Fundamentals of Functional Analysis

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Fundamentals of Functional Analysis

by

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Preface
to the English Translation

This is a concise guide to basic sections of modern functional analysis. Included are such topics as the principles of Banach and Hilbert spaces, the theory of multinormed and uniform spaces, the Riesz-Dunford holomorphic functional calculus, the Fredholm index theory, convex analysis and duality theory for locally convex spaces.

With standard provisos the presentation is self-contained, exposing about a hundred famous "named" theorems furnished with complete proofs and culminating in the Gelfand-Naĭmark-Segal construction for C^*-algebras.

The first Russian edition was printed by the Siberian Division of "Nauka" Publishers in 1983. Since then the monograph has served as the standard textbook on functional analysis at the University of Novosibirsk.

This volume is translated from the second Russian edition printed by the Sobolev Institute of Mathematics of the Siberian Division of the Russian Academy of Sciences in 1995. It incorporates new sections on Radon measures, the Schwartz spaces of distributions, and a supplementary list of theoretical exercises and problems.

This edition was typeset using AMS-TEX, the American Mathematical Society's TEX system.

To clear my conscience completely, I also confess that := stands for the *definor*, the *assignment operator*, $\triangleleft$ marks the beginning of a (possibly empty) proof, and $\triangleright$ signifies the end of the proof.

S. Kutateladze

Preface to the First Russian Edition

As the title implies, this book treats functional analysis. At the turn of the century the term "functional analysis" was coined by J. Hadamard who is famous among mathematicians for the formula of the radius of convergence of a power series. The term "functional analysis" was universally accepted then as related to the calculus of variations, standing for a new direction of analysis which was intensively developed by V. Volterra, C. Arzelà, S. Pincherle, P. Levy, and other representatives of the French and Italian mathematical schools. J. Hadamard's contribution to the present discipline should not be reduced to the invention of the word "functional" (or more precisely to the transformation of the adjective into a proper noun). J. Hadamard was fully aware of the relevance of the rising subject. Working hard, he constantly advertised problems, ideas, and methods just evolved. In particular, to one of his students, M. Fréchet, he suggested the problem of inventing something that is now generally acclaimed as the theory of metric spaces. In this connection it is worth indicating that neighborhoods pertinent to functional analysis in the sense of Hadamard and Volterra served as precursors to Hausdorff's famous research, heralding the birth of general topology.

Further, it is essential to emphasize that one of the most attractive, difficult, and important sections of classical analysis, the calculus of variations, became the first source of functional analysis.

The second source of functional analysis was provided by the study directed to creating some algebraic theory for functional equations or, stated strictly, to simplifying and formalizing the manipulations of "equations in functions" and, in particular, linear integral equations. Ascending to H. Abel and J. Liouville, the theory of the latter was considerably expanded by works of I. Fredholm, K. Neumann, F. Noether,

A. Poincaré, et al. The efforts of these mathematicians fertilized soil for D. Hilbert's celebrated research into quadratic forms in infinitely many variables. His ideas, developed further by F. Riesz, E. Schmidt, et al., were the immediate predecessors of the axiomatic presentation of Hilbert space theory which was undertaken and implemented by J. von Neumann and M. Stone. The resulting section of mathematics has vigorously influenced theoretical physics, first of all, quantum mechanics. In this regard it is instructive as well as entertaining to mention that both terms, "quantum" and "functional," originated in the same year, 1900.

The third major source of functional analysis encompassed Minkowski's geometric ideas. His invention, the apparatus for the finite-dimensional geometry of convex bodies, prepared the bulk of spatial notions ensuring the modern development of analysis. Elaborated by E. Helly, H. Hahn, C. Carathéodory, I. Radon, et al., the idea of convexity has eventually shaped the fundamentals of the theory of locally convex spaces. In turn, the latter has facilitated the spread of distributions and weak derivatives which were recognized by S. L. Sobolev as drastically changing all tools of mathematical physics. In the postwar years the geometric notion of convexity has conquered a new sphere of application for mathematics, *viz.*, social sciences and especially economics. An exceptional role in this process was performed by linear programming discovered by L. V. Kantorovich.

The above synopsis of the history of functional analysis is schematic, incomplete, and arbitrary (for instance, it casts aside the line of D. Bernoulli's superposition principle, the line of set functions and integration theory, the line of operational calculus, the line of finite differences and fractional derivation, the line of general analysis, and many others). These demerits notwithstanding, the three sources listed above reflect the main, and most principal, regularity: functional analysis has synthesized and promoted ideas, concepts, and methods from classical sections of mathematics: algebra, geometry, and analysis. Therefore, although functional analysis *verbatim* means analysis of functions and functionals, even a superficial glance at its history gives grounds to claim that functional analysis is algebra, geometry, and analysis of functions and functionals.

A more viable and penetrating explanation for the notion of functional analysis is given by the Soviet Encyclopedic Dictionary: "Functional analysis is one of the principal branches of modern mathematics. It resulted from mutual interaction, unification, and generalization of the ideas and methods stemming from all parts of classical mathematical analysis. It is characterized by the use of concepts pertaining to various abstract spaces such as vector spaces, Hilbert spaces, etc. It finds diverse applications in modern physics, especially in quantum mechanics."

The S. Banach treatise *Theórie des Operationes Linéares*, printed half a century ago, inaugurated functional analysis as an essential activity in mathematics. Its influence on the development of mathematics is seminal: Omnipresent, Banach's ideas, propounded in the book, captivate the realm of modern mathematics.

An outstanding contribution toward progress in functional analysis was made by the renowned Soviet scientists: I. M. Gelfand, L. V. Kantorovich, M. V. Keldysh, A. N. Kolmogorov, M. G. Kreĭn, L. A. Lyusternik, and S. L. Sobolev. The characteristic feature of the Soviet school is that its research on functional analysis is always conducted in connection with profound applied problems. The research has expanded the scope of functional analysis which becomes the prevailing language of the applications of mathematics.

The next fact is demonstrative: In 1948 even the title of Kantorovich's insightful article *Functional Analysis and Applied Mathematics* was considered paradoxical, but it provided a basis for the numerical mathematics of today. And in 1974 S. L. Sobolev stated that "to conceive the theory of calculations without Banach spaces is just as impossible as trying to conceive of it without the use of computers".

The exponential accumulation of knowledge within functional analysis is now observed alongside a sharp rise in demand for the tools and concepts of the discipline. The resulting conspicuous gap widens permanently between the current level of analysis and the level fixed in the literature accessible to the reading community. To alter this ominous trend is the purpose of the present book.

Preface to the Second Russian Edition

For more than a decade the monograph has served as a reference book for compulsory and optional courses in functional analysis at Novosibirsk State University. This time span proves that the principles of compiling the book are legitimate. The present edition is enlarged with sections addressing the fundamentals of distribution theory. Theoretical exercises are supplemented and the list of references is updated. Also, inaccuracies, mostly indicated by my colleagues, have been corrected.

I seize the opportunity to express my gratitude to all those who helped me in the preparation of the book. My pleasant debt is to acknowledge the financial support of the Sobolev Institute of Mathematics of the Siberian Division of the Russian Academy of Sciences, the Russian Foundation for Fundamental Research, the International Science Foundation and the American Mathematical Society during the compilation of the second edition.

March, 1995 *S. Kutateladze*

Chapter 1
An Excursion into Set Theory

1.1. Correspondences

1.1.1. DEFINITION. Let A and B be sets and let F be a subset of the *product* $A \times B := \{(a,\ b) : a \in A,\ b \in B\}$. Then F is a *correspondence* with the *set of departure* A and the *set of arrival* B or just a correspondence *from* A (*in*)*to* B.

1.1.2. DEFINITION. For a correspondence $F \subset A \times B$ the set

$$\text{dom } F := D(F) := \{a \in A : \ (\exists\, b \in B)\ (a,\ b) \in F\}$$

is the *domain* (of definition) of F and the set

$$\text{im } F := R(F) := \{b \in B : \ (\exists\, a \in A)\ (a,\ b) \in F\}$$

is the *codomain* of F, or the *range* of F, or the *image* of F.

1.1.3. EXAMPLES.

(1) If F is a correspondence from A into B then

$$F^{-1} := \{(b,\ a) \in B \times A : \ (a,\ b) \in F\}$$

is a correspondence from B into A which is called *inverse* to F or the *inverse* of F. It is obvious that F is the inverse of F^{-1}.

(2) A *relation* F on A is by definition a subset of A^2, i.e. a correspondence from A to A (in words: "F acts in A").

(3) Let $F \subset A \times B$. Then F is a *single-valued* correspondence if for all $a \in A$ the containments $(a,\ b_1) \in F$ and $(a,\ b_2) \in F$ imply $b_1 = b_2$. In particular, if $U \subset A$ and $I_U := \{(a,\ a) \in A^2 \ : \ a \in U\}$, then I_U is a single-valued correspondence acting in A and called the *identity relation* (over U on A). A single-valued correspondence $F \subset A \times B$ with dom $F = A$ is a *mapping* of A into B or a mapping from A (in)to B. The terms "function" and "map" are also in current usage. A mapping $F \subset A \times B$ is denoted by $F : A \to B$. Observe that here dom F always coincides with A whereas im F may differ from B. The identity relation I_U on A is a mapping if and only if $A = U$ and in this case I_U is called the *identity mapping* in U or the *diagonal* of U^2. The set I_U may be treated as a subset of $U \times A$. The resulting mapping is usually denoted by $\iota : U \to A$ and is called the *identical embedding* of U into A. It is said that "F is a correspondence from A *onto* B" if im $F = B$. Finally, a correspondence $F \subset X \times Y$ is *one-to-one* whenever the correspondence $F^{-1} \subset B \times A$ is single-valued.

(4) Occasionally the term "family" is used instead of "mapping." Namely, a mapping $F : A \to B$ is a *family* in B (indexed in A by F), also denoted by $(b_a)_{a \in A}$ or $a \mapsto b_a$ $(a \in A)$ or even (b_a). Specifically, $(a,\ b) \in F$ if and only if $b = b_a$. In the sequel, a subset U of A is often treated as indexed in itself by the identical embedding of U into A. It is worth recalling that in set theory a is an *element* or a *member* of A whenever $a \in A$. In this connection a family in B is also called a family of elements of B or a family of members of B. By way of expressiveness a family or a set of numbers is often addressed as *numeric*. Also, common abusage practices the identification of a family and its range. This sin is very enticing.

(5) Let $F \subset A \times B$ be a correspondence and $U \subset A$. The *restriction* of F to U, denoted by $F|_U$, is the set $F \cap (U \times B) \subset U \times B$. The set $F(U) := \operatorname{im} F|_U$ is the *image* of U under F.

If a and b are elements of A and B then $F(a) = b$ is usually written instead of $F(\{a\}) = \{b\}$. Often the parentheses in the symbol $F(a)$ are omitted or replaced with other symbols. For a subset U of B the image $F^{-1}(U)$ of U under F^{-1} is the *inverse image* of U or the *preimage* of U under F. So, inverse images are just images of inverses.

(6) Given a correspondence $F \subset A \times B$, assume that A is the product of A_1 and A_2, i.e. $A = A_1 \times A_2$. Fixing a_1 in A_1 and a_2 in A_2, consider the sets

$$F(a_1,\ \cdot) := \{(a_2,\ b) \in A_2 \times B : ((a_1,\ a_2),\ b) \in F\};$$
$$F(\cdot,\ a_2) := \{(a_1,\ b) \in A_1 \times B : ((a_1,\ a_2),\ b) \in F\}.$$

These are the *partial correspondences* of F. In this regard F itself is often symbolized as $F(\cdot,\ \cdot)$ and referred to as a *correspondence in two arguments*. This beneficial agreement is effective in similar events.

1.1.4. DEFINITION. The *composite correspondence* or *composition* of correspondences $F \subset A \times B$ and $G \subset C \times D$ is the set

$$G \circ F := \{(a,\ d) \in A \times D :\ (\exists\, b)\ (a,\ b) \in F\ \&\ (b,\ d) \in G\}.$$

The correspondence $G \circ F$ is considered as acting from A into D.

1.1.5. REMARK. The scope of the concept of composition does not diminish if it is assumed in 1.1.4 from the very beginning that $B = C$.

1.1.6. *Let F be a correspondence. Then $F \circ F^{-1} \supset I_{\operatorname{im} F}$. Moreover, the equality $F \circ F^{-1} = I_{\operatorname{im} F}$ holds if and only if $F|_{\operatorname{dom} F}$ is a mapping.* ◁▷

1.1.7. *Let $F \subset A \times B$ and $G \subset B \times C$. Further assume $U \subset A$. Then the correspondence $G \circ F \subset A \times C$ satisfies the equality $G \circ F(U) = G(F(U))$.* ◁▷

1.1.8. *Let $F \subset A \times B$, $G \subset B \times C$, and $H \subset C \times D$. Then the correspondences $H \circ (G \circ F) \subset A \times D$ and $(H \circ G) \circ F \subset A \times D$ coincide.* ◁▷

1.1.9. REMARK. By virtue of 1.1.8, the symbol $H \circ G \circ F$ and the like are defined soundly.

1.1.10. *Let F, G, and H be correspondences. Then*

$$H \circ G \circ F = \bigcup_{(b,c) \in G} F^{-1}(b) \times H(c).$$

◁ $(a,\ d) \in H \circ G \circ F \Leftrightarrow (\exists (b,\ c) \in G)\ (c,\ d) \in H \ \&\ (a,\ b) \in F \Leftrightarrow (\exists\ (b,\ c) \in G)\ a \in F^{-1}(b) \ \&\ d \in H(c)$ ▷

1.1.11. REMARK. The claim of 1.1.10, together with the calculation intended as its proof, is blatantly illegitimate from a formalistic point of view as based on ambiguous or imprecise information (in particular, on Definition 1.1.1!). Experience justifies treating such a criticism as petty. In the sequel, analogous convenient (and, in fact, inevitable) violations of formal purity are mercilessly exercised with no circumlocution.

1.1.12. *Let G and F be correspondences. Then*

$$G \circ F = \bigcup_{b \in \operatorname{im} F} F^{-1}(b) \times G(b).$$

◁ Insert $H := G$, $G := I_{\operatorname{im} F}$ and $F := F$ into 1.1.10. ▷

1.2. Ordered Sets

1.2.1. DEFINITION. Let σ be a relation on a set X, i.e. $\sigma \subset X^2$. *Reflexivity* for σ means the inclusion $\sigma \supset I_X$; *transitivity*, the inclusion $\sigma \circ \sigma \subset \sigma$; *antisymmetry*, the inclusion $\sigma \cap \sigma^{-1} \subset I_X$; and, finally, *symmetry*, the equality $\sigma = \sigma^{-1}$.

1.2.2. DEFINITION. A *preorder* is a reflexive and transitive relation. A symmetric preorder is an *equivalence*. An *order* (*partial order*, *ordering*, etc.) is an antisymmetric preorder. For a set X, the pair $(X,\ \sigma)$, with σ an order on X, is an *ordered set* or rarely a *poset*. The notation $x \le_\sigma y$ is used instead of $y \in \sigma(x)$. The terminology and notation are often simplified and even abused in common parlance: The underlying set X itself is called an ordered set, a partially ordered set, or even a poset. It is said that "x is less than y," or "y is greater than x," or "$x \le y$," or "$y \ge x$," or "x is in relation σ to y," or "x and y belong to σ," etc. Analogous agreements apply customarily to a *preordered set*, i.e. to a set furnished with a preorder. The convention is very propitious and extends often to an arbitrary relation. However, an equivalence is usually denoted by the signs like $\sim$.

1.2.3. EXAMPLES.

(1) The identity relation; each subset X_0 of a set X bearing a relation σ is endowed with the (*induced*) relation $\sigma_0 := \sigma \cap X_0 \times X_0$.

(2) If σ is an order (preorder) in X, then σ^{-1} is also an order (preorder) which is called *reverse* to σ.

(3) Let $f : X \to Y$ be a mapping and let τ be a relation on Y. Consider the relation $f^{-1} \circ \tau \circ f$ appearing on X. By 1.1.10,

$$f^{-1} \circ \tau \circ f = \bigcup_{(y_1,y_2)\in\tau} f^{-1}(y_1) \times f^{-1}(y_2).$$

Hence it follows that $(x_1,\ x_2) \in f^{-1} \circ \tau \circ f \Leftrightarrow (f(x_1),\ f(x_2)) \in \tau$. Thus, if τ is a preorder then $f^{-1} \circ \tau \circ f$ itself is a preorder called the *preimage* or *inverse image* of τ under f. It is clear that the inverse image of an equivalence is also an equivalence. Whereas the preimage of an order is not always antisymmetric. In particular, this relates to the equivalence $f^{-1} \circ f$ $(= f^{-1} \circ I_Y \circ f)$.

(4) Let X be an arbitrary set and let ω be an equivalence on X. Define a mapping $\varphi : X \to \mathscr{P}(X)$ by $\varphi(x) := \omega(x)$. Recall that $\mathscr{P}(X)$ stands for the *powerset* of X comprising all subsets of X and also denoted by 2^X. Let $\overline{X} := X/\omega := \operatorname{im} \varphi$ be the *quotient set* or *factor set* of X by ω or modulo ω. A member of X/ω is usually referred to as a *coset* or *equivalence class*. The mapping φ is the *coset mapping* (canonical projection, quotient mapping, etc.). Note that φ is treated as acting onto $\overline{X}$. Observe that

$$\omega = \varphi^{-1} \circ \varphi = \bigcup_{\overline{x}\in\overline{X}} \varphi^{-1}(\overline{x}) \times \varphi^{-1}(\overline{x}).$$

Now let $f : X \to Y$ be a mapping. Then f admits factorization through $\overline{X}$; i.e., there is a mapping $\overline{f} : \overline{X} \to Y$ (called a *quotient* of f by ω) such that $\overline{f} \circ \varphi = f$ if and only if $\omega \subset f^{-1} \circ f$. ◁▷

(5) Let $(X,\ \sigma)$ and $(Y,\ \tau)$ be two preordered sets. A mapping $f : X \to Y$ is *increasing* or *isotone* (i.e., $x \le_\sigma y \Rightarrow f(x) \le_\tau f(y)$) whenever $\sigma \subset f^{-1} \circ \tau \circ f$. That f *decreases* or is *antitone* means $\sigma \subset f^{-1} \circ \tau^{-1} \circ f$. ◁▷

1.2.4. DEFINITION. Let $(X,\ \sigma)$ be an ordered set and let $U \subset X$. An element x of X is an *upper bound* of U (write $x \ge U$) if $U \subset \sigma^{-1}(x)$. In particular,

$x \geq \varnothing$. An element x of X is a *lower bound* of U (write $x \leq U$) if x is an upper bound of U in the reverse order σ^{-1}. In particular, $x \leq \varnothing$.

1.2.5. REMARK. Throughout this book liberties are taken with introducing concepts which arise from those stated by *reversal*, i.e. by transition from a (pre)order to the reverse (pre)order. Note also that the definitions of upper and lower bounds make sense in a preordered set.

1.2.6. DEFINITION. An element x of U is *greatest* or *last* if $x \geq U$ and $x \in U$. When existent, such element is unique and is thus often referred to as *the* greatest element of U. A *least* or *first* element is defined by reversal.

1.2.7. *Let U be a subset of an ordered set $(X,\ \sigma)$ and let $\pi_\sigma(U)$ be the collection of all upper bounds of U. Suppose that a member x of X is the greatest element of U. Then, first, x is the least element of $\pi_\sigma(U)$; second, $\sigma(x) \cap U = \{x\}$.* ⊲⊳

1.2.8. REMARK. The claim of 1.2.7 gives rise to two generalizations of the concept of greatest element.

1.2.9. DEFINITION. Let X be a (preordered) set and let $U \subset X$. A *supremum* of U in X is a least upper bound of U, i.e. a least element of the set of all upper bounds of U. This element is denoted by $\sup_X U$ or in short $\sup U$. Certainly, in a poset an existing supremum of U is unique and so it is in fact *the* supremum of U. An *infimum*, a *greatest lower bound* $\inf U$ or $\inf_X U$ is defined by reversal.

1.2.10. DEFINITION. Let U be a subset of an ordered set $(X,\ \sigma)$. A member x of X is a *maximal* element of U if $\sigma(x) \cap U = \{x\}$. A *minimal* element is again defined by reversal.

1.2.11. REMARK. It is important to make clear the common properties and distinctions of the concepts of greatest element, maximal element, and supremum. In particular, it is worth demonstrating that a "typical" set has no greatest element while possibly possessing a maximal element.

1.2.12. DEFINITION. A *lattice* is an ordered set with the following property: each pair $(x_1,\ x_2)$ of elements of X has a least upper bound, $x_1 \vee x_2 := \sup\{x_1,\ x_2\}$, the *join* of x_1 and x_2, and a greatest lower bound, $x_1 \wedge x_2 := \inf\{x_1,\ x_2\}$, the *meet* of x_1 and x_2.

1.2.13. DEFINITION. A lattice X is *complete* if each subset of X has a supremum and an infimum in X.

1.2.14. *An ordered set X is a complete lattice if and only if each subset of X has a least upper bound.* ⊲⊳

1.2.15. DEFINITION. An ordered set $(X,\ \sigma)$ is *filtered upward* provided that $X^2 = \sigma^{-1} \circ \sigma$. A *downward-filtered* set is defined by reversal. A nonempty poset is a *directed* set or simply a *direction* if it is filtered upward.

1.2.16. DEFINITION. Let X be a set. A *net* or a (*generalized*) *sequence* in X is a mapping of a direction into X. A mapping of the set $\mathbb{N}$ of natural numbers, $\mathbb{N} = \{1, 2, 3 \dots\}$, furnished with the conventional order, is a (*countable*) *sequence*.

