_a^\sim(E,F) \ \leftrightarrow\ [\![\, \tau \in (E^{\wedge\sim})_a \,]\!] = \mathbb{1}$;

(6) $T \in L_d^\sim(E,F) \ \leftrightarrow\ [\![\, \tau \in (E^{\wedge\sim})_d \,]\!] = \mathbb{1}$.

Since τ, the ascent of an order bounded operator T, is defined up to a scalar from $\ker(\tau)$, we infer the following analog of the Sard Theorem.

5.4. Theorem. *Let S and T be linear operators from E to F. Then $\ker(bS) \supset \ker(bT)$ for all $b \in \mathbb{B}$ if and only if there is an orthomorphism α of F such that $S = \alpha T$.*

We see that a linear operator T is, in a sense, determined up to an orthomorphism from the family of the kernels of the *strata* bT of T. This remark opens a possibility of studying some properties of T in terms of the kernels of the strata of T.

5.5. Theorem. *An order bounded operator T from E to F may be presented as the difference of some lattice homomorphisms if and only if the kernel of each stratum bT of T is a vector sublattice of E for all $b \in \mathbb{B}$.*

Straightforward calculations of truth values show that $T_+\!\uparrow = \tau_+$ and $T_-\!\uparrow = \tau_-$ inside $\mathbb{V}^{(\mathbb{B})}$. Moreover, $[\![\ker(\tau)$ is a vector sublattice of $E^\wedge]\!] = \mathbb{1}$ whenever so are $\ker(bT)$ for all $b \in \mathbb{B}$. Since the ascent of a sum is the sum of the ascents of the summands, we reduce the proof of Theorem 5.5 to the case of the functionals on using 5.3 (3).

5.6. Recall that a subspace H of a vector lattice E is a *G-space* or *Grothendieck subspace* (cp. [6, 18]) provided that H enjoys the following property:

$$(\forall x, y \in H)\ (x \vee y \vee 0 + x \wedge y \wedge 0 \in H).$$

By simple calculations of truth values we infer that $[\![\ker(\tau)$ is a Grothendieck subspace of $E^\wedge]\!] = \mathbb{1}$ if and only if the kernel of each stratum bT is a Grothendieck subspace of E. We may now assert that the following appears as a result of “descending” its scalar analog.

5.7. Theorem. *The modulus of an order bounded operator $T : E \to F$ is the sum of some pair of lattice homomorphisms if and only if the kernel of each stratum bT of T with $b \in \mathbb{B}$ is a Grothendieck subspace of the ambient vector lattice E.*

To prove the relevant scalar versions of Theorems 5.5 and 5.7, we use one of the formulas of subdifferential calculus (cp. [14]):

5.8. Theorems 5.5 and 5.7 were obtained by S. S. Kutateladze in [16, 17]. Note that the sums of lattice homomorphisms were first described by S. J. Bernau, C. B. Huijsmans, and B. de Pagter in terms of n-disjoint operators in [3]. A survey of some conceptually close results on n-disjoint operators is given in [12].

References

[1] Abramovich Yu. A., Veksler A. I., and Koldunov A. V., "On disjointness preserving operators," Dokl. Akad. Nauk SSSR, **289**, No. 5, 1033–1036 (1979).

[2] Abramovich Yu. A., Veksler A. I., and Koldunov A. V., "Disjointness preserving operators, their continuity, and multiplicative representation," in: Linear Operators and Their Applications [in Russian], Leningrad: Leningrad Ped. Inst., 1981.

[3] Bernau S. J., Huijsmans C. B., and de Pagter B., "Sums of lattice homomorphisms," Proc. Amer. Math. Soc., **115**, No. 1, 51–156 (1992).

[4] Gordon E. I., "Real numbers in Boolean valued models of set theory and K-spaces," Dokl. Akad. Nauk SSSR, **237**, No. 4, 773–775 (1977).

[5] Gordon E. I., "K-spaces in Boolean valued models of set theory," Dokl. Akad. Nauk SSSR, **258**, No. 4, 777–780 (1981).

[6] Grothendieck A., "Une caractérisation vectorielle-métrique des espaces L^1," Canad. J. Math., **4**, 552–561 (1955).

[7] Gutman A. E., "Locally one-dimensional K-spaces and σ-distributive Boolean algebras," Siberian Adv. Math., **5**, No. 2, 99–121 (1995).

[8] Jech T. J., "Boolean-linear spaces," Adv. in Math., **81**, No. 2, 117–197 (1990).

[9] Jech T., *Lectures in Set Theory with Particular Emphasis on the Method of Forcing*, Berlin: Springer-Verlag, 1971.

[10] Kusraev A. G., "Numeric systems in Boolean valued models of set theory," in: Proceedings of the VIII All-Union Conference in Mathematical Logic (Moscow), Moscow, 1986, p. 99.

[11] Kusraev A. G., "On band preserving operators," Vladikavkaz Math. J., **6**, No. 3, 48–58 (2004).

[12] Kusraev A. G., *Dominated Operators*, Dordrecht: Kluwer Academic Publishers, 2000.

[13] Kusraev A. G., "Automorphisms and derivations in extended complex f-algebras," Siberian Math. J., **46**, No. 6 (2005).

[14] Kusraev A. G. and Kutateladze S. S., *Subdifferentials: Theory and Applications*, Novosibirk: Nauka, 1992; Dordrecht: Kluwer Academic Publishers, 1995.

[15] Kusraev A. G. and Kutateladze S. S., *Introduction to Boolean Valued Analysis* [in Russian], Moscow: Nauka, 2005.

[16] Kutateladze S. S., "On differences of Riesz homomorphisms," Siberian Math. J. **46**, No. 2, 305–307 (2005).

[17] Kutateladze S. S., "On Grothendieck subspaces," Siberian Math. J., **46**, No. 3, 489–493 (2005).

[18] Lindenstrauss J. and Wulbert D. E., "On the classification of the Banach spaces whose duals are L_1-spaces," J. Funct. Anal., **4**, No. 3, 322–249 (1969).

[19] Luxemburg W. A. J. and Zaanen A. C., *Riesz Spaces*. Vol. 1, Amsterdam; London: North-Holland, 1971.

[20] Maharam D., "On positive operators," Contemporary Math., **26**, 263–277 (1984).

[21] McPolin P. T. N. and Wickstead A. W., "The order boundedness of band preserving operators on uniformly complete vector lattices," Math. Proc. Cambridge Philos. Soc., **97**, No. 3, 481–487 (1985).

[22] Wickstead A. W., "Representation and duality of multiplication operators on Archimedean Riesz spaces," Compositio Math., **35**, No. 3, 225–238 (1977).

[23] Zaanen A. C., *Riesz Spaces*. Vol. 2, Amsterdam etc.: North-Holland, 1983.

Anatoly Kusraev
Institute of Applied Mathematics and Informatics
Vladikavkaz, Russia

Semen Kutateladze
Sobolev Institute
Novosibirsk, Russia