1.2.17. *A lattice X is complete if and only if each upward-filtered subset of X has a least upper bound.* ◁▷

1.2.18. REMARK. The claim of 1.2.17 implies that for calculating a supremum of each subset it suffices to find suprema of pairs and increasing nets.

1.2.19. DEFINITION. Let (X, σ) be an ordered set. It is said that X is *ordered linearly* whenever $X^2 = \sigma \cup \sigma^{-1}$. A nonempty linearly-ordered subset of X is a *chain* in X. A nonempty ordered set is called *inductive* whenever its every chain is *bounded above* (i.e., has an upper bound).

1.2.20. ***Kuratowski–Zorn Lemma.*** *Each inductive set contains a maximal element.*

1.2.21. REMARK. The Kuratowski–Zorn Lemma is equivalent to the axiom of choice which is accepted in set theory.

1.3. Filters

1.3.1. DEFINITION. Let X be a set and let $\mathscr{B}$, a nonempty subset of $\mathscr{P}(X)$, consist of nonempty elements. Such $\mathscr{B}$ is said to be a *filterbase* (on X) if $\mathscr{B}$ is filtered downward. Recall that $\mathscr{P}(X)$ is *ordered by inclusion*. It means that a greater subset includes a smaller subset by definition; this order is always presumed in $\mathscr{P}(X)$.

1.3.2. *A subset $\mathscr{B}$ of $\mathscr{P}(X)$ is a filterbase if and only if*

(1) $\mathscr{B} \neq \varnothing$ and $\varnothing \notin \mathscr{B}$;

(2) $B_1,\ B_2 \in \mathscr{B} \Rightarrow (\exists\, B \in \mathscr{B})\ B \subset B_1 \cap B_2$.

1.3.3. DEFINITION. A subset $\mathscr{F}$ of $\mathscr{P}(X)$ is a *filter* (on X) if there is a filterbase $\mathscr{B}$ such that $\mathscr{F}$ is the set of all *supersets* of $\mathscr{B}$; i.e.,

$$\mathscr{F} = \operatorname{fil} \mathscr{B} := \{C \in \mathscr{P}(X) :\ (\exists\, B \in \mathscr{B})\ B \subset C\}.$$

In this case $\mathscr{B}$ is said to be a *base* for the filter $\mathscr{F}$ (so each filterbase is a filter base).

1.3.4. *A subset $\mathscr{F}$ in $\mathscr{P}(X)$ is a filter if and only if*

(1) $\mathscr{F} \neq \varnothing$ and $\varnothing \notin \mathscr{F}$;

(2) $(A \in \mathscr{F}\ \&\ A \subset B \subset X) \Rightarrow B \in \mathscr{F}$;

(3) $A_1,\ A_2 \in \mathscr{F} \Rightarrow A_1 \cap A_2 \in \mathscr{F}$. ◁▷

1.3.5. EXAMPLES.

(1) Let $F \subset X \times Y$ be a correspondence and let $\mathscr{B}$ be a downward-filtered subset of $\mathscr{P}(X)$. Put $F(\mathscr{B}) := \{F(B) : B \in \mathscr{B}\}$. It is easy to see that $F(\mathscr{B})$ is filtered downward. The notation is alleviated by putting $F(\mathscr{B}) :=$ fil $F(\mathscr{B})$. If $\mathscr{F}$ is a filter on X and $B \cap \operatorname{dom} F \neq \varnothing$ for all $B \in \mathscr{F}$, then $F(\mathscr{F})$ is a filter (on Y) called the *image* of $\mathscr{F}$ under F. In particular, if F is a mapping then the image of a filter on X is a filter on Y.

(2) Let (X, σ) be a direction. Clearly, $\mathscr{B} := \{\sigma(x) : x \in X\}$ is a filterbase. For a net $F : X \to Y$, the filter fil $F(\mathscr{B})$ is the *tail filter* of F. Let $(\overline{X}, \overline{\sigma})$ and $\overline{F} : \overline{X} \to Y$ be another direction and another net in Y. If the tail filter of $\overline{F}$ includes the tail filter of F then $\overline{F}$ is a *subnet* (in a broad sense) of the net F. If there is a subnet (in a broad sense) $G : \overline{X} \to X$ of the identity net $((x)_{x \in X}$ in the direction $(X, \sigma))$ such that $\overline{F} = F \circ G$, then $\overline{F}$ is a *subnet* of F (sometimes $\overline{F}$ is addressed as a *Moore subnet* or a *strict subnet* of F). Every subnet is a subnet in a broad sense. It is customary to speak of a *net having* or *lacking a subnet* with some property.

1.3.6. DEFINITION. Let $\mathscr{F}(X)$ be the collection of all filters on X. Take $\mathscr{F}_1, \mathscr{F}_2 \in \mathscr{F}(X)$. Say that $\mathscr{F}_1$ is *finer* than $\mathscr{F}_2$ or $\mathscr{F}_2$ *refines* $\mathscr{F}_1$ (in other words, $\mathscr{F}_1$ is *coarser* than $\mathscr{F}_2$ or $\mathscr{F}_1$ *coarsens* $\mathscr{F}_2$) whenever $\mathscr{F}_1 \supset \mathscr{F}_2$.

1.3.7. *The set $\mathscr{F}(X)$ with the relation "to be finer" is a poset.* ◁▷

1.3.8. *Let $\mathscr{N}$ be a direction in $\mathscr{F}(X)$. Then $\mathscr{N}$ has a supremum $\mathscr{F}_0 := \sup \mathscr{N}$. Moreover, $\mathscr{F}_0 = \cup\{\mathscr{F} : \mathscr{F} \in \mathscr{N}\}$.*

◁ To prove this, it is necessary to show that $\mathscr{F}_0$ is a filter. Since $\mathscr{N}$ is not empty it is clear that $\mathscr{F}_0 \neq \varnothing$ and $\varnothing \notin \mathscr{F}_0$. If $A \in \mathscr{F}_0$ and $B \supset A$ then, choosing $\mathscr{F}$ in $\mathscr{N}$ for which $A \in \mathscr{F}$, conclude that $B \in \mathscr{F} \subset \mathscr{F}_0$. Given $A_1, A_2 \in \mathscr{F}_0$, find an element $\mathscr{F}$ of $\mathscr{N}$ satisfying $A_1, A_2 \in \mathscr{F}$, which is possible because $\mathscr{N}$ is a direction. By 1.3.4, $A_1 \cap A_2 \in \mathscr{F} \subset \mathscr{F}_0$. ▷

1.3.9. DEFINITION. An *ultrafilter* is a maximal element of the ordered set $\mathscr{F}(X)$ of all filters on X.

1.3.10. *Each filter is coarser than an ultrafilter.*

◁ By 1.3.8, the set of filters finer than a given filter is inductive. Recalling the Kuratowski–Zorn Lemma completes the proof. ▷

1.3.11. *A filter $\mathscr{F}$ is an ultrafilter if and only if for all $A \subset X$ either $A \in \mathscr{F}$ or $X \setminus A \in \mathscr{F}$.*

◁ $\Rightarrow$: Suppose that $A \notin \mathscr{F}$ and $B := X \setminus A \notin \mathscr{F}$. Note that $A \neq \varnothing$ and $B \neq \varnothing$. Put $\mathscr{F}_1 := \{C \in \mathscr{P}(X) : A \cup C \in \mathscr{F}\}$. Then $A \notin \mathscr{F} \Rightarrow \varnothing \notin \mathscr{F}_1$ and $B \in \mathscr{F}_1 \Rightarrow \mathscr{F}_1 \neq \varnothing$. The checking of 1.3.4 (2) and 1.3.4 (3) is similar. Hence, $\mathscr{F}_1$ is an ultrafilter. By definition, $\mathscr{F}_1 \supset \mathscr{F}$. In addition, $\mathscr{F}$ is an ultrafilter and so $\mathscr{F}_1 = \mathscr{F}$. Observe that $B \notin \mathscr{F}$ and $B \in \mathscr{F}$, a contradiction.

$\Leftarrow$: Take $\mathscr{F}_1 \in \mathscr{F}(X)$ and let $\mathscr{F}_1 \supset \mathscr{F}$. If $A \in \mathscr{F}_1$ and $A \notin \mathscr{F}$ then $X \setminus A \in \mathscr{F}$ by hypothesis. Hence, $X \setminus A \in \mathscr{F}_1$; i.e., $\varnothing = A \cap (X \setminus A) \in \mathscr{F}_1$, which is impossible. ▷

1.3.12. *If f is a mapping from X into Y and $\mathscr{F}$ is an ultrafilter on X then $f(\mathscr{F})$ is an ultrafilter on Y.* ◁▷

1.3.13. *Let $\mathscr{X} := \mathscr{X}_{\mathscr{F}_0} := \{\mathscr{F} \in \mathscr{F}(X) : \mathscr{F} \subset \mathscr{F}_0\}$ for $\mathscr{F}_0 \in \mathscr{F}(X)$. Then $\mathscr{X}$ is a complete lattice.*

◁ It is obvious that $\mathscr{F}_0$ is the greatest element of $\mathscr{X}$ and $\{X\}$ is the least element of $\mathscr{X}$. Therefore, the empty set has a supremum and an infimum in $\mathscr{X}$: in fact, $\sup \varnothing = \inf \mathscr{X} = \{X\}$ and $\inf \varnothing = \sup \mathscr{X} = \mathscr{F}_0$. By 1.2.17 and 1.3.8, it suffices to show that the join $\mathscr{F}_1 \vee \mathscr{F}_2$ is available for all $\mathscr{F}_1, \mathscr{F}_2 \in \mathscr{X}$. Consider $\mathscr{F} := \{A_1 \cap A_2 : A_1 \in \mathscr{F}_1,\ A_2 \in \mathscr{F}_2\}$. Clearly, $\mathscr{F} \subset \mathscr{F}_0$ while $\mathscr{F} \supset \mathscr{F}_1$ and $\mathscr{F} \supset \mathscr{F}_2$. Thus to verify the equality $\mathscr{F} = \mathscr{F}_1 \vee \mathscr{F}_2$, it is necessary to establish that $\mathscr{F}$ is a filter.

Plainly, $\mathscr{F} \neq \varnothing$ and $\varnothing \notin \mathscr{F}$. It is also immediate that $(B_1, B_2 \in \mathscr{F} \Rightarrow B_1 \cap B_2 \in \mathscr{F})$. Moreover, if $C \supset A_1 \cap A_2$ where $A_1 \in \mathscr{F}_1$ and $A_2 \in \mathscr{F}_2$, then $C = \{A_1 \cap A_2\} \cup C = (A_1 \cup C) \cap (A_2 \cup C)$. Since $A_1 \cup C \in \mathscr{F}_1$ and $A_2 \cup C \in \mathscr{F}_2$, conclude that $C \in \mathscr{F}$. Appealing to 1.3.4, complete the proof. ▷

Exercises

1.1. Give examples of sets and nonsets as well as set-theoretic properties and non-set-theoretic properties.

1.2. Is it possible for the interval $[0, 1]$ to be a member of the interval $[0, 1]$? For the interval $[0, 2]$?

1.3. Find compositions of the simplest correspondences and relations: squares, disks and circles with coincident or distinct centers in $\mathbb{R}^M \times \mathbb{R}^N$ for all feasible values of M and N.

1.4. Given correspondences R, S, and T, demonstrate that

$$\begin{aligned} (R \cup S)^{-1} &= R^{-1} \cup S^{-1}; & (R \cap S)^{-1} &= R^{-1} \cap S^{-1}; \\ (R \cup S) \circ T &= (R \circ T) \cup (S \circ T); & R \circ (S \cup T) &= (R \circ S) \cup (R \circ T); \\ (R \cap S) \circ T &\subset (R \circ T) \cap (S \circ T); & R \circ (S \cap T) &\subset (R \circ S) \cap (R \circ T). \end{aligned}$$

1.5. Assume $X \subset X \times X$. Prove that $X = \varnothing$.

1.6. Find conditions for the equations $\mathscr{X} A = B$ and $A \mathscr{X} = B$ to be solvable for $\mathscr{X}$ in correspondences or in functions.

1.7. Find the number of equivalences on a finite set.

1.8. Is the intersection of equivalences also an equivalence? And the union of equivalences?

1.9. Find conditions for commutativity of equivalences (with respect to composition).

1.10. How many orders and preorders are there on two-element and three-element sets? List all of them. What can you say about the number of preorders on a finite set?

1.11. Let F be an increasing idempotent mapping of a set X into itself. Assume that F dominates the identity mapping: $F \geq I_X$. Such an F is an abstract ***closure operator*** or, briefly, an (upper) ***envelope***. Study fixed points of a closure operator. (Recall that an element x is a ***fixed point*** of F if $F(x) = x$.)

1.12. Let X and Y be ordered sets and $M(X, Y)$, the set of increasing mappings from X to Y with the natural order (specify the latter). Prove that

(1) ($M(X, Y)$ is a lattice) $\Leftrightarrow$ (Y is a lattice);
(2) ($M(X, Y)$ is a complete lattice) $\Leftrightarrow$ (Y is a complete lattice).

1.13. Given ordered sets X, Y, and Z, demonstrate that

(1) $M(X, Y \times Z)$ is isomorphic with $M(X, Y) \times M(Y, Z)$;
(2) $M(X \times Y, Z)$ is isomorphic with $M(X, M(Y, Z))$.

1.14. How many filters are there on a finite set?

1.15. How do the least upper and greatest lower bounds of a set of filters look like?

1.16. Let f be a mapping from X onto Y. Prove that each ultrafilter on Y is the image of some ultrafilter on X under f.

1.17. Prove that an ultrafilter refining the intersection of two filters is finer than either of them.

1.18. Prove that each filter is the intersection of all ultrafilters finer than it.

1.19. Let $\mathscr{A}$ be an ultrafilter on $\mathbb{N}$ containing cofinite subsets (a ***cofinite*** subset is a subset with finite complement). Given $x, y \in s := \mathbb{R}^{\mathbb{N}}$, put $x \sim_{\mathscr{A}} y := (\exists A \in \mathscr{A})\ x|_A = y|_A$. Denote $*\mathbb{R} := \mathbb{R}^{\mathbb{N}}/\sim_{\mathscr{A}}$. For $t \in \mathbb{R}$ the notation $*t$ symbolizes the coset with the constant sequence $\bar{t}$ defined as $\bar{t}(n) := t\ (n \in \mathbb{N})$. Prove that $*\mathbb{R} \setminus \{*t : t \in \mathbb{R}\} \neq \varnothing$. Furnish $*\mathbb{R}$ with algebraic and order structures. How are the properties of $\mathbb{R}$ and $*\mathbb{R}$ related to each other?

Chapter 2
Vector Spaces

2.1. Spaces and Subspaces

2.1.1. REMARK. In algebra, in particular, modules over rings are studied. A *module X over a ring A* is defined by an abelian group $(X,\ +)$ and a representation of the ring A in the endomorphism ring of X which is considered as left multiplication $\cdot : A \times X \to X$ by elements of A. Moreover, a natural agreement is presumed between addition and multiplication. With this in mind, the following phrase is interpreted: "A module X over a ring A is described by the quadruple $(X,\ A,\ +,\ \cdot)$." Note also that A is referred to as the *ground ring* of X.

2.1.2. DEFINITION. A *basic field* is the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. The symbol $\mathbb{F}$ stands for a basic field. Observe that $\mathbb{R}$ is treated as embedded into $\mathbb{C}$ in a standard (and well-known) fashion so that the operation Re of taking the *real part* of a number sends $\mathbb{C}$ onto the *real axis*, $\mathbb{R}$.

2.1.3. DEFINITION. Let $\mathbb{F}$ be a field. A module X over $\mathbb{F}$ is a *vector space* (over $\mathbb{F}$). An element of the *ground field* $\mathbb{F}$ is a *scalar* in X and an element of X is a *vector* in X or a *point* in X. So, X is a *vector space with scalar field* $\mathbb{F}$. The operation $+ : X \times X \to X$ is *addition* in X and $\cdot : \mathbb{F} \times X \to X$ is *scalar multiplication* in X. We refer to X as a *real vector space* in case $\mathbb{F} = \mathbb{R}$ and as a *complex vector space*, in case $\mathbb{F} = \mathbb{C}$. A more complete nomenclature consists of $(X,\ \mathbb{F},\ +,\ \cdot)$, $(X,\ \mathbb{R},\ +,\ \cdot)$, and $(X,\ \mathbb{C},\ +,\ \cdot)$. Neglecting these subtleties, allow X to stand for every vector space associated with the *underlying set* X.

2.1.4. EXAMPLES.

(1) A field $\mathbb{F}$ is a vector space over $\mathbb{F}$.

(2) Let $(X,\ \mathbb{F},\ +,\ \cdot)$ be a vector space. Consider $(X,\ \mathbb{F},\ +,\ \cdot_*)$, where $\cdot_* : (\lambda,\ x) \mapsto \lambda^* x$ for $\lambda \in \mathbb{F}$ and $x \in X$, the symbol λ^* standing for the conventional *complex conjugate* of λ. The so-defined vector space is the *twin* of X denoted by X_*. If $\mathbb{F} := \mathbb{R}$ then the space X and the twin of X, the space X_*, coincide.

(3) A vector space $(X_0,\ \mathbb{F},\ +,\ \cdot)$ is called a *subspace* of a vector space $(X,\ \mathbb{F},\ +,\ \cdot)$, if X_0 is a subgroup of X and scalar multiplication in X_0 is the restriction of that in X to $\mathbb{F} \times X_0$. Such a set X_0 is a *linear set* in X, whereas X is referred to as an *ambient space* (for X_0). It is convenient although not perfectly puristic to treat X_0 itself as a subspace of X. Observe the important particularity of terminology: a linear set is a subset of a vector space (whose obsolete title is a *linear space*). To call a subset of whatever space X a *set in* X is a mathematical idiom of long standing. The same applies to calling a member of X a *point in* X. Furthermore, the *neutral element* of X, the *zero vector* of X, or simply *zero* of X, is considered as a subspace of X and is denoted by 0. Since 0 is not explicitly related to X, all vector spaces including the basic fields may seem to have a point in common, zero.

(4) Take $(X_\xi)_{\xi\in\Xi}$, a family of vector spaces over $\mathbb{F}$, and let $\mathscr{X} := \prod_{\xi\in\Xi} X_\xi$ be the *product* of the underlying sets, i.e. the collection of mappings $x : \Xi \to \cup_{\xi\in\Xi} X_\xi$ such that $x_\xi := x(\xi) \in X_\xi$ as $\xi \in \Xi$ (of course, here Ξ is not empty). Endow $\mathscr{X}$ with the *coordinatewise* or *pointwise* operations of addition and scalar multiplication:

$$(x_1 + x_2)(\xi) := x_1(\xi) + x_2(\xi) \quad (x_1,\ x_2 \in \mathscr{X},\ \xi \in \Xi);$$
$$(\lambda \cdot x)(\xi) := \lambda \cdot x(\xi) \quad (x \in \mathscr{X},\ \lambda \in \mathbb{F},\ \xi \in \Xi)$$

(below, as a rule, we write λx and sometimes $x\lambda$ rather than $\lambda \cdot x$). The so-constructed vector space $\mathscr{X}$ over $\mathbb{F}$ is the *product* of $(X_\xi)_{\xi\in\Xi}$. If $\Xi := \{1, 2, \dots, N\}$ then $X_1 \times X_2 \times \ldots \times X_N := \mathscr{X}$. In the case $X_\xi = X$ for all $\xi \in \Xi$, the designation $X^\Xi := \mathscr{X}$ is used. Given $\Xi := \{1, 2, \dots, N\}$, put $X^N := \mathscr{X}$.

(5) Let $(X_\xi)_{\xi\in\Xi}$ be a family of vector spaces over $\mathbb{F}$. Consider their direct sum $\mathscr{X}_0 := \sum_{\xi\in\Xi} X_\xi$. By definition, $\mathscr{X}_0$ is the subset of $\mathscr{X} := \prod_{\xi\in\Xi} X_\xi$ which comprises all x_0 such that $x_0(\Xi \setminus \Xi_0) \subset 0$ for a finite subset $\Xi_0 \subset \Xi$ (routinely speaking, Ξ_0 is dependent on x_0). It is easily seen that $\mathscr{X}_0$ is a linear set in $\mathscr{X}$. The vector space associated with $\mathscr{X}_0$ presents a subspace of the product of $(X_\xi)_{\xi\in\Xi}$ and is the *direct sum* of $(X_\xi)_{\xi\in\Xi}$.

(6) Given a subspace $(X,\ \mathbb{F},\ +,\ \cdot)$ of a vector space $(X_0,\ \mathbb{F},\ +,\ \cdot)$, introduce

$$\sim_{X_0} := \{(x_1,\ x_2) \in X^2 :\ x_1 - x_2 \in X_0\}.$$

Then $\sim_{X_0}$ is an equivalence on X. Denote $\mathscr{X} := X/{\sim_{X_0}}$ and let $\varphi : X \to \mathscr{X}$ be the coset mapping. Define operations on $\mathscr{X}$ by letting

$$x_1 + x_2 := \varphi(\varphi^{-1}(x_1) + \varphi^{-1}(x_2)) \quad (x_1,\ x_2 \in \mathscr{X});$$
$$\lambda x := \varphi(\lambda\varphi^{-1}(x)) \quad (x \in \mathscr{X},\ \lambda \in \mathbb{F}).$$

Here, as usual, for subsets S_1 and S_2 of X, a subset Λ of $\mathbb{F}$, and a scalar λ, a member of $\mathbb{F}$, it is assumed that

$$S_1 + S_2 := +\{S_1 \times S_2\};$$
$$\Lambda S_1 := \cdot\ (\Lambda \times S_1); \quad \lambda S_1 := \{\lambda\} S_1.$$

Thus $\mathscr{X}$ is furnished with the structure of a vector space. This space, denoted by X/X_0, is the *quotient* (*space*) of X by X_0 or the *factor space* of X modulo X_0.

2.1.5. *Let X be a vector space and let* $\operatorname{Lat}(X)$ *stand for the collection of all subspaces of X. Ordered by inclusion,* $\operatorname{Lat}(X)$ *presents a complete lattice.*

◁ It is clear that $\inf \operatorname{Lat}(X) = 0$ and $\sup \operatorname{Lat}(X) = X$. Further, the intersection of a nonempty set of subspaces is also a subspace. By 1.2.17, the proof is complete. ▷

2.1.6. REMARK. With $X_1,\ X_2 \in \operatorname{Lat}(X)$, the equality $X_1 \vee X_2 = X_1 + X_2$ holds. It is evident that $\inf \mathscr{E} = \cap\{X_0 :\ X_0 \in \mathscr{E}\}$ for a nonempty subset $\mathscr{E}$ of $\operatorname{Lat}(X)$. Provided that $\mathscr{E}$ is filtered upward, $\sup \mathscr{E} = \cup\{X_0 :\ X_0 \in \mathscr{E}\}$. ◁▷

2.1.7. DEFINITION. Subspaces X_1 and X_2 of a vector space X *split X into* (*algebraic*) *direct sum decomposition* (in symbols, $X = X_1 \oplus X_2$), if $X_1 \wedge X_2 = 0$ and $X_1 \vee X_2 = X$. In this case X_2 is an (*algebraic*) *complement* of X_1 to X, and X_1 is an (algebraic) complement of X_2 to X. It is also said that X_1 and X_2 are (*algebraically*) *complementary* to one another.

2.1.8. *Each subspace of a vector space has an algebraic complement.*

◁ Take a subspace X_1 of X. Put

$$\mathscr{E} := \{X_0 \in \operatorname{Lat}(X) :\ X_0 \wedge X_1 = 0\}.$$

Obviously, $0 \in \mathscr{E}$. Given a chain $\mathscr{E}_0$ in $\mathscr{E}$, from 2.1.6 infer that $X_1 \wedge \sup \mathscr{E}_0 = 0$, i.e. $\sup \mathscr{E}_0 \in \mathscr{E}$. Thus $\mathscr{E}$ is inductive and, by 1.2.20, $\mathscr{E}$ has a maximal element, say, X_2. If $x \in X \setminus (X_1 + X_2)$ then

$$(X_2 + \{\lambda x :\ \lambda \in \mathbb{F}\}) \wedge X_1 = 0.$$

Indeed, if $x_2 + \lambda x = x_1$ with $x_1 \in X_1,\ x_2 \in X_2$ and $\lambda \in \mathbb{F}$, then $\lambda x \in X_1 + X_2$ and so $\lambda = 0$. Hence, $x_1 = x_2 = 0$ as $X_1 \wedge X_2 = 0$. Therefore, $X_2 + \{\lambda x :\ \lambda \in \mathbb{F}\} = X_2$ because X_2 is maximal. It follows that $x = 0$. At the same time, it is clear that $x \neq 0$. Finally, $X_1 \vee X_2 = X_1 + X_2 = X$. ▷

2.2. Linear Operators

2.2.1. DEFINITION. Let X and Y be vector spaces over $\mathbb{F}$. A correspondence $T \subset X \times Y$ is *linear*, if T is a linear set in $X \times Y$. A *linear operator* on X (or simply an *operator*, with linearity apparent from the context) is a mapping of X and a linear correspondence simultaneously. If need be, we distinguish such a T from a linear single-valued correspondence S with $\operatorname{dom} S \neq X$ and say that T is given on X or T is *defined everywhere* or even T is a *total* operator, whereas S is referred to as *not-everywhere-defined* or *partially-defined operator* or even a *partial operator*. In the case $X = Y$, a linear operator from X to Y is also called an operator in X or an *endomorphism* of X.

2.2.2. *A correspondence $T \subset X \times Y$ is a linear operator from X to Y if and only if* $\operatorname{dom} T = X$ *and*

$$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T x_1 + \lambda_2 T x_2 \quad (\lambda_1,\ \lambda_2 \in \mathbb{F};\ x_1,\ x_2 \in X).\ ⊲⊳$$

2.2.3. *The set $\mathscr{L}(X,\ Y)$ of all linear operators carrying X into Y constitutes a vector space, a subspace of Y^X.* ⊲⊳

2.2.4. DEFINITION. A member of $\mathscr{L}(X,\ \mathbb{F})$ is a *linear functional* on X, and the space $X^{\#} := \mathscr{L}(X,\ \mathbb{F})$ is the (*algebraic*) *dual* of X. A linear functional on X_* is a **-linear* or *conjugate-linear functional* on X. If the nature of $\mathbb{F}$ needs specifying, then we speak of real linear functionals, complex duals, etc. Evidently, when $\mathbb{F} = \mathbb{R}$ the term "*-linear functional" is used rarely, if ever.

2.2.5. DEFINITION. A linear operator T, a member of $\mathscr{L}(X,\ Y)$, is an (*algebraic*) *isomorphism* (of X and Y, or between X and Y) if the correspondence T^{-1} is a linear operator, a member of $\mathscr{L}(Y,\ X)$.

2.2.6. DEFINITION. Vector spaces X and Y are (*algebraically*) *isomorphic*, in symbols $X \simeq Y$, provided that there is an isomorphism between X and Y.

2.2.7. *Vector spaces X and Y are isomorphic if and only if there are operators $T \in \mathscr{L}(X,\ Y)$ and $S \in \mathscr{L}(Y,\ X)$ such that $S \circ T = I_X$ and $T \circ S = I_Y$ (in this event $S = T^{-1}$ and $T = S^{-1}$).* ⊲⊳

2.2.8. REMARK. Given vector spaces X, Y, and Z, take $T \in \mathscr{L}(X,\ Y)$ and $S \in \mathscr{L}(Y,\ Z)$. The correspondence $S \circ T$ is undoubtedly a member of $\mathscr{L}(X,\ Z)$. For simplicity every composite operator $S \circ T$ is denoted by *juxtaposition* ST. Observe also that the taking of composition $(S,\ T) \mapsto ST$ is usually treated as the mapping $\circ : \mathscr{L}(Y,\ Z) \times \mathscr{L}(X,\ Y) \to \mathscr{L}(X,\ Z)$. In particular, if $\mathscr{E} \subset \mathscr{L}(Y,\ Z)$ and $T \in \mathscr{L}(X,\ Y)$ then we let $\mathscr{E} \circ T := \circ(\mathscr{E} \times \{T\})$. One of the reasons behind the convention is that juxtaposition in the *endomorphism space* $\mathscr{L}(X) := \mathscr{L}(X,\ X)$ of X which comprises all endomorphisms of X transforms $\mathscr{L}(X)$ into a ring (and even into an algebra, the *endomorphism algebra* of X, cf. 5.6.2).

2.2.9. EXAMPLES.

(**1**) If T is a linear correspondence then T^{-1} is also a linear correspondence.

(**2**) If X_1 is a subspace of a vector space X and X_2 is an algebraic complement of X_1 then X_2 is isomorphic with X/X_1. Indeed, if $\varphi : X \to X/X_1$ is the coset mapping then its restriction to X_2, i.e. the operator $x_2 \mapsto \varphi(x_2)$ with $x_2 \in X_2$, implements a desired isomorphism. ⊲⊳

(**3**) Consider $\mathscr{X} := \prod_{\xi \in \Xi} X_\xi$, the product of vector spaces $(X_\xi)_{\xi \in \Xi}$. Take a *coordinate projection*, i.e. a mapping $\operatorname{Pr}_\xi : \mathscr{X} \to X_\xi$ defined by $\operatorname{Pr}_\xi x := x_\xi$. Clearly, Pr_ξ is a linear operator, $\operatorname{Pr}_\xi \in \mathscr{L}(\mathscr{X},\ X_\xi)$. Such an operator is often treated as an endomorphism of $\mathscr{X}$, a member of $\mathscr{L}(\mathscr{X})$, on implying a natural isomorphism between $\mathscr{X}_\xi$ and X_ξ, where $\mathscr{X}_\xi := \prod_{\eta\in\Xi} \overline{X}_\eta$ with $\overline{X}_\eta := 0$ if $\eta \neq \xi$ and $\overline{X}_\xi := X_\xi$.

(4) Let $X := X_1 \oplus X_2$. Since $+^{-1}$ is an isomorphism between X and $X_1 \times X_2$, we may define $P_1, P_2 \in \mathscr{L}(X)$ as $P_1 := P_{X_1||X_2} := \mathrm{Pr}_1 \circ (+^{-1})$ and $P_2 := P_{X_2||X_1} := \mathrm{Pr}_2 \circ (+^{-1})$. The operator P_1 is the *projection of X onto X_1 along X_2* and P_2 is the *complementary projection* to P_1 or the *complement* of P_1 (in symbols, $P_2 = P_1^d$). In turn, P_1 is complementary to P_2, and P_2 projects X onto X_2 along X_1. Observe also that $P_1 + P_2 = I_X$. Moreover, $P_1^2 := P_1P_1 = P_1$, and so a projection is an *idempotent operator.* Conversely, every idempotent P belonging to $\mathscr{L}(X)$ projects X onto $P(X)$ along $P^{-1}(0)$.

For $T \in \mathscr{L}(X)$ and P a projection, the equality $PTP = TP$ holds if and only if $T(X_0) \subset X_0$ with $X_0 = \mathrm{im}\ P$ (read: X_0 is *invariant* under T). ◁▷

The equality $TP_{X_1||X_2} = P_{X_1||X_2}T$ holds whenever both X_1 and X_2 are invariant under T, and in this case the *direct sum decomposition* $X = X_1 \oplus X_2$ *reduces* T. The restriction of T to X_1 is acknowledged as an element T_1 of $\mathscr{L}(X_1)$ which is called the *part* of T in X_1. If $T_2 \in \mathscr{L}(X_2)$ is the part of T in X_2, then T is expressible in *matrix form*

$$T \sim \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}.$$

Namely, an element x of $X_1 \oplus X_2$ is regarded as a "column vector" with components x_1 and x_2, where $x_1 = P_{X_1||X_2}x$ and $x_2 = P_{X_1||X_2}x$; matrices are multiplied according to the usual rule, "rows by columns." The product of T and the column vector x, i.e. the vector with components T_1x_1 and T_2x_2, is certainly considered as Tx (in this case, we also write Tx_1 and Tx_2). In other words, T is identified with the mapping from $X_1 \times X_2$ to $X_1 \times X_2$ acting as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

In a similar way we can introduce matrix presentation for general operators contained in $\mathscr{L}(X_1 \oplus X_2,\ Y_1 \oplus Y_2)$. ◁▷

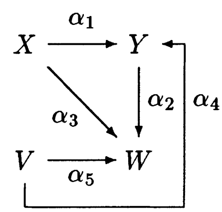
(5) A finite subset $\mathscr{E}$ of X is *linearly independent* (in X) provided that $\sum_{e\in\mathscr{E}} \lambda_e e = 0$, with $\lambda_e \in \mathbb{F}$ $(e \in \mathscr{E})$, implies $\lambda_e = 0$ for all $e \in \mathscr{E}$. An arbitrary subset $\mathscr{E}$ of X is *linearly independent* if every finite subset of $\mathscr{E}$ is linearly independent. A *Hamel basis* (or an *algebraic basis*) for X is a linearly independent set in X maximal by inclusion. Each linearly independent set is contained in a Hamel basis. All Hamel bases have the same cardinality called the *dimension* of X and denoted by $\dim X$. Every vector space is isomorphic to the direct sum of a family $(\mathbb{F})_{\xi\in\Xi}$ with Ξ of cardinality $\dim X$. Suppose that X_1 is a subspace of X. The *codimension* of X_1 is the dimension of X/X_1, with codim X_1 standing for the former. If $X = X_1 \oplus X_2$ then codim $X_1 = \dim X_2$ and $\dim X = \dim X_1 +$ codim X_1. ◁▷

2.3. Equations in Operators

2.3.1. DEFINITION. Given $T \in \mathscr{L}(X, Y)$, define the *kernel* of T as $\ker T := T^{-1}(0)$, the *cokernel* of T as $\operatorname{coker} T := Y/\operatorname{im} T$, and the *coimage* of T as $\operatorname{coim} T := X/\ker T$. Agree that an operator T is an *monomorphism* whenever $\ker T = 0$. An operator T is an *epimorphism* in the case of the equality $\operatorname{im} T = Y$.

2.3.2. *An operator is an isomorphism if and only if it is a monomorphism and an epimorphism simultaneously.* ◁▷

2.3.3. REMARK. Below use is made of the concept of *commutative diagram.* So the phrase, "The following diagram commutes,"



encodes the containments $\alpha_1 \in \mathscr{L}(X, Y)$, $\alpha_2 \in \mathscr{L}(Y, W)$, $\alpha_3 \in \mathscr{L}(X, W)$, $\alpha_4 \in \mathscr{L}(V, Y)$ and $\alpha_5 \in \mathscr{L}(V, W)$, as well as the equalities $\alpha_2\alpha_1 = \alpha_3$ and $\alpha_5 = \alpha_2\alpha_4$.

2.3.4. DEFINITION. A diagram $X \xrightarrow{T} Y \xrightarrow{S} Z$ is an *exact sequence* (at the term Y), if $\ker S = \operatorname{im} T$. A sequence $\ldots \to X_{k-1} \to X_k \to X_{k+1} \to \ldots$ is *exact at* X_k, if for all k the subsequence $X_{k-1} \to X_k \to X_{k+1}$ (symbols of operators are omitted), and is *exact* if it is exact at every term (except the first and the last, if any).

2.3.5. EXAMPLES.

(1) An exact sequence $X \xrightarrow{T} Y \xrightarrow{S} Z$ is *semi-exact*, i.e. $ST = 0$. The converse is not true.

(2) A sequence $0 \to X \xrightarrow{T} Y$ is exact if and only if T is a monomorphism. (Throughout the book $0 \to X$ certainly denotes the sole element of $\mathscr{L}(0, X)$, zero (cf. 2.1.4 (3)).)

(3) A sequence $X \xrightarrow{T} Y \to 0$ is exact if and only if T is an epimorphism. (Plainly, the symbol $Y \to 0$ again stands for zero, the single element of $\mathscr{L}(Y, 0)$.)

(4) An operator T from X to Y, a member of $\mathscr{L}(X, Y)$, is an isomorphism if and only if $0 \to X \xrightarrow{T} Y \to 0$ is exact.

(5) Suppose that X_0 is a subspace of X. Let $\iota : X_0 \to X$ stand for the identical embedding of X_0 into X. Consider the quotient space X/X_0 and let $\varphi : X \to X/X_0$ be the corresponding coset mapping. Then the sequence

$$0 \to X_0 \xrightarrow{\iota} X \xrightarrow{\varphi} X/X_0 \to 0$$

is exact. (On similar occasions the letters ι and φ are usually omitted.) In a sense, this sequence is unique. Namely, consider a so-called *short sequence*

$$0 \to X \xrightarrow{T} Y \xrightarrow{S} Z \to 0$$

and assume that it is exact. Putting $Y_0 := \operatorname{im} T$, arrange the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{T} & Y & \xrightarrow{S} & Z & \longrightarrow & 0 \\
 & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\
0 & \longrightarrow & Y_0 & \longrightarrow & Y & \longrightarrow & Y/Y_0 & \longrightarrow & 0
\end{array}$$

where α, β, and γ are some isomorphisms. In other words, a short exact sequence actually presents a subspace and the corresponding quotient space. ◁▷

(6) Every T in $\mathscr{L}(X, Y)$ generates the exact sequence

$$0 \to \ker T \to X \xrightarrow{T} Y \to \operatorname{coker} T \to 0$$

which is called the *canonical exact sequence* for T.

2.3.6. Definition. Given $T_0 \in \mathscr{L}(X_0, Y)$, with X_0 a subspace of X, call an operator T from X to Y an *extension* of T_0 (onto X, in symbols, $T \supset T_0$), provided that $T_0 = T\iota$, where $\iota : X_0 \to X$ is the identical embedding of X_0 into X.

2.3.7. *Let X and Y be vector spaces and let X_0 be a subspace of X. Then each T_0 in $\mathscr{L}(X_0, Y)$ has an extension T in $\mathscr{L}(X, Y)$.*

◁ Putting $T := T_0 P_{X_0}$, where P_{X_0} is a projection onto X_0, settles the claim. ▷

2.3.8. Theorem. *Let X, Y, and Z be vector spaces. Take $A \in \mathscr{L}(X, Y)$ and $B \in \mathscr{L}(X, Z)$. The diagram*

$$\begin{array}{ccc}
X & \xrightarrow{A} & Y \\
 & {\scriptstyle B}\searrow & \downarrow{\scriptstyle \mathscr{X}} \\
 & & Z
\end{array}$$

is commutative for some $\mathscr{X}$ in $\mathscr{L}(Y, Z)$ if and only if $\ker A \subset \ker B$.

◁ $\Rightarrow$: It is evident that $\ker A \subset \ker B$ in case $B = \mathscr{X} A$.

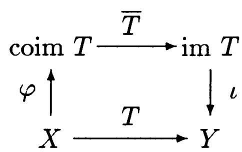
$\Leftarrow$: Set $\overline{\mathscr{X}} := B \circ A^{-1}$. Clearly, $\overline{\mathscr{X}} \circ A(x) = B \circ (A^{-1} \circ A)x = B(x + \ker A) = Bx$. Show that $\mathscr{X}_0 := \overline{\mathscr{X}}|_{\operatorname{im} A}$ is a linear operator. It suffices to check that $\overline{\mathscr{X}}$ is single-valued. Suppose that $y \in \operatorname{im} A$ and $z_1, z_2 \in \overline{\mathscr{X}}(y)$. Then $z_1 = Bx_1$, $z_2 = Bx_2$ and $Ax_1 = Ax_2 = y$. By hypothesis $B(x_1 - x_2) = 0$; therefore, $z_1 = z_2$. Applying 2.3.7, find an extension $\mathscr{X}$ of $\mathscr{X}_0$ to Y. ▷

2.3.9. REMARK. Provided that the operator A is an epimorphism, there is a unique solution $\mathscr{X}$ in 2.3.8. It is the right place to emphasize that the phrase, "There is a unique $\mathscr{X}$," implies that $\mathscr{X}$ is available as well as unique.

2.3.10. *Every linear operator T admits a unique factorization through its coimage; i.e., there is a unique quotient $\overline{T}$ of T by the equivalence $\sim_{\mathrm{codim}\ T}$.*

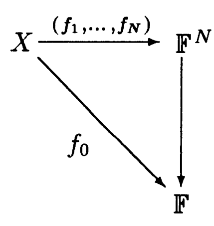
◁ Immediate from 2.3.8 and 2.3.9. ▷

2.3.11. REMARK. The operator $\overline{T}$ is sometimes called the *monoquotient* of T and treated as acting onto im T. In this connection, observe that T is expressible as the composition $T = \iota\overline{T}\varphi$, with φ an epimorphism, $\overline{T}$ an isomorphism, and ι a monomorphism; i.e., the following diagram commutes:



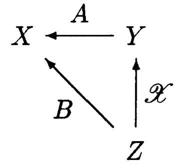
2.3.12. *Let X be a vector space and let f_0, $f_1, \dots, f_N$ belong to $X^{\#}$. The functional f_0 is a linear combination of $f_1, \dots, f_N$ (i.e., $f_0 = \sum_{j=1}^{N} \lambda_j f_j$ with λ_j in $\mathbb{F}$) if and only if* $\ker f_0 \supset \cap_{j=1}^{N} \ker f_j$.

◁ Define the linear operator $(f_1, \dots, f_N) : X \to \mathbb{F}^N$ by $(f_1, \dots, f_N)x := (f_1(x), \dots, f_N(x))$. Obviously, $\ker(f_1, \dots, f_N) = \cap_{j=1}^{N} \ker f_j$. Now apply 2.3.8 to the problem



on recalling what $\mathbb{F}^{N\#}$ is. ▷

2.3.13. Theorem. *Let X, Y, and Z be vector spaces. Take $A \in \mathscr{L}(Y, X)$ and $B \in \mathscr{L}(Z, X)$. The diagram*



is commutative for some $\mathscr{X}$ in $\mathscr{L}(Z, Y)$ if and only if im $A \supset$ im B.

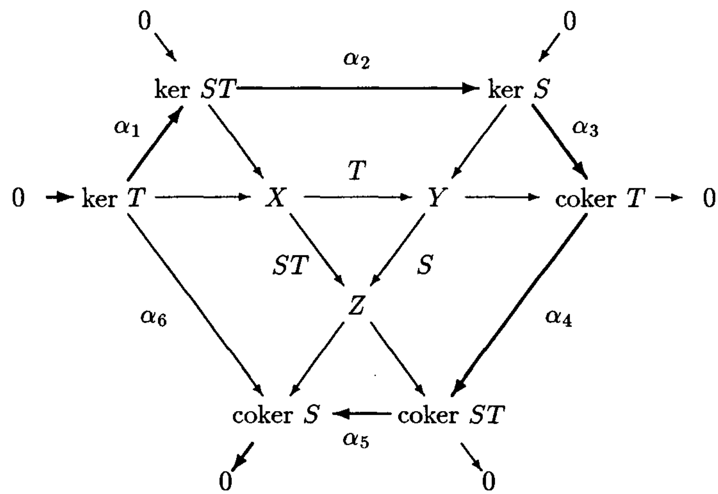
◁ $\Rightarrow$: $\operatorname{im} B = B(Z) = A(\mathscr{X}(Z)) \subset A(Y) = \operatorname{im} A$.

$\Leftarrow$: Let Y_0 be an algebraic complement of $\ker A$ to Y and $A_0 := A|_{Y_0}$. Then A_0 is an isomorphism between Y_0 and $\operatorname{im} A$. The operator $\mathscr{X} := \iota A_0^{-1} B$ is obviously a sought solution, with ι the identical embedding of Y_0 into Y. ▷

2.3.14. Remark. Provided that the operator A is a monomorphism, there is a unique operator $\mathscr{X}$ in 2.3.13. ◁▷

2.3.15. Remark. Theorems 2.3.8 and 2.3.13 are in "formal duality." One results from the other by "reversing arrows," "interchanging kernels and images," and "passing to reverse inclusion."

2.3.16. ***Snowflake Lemma.*** *Let $S \in \mathscr{L}(Y, Z)$ and $T \in \mathscr{L}(X, Y)$. There are unique operators $\alpha_1, \dots, \alpha_6$ such that the following diagram commutes:*



Moreover, the sequence

$$0 \to \ker T \xrightarrow{\alpha_1} \ker ST \xrightarrow{\alpha_2} \ker S \xrightarrow{\alpha_3} \operatorname{coker} T \xrightarrow{\alpha_4} \operatorname{coker} ST \xrightarrow{\alpha_5} \operatorname{coker} S \to 0$$

is exact. ◁▷

Exercises

2.1. Give examples of vector spaces and nonvector spaces. Which constructions lead to vector spaces?

2.2. Study vector spaces over the two-element field $\mathbb{Z}_2$.

2.3. Describe vector spaces with a countable Hamel basis.

2.4. Prove that there is a discontinuous solution $f : \mathbb{R} \to \mathbb{R}$ to the function equation

$$f(x + y) = f(x) + f(y) \quad (x,\ y \in \mathbb{R}).$$

How to visualize such an f graphically?

2.5. Prove that the algebraic dual to the direct sum of (X_ξ) is presentable as the product of the algebraic duals of (X_ξ).

2.6. Let $X \supset X_0 \supset X_{00}$. Prove that the spaces X/X_{00} and $(X/X_0)/(X_{00}/X_0)$ are isomorphic.

2.7. Define the "double sharp" mapping by the rule

$$x^{\#\#} : x^{\#} \mapsto \langle x \mid x^{\#} \rangle \quad (x \in X,\ x^{\#} \in X^{\#}).$$

Show that this mapping embeds a vector space X into the second dual $X^{\#\#}$.

2.8. Prove that finite-dimensional spaces and only such spaces are ***algebraically reflexive***; i.e.,

$$^{\#\#}(X) = X^{\#\#} \Leftrightarrow \dim X < +\infty.$$

2.9. Are there any analogs for a Hamel basis in general modules?

2.10. When does a sum of projections present a projection itself?

2.11. Let T be an endomorphism of some vector space which satisfies the conditions $T^{n-1} \neq 0$ and $T^n = 0$ for a natural n. Prove that the operators $T^0,\ T, \dots, T^{n-1}$ are linearly independent.

2.12. Describe the structure of a linear operator defined on the direct sum of spaces and acting into the product of spaces.

2.13. Find conditions for unique solvability of the following equations in operators $\mathscr{X}A = B$ and $A\mathscr{X} = B$ (here the operator $\mathscr{X}$ is unknown).

2.14. What is the structure of the spaces of bilinear operators?

2.15. Characterize the real vector space that results from neglecting multiplication by imaginary scalars in a complex vector space (cf. 3.7.1).

2.16. Given a family of linearly independent vectors $(x_e)_{e \in \mathscr{E}}$, find a family of functionals $(x_e^{\#})_{e \in \mathscr{E}}$ satisfying the following conditions:

$$\langle x_e \mid x_e^{\#} \rangle = 1 \quad (e \in \mathscr{E});$$
$$\langle x_e \mid x_{e'}^{\#} \rangle = 0 \quad (e,\ e' \in \mathscr{E},\ e \neq e').$$

2.17. Given a family of linearly independent functionals $(x_e^{\#})_{e \in \mathscr{E}}$, find a family of vectors $(x_e)_{e \in \mathscr{E}}$ satisfying the following conditions:

$$\langle x_e \mid x_e^{\#} \rangle = 1 \quad (e \in \mathscr{E});$$
$$\langle x_e \mid x_{e'}^{\#} \rangle = 0 \quad (e,\ e' \in \mathscr{E},\ e \neq e').$$

2.18. Find compatibility conditions for simultaneous linear equations and linear inequalities in real vector spaces.

2.19. Consider the commutative diagram

$$\begin{array}{ccccccc} W & \longrightarrow & X & \xrightarrow{T} & Y & \longrightarrow & Z \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow \\ \overline{W} & \longrightarrow & \overline{X} & \xrightarrow{\overline{T}} & \overline{Y} & \longrightarrow & \overline{Z} \end{array}$$

with exact rows and such that α is an epimorphism, and δ is a monomorphism. Prove that $\ker \gamma = T(\ker \beta)$ and $\overline{T}^{-1}(\operatorname{im} \gamma) = \operatorname{im} \beta$.

Chapter 3
Convex Analysis

3.1. Sets in Vector Spaces

3.1.1. DEFINITION. Let Γ be a subset of $\mathbb{F}^2$. A subset U of a vector space is a Γ-*set* in this space (in symbols, $U \in (\Gamma)$) if $(\lambda_1, \lambda_2) \in \Gamma \Rightarrow \lambda_1 U + \lambda_2 U \subset U$.

3.1.2. EXAMPLES.

(1) Every set is in $(\varnothing)$. (Hence, $(\varnothing)$ is not a set.)

(2) If $\Gamma := \mathbb{F}^2$ then a nonempty Γ-set is precisely a linear set in a vector space.

(3) If $\Gamma := \mathbb{R}^2$ then a nonempty Γ-set in a vector space X is a *real subspace* of X.

(4) By definition, a *cone* is a nonempty Γ-set with $\Gamma := \mathbb{R}^2_+$. In other words, a nonempty set K is a cone if and only if $K + K \subset K$ and $\alpha K \subset K$ for all $\alpha \in \mathbb{R}_+$. A nonempty $\mathbb{R}^2_+ \setminus 0$-set is sometimes referred to as a *nonpointed cone*; and a nonempty $\mathbb{R}_+ \times 0$-set, as a *nonconvex cone*. (From now on we use the convenient notation $\mathbb{R}_+ := \{t \in \mathbb{R} : t \geq 0\}$.)

(5) A nonempty Γ-set, for $\Gamma := \{(\lambda_1, \lambda_2) \in \mathbb{F}^2 : \lambda_1 + \lambda_2 = 1\}$, is an *affine variety* or a *flat*. If X_0 is a subspace of X and $x \in X$ then $x + X_0 := \{x\} + X_0$ is an affine variety in X. Conversely, if L is an affine variety in X and $x \in L$ then $L - x := L + \{-x\}$ is a linear set in X. ◁▷

(6) If $\Gamma := \{(\lambda_1, \lambda_2) \in \mathbb{F}^2 : |\lambda_1| + |\lambda_2| \leq 1\}$ then a nonempty Γ-set is *absolutely convex*.

(7) If $\Gamma := \{(\lambda, 0) \in \mathbb{F}^2 : |\lambda| \leq 1\}$ then a nonempty Γ-set is *balanced*. (In the case $\mathbb{F} := \mathbb{R}$ the term "star-shaped" is occasionally employed; the word "symmetric" can also be found.)

(8) A set is *convex* if it is a Γ-set for $\Gamma := \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1\}$.

(9) A *conical segment* or *conical slice* is by definition a nonempty Γ-set with $\Gamma := \{(\lambda_1, \lambda_2) \in \mathbb{R}^2_+ : \lambda_1 + \lambda_2 \leq 1\}$. A set is a conical segment if and only if it is a convex set containing zero. ◁▷

(10) Given $\Gamma \subset \mathbb{F}^2$, observe that $X \in (\Gamma)$ for every vector space X over $\mathbb{F}$. Note also that in 3.1.2 (1)–3.1.2 (9) the set Γ is itself a Γ-set.

3.1.3. *Let X be a vector space and let $\mathscr{E}$ be a family of Γ-sets in X. Then* $\cap\{U : U \in \operatorname{im} \mathscr{E}\} \in (\Gamma)$. *Provided that* $\operatorname{im} \mathscr{E}$ *is filtered upward (by inclusion),* $\cup\{U : U \in \operatorname{im} \mathscr{E}\} \in (\Gamma)$. ◁▷

3.1.4. REMARK. The claim of 3.1.3 means in particular that the collection of Γ-sets of a vector space, ordered by inclusion, presents a complete lattice.

3.1.5. *Let X and Y be vector spaces and let U and V be Γ-sets, with $U \subset X$ and $V \subset Y$. Then $U \times V \in (\Gamma)$.*

◁ If U or V is nonempty then $U \times V = \varnothing$, and there is nothing to prove. Now take $u_1,\ u_2 \in U$, $v_1,\ v_2 \in V$, and $(\lambda_1,\ \lambda_2) \in \Gamma$. Find $\lambda_1 u_1 + \lambda_2 u_2 \in U$ and $\lambda_1 v_2 + \lambda_2 v_1 \in V$. Hence, $(\lambda_1 u_1 + \lambda_2 u_2,\ \lambda_1 v_1 + \lambda_2 v_2) \in U \times V$. ▷

3.1.6. DEFINITION. Let X and Y be vector spaces and $\Gamma \subset \mathbb{F}^2$. A correspondence $T \subset X \times Y$ is a Γ-*correspondence* provided that $T \in (\Gamma)$.

3.1.7. REMARK. When a Γ-set bears a specific attribute, the latter is preserved for naming a Γ-correspondence. With this in mind, we speak about *linear* and *convex correspondences*, *affine mappings*, etc. The next particularity of the nomenclature is worth memorizing: a convex function of one variable is not a convex correspondence, save trivial cases (cf. 3.4.2).

3.1.8. *Let $T \subset X \times Y$ be a Γ_1-correspondence and let $U \subset X$ be a Γ_2-set. If $\Gamma_2 \subset \Gamma_1$ then $T(U) \in (\Gamma_2)$.*

◁ Take $y_1,\ y_2 \in T(U)$. Then $(x_1,\ y_1) \in T$ and $(x_2,\ y_2) \in T$ with some $x_1,\ x_2 \in U$. Given $(\lambda_1,\ \lambda_2) \in \Gamma_2$, observe that $\lambda_1(x_1,\ y_1) + \lambda_2(x_2,\ y_2) \in T$, because by hypothesis $(\lambda_1,\ \lambda_2) \in \Gamma_1$. Finally, infer that $\lambda_1 y_1 + \lambda_2 y_2 \in T(U)$. ▷

3.1.9. *The composition of Γ-correspondences is also a Γ-correspondence.*

◁ Let $F \subset X \times V$ and $G \subset W \times Y$, with $F,\ G \in (\Gamma)$. Note that

$$(x_1,\ y_1) \in G \circ F \Leftrightarrow (\exists v_1) \quad (x_1,\ v_1) \in F \ \&\ (v_1,\ y_1) \in G;$$
$$(x_2,\ y_2) \in G \circ F \Leftrightarrow (\exists v_2) \quad (x_2,\ v_2) \in F \ \&\ (v_2,\ y_2) \in G.$$

To complete the proof, "multiply the first row by λ_1; the second, by λ_2, where $(\lambda_1,\ \lambda_2) \in \Gamma$; and sum the results." ▷

3.1.10. *If subsets U and V of a vector space are Γ-sets for some $\Gamma \subset \mathbb{F}^2$ then $\alpha U + \beta V \in (\Gamma)$ for all $\alpha,\ \beta \in \mathbb{F}$.*

◁ The claim is immediate from 3.1.5, 3.1.8, and 3.1.9. ▷

3.1.11. DEFINITION. Let X be a vector space and let U be a subset of X. For $\Gamma \subset \mathbb{F}^2$ the Γ-*hull* of U is the set

$$H_\Gamma(U) := \cap\{V \subset X : V \in (\Gamma),\ V \supset U\}.$$

3.1.12. *The following statements are valid:*

(1) $H_\Gamma(U) \in (\Gamma)$;

(2) $H_\Gamma(U)$ *is the least* Γ*-set including* U;

(3) $U_1 \subset U_2 \Rightarrow H_\Gamma(U_1) \subset H_\Gamma(U_2)$;

(4) $U \in (\Gamma) \Leftrightarrow U = H_\Gamma(U)$;

(5) $H_\Gamma(H_\Gamma(U)) = H_\Gamma(U)$. ◁▷

3.1.13. *The Motzkin formula holds:*

$$H_\Gamma(U) = \cup\{H_\Gamma(U_0) : \ U_0 \text{ is a finite subset of } U\}.$$

◁ Denote the right side of the Motzkin formula by V. Since $U_0 \subset U$; applying 3.1.12 (3), deduce that $H_\Gamma(U_0) \subset H_\Gamma(U)$; and, hence, $H_\Gamma(U) \supset V$. By 3.1.12 (2), it is necessary (and, surely, sufficient) to verify that $V \in (\Gamma)$. The last follows from 3.1.3 and the inclusion $H_\Gamma(U_0) \cup H_\Gamma(U_1) \subset H_\Gamma(U_0 \cup U_1)$. ▷

3.1.14. Remark. The Motzkin formula reduces the problem of describing arbitrary Γ-hulls to calculating Γ-hulls of finite sets. Observe also that Γ-hulls for concrete Γs have special (but natural) designations. For instance, if $\Gamma := \{(\lambda_1, \lambda_2) \in \mathbb{R}^2_+ : \ \lambda_1 + \lambda_2 = 1\}$, then the term "convex hull" is used and $\operatorname{co}(U)$ stands for $H_\Gamma(U)$. If $U \neq \varnothing$, the notation for $H_{\mathbb{F}^2}(U)$ is $\operatorname{lin}(U)$; moreover, it is natural and convenient to put $\operatorname{lin}(\varnothing) := 0$. The subspace $\operatorname{lin}(U)$ is called the *linear span* of U. The concepts of *affine hull*, *conical hull*, etc. are introduced in a similar fashion. Note also that the convex hull of a finite set of points comprises their *convex combinations*; i.e.,

$$\operatorname{co}(\{x_1, \dots, x_N\}) = \left\{ \sum_{k=1}^{N} \lambda_k x_k : \ \lambda_k \geq 0, \ \lambda_1 + \dots + \lambda_N = 1 \right\}. \ ◁▷$$

3.2. Ordered Vector Spaces

3.2.1. Definition. Let $(X, \mathbb{R}, +, \cdot)$ be a vector space. A preorder σ on X is *compatible with vector structure* if σ is a cone in X^2; in this case X is an *ordered vector space.* (It is more precise to call $(X, \mathbb{R}, +, \cdot, \sigma)$ a *preordered vector space*, reserving the term "ordered vector space" for the case in which σ is an order.)

3.2.2. *If X is an ordered vector space and σ is the corresponding preorder then $\sigma(0)$ is a cone and $\sigma(x) = x + \sigma(0)$ for all $x \in X$.*

◁ By 3.1.3, $\sigma(0)$ is a cone. The equality $(x, y) = (x, x) + (0, y - x)$ yields the equivalence $(x, y) \in \sigma \Leftrightarrow y - x \in \sigma(0)$. ▷

3.2.3. *Let K be a cone in a vector space X. Denote*

$$\sigma := \{(x, y) \in X^2 : \ y - x \in K\}.$$

Then σ is a preorder compatible with vector structure and K coincides with the cone $\sigma(0)$ of positive elements of X. The relation σ is an order if and only if $K \cap (-K) = 0$.

$\triangleleft$ It is clear that $0 \in K \Rightarrow I_X \subset \sigma$ and $K + K \subset K \Rightarrow \sigma \circ \sigma \subset \sigma$. Furthermore, $\sigma^{-1} = \{(x,\ y) \in X^2 :\ x - y \in K\}$. Therefore, $\sigma \cap \sigma^{-1} \subset I_X \Leftrightarrow K \cap (-K) = 0$. To show that σ is a cone, take $(x_1,\ y_1),\ (x_2,\ y_2) \in \sigma$ and $\alpha_1,\ \alpha_2 \in \mathbb{R}_+$. Find $\alpha_1 y_1 + \alpha_2 y_2 - (\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1(y_1 - x_1) + \alpha_2(y_2 - x_2) \in \alpha_1 K + \alpha_2 K \subset K$. $\triangleright$

3.2.4. DEFINITION. A cone K is an *ordering cone* or a *salient cone* provided that $K \cap (-K) = 0$.

3.2.5. REMARK. By virtue of 3.2.2 and 3.2.3, assigning a preorder to X is equivalent to distinguishing some cone of positive elements in it which is the *positive cone* of X. The structure of an ordered vector space arises from selecting an ordering cone. Keeping this in mind, we customarily call a pair $(X,\ X_+)$ with positive cone X_+, as well as X itself, a (pre)ordered vector space.

3.2.6. EXAMPLES.

(1) The space of real-valued functions $\mathbb{R}^\Xi$ with the positive cone $\mathbb{R}^\Xi_+ := (\mathbb{R}_+)^\Xi$ constituted by the functions assuming only positive values has the "natural order."

(2) Let X be an ordered vector space with positive cone X_+. If X_0 is a subspace of X then the order induced in X_0 is defined by the cone $X_0 \cap X_+$. In this way X_0 is considered as an ordered vector space, a *subspace* of X.

(3) If X and Y are preordered vector spaces then $T \in \mathscr{L}(X,\ Y)$ is a *positive operator* (in symbols, $T \geq 0$) whenever $T(X_+) \subset Y_+$. The set $\mathscr{L}_+(X,\ Y)$ of all positive operators is a cone. The linear span of $\mathscr{L}_+(X,\ Y)$ is denoted by $\mathscr{L}_r(X,\ Y)$, and a member of $\mathscr{L}_r(X,\ Y)$ is called a *regular operator.*

3.2.7. DEFINITION. An ordered vector space is a *vector lattice* or a *Riesz space* if the ordered set of its vectors presents a lattice.

3.2.8. DEFINITION. A vector lattice X is called a *Kantorovich space* or briefly a *K-space* if X is *boundedly order complete* or *Dedekind complete*, which means that each nonempty bounded above subset of X has a least upper bound.

3.2.9. *Each nonempty bounded below subset of a Kantorovich space has a greatest lower bound.*

$\triangleleft$ Let U be bounded below: $x \leq U$ for some x. So, $-x \geq -U$. Applying 3.2.8, find $\sup(-U)$. Obviously, $-\sup(-U) = \inf U$. $\triangleright$

3.2.10. *If U and V are nonempty bounded above subsets of a K-space then*

$$\sup(U + V) = \sup U + \sup V.$$

◁ If U or V is a singleton then the equality is plain. The general case follows from the *associativity of least upper bounds*. Namely,

$$\sup(U+V) = \sup\{\sup(u+V) : \ u \in U\}$$
$$= \sup\{u + \sup V : \ u \in U\} = \sup V + \sup\{u : \ u \in U\}$$
$$= \sup V + \sup U. \ \triangleright$$

3.2.11. Remark. The derivation of 3.2.10 is valid for an arbitrary ordered vector space provided that the given sets have least upper bounds. The equality $\sup \lambda U = \lambda \sup U$ for $\lambda \in \mathbb{R}_+$ is comprehended by analogy.

3.2.12. Definition. For an element x of a vector lattice, the *positive part* of x is $x_+ := x \vee 0$, the *negative part* of x is $x_- := (-x)_+$, and the *modulus* of x is $|x| := x \vee (-x)$.

3.2.13. *If x and y are elements of a vector lattice then*

$$x + y = x \vee y + x \wedge y.$$

◁ $x + y - x \wedge y = x + y + (-x) \vee (-y) = y \vee x$ ▷

3.2.14. $x = x_+ - x_-$; $|x| = x_+ + x_-$.

◁ The first equality follows from 3.2.13 on letting $y := 0$. Furthermore, $|x| = x \vee (-x) = -x + (2x) \vee 0 = -x + 2x_+ = -x(x_+ - x_-) + 2x_+ = x_+ + x_-$. ▷

3.2.15. *Interval Addition Lemma.* *If x and y are positive elements of a vector lattice X then*

$$[0,\ x+y] = [0,\ x] + [0,\ y].$$

(As usual, $[u,\ v] := \sigma(u) \cap \sigma^{-1}(v)$ is the (order) interval with endpoints u and v.)

◁ The inclusion $[0,\ x]+[0,\ y] \subset [0,\ x+y]$ is obvious. Assume that $0 \le z \le x+y$ and put $z_1 := z \wedge x$. It is easy that $z_1 \in [0,\ x]$. Now if $z_2 := z - z_1$ then $z_2 \ge 0$ and $z_2 = z - z \wedge x = z + (-z) \vee (-x) = 0 \vee (z - x) \le 0 \vee (x + y - x) = 0 \vee y = y$. ▷

3.2.16. Remark. The conclusion of the Interval Addition Lemma is often referred to as the *Riesz Decomposition Property*.

3.2.17. *Riesz–Kantorovich Theorem.* *Let X be a vector space and let Y be a K-space. The space of regular operators $\mathscr{L}_r(X,\ Y)$, ordered by the cone of positive operators $\mathscr{L}_+(X,\ Y)$, is a K-space.* ◁▷

3.3. Extension of Positive Functionals and Operators

3.3.1. COUNTEREXAMPLES.

(1) Let X be $B([0,\ 1],\ \mathbb{R})$, the space of the bounded real-valued functions given on $[0,\ 1]$; and let X_0 be $C([0,\ 1],\ \mathbb{R})$, the subspace of X comprising all continuous functions. Put $Y := X_0$ and equip X_0, X, and Y with the natural order (cf. 3.2.6 (1) and 3.2.6 (2)). Consider the problem of extending the identical embedding $T_0 : X_0 \to Y$ to a positive operator T in $\mathscr{L}_+(X,\ Y)$. If the problem had a solution T, then each nonempty bounded set $\mathscr{E}$ in X_0 would have a least upper bound $\sup_{X_0} \mathscr{E}$ calculated in X_0. Namely, $\sup_{X_0} \mathscr{E} = T \sup_X \mathscr{E}$, where $\sup_X \mathscr{E}$ is the least upper bound of $\mathscr{E}$ in X; whereas, undoubtedly, Y fails to be a K-space.

(2) Denote by $s := \mathbb{R}^N$ the *sequence space* and furnish s with the natural order. Let c be the subspace of s comprising all convergent sequences, the *convergent sequence space.* Demonstrate that the positive functional $f_0 : c \to \mathbb{R}$ defined by $f_0(x) := \lim x(n)$ has no positive extension to s. Indeed, assume that $f \in s^{\#}$, $f \geq 0$ and $f \supset f_0$. Put $x_0(n) := n$ and $x_k(n) := k \wedge n$ for $k,\ n \in \mathbb{N}$. Plainly, $f_0(x_k) = k$; moreover, $f(x_0) \geq f(x_k) \geq 0$, since $x_0 \geq x_k \geq 0$, a contradiction.

3.3.2. DEFINITION. A subspace X_0 of an ordered vector space X with positive cone X_+ is *massive* (in X) if $X_0 + X_+ = X$. The terms "coinitial" or "minorizing" are also in common parlance.

3.3.3. *A subspace X_0 is massive in X if and only if for all $x \in X$ there are elements x_0 and x^0 in X_0 such that $x_0 \leq x \leq x^0$.* ⊲⊳

3.3.4. ***Kantorovich Theorem.*** *If X is an ordered vector space, X_0 is massive in X, and Y is a K-space; then each positive operator T_0, a member of $\mathscr{L}_+(X_0,\ Y)$, has a positive extension T, a member of $\mathscr{L}_+(X,\ Y)$.*

⊲ STEP I. First, let $X := X_0 \oplus X_1$, where X_1 is a one-dimensional subspace, $X_1 := \{\alpha\overline{x} :\ \alpha \in \mathbb{R}\}$. Since X_0 is massive and T_0 is positive, the set $U := \{T_0 x^0 :\ x^0 \in X_0,\ x^0 \geq \overline{x}\}$ is nonempty and bounded below. Consequently, there is some $\overline{y}$ such that $\overline{y} := \inf U$. Assign $Tx := \{T_0 x_0 + \alpha\overline{y} :\ x = x_0 + \alpha\overline{x},\ x_0 \in X_0,\ \alpha \in \mathbb{R}\}$. It is clear that T is a single-valued correspondence. Further, $T \supset T_0$ and dom T $= X$. So, only the positivity of T needs checking. If $x = x_0 + \alpha\overline{x}$ and $x \geq 0$, then the case of α equal to 0 is trivial. If $\alpha > 0$ then $\overline{x} \geq -x_0/\alpha$. This implies that $-T_0 x_0/\alpha \leq \overline{y}$, i.e., $Tx \in Y_+$. In a similar way for $\alpha < 0$ observe that $\overline{x} \leq -x_0/\alpha$. Thus, $\overline{y} \leq -T_0 x_0/\alpha$ and, finally, $Tx = T_0 x_0 + \alpha\overline{y} \in Y_+$.

STEP II. Now let $\mathscr{E}$ be the collection of single-valued correspondences $S \subset X \times Y$ such that $S \supset T_0$ and $S(X_+) \subset Y_+$. By 3.1.3, $\mathscr{E}$ is inductive in order by inclusion and so, by the Kuratowski–Zorn Lemma, $\mathscr{E}$ has a maximal element T. If $\overline{x} \in X \setminus \operatorname{dom} T$, apply the result of Step I with $X := \operatorname{dom} T \oplus X_1$, $X_0 := \operatorname{dom} T$, $T_0 := T$ and $X_1 := \{\alpha\overline{x} :\ \alpha \in \mathbb{R}\}$ to obtain an extension of T. But T is maximal, a contradiction. Thus, T is a sought operator. ⊳

3.3.5. REMARK. When $Y := \mathbb{R}$, Theorem 3.3.4 is sometimes referred to as the *Kreĭn–Rutman Theorem.*

3.3.6. DEFINITION. A positive element x is *discrete* whenever $[0,\ x] = [0,\ 1]x$.

3.3.7. *If there is a discrete functional on $(X,\ X_+)$ then $X = X_+ - X_+$.*

◁ Let T be such a functional and $\mathscr{X} := X_+ - X_+$. Take $f \in X^\#$. It suffices to check that $\ker f \supset \mathscr{X} \Rightarrow f = 0$. Evidently, $T + f \in [0,\ T]$; i.e., $T + f = \alpha T$ for some $\alpha \in [0,\ 1]$. If $T|_{\mathscr{X}} = 0$ then $2T \in [0,\ T]$. Hence, $T = 0$ and $f = 0$. Now if $T(x_0) \neq 0$ for some $x_0 \in \mathscr{X}$, then $\alpha = 1$ and $f = 0$. ▷

3.3.8. ***Discrete Kreĭn–Rutman Theorem.*** *Let X be a massive subspace of an ordered vector space X and let T_0 be a discrete functional on X_0. Then there is a discrete functional T on X extending T_0.*

◁ Adjust the proof of 3.3.4.

STEP I. Observe that the exhibited functional T is discrete. For, if $T' \in [0,\ T]$ then there is some $\alpha \in [0,\ 1]$ such that $T'(x_0) = \alpha T(x_0)$ for all $x_0 \in X_0$ and so $(T - T')(x_0) = (1 - \alpha)T(x_0)$. Estimating, find

$$T'(\overline{x}) \leq \inf\{T'(x^0) :\ x^0 \geq \overline{x},\ x^0 \in X_0\} = \alpha T(\overline{x});$$
$$(T - T')(\overline{x}) \leq \inf\{(T - T')(x^0) :\ x^0 \geq \overline{x},\ x^0 \in X_0\} = (1 - \alpha)T(\overline{x}).$$

Therefore, $T' = \alpha T$ and $[0,\ T] \subset [0,\ 1]T$. The reverse inclusion is always true. Thus, T is discrete.

STEP II. Let $\mathscr{E}$ be the same as in the proof of 3.3.4. Consider $\mathscr{E}_d$, the set comprising all S in $\mathscr{E}$ such that the restriction $S|_{\mathrm{dom}\ S}$ is a discrete functional on dom S. Show that $\mathscr{E}_d$ is inductive. To this end, take a chain $\mathscr{E}_0$ in $\mathscr{E}_d$ and put $S := \cup\{S_0 :\ S_0 \in \mathscr{E}_0\}$; obviously, $S \in \mathscr{E}$. Verifying the discreteness of S will complete the proof.

Suppose that $0 \leq S'(x_0) \leq S(x_0)$ for all $x_0 \in (\mathrm{dom}\ S)_+$ and $S' \in (\mathrm{dom}\ S)^\#$. If $S(x_0) = 0$ for all x_0 then $S' = 0S$, as was needed. In the case $S(x_0) \neq 0$ for some $x_0 \in (\mathrm{dom}\ S)_+$ choose $S_0 \in \mathscr{E}_0$ such that $S_0(x_0) = S(x_0)$. Since S_0 is discrete, deduce that $S'(x') = \alpha S(x')$ for all $x' \in \mathrm{dom}\ S_0$. Furthermore, $\alpha = S'(x_0)/S(x_0)$; i.e., α does not depend on the choice of S_0. Since $\mathscr{E}_0$ is a chain, infer that $S' = \alpha S$. ▷

3.4. Convex Functions and Sublinear Functionals

3.4.1. DEFINITION. The *semi-extended real line* $\mathbb{R}^{\cdot}$ is the set $\mathbb{R}^{\cdot}$ with some greatest element $+\infty$ adjoined formally. The following agreements are effective: $\alpha(+\infty) := +\infty$ $(\alpha \in \mathbb{R}_+)$ and $+\infty + x := x + (+\infty) := +\infty$ $(x \in \mathbb{R}^{\cdot})$.

3.4.2. DEFINITION. Let $f : X \to \mathbb{R}^{\cdot}$ be a mapping (also called an *extended function*). The *epigraph* of f is the set

$$\mathrm{epi}\ f := \{(x,\ t) \in X \times \mathbb{R} :\ t \geq f(x)\}.$$

The *effective domain of definition* or simply the *domain* of f is the set

$$\operatorname{dom} f := \{x \in X : \ f(x) < +\infty\}.$$

3.4.3. REMARK. Inconsistency in overusing the symbol dom f is illusory. Namely, the effective domain of definition of $f : X \to \mathbb{R}^{\cdot}$ coincides with the domain of definition of the single-valued correspondence $f \cap X \times \mathbb{R}$. We thus continue to write $f : X \to \mathbb{R}$, omitting the dot in the symbol $\mathbb{R}^{\cdot}$ whenever $\operatorname{dom} f = X$.

3.4.4. DEFINITION. If X is a real vector space then a mapping $f : X \to \mathbb{R}^{\cdot}$ is a *convex function* provided that the epigraph epi f is convex.

3.4.5. *A function $f : X \to \mathbb{R}^{\cdot}$ is convex if and only if the Jensen inequality holds:*

$$f(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

for all $\alpha_1, \ \alpha_2 \geq 0, \ \alpha_1 + \alpha_2 = 1$ and $x_1, \ x_2 \in X$.

◁ $\Rightarrow$: Take $\alpha_1, \ \alpha_2 \geq 0, \ \alpha_1 + \alpha_2 = 1$. If either of x_1 and x_2 fails to belong to dom f then the Jensen inequality is evident. Let $x_1, \ x_2 \in \operatorname{dom} f$. Then $(x_1, \ f(x_1)) \in \operatorname{epi} f$ and $(x_2, \ f(x_2)) \in \operatorname{epi} f$. Appealing to 3.1.2 (8), find $\alpha_1(x_1, \ f(x_1)) + \alpha_2(x_2, \ f(x_2)) \in \operatorname{epi} f$.

$\Leftarrow$: Take $(x_1, \ t_1) \in \operatorname{epi} f$ and $(x_2, \ t_2) \in \operatorname{epi} f$, i.e. $t_1 \geq f(x_1)$ and $t_2 \geq f(x_2)$ (if $\operatorname{dom} f = \varnothing$ then $f(x) = +\infty$ $(x \in X)$ and $\operatorname{epi} f = \varnothing$). Applying the Jensen inequality, observe the containment $(\alpha_1 x_1 + \alpha_2 x_2, \ \alpha_1 t_1 + \alpha_2 t_2) \in \operatorname{epi} f$ for all $\alpha_1, \ \alpha_2 \geq 0, \ \alpha_1 + \alpha_2 = 1$. ▷

3.4.6. DEFINITION. A mapping $p : X \to \mathbb{R}^{\cdot}$ is a *sublinear functional* provided that epi p is a cone.

3.4.7. *If* dom $p \neq 0$ *then the following statements are equivalent:*

(1) *p is a sublinear functional;*

(2) *p is a convex function that is positively homogeneous:*
$p(\alpha x) = \alpha p(x)$ for all $\alpha \geq 0$ and $x \in X$;

(3) *if $\alpha_1, \ \alpha_2 \in \mathbb{R}_+$ and $x_1, \ x_2 \in X$, then*
$p(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 p(x_1) + \alpha_2 p(x_2)$;

(4) *p is a positively homogeneous functional that is subadditive:*
$p(x_1 + x_2) \leq p(x_1) + p(x_2)$ $(x_1, \ x_2 \in X)$. ◁▷

3.4.8. EXAMPLES.

(1) A linear functional is sublinear; an affine functional is a convex function.

(2) Let U be a convex set in X. Define the *indicator function* $\delta(U) : X \to \mathbb{R}^{\cdot}$ of U as the mapping

$$\delta(U)(x) := \begin{cases} 0, & \text{if } \ x \in U \\ +\infty, & \text{if } \ x \notin U. \end{cases}$$

It is clear that $\delta(U)$ is a convex function. If U is a cone then $\delta(U)$ is a sublinear function. If U is an affine set then $\delta(U)$ is an affine function.

(3) The sum of finitely many convex functions and the supremum (or *upper envelope*) of a family of convex functions (calculated pointwise, i.e. in $(\mathbb{R}^{\cdot})^X$) are convex functionals. Sublinear functionals have analogous properties.

(4) The composition of a convex function with an *affine operator*, i.e. an everywhere-defined single-valued affine correspondence, is a convex function. The composition of a sublinear functional with a linear operator is sublinear.

3.4.9. Definition. If U and V are subsets of a vector space X then U *absorbs* V if $V \subset nU$ for some $n \in \mathbb{N}$. A set U is *absorbing* (in X) if it absorbs every point of X; i.e., $X = \cup_{n \in \mathbb{N}} nU$.

3.4.10. *Let $T \subset X \times Y$ be a linear correspondence with* $\operatorname{im} T = Y$. *If U is absorbing (in X) then $T(U)$ is absorbing (in Y).*

◁ $Y = T(X) = T(\cup_{n\in\mathbb{N}} nU) = \cup_{n\in\mathbb{N}} T(nU) = \cup_{n\in\mathbb{N}} nT(U)$ ▷

3.4.11. Definition. Let U be a subset of a vector space X. An element x belongs to the *core* of U (or x is an *algebraically interior point* of U) if $U - x$ is absorbing in X.

3.4.12. *If $f : X \to \mathbb{R}^{\cdot}$ is a convex function and $x \in$ core dom f then for all $h \in X$ there is a limit*

$$f'(x)(h) := \lim_{\alpha\downarrow 0} \frac{f(x+\alpha h) - f(x)}{\alpha} = \inf_{\alpha>0} \frac{f(x+\alpha h) - f(x)}{\alpha}.$$

*Moreover, the mapping $f'(x) : h \mapsto f'(x)h$ is a sublinear functional $f'(x) : X \to \mathbb{R}$, the **directional derivative** of f at x.*

◁ Set $\varphi(\alpha) := f(x + \alpha h)$. By 3.4.8 (4), $\varphi : \mathbb{R} \to \mathbb{R}^{\cdot}$ is a convex function and $0 \in$ core dom φ. The mapping $\alpha \mapsto (\varphi(\alpha) - \varphi(0))/\alpha \quad (\alpha > 0)$ is increasing and bounded above; i.e., $\varphi'(0)(1)$ is available. By definition, $f'(x)(h) = \varphi'(0)(1)$.

Given $\beta > 0$ and $h \in H$, successively find

$$f'(x)(\beta h) = \inf \frac{f(x+\alpha\beta h) - f(x)}{\alpha} = \beta \inf \frac{f(x+\alpha\beta h) - f(x)}{\alpha\beta} = \beta f'(x)(h).$$

Moreover, using the above result, for $h_1,\ h_2 \in X$ infer that

$$\begin{aligned} f'(x)(h_1+h_2) &= 2\lim_{\alpha\downarrow 0} \frac{f\left(x + {}^1/_2\alpha(h_1+h_2)\right) - f(x)}{\alpha} \\ &= 2\lim_{\alpha\downarrow 0} \frac{f\left({}^1/_2(x+\alpha h_1) + {}^1/_2(x+\alpha h_2)\right) - f(x)}{\alpha} \\ &\le \lim_{\alpha\downarrow 0} \frac{f(x+\alpha h_1) - f(x)}{\alpha} + \lim_{\alpha\downarrow 0} \frac{f(x+\alpha h_2) - f(x)}{\alpha} \\ &= f'(x)(h_1) + f'(x)(h_2). \end{aligned}$$

Appealing to 3.4.7, end the proof. ▷

3.5. The Hahn–Banach Theorem

3.5.1. DEFINITION. Let X be a real vector space and let $f : X \to \mathbb{R}^{\cdot}$ be a convex function. The *subdifferential* of f at a point x in dom f is the set

$$\partial_x(f) := \{l \in X^{\#} : (\forall y \in X) \quad l(y) - l(x) \leq f(y) - f(x)\}.$$

3.5.2. EXAMPLES.

(1) Let $p : X \to \mathbb{R}^{\cdot}$ be a sublinear functional. Put $\partial(p) := \partial_0(p)$. Then

$$\partial(p) = \{l \in X^{\#} : (\forall x \in X) \quad l(x) \leq p(x)\};$$
$$\partial_x(p) = \{l \in \partial(p) : \ l(x) = p(x)\}.$$

(2) If $l \in X^{\#}$ then $\partial(l) = \partial_x(l) = \{l\}$.

(3) Let X_0 be a subspace of X. Then

$$\partial(\delta(X_0)) = \{l \in X^{\#} : \ \ker l \supset X_0\}.$$

(4) If $f : X \to \mathbb{R}^{\cdot}$ is a convex function then

$$\partial_x(f) = \partial(f'(x))$$

whenever $x \in$ core dom f. ⊲⊳

3.5.3. ***Hahn–Banach Theorem.*** *Let $T \in \mathscr{L}(X, Y)$ be a linear operator and let $f : Y \to \mathbb{R}^{\cdot}$ be a convex function. If $x \in X$ and $Tx \in$ core dom f then*

$$\partial_x(f \circ T) = \partial_{Tx}(f) \circ T.$$

⊲ By 3.4.10 it follows that $x \in$ core dom f. From 3.5.2 (4) obtain $\partial_x(f \circ T) = \partial((f \circ T)'(x))$. Moreover,

$$(f \circ T)'(x)(h) = \lim_{\alpha \downarrow 0} \frac{(f \circ T)(x + \alpha h) - (f \circ T)(x)}{\alpha}$$
$$= \lim_{\alpha \downarrow 0} \frac{f(Tx + \alpha Th) - f(Tx)}{\alpha} = f'(Tx)(Th)$$

for $h \in X$. Put $p := f'(Tx)$. By 3.4.12, p is a sublinear functional. Whence, on appealing again to 3.5.2 (4), infer that

$$\partial(p) = \partial(f'(Tx)) = \partial_{Tx}(f);$$
$$\partial(p \circ T) = \partial((f \circ T)'(x)) = \partial_x(f \circ T).$$

Thereby, the only claim left unproven is the equality

$$\partial(p \circ T) = \partial(p) \circ T.$$

If $l \in \partial(p) \circ T$ (i.e., $l = l_1 \circ T$, where $l_1 \in \partial(p)$) then $l_1(y) \leq p(y)$ for all $y \in Y$. In particular, $l(x) \in l_1(Tx) \leq p(Tx) = p \circ T(x)$ for all $x \in X$ and so $l \in \partial(p \circ T)$. This argument yields the inclusion $\partial(p) \circ T \subset \partial(p \circ T)$.

Now take $l \in \partial(p \circ T)$. If $Tx = 0$ then $l(x) \leq p(Tx) = p(0) = 0$; i.e., $l(x) \leq 0$. The same holds for $-x$. Therefore, $l(x) = 0$; in other words, $\ker l \supset \ker T$. Hence, by 2.3.8, $l = l_1 \circ T$ for some $l_1 \in Y^{\#}$. Putting $Y_0 := T(X)$, observe that the functional $l_1 \circ \iota$ belongs to $\partial(p \circ \iota)$, with ι the identical embedding of Y_0 into Y. With $\partial(p \circ \iota) \subset \partial(p) \circ \iota$ proven, observe that $l_1 \circ \iota = l_2 \circ \iota$ for some $l_2 \in \partial(p)$, which consequently yields $l = l_1 \circ T = l_1 \circ \iota \circ T = l_2 \circ \iota \circ T = l_2 \circ T$; i.e., $l \in \partial(p) \circ T$.

Thus, to complete the proof of the Hahn–Banach Theorem, showing that $\partial(p \circ \iota) \subset \partial(p) \circ \iota$ is in order.

Take an element l_0 in $\partial(p \circ \iota)$ and consider the functional $T_0 : (y_0, t) \mapsto t - l_0(y_0)$ given on the subspace $\mathfrak{Y}_0 := Y_0 \times \mathbb{R}$ of $\mathfrak{Y} := Y \times \mathbb{R}$. Equip $\mathfrak{Y}$ with the cone $\mathfrak{Y}_+ := \operatorname{epi} p$. Note that, first, $\mathfrak{Y}_0$ is massive since

$$(y, t) = (0, t - p(y)) + (y, p(y)) \quad (y \in Y, t \in \mathbb{R}).$$

Second, by 3.4.2, $t \geq p(y_0)$ for $(y_0, t) \in \mathfrak{Y}_0 \cap \mathfrak{Y}_+$, and so $T_0(y_0, t) = t - l_0(y_0) \geq 0$; i.e., T_0 is a positive functional on $\mathfrak{Y}_0$. By 3.3.4, there is a positive functional T defined on $\mathfrak{Y}$ which is an extension of T_0. Put $l(y) := T(-y, 0)$ for $y \in Y$. It is clear that $l \circ \iota = l_0$. Furthermore, $T(0, t) = T_0(0, t) = t$. Hence, $0 \leq T(y, p(y)) = p(y) - l(y)$, i.e., $l \in \partial(p)$. ▷

3.5.4. REMARK. The claim of Theorem 3.5.3 is also referred to as the formula for a *linear change-of-variable under the subdifferential sign* or the *Hahn–Banach Theorem in subdifferential form.* Observe that the inclusion $\partial(p \circ \iota) \subset \partial(p) \circ \iota$ is often referred to as the *Hahn–Banach Theorem in analytical form* or the *Dominated Extension Theorem* and verbalized as follows: "A linear functional given on a subspace of a vector space and dominated by a sublinear functional on this subspace has an extension also dominated by the same sublinear functional."

3.5.5. Corollary. *If X_0 is a subspace of a vector space X and $p : X \to \mathbb{R}$ is a sublinear functional then the (asymmetric) Hahn–Banach formula holds:*

$$\partial(p + \delta(X_0)) = \partial(p) + \partial(\delta(X_0)).$$

◁ It is obvious that the left side includes the right side. To prove the reverse inclusion, take $l \in \partial(p + \delta(X_0))$. Then $l \circ \iota \in \partial(p \circ \iota)$, where ι is the identical embedding of X_0 into X. By 3.5.3, $l \circ \iota \in \partial(p) \circ \iota$, i.e., $l \circ \iota = l_1 \circ \iota$ for some $l_1 \in \partial(p)$. Put $l_2 := l - l_1$. By definition, find $l_2 \circ \iota = (l - l_1) \circ \iota = l \circ \iota - l_1 \circ \iota = 0$, i.e., $\ker l_2 \supset X_0$. As was mentioned in 3.5.2 (3), this means $l_2 \in \partial(\delta(X_0))$. ▷

3.5.6. Corollary. *If $f : X \to \mathbb{R}^{\cdot}$ is a linear functional and $x \in$ core dom f then $\partial_x(f) \neq \varnothing$.*

◁ Let $p := f'(x)$ and let $\iota : 0 \to X$ be the identical embedding of 0 to X. Plainly, $0 \in \partial(p \circ \iota)$, i.e., $\partial(p \circ \iota) \neq \varnothing$. By the Hahn–Banach Theorem, $\partial(p) \neq \varnothing$ (otherwise, $\varnothing = \partial(p) \circ \iota = \partial(p \circ \iota)$). To complete the proof, apply 3.5.2 (4). ▷

3.5.7. Corollary. *Let $f_1,\ f_2 : X \to \mathbb{R}^{\cdot}$ be convex functions. Assume further that $x \in$ core dom $f_1 \cap$ core dom f_2. Then*

$$\partial_x(f_1 + f_2) = \partial_x(f_1) + \partial_x(f_2).$$

◁ Let $p_1 := f_1'(x)$ and $p_2 := f_2'(x)$. Given $x_1,\ x_2 \in X$, define $p(x_1,\ x_2) := p_1(x_1) + p_2(x_2)$ and $\iota(x_1) := (x_1,\ x_1)$. Using 3.5.2 (4) and 3.5.3, infer that

$$\partial_x(f_1 + f_2) = \partial(p_1 + p_2) = \partial(p \circ \iota)$$
$$= \partial(p) \circ \iota = \partial(p_1) + \partial(p_2) = \partial_x(f_1) + \partial_x(f_2). \text{ ▷}$$

3.5.8. REMARK. Corollary 3.5.6 is sometimes called the *Nonempty Subdifferential Theorem.* On the one hand, it is straightforward from the Kuratowski–Zorn Lemma. On the other hand, with Corollary 3.5.6 available, demonstrate $\partial(p \circ T) = \partial(p) \circ T$ as follows: Define $p_T(y) := \inf\{p(y + Tx) - l(x) : \ x \in X\}$, where $l \in \partial(p)$ and the notation of 3.5.3 is employed. Check that p_T is sublinear and every l_1 in $\partial(p_T)$ satisfies the equality $l = l_1 \circ T$. Thus, the Nonempty Subdifferential Theorem and the Hahn–Banach Theorem in subdifferential form constitute a precious (rather than vicious) circle.

3.6. The Kreĭn–Milman Theorem

3.6.1. DEFINITION. Let X be a real vector space. Putting

$$\operatorname{seg}(x_1,\ x_2) := \{\alpha_1 x_1 + \alpha_2 x_2 :\ \alpha_1,\ \alpha_2 > 0,\ \alpha_1 + \alpha_2 = 1\},$$

define the correspondence seg $\subset X^2 \times X$ that assigns to each pair of points the *open segment* joining them. If U is a convex set in X and seg_U is the restriction of seg to U^2, then a convex subset V of U is *extreme* in U if $\operatorname{seg}_U^{-1}(V) \subset V^2$. An extreme subset of U is sometimes called a *face* in U. A member x of U is an *extreme point* in U if $\{x\}$ is an extreme subset of U. The set of extreme points of U is usually denoted by ext U.

3.6.2. *A convex subset V is extreme in U if and only if for all $u_1,\ u_2 \in U$ and $\alpha_1,\ \alpha_2 > 0,\ \alpha_1 + \alpha_2 = 1$, the containment $\alpha_1 u_1 + \alpha_2 u_2 \in V$ implies that $u_1 \in V$ and $u_2 \in V$.* ◁▷

3.6.3. EXAMPLES.

(1) *Let $p : X \to \mathbb{R}^{\cdot}$ be a sublinear functional and let a point x of X belong to* dom p. *Then $\partial_x(p)$ is an extreme subset of $\partial(p)$.*

◁ For, if $\alpha_1 l_1 + \alpha_2 l_2 \in \partial_x(p)$ and $l_1,\ l_2 \in \partial(p)$, where $\alpha_1,\ \alpha_2 > 0$, $\alpha_1 + \alpha_2 = 1$; then $0 = p(x) - (\alpha_1 l_1(x) + \alpha_2 l_2(x)) = \alpha_1(p(x) - l_1(x)) + \alpha_2(p(x) - l_2(x)) \geq 0$. Moreover, $p(x) - l_1(x) \geq 0$ and $p(x) - l_2(x) \geq 0$. Hence, $l_1 \in \partial_x(p)$ and $l_2 \in \partial_x(p)$. ▷

(2) *Let W be an extreme set in V and let V be in turn an extreme set in U. Then W is an extreme set in U.* ◁▷

(3) *If X is an ordered vector space then a positive element x of X is discrete if and only if the cone $\{\alpha x :\ \alpha \in \mathbb{R}_+\}$ is an extreme subset of X_+.*

◁ $\Leftarrow$: If $0 \leq y \leq x$ then $x = {}^1/_2(2y) + {}^1/_2(2(x-y))$. Therefore, by 3.6.2, $2y = \alpha x$ and $2(x-y) = \beta x$ for some $\alpha,\ \beta \in \mathbb{R}_+$. Thus, $2x = (\alpha + \beta)x$. The case $x = 0$ is trivial. Now if $x \neq 0$ then ${}^\alpha/_2 \in [0,\ 1]$ and, consequently, $[0,\ x] \subset [0,\ 1]x$. The reverse inclusion is evident.

$\Rightarrow$: Let $[0,\ x] = [0,\ 1]x$, and suppose that $\alpha x = \alpha_1 y_1 + \alpha_2 y_2$ for $\alpha \geq 0$; $\alpha_1,\ \alpha_2 > 0$, $\alpha_1 + \alpha_2 = 1$ and $y_1,\ y_2 \in X_+$. If $\alpha = 0$ then $\alpha_1 y_1 \in [0,\ x]$ and $\alpha_2 y_2 \in [0,\ x]$; hence, y_1 is a positive multiple of x; the same is valid for y_2. In the case $\alpha > 0$ observe that $({}^{\alpha_1}/_{\alpha})y_1 = tx$ for some $t \in [0,\ 1]$. Finally, $({}^{\alpha_2}/_{\alpha})y_2 = (1-t)x$. ▷

(4) Let U be a convex set. A convex subset V of U is a *cap* of U, if $U \setminus V$ is convex. A point x in U is extreme if and only if $\{x\}$ is a cap of U. ◁▷

3.6.4. ***Extreme and Discrete Lemma.*** *Let $p : X \to \mathbb{R}$ be a sublinear functional and $l \in \partial(p)$. Assign $\mathscr{X} := X \times \mathbb{R}$, $\mathscr{X}_+ :=$ epi p, and $T_l : (x,\ t) \mapsto t - l(x)$ $(x \in X,\ t \in \mathbb{R})$. Then the functional l is an extreme point of $\partial(p)$ if and only if T_l is a discrete functional.*

◁ $\Rightarrow$: Consider $T' \in \mathscr{X}^{\#}$ such that $T' \in [0,\ T_l]$. Put

$$t_1 := T'(0,\ 1),\quad l_1(x) := T'(-x,\ 0);$$
$$t_2 := (T_l - T')(0,\ 1),\quad l_2(x) := (T_l - T')(-x,\ 0).$$

It is clear that $t_1 \geq 0$, $t_2 \geq 0$, $t_1 + t_2 = 1$; $l_1 \in \partial(t_1 p)$, $l_2 \in \partial(t_2 p)$, and $l_1 + l_2 = l$. If $t_1 = 0$ then $l_1 = 0$; i.e., $T' = 0$ and $T' \in [0,\ 1]T_l$. Now if $t_2 = 0$ then $t_1 = 1$; i.e., $T' = T_l$, and so $T' \in [0,\ 1]T_l$. Assume that $t_1,\ t_2 > 0$. In this case ${}^1/_{t_1} l_1 \in \partial(p)$ and ${}^1/_{t_2} l_2 \in \partial(p)$; moreover, $l = t_1\left({}^1/_{t_1} l_1\right) + t_2\left({}^1/_{t_2} l_2\right)$. Since, by hypothesis, $l \in \text{ext}\ \partial(p)$, from 3.6.2 it follows that $l_1 = t_1 l$; i.e., $T' = t_1 T_l$. ▷

$\Leftarrow$: Let $l = \alpha_1 l_1 + \alpha_2 l_2$, where $l_1,\ l_2 \in \partial(p)$ and $\alpha_1,\ \alpha_2 > 0$, $\alpha_1 + \alpha_2 = 1$. The functionals $T' := \alpha_1 T_{l_1}$ and $T'' := \alpha_2 T_{l_2}$ are positive; moreover, $T' \in [0,\ T_l]$, since $T' + T'' = T_l$. Therefore, there is some $\beta \in [0,\ 1]$ such that $T' = \beta T_l$. At the point $(0,1)$, find $\alpha_1 = \beta$. Hence, $l_1 = l$. By analogy, $l_2 = l$. ◁▷

3.6.5. *Kreĭn–Milman Theorem.* *Let $T \in \mathscr{L}(X,\ Y)$ be a linear operator and let $f : Y \to \mathbb{R}^{\cdot}$ be a convex function. If $x \in X$ and $Tx \in \operatorname{core} \operatorname{dom} f$ then*

$$\operatorname{ext} \partial_x(f \circ T) \subset (\operatorname{ext} \partial_{Tx}(f)) \circ T.$$

◁ Start arguing as in the proof of the Hahn–Banach Theorem: Put $p := f'(Tx)$ and observe that the only claim left to checking is the inclusion $\operatorname{ext} \partial(p \circ T) \subset (\operatorname{ext} \partial(p)) \circ T$. Take $l \in \operatorname{ext} \partial(p \circ T)$. Since $\operatorname{ext} \partial(p \circ T) \subset \partial(p \circ T) \subset \partial(p) \circ T$, there is some f in $\partial(p)$ such that $l = f \circ T$. Denote by f_0 the restriction of f to $Y_0 := \operatorname{im} T$ and notice that $f_0 \in \operatorname{ext} \partial(p \circ \iota)$, with ι the identical imbedding of Y_0 into Y.

Now in $\mathfrak{Y} := Y \times \mathbb{R}$ consider $\mathfrak{Y}_+ := \operatorname{epi} p$ and introduce the space $\mathfrak{Y}_0 := Y_0 \times \mathbb{R}$. Note that $\mathfrak{Y}_+ \cap \mathfrak{Y}_0 = \operatorname{epi}(p \circ \iota)$. Applying 3.6.4 with $X := \mathfrak{Y}_0$, $l := f_0$, and $p := p \circ \iota$, observe that T_{f_0} is a discrete functional on $\mathfrak{Y}_0$; moreover, $\mathfrak{Y}_0$ is massive in $\mathfrak{Y}$ (cf. the proof of 3.5.3). By 3.3.8, find a discrete extension $S \in \mathfrak{Y}^{\#}$ of T_{f_0}. Evidently, $S = T_g$, where $g(y) := S(-y,\ 0)$ for $y \in Y$. Appealing again to 3.6.4, infer that $g \in \operatorname{ext} \partial(p)$. By construction, $l(x) = f(Tx) = f_0(Tx) = g(Tx)$ for all $x \in X$. Finally, $l \in (\operatorname{ext} \partial(p)) \circ T$. ▷

3.6.6. *Corollary.* *If $p : X \to \mathbb{R}$ is a sublinear functional then for every x in X there is an extreme functional l, a member of $\operatorname{ext} \partial(p)$, such that $l(x) = p(x)$.*

◁ From 3.6.5 it is easy that $\operatorname{ext} \partial(p) \neq \varnothing$ for every p (cf. 3.5.6). Using this and 3.4.12, choose l in $\operatorname{ext} \partial_x(p'(x))$. Applying 3.5.2 (2) and 3.5.2 (4), obtain $l \in \operatorname{ext} \partial_x(p)$. By 3.6.3 (1), $\partial_x(p)$ is extreme in $\partial(p)$. Finally, 3.6.3 (2) implies that l is an extreme point of $\partial(p)$. ▷

3.7. The Balanced Hahn–Banach Theorem

3.7.1. Definition. Let $(X,\ \mathbb{F},\ +,\ \cdot\,)$ be a vector space over a basic field $\mathbb{F}$. The vector space $(X,\ \mathbb{R},\ +,\ \cdot\,|_{\mathbb{R}\times X})$ is called the *real carrier* or *realification* of $(X,\ \mathbb{F},\ +,\ \cdot\,)$ and is denoted by $X_{\mathbb{R}}$.

3.7.2. Definition. Given a linear functional f on a vector space X, define the mapping $\mathbb{R}\mathrm{e} f : x \mapsto \operatorname{Re} f(x)$ $(x \in X)$. The *real part map* or *realifier* is the mapping $\mathbb{R}\mathrm{e} : (X^{\#})_{\mathbb{R}} \to (X_{\mathbb{R}})^{\#}$.

3.7.3. *The real part map $\mathbb{R}\mathrm{e}$ is a linear operator and, moreover, an isomorphism of $(X^{\#})_{\mathbb{R}}$ onto $(X_{\mathbb{R}})^{\#}$.*

◁ Only the case $\mathbb{F} := \mathbb{C}$ needs verifying, because $\mathbb{R}\mathrm{e}$ is the identity mapping when $\mathbb{F} := \mathbb{R}$.

Undoubtedly, $\mathbb{R}\mathrm{e}$ is a linear operator. Check that $\mathbb{R}\mathrm{e}$ is a monomorphism and an epimorphism simultaneously (cf. 2.3.2).

If $\mathbb{R}\mathrm{e} f = 0$ then $0 = \operatorname{Re} f(ix) = \operatorname{Re}(if(x)) = \operatorname{Re}(i(\operatorname{Re} f(x) + i\operatorname{Im} f(x))) = -\operatorname{Im} f(x)$. Hence, $f = 0$ and so $\mathbb{R}\mathrm{e}$ is a monomorphism.

Now take $g \in (X_{\mathbb{R}})^{\#}$ and put $f(x) := g(x) - ig(ix)$. Evidently, $f \in \mathscr{L}(X_{\mathbb{R}}, \mathbb{C}_{\mathbb{R}})$ and $\operatorname{Re} f(x) = g(x)$ for $x \in X$. It is sufficient to show that $f(ix) = if(x)$, because this equality implies $f \in X^{\#}$. Straightforward calculation shows $f(ix) = g(ix) + ig(x) = i(g(x) - ig(ix)) = if(x)$, which enables us to conclude that $\mathbb{R}\mathrm{e}$ is also an epimorphism. ▷

3.7.4. DEFINITION. The *lear trap map* or *complexifier* is the inverse $\mathbb{R}\mathrm{e}^{-1} : (X_{\mathbb{R}})^{\#} \to (X^{\#})_{\mathbb{R}}$ of the real part map.

3.7.5. REMARK. By 3.7.3, for the complex scalar field

$$\mathbb{R}\mathrm{e}^{-1} g : x \mapsto g(x) - ig(ix) \quad (g \in (X_{\mathbb{R}})^{\#},\ x \in X).$$

In the case of the reals, $\mathbb{F} := \mathbb{R}$, the complexifier $\mathbb{R}\mathrm{e}^{-1}$ is the identity operator.

3.7.6. DEFINITION. Let $(X, \mathbb{F}, +, \cdot)$ be a vector space over $\mathbb{F}$. A *seminorm* on X is a function $p : X \to \mathbb{R}^{\cdot}$ such that $\operatorname{dom} p \neq \varnothing$ and

$$p(\lambda_1 x_1 + \lambda_2 x_2) \le |\lambda_1| p(x_1) + |\lambda_2| p(x_2)$$

for $x_1,\ x_2 \in X$ and $\lambda_1,\ \lambda_2 \in \mathbb{F}$.

3.7.7. REMARK. A seminorm is a sublinear functional (on the real carrier of the space in question).

3.7.8. DEFINITION. If $p : X \to \mathbb{R}^{\cdot}$ is a seminorm then the *balanced subdifferential* of p is the set

$$|\partial|(p) := \{l \in X^{\#} :\ |l(x)| \le p(x) \text{ for all } x \in X\}.$$

3.7.9. ***Balanced Subdifferential Lemma.*** *Let $p : X \to \mathbb{R}^{\cdot}$ be a seminorm. Then*

$$|\partial|(p) = \mathbb{R}\mathrm{e}^{-1}(\partial(p)); \quad \mathbb{R}\mathrm{e}(|\partial|(p)) = \partial(p)$$

for the subdifferentials $|\partial|(p)$ and $\partial(p)$ of p.

◁ If $\mathbb{F} := \mathbb{R}$ then the equality $|\partial|(p) = \partial(p)$ is easy. Furthermore, in this case $\mathbb{R}\mathrm{e}$ is the identity operator.

Let $\mathbb{F} := \mathbb{C}$. If $l \in |\partial|(p)$ then $(\mathbb{R}\mathrm{e}\, l)(x) = \operatorname{Re} l(x) \le |l(x)| \le p(x)$ for all $x \in X$; i.e., $\mathbb{R}\mathrm{e}(|\partial|(p)) \subset \partial(p)$. Take $g \in \partial(p)$ and put $f := \mathbb{R}\mathrm{e}^{-1} g$. If $f(x) = 0$ then $|f(x)| \le p(x)$; for $f(x) \neq 0$ set $\theta := |f(x)|/f(x)$. Thereby $|f(x)| = \theta f(x) = f(\theta x) = \operatorname{Re} f(\theta x) = g(\theta x) \le p(\theta x) = |\theta| p(x) = p(x)$, since $|\theta| = 1$. Finally, $f \in |\partial|(p)$. ▷

3.7.10. *Let X be a vector space, let $p : X \to \mathbb{R}$ be a seminorm, and let X_0 be a subspace of X. The* asymmetric balanced Hahn–Banach formula *holds:*

$$|\partial|(p + \delta(X_0)) = |\partial|(p) + |\partial|(\delta(X_0)).$$

◁ From 3.7.9, 3.5.5, and the results of Section 3.1 obtain

$$|\partial|(p+\delta(X_0)) = \mathbb{R}\mathrm{e}^{-1}(\partial(p+\delta(X_0))) = \mathbb{R}\mathrm{e}^{-1}(\partial(p)+\partial(\delta(X_0)))$$
$$= \mathbb{R}\mathrm{e}^{-1}(\partial(p)) + \mathbb{R}\mathrm{e}^{-1}(\partial(\delta(X_0))) = |\partial|(p) + |\partial|(\delta(X_0)). \ \triangleright$$

3.7.11. *If X and Y are vector spaces, $T \in \mathscr{L}(X, Y)$ is a linear operator, and $p : Y \to \mathbb{R}$ is a seminorm; then $p \circ T$ is also a seminorm and, moreover,*

$$|\partial|(p \circ T) = |\partial|(p) \circ T.$$

◁ Applying 2.3.8 and 3.7.10, successively infer that

$$|\partial|(p \circ T) = |\partial|(p + \delta(\mathrm{im}\ T)) \circ T = (|\partial|(p) + |\partial|(\delta(\mathrm{im}\ T))) \circ T$$
$$= |\partial|(p) \circ T + |\partial|(\delta(\mathrm{im}\ T)) \circ T = |\partial|(p) \circ T. \ \triangleright$$

3.7.12. REMARK. If T is the identical embedding of a subspace and the ground field is $\mathbb{C}$ then 3.7.11 is referred to as the *Sukhomlinov–Bohnenblust–Sobczyk Theorem.*

3.7.13. ***Balanced Hahn–Banach Theorem.*** *Let X be a vector space. Assume further that $p : X \to \mathbb{R}$ is a seminorm and X_0 is a subspace of X. If l_0 is a linear functional given on X_0 such that $|l_0(x_0)| \le p(x_0)$ for $x_0 \in X_0$, then there is a linear functional l on X such that $l(x_0) = l_0(x_0)$ whenever $x_0 \in X_0$ and $|l(x)| \le p(x)$ for all $x \in X$.* ◁▷

3.8. The Minkowski Functional and Separation

3.8.1. DEFINITION. Let $\overline{\mathbb{R}}$ stand for the *extended real axis* or *extended reals* (i.e., $\overline{\mathbb{R}}$ denotes $\mathbb{R}^{\cdot}$ with the least element $-\infty$ adjoined formally). If X is an arbitrary set and $f : X \to \overline{\mathbb{R}}$ is a mapping; then, given $t \in \overline{\mathbb{R}}$, put

$$\{f \le t\} := \{x \in X :\ f(x) \le t\};$$
$$\{f = t\} := f^{-1}(t);$$
$$\{f < t\} := \{f \le t\} \setminus \{f = t\}.$$

Every set of the form $\{f \le t\}$, $\{f = t\}$, and $\{f < t\}$ is a *level set* or *Lebesque set* of f.

3.8.2. ***Function Recovery Lemma.*** *Let $T \subset \overline{\mathbb{R}}$ and let $t \mapsto U_t$ $(t \in T)$ be a family of subsets of X. There is a function $f : X \to \overline{\mathbb{R}}$ such that*

$$\{f < t\} \subset U_t \subset \{f \le t\} \quad (t \in T)$$

if and only if the mapping $t \mapsto U_t$ increases.

◁ ⇒: Suppose that T contains at least two elements s and t. (Otherwise there is nothing left to proof.) If $s < t$ then

$$U_s \subset \{f \leq s\} \subset \{f < t\} \subset U_t.$$

⇐: Define a mapping $f : X \to \overline{\mathbb{R}}$ by setting $f(x) := \inf\{t \in T : x \in U_t\}$. If $\{f < t\}$ is empty for $t \in T$ then all is clear. If $x \in \{f < t\}$ then $f(x) < +\infty$, and so there is some $s \in T$ such that $x \in U_s$ and $s < t$. Thus, $\{f < t\} \subset U_s \subset U_t$. Continuing, for $x \in U_t$ find $f(x) \leq t$; i.e., $U_t \subset \{f \leq t\}$. ▷

3.8.3. *Function Comparison Lemma.* *Let $f,\ g : X \to \overline{\mathbb{R}}$ be functions defined by $(U_t)_{t\in T}$ and $(V_t)_{t\in T}$ as follows:*

$$\{f < t\} \subset U_t \subset \{f \leq t\};$$

$$\{g < t\} \subset V_t \subset \{g \leq t\} \quad (t \in T).$$

If T is dense in $\overline{\mathbb{R}}$ (i.e., $(\forall r,\ t \in \overline{\mathbb{R}},\ r < t)\ (\exists\, s \in T)\ (r < s < t)$) then the inequality $f \leq g$ (in $\overline{\mathbb{R}}^X$; i.e., $f(x) \leq g(x)$ for $x \in X$) holds if and only if

$$(t_1,\ t_2 \in T\ \&\ t_1 < t_2) \Rightarrow V_{t_1} \subset U_{t_2}.$$

◁ ⇒: This is immediate from the inclusions

$$V_{t_1} \subset \{g \leq t_1\} \subset \{f \leq t_1\} \subset \{f < t_2\} \subset U_{t_2}.$$

⇐: Assume that $g(x) \neq +\infty$ (if not, $f(x) \leq g(x)$ for obvious reasons). Given $t \in \mathbb{R}$ such that $g(x) < t < +\infty$, choose $t_1,\ t_2 \in T$ so as to satisfy the conditions $g(x) < t_1 < t_2 < t$. Now

$$x \in \{g < t_1\} \subset V_{t_1} \subset U_{t_2} \subset \{f \leq t_2\} \subset \{f < t\}.$$

Thus, $f(x) < t$. Since t is arbitrary, obtain $f(x) \leq g(x)$. ▷

3.8.4. *Corollary.* *If T is dense in $\overline{\mathbb{R}}$ and the mapping $t \mapsto U_t$ $(t \in T)$ increases then there is a unique function $f : X \to \overline{\mathbb{R}}$ such that*

$$\{f < t\} \subset U_t \subset \{f \leq t\} \quad (t \in T).$$

Moreover, the level sets of f may be presented as follows:

$$\{f < t\} = \cup\, \{U_s : \ s < t,\ s \in T\};$$

$$\{f \leq t\} = \cap \{U_r : \ t < r,\ r \in T\} \quad (t \in \overline{\mathbb{R}}).$$

◁ It is immediate from 3.8.2 and 3.8.3 that f exists and is unique. If $s < t$ and $s \in T$ then $U_s \subset \{f \leq s\} \subset \{f < t\}$. If now $f(x) < t$ then, since T is dense, there is an element s in T such that $f(x) < s < t$. Therefore, $f \in \{f < s\} \subset U_s$, which proves the formula for $\{f < t\}$.

Suppose that $r > t,\ r \in T$. Then $\{f \leq t\} \subset \{f < r\} \subset U_r$. In turn, if $x \in U_r$ for $r \in T,\ r > t$; then $f(x) \leq r$ for all $r > t$. Hence, $f(x) \leq t$. ▷

3.8.5. *Let X be a vector space and let S be a conical segment in X. Given $t \in \mathbb{R}$, put $U_t := \varnothing$ if $t < 0$ and $U_t := tS$ if $t \geq 0$. The mapping $t \mapsto U_t$ $(t \in \mathbb{R})$ increases.*

$\triangleleft$ If $0 \leq t_1 < t_2$ and $x \in t_1 S$ then $x \in ({}^{t_1}/_{t_2}) t_2 S$. Hence, $x \in t_2 S$. $\triangleright$

3.8.6. DEFINITION. The *Minkowski functional* or the *gauge function* or simply the *gauge* of a conical segment S is a functional $p_S : X \to \mathbb{R}$ such that

$$\{p_S < t\} \subset tS \subset \{p_S \leq t\} \quad (t \in \mathbb{R}_+)$$

and $\{p < 0\} = \varnothing$. (Such a functional exists and is unique by 3.8.2, 3.8.4, and 3.8.5.) In other words,

$$p_S(x) = \inf\{t > 0 : \ x \in tS\} \quad (x \in X).$$

3.8.7. Gauge Theorem. *The Minkowski functional of a conical segment is a sublinear functional assuming positive values. Conversely, if p is a sublinear functional assuming positive values then the sets $\{p < 1\}$ and $\{p \leq 1\}$ are conical segments. Moreover, p is the Minkowski functional of each conical segment S such that $\{p < 1\} \subset S \subset \{p \leq 1\}$.*

$\triangleleft$ Consider a conical segment S and its Minkowski functional p_S. Let $x \in X$. The inequality $p_S(x) \geq 0$ is evident. Take $\alpha > 0$. Then

$$\begin{aligned} p_S(\alpha x) &= \inf\{t > 0 : \ \alpha x \in tS\} = \inf\left\{t > 0 : \ x \in {}^t/_\alpha S\right\} \\ &= \inf\{\alpha\beta > 0 : \ x \in \beta S, \ \beta > 0\} \\ &= \alpha \inf\{\beta > 0 : \ x \in \beta S\} = \alpha p_S(x). \end{aligned}$$

To start checking that p_S is subadditive, take $x_1, \ x_2 \in X$. Noting the inclusion $t_1 S + t_2 S \subset (t_1 + t_2) S$ for $t_1, \ t_2 > 0$, in view of the identity

$$t_1 x_1 + t_2 x_2 = (t_1 + t_2) \left(\frac{t_1}{t_1 + t_2} x_1 + \frac{t_2}{t_1 + t_2} x_2 \right),$$

successively infer that

$$\begin{aligned} p_S(x_1 + x_2) &= \inf\{t > 0 : \ x_1 + x_2 \in tS\} \\ &\leq \inf\{t : \ t = t_1 + t_2; \ t_1, \ t_2 > 0, \ x_1 \in t_1 S, \ x_2 \in t_2 S\} \\ &= \inf\{t_1 > 0 : \ x_1 \in t_1 S\} + \inf\{t_2 > 0 : \ x_2 \in t_2 S\} = p_S(x_1) + p_S(x_2). \end{aligned}$$

Let $p : X \to \mathbb{R}^{\cdot}$ be an arbitrary sublinear functional with positive values and let $\{p < 1\} \subset S \subset \{p \leq 1\}$. Given $t \in \mathbb{R}_+$, put $V_t := \{p < t\}$ and $U_t := tS$; given $t < 0$, put $V_t := U_t := \varnothing$. Plainly,

$$\{p_S < t\} \subset U_t \subset \{p_S \leq t\}; \quad \{p < t\} \subset V_t \subset \{p \leq t\}$$

for $t \in \mathbb{R}$. If $0 \leq t_1 < t_2$ then $V_{t_1} = \{p < t_1\} = t_1\{p < 1\} \subset t_1 S = U_{t_1} \subset U_{t_2}$. Moreover, $U_{t_1} \subset t_1\{p \leq 1\} \subset \{p \leq t_1\} \subset \{p < t_2\} \subset V_{t_2}$. Therefore, by 3.8.3 and 3.8.4, $p = p_S$. ▷

3.8.8. REMARK. A conical segment S in X is absorbing in X if and only if $\operatorname{dom} p_S = X$. Also, p_S is a seminorm whenever S is absolutely convex. Conversely, for every seminorm p the sets $\{p < 1\}$ and $\{p \leq 1\}$ are absolutely convex. ◁▷

3.8.9. DEFINITION. A subspace H of a vector space X is a *hypersubspace* in X if X/H is isomorphic to the ground field of X. An element of X/H is called a *hyperplane* in X *parallel* to H. By a *hyperplane* in X we mean an affine variety parallel to some hypersubspace of X. An affine variety H is a hyperplane if $H - h$ is a hypersubspace for some (and, hence, for every) h in H. If necessary, a hyperplane in the real carrier of X is referred to as a *real hyperplane* in X.

3.8.10. *Hyperplanes in X are exactly level sets of nonzero elements of $X^{\#}$.* ◁▷

3.8.11. *Separation Theorem.* *Let X be a vector space. Assume further that U is a nonempty convex set in X and L is an affine variety in X. If $L \cap U = \varnothing$ then there is a hyperplane H in X such that $H \supset L$ and $H \cap \operatorname{core} U = \varnothing$.*

◁ Without loss of generality, it may be supposed that $\operatorname{core} U \neq \varnothing$ (otherwise there is nothing left unproven) and, moreover, $0 \in \operatorname{core} U$. Take $x \in L$ and put $X_0 := L - x$. Consider the quotient space X/X_0 and the corresponding coset mapping $\varphi : X \to X/X_0$. Applying 3.1.8 and 3.4.10, observe that $\varphi(U)$ is an absorbing conical segment. Hence, by 3.8.7 and 3.8.8, the domain of the Minkowski functional $p := p_{\varphi(U)}$ is the quotient X/X_0; moreover,

$$\varphi(\operatorname{core} U) \subset \operatorname{core} \varphi(U) \subset \{p < 1\} \subset \varphi(U).$$

In particular, this entails the inequality $p(\varphi(x)) \geq 1$, since $\varphi(x) \notin \varphi(U)$.

Using 3.5.6, find a functional $\overline{f}$ in $\partial_x(p \circ \varphi)$; now the Hahn–Banach Theorem implies

$$\overline{f} \in \partial_x(p \circ \varphi) = \partial_{\varphi(x)}(p) \circ \varphi.$$

Put $\overline{H} := \{\overline{f} = p \circ \varphi(x)\}$. It is clear that $\overline{H}$ is a real hyperplane and, undoubtedly, $\overline{H} \supset L$. Appealing to 3.5.2 (1), conclude that $\overline{H} \cap \operatorname{core} U = \varnothing$. Now let $f := \mathbb{R}e^{-1}\overline{f}$ and $H := \{f = f(x)\}$. There is no denying that $L \subset H \subset \overline{H}$. Thus, the hyperplane H provides us with what was required. ▷

3.8.12. REMARK. Under the hypotheses of the Separation Theorem, it may be assumed that $\operatorname{core} U \cap L = \varnothing$. Note also that Theorem 3.8.11 is often referred to as the *Hahn–Banach Theorem in geometric form* or as the *Minkowski–Ascoli–Mazur Theorem.*

3.8.13. Definition. Let U and V be sets in X. A real hyperplane H in X *separates* U and V if these sets lie in the different halfspaces defined by H; i.e., if there is a presentation $H = \{f \le t\}$, where $f \in (X_{\mathbb{R}})^{\#}$ and $t \in \mathbb{R}$, such that $V \subset \{f \le t\}$ and $U \subset \{f \ge t\} := \{-f \le -t\}$.

3.8.14. ***Eidelheit Separation Theorem.*** *If U and V are convex sets such that* core $V \ne \varnothing$ *and* $U \cap$ core $V = \varnothing$, *then there is a hyperplane separating U and V and disjoint from* core V. ⊲⊳

Exercises

3.1. Show that a hyperplane is precisely an affine set maximal by inclusion and other than the whole space.

3.2. Prove that every affine set is an intersection of hyperplanes.

3.3. Prove that the complement of a hyperplane to a real vector space consists of two convex sets each of which coincides with its own core. The sets are named *open halfspaces*. The union of an open halfspace with the corresponding hyperplane is called a *closed halfspace*. Find out how a halfspace can be prescribed.

3.4. Find possible presentations of the elements of the convex hull of a finite set. What use can be made of finite dimensions?

3.5. Given sets S_1 and S_2, let $S := \bigcup_{0 \le \lambda \le 1} \lambda S_1 \cap (1-\lambda)S_2$. Prove that S is convex whenever so are S_1 and S_2.

3.6. Calculate the Minkowski functional of a halfspace or a cone and of the convex hull of the union or intersection of conical segments.

3.7. Let $S := \{p + q \le 1\}$, where p and q are the Minkowski functionals of the conical segments S_p and S_q. Express S via S_p and S_q.

3.8. Describe sublinear functionals with domain $\mathbb{R}^N$.

3.9. Calculate the subdifferential of the upper envelope of a finite set of linear or sublinear functionals.

3.10. Let p and q be sublinear functionals in *general position*, i.e. such that dom p − dom q = dom q − dom p. Prove the *symmetric Hahn–Banach formula* (cf. 3.5.7): $\partial(p+q) = \partial(p) + \partial(q)$.

3.11. Let $p,\ q : X \to \mathbb{R}$ be total (= everywhere-defined) sublinear functionals on X. Then the equality holds: $\partial(p \vee q) = \operatorname{co}(\partial(p) \cup \partial(q))$.

3.12. Find the Minkowski functional of a ball in a Hilbert space whose center of symmetry is not necessarily the zero of the space.

3.13. A symmetric square 2×2-matrix is called *positive*, provided that its eigenvalues are positive. Does the resulting order in the space of such matrices agree with vector structure? Does it define the structure of a Kantorovich space?

3.14. Each ordered vector space admits a nonzero positive linear functional, doesn't it?

3.15. What are the means for transforming $\mathbb{R}^N$ into a Kantorovich space?

3.16. Under what conditions does the claim of the Hahn–Banach Theorem in analytical form hold for a partial (= not-everywhere-defined) sublinear functional?

3.17. Find the extreme points of the subdifferential of the conventional norm on l_∞.

3.18. Find possible generalizations of the Hahn–Banach Theorem for a mapping acting into a Kantorovich space.

3.19. Given a set C in a space X, define the *Hörmander transform* $H(C)$ of C as

$$H(C) = \{(x,\ t) \in X \times \mathbb{R} :\ x \in tC\}.$$

Study the properties of the Hörmander transform on the collection of all convex sets.

Chapter 4
An Excursion into Metric Spaces

4.1. The Uniformity and Topology of a Metric Space

4.1.1. DEFINITION. A mapping $d : X^2 \to \mathbb{R}_+$ is a *metric* on X if

(1) $d(x, y) = 0 \Leftrightarrow x = y$;

(2) $d(x, y) = d(y, x)$ $(x, y \in X)$;

(3) $d(x, y) \le d(x, z) + d(z, y)$ $(x, y, z \in X)$.

The real $d(x, y)$ is usually referred to as the *distance* between x and y. The pair (X, d) is a *metric space*. In this situation, it is convenient to take the liberty of calling the underlying set X a metric space. An element of a metric space X is also called a *point* of X.

4.1.2. *A mapping $d : X^2 \to \mathbb{R}_+$ is a metric if and only if*

(1) $\{d \le 0\} = I_X$;

(2) $\{d \le t\} = \{d \le t\}^{-1}$ $(t \in \mathbb{R}_+)$;

(3) $\{d \le t_1\} \circ \{d \le t_2\} \subset \{d \le t_1 + t_2\}$ $(t_1, t_2 \in \mathbb{R}_+)$.

◁ Items 4.1.2 (1)–4.1.2 (3) reformulate 4.1.1 (1)–4.1.1 (3). ▷

4.1.3. DEFINITION. Let (X, d) be a metric space and take $\varepsilon \in \mathbb{R}_+ \setminus 0$, a *strictly positive real*. The relation $B_\varepsilon := B_{d,\varepsilon} := \{d \le \varepsilon\}$ is the *closed cylinder* of size ε. The set $\overset{\circ}{B}_\varepsilon := \overset{\circ}{B}_{d,\varepsilon} := \{d < \varepsilon\}$ is the *open cylinder* of size ε. The image $B_\varepsilon(x)$ of a point x under the relation B_ε is called the *closed ball* with center x and radius ε. By analogy, the set $\overset{\circ}{B}_\varepsilon(x)$ is the *open ball* with center x and radius ε.

4.1.4. *In a nonempty metric space open cylinders as well as closed cylinders form bases for the same filter.* ◁▷

4.1.5. DEFINITION. The filter on X^2 with the filterbase of all cylinders of a nonempty metric space (X, d) is the *metric uniformity* on X. It is denoted by $\mathscr{U}_X$, $\mathscr{U}_d$, or even $\mathscr{U}$, if the space under consideration is implied. Given $X := \varnothing$, put $\mathscr{U}_X := \{\varnothing\}$. An element of $\mathscr{U}_X$ is an *entourage* on X.

4.1.6. *If $\mathscr{U}$ is a metric uniformity then*

(1) $\mathscr{U} \subset \text{fil}\,\{I_X\}$;

(2) $U \in \mathscr{U} \Rightarrow U^{-1} \in \mathscr{U}$;

(3) $(\forall U \in \mathscr{U})\,(\exists V \in \mathscr{U})\ V \circ V \subset U$;

(4) $\cap\{U :\ U \in \mathscr{U}\} = I_X$. ◁▷

4.1.7. Remark. Property 4.1.6 (4) reflecting 4.1.1 (1) is often expressed as follows: "$\mathscr{U}$ is a Hausdorff or separated uniformity."

4.1.8. *Given a space X with uniformity $\mathscr{U}_X$, put*

$$\tau(x) := \{U(x) :\ U \in \mathscr{U}\}.$$

Then $\tau(x)$ is a filter for every x in X. Moreover,

(1) $\tau(x) \subset \text{fil}\,\{x\}$;

(2) $(\forall U \in \tau(x))\,(\exists V \in \tau(x)\ \&\ V \subset U)\,(\forall y \in V)\ U \in \tau(y)$. ◁▷

4.1.9. Definition. The mapping $\tau : x \mapsto \tau(x)$ is the *metric topology* on X. An element of $\tau(x)$ is a *neighborhood* of x or about x. More complete designations for the topology are also in use: τ_X, $\tau(\mathscr{U})$, etc.

4.1.10. Remark. All closed balls centered at x form a base for the *neighborhood filter* of x. The same is true of open balls. Note also that there are *disjoint* (= nonintersecting) neighborhoods of different points in X. This property, ciphered in 4.1.6 (4), reads: "τ_X is a Hausdorff or separated topology."

4.1.11. Definition. A subset G of X is an *open set* in X whenever G is a neighborhood of its every point (i.e., $G \in \text{Op}(\tau) \Leftrightarrow ((\forall x \in G)\ G \in \tau(x))$). A subset F of X is a *closed set* in X whenever its complement to X is open (in symbols, $F \in \text{Cl}(\tau) \Leftrightarrow (X \setminus F \in \text{Op}(\tau))$).

4.1.12. *The union of a family of open sets and the intersection of a finite family of open sets are open. The intersection of a family of closed sets and the union of a finite family of closed sets are closed.* ◁▷

4.1.13. Definition. Given a subset U of X, put

$$\text{int}\ U := \overset{\circ}{U} := \cup\{G \in \text{Op}(\tau_X) :\ G \subset U\};$$
$$\text{cl}\ U := \overline{U} := \cap\{F \in \text{Cl}(\tau_X) :\ F \supset U\}.$$

The set int U is the *interior* of U and its elements are *interior* points of U. The set cl U is the *closure* of U and its elements are *adherent* to U. The *exterior* of U is the interior of $X \setminus U$; the elements of the former are *exterior* to U. A *boundary point* of U is by agreement a point of X neither interior nor exterior to U. The collection of all boundary points of U is called the *boundary* of U or the *frontier* of U and denoted by fr U or ∂U.

4.1.14. *A set U is a neighborhood about x if and only if x is an interior point of U.* ◁▷

4.1.15. REMARK. In connection with 4.1.14, the set $\mathrm{Op}(\tau)$ is also referred to as the topology of U, since τ is uniquely determined from $\mathrm{Op}(\tau)$. The same relates also to $\mathrm{Cl}(\tau)$, the collection of all closed sets in X.

4.1.16. DEFINITION. A filterbase $\mathscr{B}$ on X *converges* to x in X or x is a *limit* of $\mathscr{B}$ (in symbols, $\mathscr{B} \to x$) if fil $\mathscr{B}$ is finer than the neighborhood filter of x; i.e., fil $\mathscr{B} \supset \tau(x)$.

4.1.17. DEFINITION. A net or (generalized) sequence $(x_\xi)_{\xi\in\Xi}$ *converges* to x (in symbols, $x_\xi \to x$) if the tail filter of (x_ξ) converges to x. Other familiar terms and designations are freely employed. For instance, $x = \lim_\xi x_\xi$ and x is a limit of (x_ξ) as ξ ranges over Ξ.

4.1.18. REMARK. A limit of a filter, as well as a limit of a net, is unique in a metric space X. This is another way of expressing the fact that the topology of X is separated. ◁▷

4.1.19. *For a nonempty set U and a point x the following statements are equivalent:*

(1) *x is an adherent point of U;*
(2) *there is a filter $\mathscr{F}$ such that $\mathscr{F} \to x$ and $U \in \mathscr{F}$;*
(3) *there is a sequence $(x_\xi)_{\xi\in\Xi}$ whose entries are in U and which converges to x.*

◁ (1) $\Rightarrow$ (2): Since x is not exterior to U, the join $\mathscr{F} := \tau(x) \vee \mathrm{fil}\,\{U\}$ is available of the pair of the filters $\tau(x)$ and fil $\{U\}$.

(2) $\Rightarrow$ (3): Let $\mathscr{F} \to x$ and $U \in \mathscr{F}$. Direct $\mathscr{F}$ by reverse inclusion. Take $x_V \in V \cap U$ for $V \in \mathscr{F}$. It is clear that $x_V \to x$.

(3) $\Rightarrow$ (1): Let V be a closed set. Take a sequence $(x_\xi)_{\xi\in\Xi}$ in V such that $x_\xi \to x$. In this case it suffices to show that $x \in V$, which is happily evident. Indeed, were x in $X \setminus V$ we would find at least one $\xi \in \Xi$ such that $x_\xi \in X \setminus V$. ▷

4.1.20. REMARK. It may be assumed that $\mathscr{F}$ has a countable base in 4.1.19 (2), and $\Xi := \mathbb{N}$ in 4.1.19 (3). This property is sometimes formulated as follows: "In metric spaces the first axiom of countability is fulfilled."

4.2. Continuity and Uniform Continuity

4.2.1. *If $f : X \to Y$ and τ_X and τ_Y are topologies on X and Y then the following conditions are equivalent:*

(1) $G \in \mathrm{Op}(\tau_Y) \Rightarrow f^{-1}(G) \in \mathrm{Op}(\tau_X)$;
(2) $F \in \mathrm{Cl}(\tau_Y) \Rightarrow f^{-1}(F) \in \mathrm{Cl}(\tau_X)$;
(3) $f(\tau_X(x)) \supset \tau_Y(f(x))$ *for all* $x \in X$;
(4) $(x \in X \ \&\ \mathscr{F} \to x) \Rightarrow (f(\mathscr{F}) \to f(x))$ *for a filter* $\mathscr{F}$;
(5) $f(x_\xi) \to f(x)$ *for every point x and every sequence (x_ξ) convergent to x.*

◁ The equivalence (1) ⇔ (2) follows from 4.1.11. It remains to demonstrate that (1) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (2).

(1) ⇒ (3): If $V \in \tau_Y(f(x))$ then $W := \operatorname{int} V \in \operatorname{Op}(\tau_Y)$ and $f(x) \in W$. Hence, $f^{-1}(W) \in \operatorname{Op}(\tau_X)$ and $x \in f^{-1}(W)$. In other words, $f^{-1}(W) \in \tau_X(x)$ (see 4.1.14). Moreover, $f^{-1}(V) \supset f^{-1}(W)$ and, consequently, $f^{-1}(V) \in \tau_X(x)$. Finally, $V \supset f(f^{-1}(V))$.

(3) ⇒ (4): Given $\mathscr{F} \to x$, by Definition 4.1.16 fil $\mathscr{F} \supset \tau_X(x)$. From the hypothesis infer that $f(\mathscr{F}) \supset f(\tau_X(x)) \supset \tau_Y(f(x))$. Appealing to 4.1.16, again reveal the sought instance of convergence, $f(\mathscr{F}) \to f(x)$.

(4) ⇒ (5): The image of the tail filter of $(x_\xi)_{\xi \in \Xi}$ under f is coarser than the tail filter of $(f(x_\xi))_{\xi \in \Xi}$.

(5) ⇒ (2): Take a closed set F in Y. If $F = \varnothing$ then $f^{-1}(F)$ is also empty and, hence, closed. Assume F nonempty and let x be an adherent point of $f^{-1}(F)$. Consider a sequence $(x_\xi)_{\xi \in \Xi}$ converging to x and consisting of points in $f^{-1}(F)$ (the claim of 4.1.19 yields existence). Then $f(x_\xi) \in F$ and $f(x_\xi) \to f(x)$. Another citation of 4.1.19 guarantees that $f(x) \in F$ and, consequently, $x \in f^{-1}(F)$. ▷

4.2.2. DEFINITION. A mapping satisfying one (and hence all) of the equivalent conditions 4.2.1 (1)–4.2.1 (5) is *continuous*. If 4.2.1 (5) holds at a fixed point x then f is said to be *continuous at* x. Thus, f is continuous on X whenever f is continuous at every point of X.

4.2.3. *Every composition of continuous mappings is continuous.*

◁ Apply 4.2.1 (5) thrice. ▷

4.2.4. *Let $f : X \to Y$ and let $\mathscr{U}_X$ and $\mathscr{U}_Y$ be uniformities on X and Y. The following statements are equivalent:*

(1) $(\forall V \in \mathscr{U}_Y)\,(\exists U \in \mathscr{U}_X)\,((\forall x,\ y)(x,\ y) \in U \Rightarrow (f(x),\ f(y)) \in V)$;

(2) $(\forall V \in \mathscr{U}_Y)\ f^{-1} \circ V \circ f \in \mathscr{U}_X$;

(3) $f^\times(\mathscr{U}_X) \supset \mathscr{U}_Y$, *with* $f^\times : X^2 \to Y$ *defined as* $f^\times : (x,\ y) \mapsto (f(x),\ f(y))$;

(4) $(\forall V \in \mathscr{U}_Y)\ f^{\times -1}(V) \in \mathscr{U}_X$; i.e., $f^{\times -1}(\mathscr{U}_Y) \subset \mathscr{U}_X$.

◁ By 1.1.10, given $U \subset X^2$ and $V \subset Y^2$, observe that

$$f^{-1} \circ V \circ f = \bigcup_{(v_1,v_2) \in V} f^{-1}(v_1) \times f^{-1}(v_2)$$
$$= \{(x,y) \in X^2 :\ (f(x),\ f(y)) \in V\} = f^{\times -1}(V);$$
$$f \circ U \circ f^{-1} = \bigcup_{(u_1,u_2) \in U} f(u_1) \times f(u_2)$$
$$= \{(f(u_1),\ f(u_2)) :\ (u_1,\ u_2) \in U\} = f^\times(U). \ \triangleright$$

4.2.5. DEFINITION. A mapping $f : X \to Y$ satisfying one (and hence all) of the equivalent conditions 4.2.4 (1)–4.2.4 (4) is called *uniformly continuous* (the term "uniform continuous" is also in common parlance).

4.2.6. *Every composition of uniformly continuous mappings is uniformly continuous.*

◁ Consider $f : X \to Y$, $g : Y \to Z$ and $h := g \circ f : X \to Z$. Plainly,

$$h^{\times}(x,\ y) = (h(x),\ h(y)) = (g(f(x)),\ g(f(y))) = g^{\times}(f(x),\ f(y)) = g^{\times} \circ f^{\times}(x,\ y)$$

for all $x,\ y \in X$. Hence, by 4.2.4 (3) $h^{\times}(\mathscr{U}_X) = g^{\times}(f^{\times}(\mathscr{U}_X)) \supset g^{\times}(\mathscr{U}_Y) \supset \mathscr{U}_Z$, meaning that h is uniformly continuous. ▷

4.2.7. *Every uniformly continuous mapping is continuous.* ◁▷

4.2.8. DEFINITION. Let $\mathscr{E}$ be a set of mappings from X into Y and let $\mathscr{U}_X$ and $\mathscr{U}_Y$ be the uniformities in X and Y. The set $\mathscr{E}$ is *equicontinuous* if

$$(\forall V \in \mathscr{U}_Y) \bigcap_{f \in \mathscr{E}} f^{-1} \circ V \circ f \in \mathscr{U}_X.$$

4.2.9. *Every equicontinuous set consists of uniformly continuous mappings. Every finite set of uniformly continuous mappings is equicontinuous.* ◁▷

4.3. Semicontinuity

4.3.1. *Let $(X_1,\ d_1)$ and $(X_2,\ d_2)$ be metric spaces, and $\mathscr{X} := X_1 \times X_2$. Given $\overline{x} := (x_1,\ x_2)$ and $\overline{y} := (y_1,\ y_2)$, put*

$$d(\overline{x},\ \overline{y}) := d_1(x_1,\ y_1) + d_2(x_2,\ y_2).$$

Then d is a metric on $\mathscr{X}$. Moreover, for every $\overline{x} := (x_1,\ x_2)$ in $\mathscr{X}$ the presentation holds:

$$\tau_{\mathscr{X}}(\overline{x}) = \operatorname{fil}\{U_1 \times U_2 :\ U_1 \in \tau_{X_1}(x_1),\ U_2 \in \tau_{X_2}(x_2)\}.$$ ◁▷

4.3.2. DEFINITION. The topology $\tau_{\mathscr{X}}$ is called the *product* of τ_{X_1} and τ_{X_2} or the *product topology* of $X_1 \times X_2$. This topology on $X_1 \times X_2$ is denoted by $\tau_{X_1} \times \tau_{X_2}$.

4.3.3. DEFINITION. A function $f : X \to \mathbb{R}^{\cdot}$ is *lower semicontinuous* if its epigraph epi f is closed in the product topology of $X \times \mathbb{R}$.

4.3.4. EXAMPLES.

(1) A continuous real-valued function $f : X \to \mathbb{R}$ is lower semicontinuous.

(2) If $f_\xi : X \to \mathbb{R}^{\cdot}$ is lower semicontinuous for all $\xi \in \Xi$, then the upper envelope $f(x) := \sup\{f_\xi(x) :\ \xi \in \Xi\}$ $(x \in X)$ is also lower semicontinuous. A simple reason behind this is the equality epi $f = \bigcap_{\xi \in \Xi}$ epi f_ξ.

4.3.5. *A function $f : X \to \mathbb{R}^{\cdot}$ is lower semicontinuous if and only if*

$$x \in X \Rightarrow f(x) = \liminf_{y \to x} f(y).$$

Here, as usual,

$$\liminf_{y \to x} f(y) := \varliminf_{y \to x} f(y) := \sup_{U \in \tau(x)} \inf f(U)$$

is the *lower limit* of f at x (with respect to $\tau(x)$).

◁ $\Rightarrow$: If $x \notin \operatorname{dom} f$ then $(x,\ t) \notin \operatorname{epi} f$ for all $t \in \mathbb{R}$. Hence, there is a neighborhood U_t of x such that $\inf f(U_t) > t$. This implies that $\lim_{y \to x} \inf f(y) = +\infty = f(x)$. Suppose that $x \in \operatorname{dom} f$. Then $\inf f(V) > -\infty$ for some neighborhood V of x. Choose $\varepsilon > 0$ and for an arbitrary U in $\tau(x)$ included in V find x_U in U so that $\inf f(U) \geq f(x_U) - \varepsilon$. By construction, $x_U \in \operatorname{dom} f$ and, moreover, $x_U \to x$ (by implication, the set of neighborhoods of x is endowed with the conventional order by inclusion, cf. 1.3.1). Put $t_U := \inf f(U) + \varepsilon$. It is clear that $t_U \to t := \lim_{y \to x} \inf f(y) + \varepsilon$. Since $(x_U,\ t_U) \in \operatorname{epi} f$, from the closure property of epi f obtain $(x,\ t) \in \operatorname{epi} f$. Thus,

$$\liminf_{y \to x} f(y) + \varepsilon \geq f(x) \geq \liminf_{y \to x} f(y).$$

$\Leftarrow$: If $(x,\ t) \notin \operatorname{epi} f$ then

$$t < \liminf_{y \to x} f(y) = \sup \inf_{U \in \tau(x)} f(U).$$

Therefore, $\inf f(U) > t$ for some neighborhood U of x. It follows that the complement of epi f to $X \times \mathbb{R}$ is open. ▷

4.3.6. Remark. The property, stated in 4.3.5, may be accepted as an initial definition of lower semicontinuity at a point.

4.3.7. *A function $f : X \to \mathbb{R}$ is continuous if and only if both f and $-f$ are lower semicontinuous.* ◁▷

4.3.8. *A function $f : X \to \mathbb{R}^{\cdot}$ is lower semicontinuous if and only if for every $t \in \mathbb{R}$ the level set $\{f \leq t\}$ is closed.*

◁ $\Rightarrow$: If $x \notin \{f \leq t\}$ then $t < f(x)$. By 4.3.5, $t < \inf f(U)$ in a suitable neighborhood U about x. In other words, the complement of $\{f \leq t\}$ to X is open.

$\Leftarrow$: Given $\lim_{y \to x} \inf f(y) \leq t < f(x)$ for some $x \in X$ and $t \in \mathbb{R}$, choose $\varepsilon > 0$ such that $t + \varepsilon < f(x)$. Repeating the argument of 4.3.5, given $U \in \tau(x)$ take a point x_U in $U \cap \{f \leq \inf f(U) + \varepsilon\}$. Undoubtedly $x_U \in \{f \leq t + \varepsilon\}$ and $x_U \to x$, a contradiction. ▷

4.4. Compactness

4.4.1. DEFINITION. A subset C of X is called a *compact set* whenever for every subset $\mathscr{E}$ of $\operatorname{Op}(\tau_X)$ with the property $C \subset \cup\{G : G \in \mathscr{E}\}$ there is a finite subset $\mathscr{E}_0$ in $\mathscr{E}$ such that still $C \subset \cup\{G : G \in \mathscr{E}_0\}$.

4.4.2. REMARK. Definition 4.4.1 is verbalized as follows: "A set is compact if its every open cover has a finite subcover." The terms "covering" and "subcovering" are also in current usage.

4.4.3. *Every closed subset of a compact set is also compact. Every compact set is closed.* ◁▷

4.4.4. REMARK. With regard to 4.4.3, it stands to reason to use the term "relatively compact set" for a set whose closure is compact.

4.4.5. ***Weierstrass Theorem.*** *The image of a compact set under a continuous mapping is compact.*

◁ The inverse images of sets in an open cover of the image compose an open cover of the original set. ▷

4.4.6. *Each lower semicontinuous real-valued function, defined on a nonempty compact set, assumes its least value; i.e., the image of a compact domain has a least element.*

◁ Suppose that $f : X \to \mathbb{R}^{\cdot}$ and X is compact. Let $t_0 := \inf f(X)$. In the case $t_0 = +\infty$ there is nothing left to proof. If $t_0 < +\infty$ then put $T := \{t \in \mathbb{R} : t > t_0\}$. The set $U_t := \{f \le t\}$ with $t \in T$ is nonempty and closed. Check that $\cap\{U_t : t \in T\}$ is not empty (then every element x of the intersection meets the claim: $f(x) = \inf f(X)$). Suppose the contrary. Then the sets $\{X \setminus U_t : t \in T\}$ compose an open cover of X. Refining a finite subcover, deduce $\cap\{U_t : t \in T_0\} = \varnothing$. However, this equality is false, since $U_{t_1} \cap U_{t_2} = U_{t_1 \wedge t_2}$ for $t_1,\ t_2 \in T$. ▷

4.4.7. ***Bourbaki Criterion.*** *A space is compact if and only if every ultrafilter on it converges* (*cf.* 9.4.4).

4.4.8. *The product of compact sets is compact.*

◁ It suffices to apply the Bourbaki Criterion twice. ▷

4.4.9. ***Cantor Theorem.*** *Every continuous mapping on a compact set is uniformly continuous.* ◁▷

4.5. Completeness

4.5.1. *If $\mathscr{B}$ is a filterbase on X then $\{B^2 : B \in \mathscr{B}\}$ is a filterbase (and a base for the filter $\mathscr{B}^{\times}$) on X^2.*

◁ $(B_1 \times B_1) \cap (B_2 \times B_2) \supset (B_1 \cap B_2) \times (B_1 \cap B_2)$ ▷

4.5.2. Definition. Let $\mathscr{U}_X$ be a uniformity on X. A filter $\mathscr{F}$ is called a *Cauchy filter* or even *Cauchy* (the latter might seem preposterous) if $\mathscr{F}^\times \supset \mathscr{U}_X$. A net in X is a *Cauchy net* or a *fundamental net* if its tail filter is a Cauchy filter. The term "fundamental sequence" is treated in a similar fashion.

4.5.3. Remark. If V is an entourage on X and U is a subset of X then U is V*-small* whenever $U^2 \subset V$. In particular, U is B_ε-small (or simply ε-small) if and only if the *diameter* of U, $\operatorname{diam} U := \sup(U^2)$, is less than or equal to ε. In this connection, the definition of a Cauchy filter is expressed as follows: "A filter is a Cauchy filter if and only if it contains arbitrarily small sets."

4.5.4. *In a metric space the following conditions are equivalent:*

(1) *every Cauchy filter converges;*

(2) *each Cauchy net has a limit;*

(3) *every fundamental sequence converges.*

◁ The implications (1) ⇒ (2) ⇒ (3) are obvious; therefore, we are left with establishing (3) ⇒ (1).

Given a Cauchy filter $\mathscr{F}$, let U_n in $\mathscr{F}$ be a $B_{1/n}$-small set. Put $V_n := U_1 \cap \ldots \cap U_n$ and take $x_n \in V_n$. Observe that $V_1 \supset V_2 \supset \ldots$ and $\operatorname{diam} V_n \leq {}^1/_n$. Hence, (x_n) is a fundamental sequence. By hypothesis it has a limit, $x := \lim x_n$. Check that $\mathscr{F} \to x$. To this end, choose $n_0 \in \mathbb{N}$ such that $d(x_m,\ x) \leq {}^1/_{2n}$ as $m \geq n_0$. Then for all $n \in \mathbb{N}$ deduce that $d(x_p,\ y) \leq \operatorname{diam} V_p \leq {}^1/_{2n}$ and $d(x_p,\ x) \leq {}^1/_{2n}$ whenever $p := n_0 \vee 2n$ and $y \in V_p$. It follows that $y \in V_p \Rightarrow d(x,\ y) \leq {}^1/_n$; i.e., $V_p \subset B_{1/n}(x)$. In conclusion, $\mathscr{F} \supset \tau(x)$. ▷

4.5.5. Definition. A metric space satisfying one (and hence all) of the equivalent conditions 4.5.4 (1)–4.5.4 (3) is called *complete.*

4.5.6. Cantor Criterion. *A metric space X is complete if and only if every nonempty downward-filtered family of nonempty closed subsets of X whose diameters tend to zero has a point of intersection.*

◁ ⇒: If $\mathscr{B}$ is such a family then by Definition 1.3.1 $\mathscr{B}$ is a filterbase. By hypothesis, $\mathscr{B}$ is a base for a Cauchy filter. Therefore, there is a limit: $\mathscr{B} \to x$. The point x meets the claim.

⇐: Let $\mathscr{F}$ be a Cauchy filter. Put $\mathscr{B} := \{\operatorname{cl} V : V \in \mathscr{F}\}$. The diameters of the sets in $\mathscr{B}$ tend to zero. Hence, there is a point x such that $x \in \operatorname{cl} V$ for all $V \in \mathscr{F}$. Plainly, $\mathscr{F} \to x$. Indeed, let V be an ${}^\varepsilon/_2$-small member of $\mathscr{F}$ and $y \in V$. Some y' in V is such that $d(x,\ y') \leq {}^\varepsilon/_2$. Therefore, $d(x,\ y) \leq d(x,\ y') + d(y',\ y) \leq \varepsilon$. Consequently, $V \subset B_\varepsilon(x)$ and so $B_\varepsilon(x) \in \mathscr{F}$. ▷

4.5.7. Nested Ball Theorem. *A metric space is complete if and only if every nested (= decreasing by inclusion) sequence of balls whose radii tend to zero has a unique point of intersection.* ◁▷

4.5.8. *The image of a Cauchy filter under a uniformly continuous mapping is a Cauchy filter.*

◁ Let a mapping f act from a space X with uniformity $\mathscr{U}_X$ into a space Y with uniformity $\mathscr{U}_Y$ and let $\mathscr{F}$ be a Cauchy filter on X. If $V \in \mathscr{U}_Y$ then, by Definition 4.2.5, $f^{-1} \circ V \circ f \in \mathscr{U}_X$ (cf. 4.2.4 (2)). Since $\mathscr{F}$ is a Cauchy filter, $U^2 \subset f^{-1} \circ V \circ f$ for some $U \in \mathscr{F}$. It turns out that $f(U)$ is V-small. Indeed,

$$f(U)^2 = \bigcup_{(u_1,u_2)\in U^2} f(u_1) \times f(u_2)$$

$$= f \circ U^2 \circ f^{-1} \subset f \circ (f^{-1} \circ V \circ f) \circ f^{-1} = (f \circ f^{-1}) \circ V \circ (f \circ f^{-1}) \subset U,$$

because, by 1.1.6, $f \circ f^{-1} = I_{\operatorname{im} f} \subset I_Y$. ▷

4.5.9. *The product of complete spaces is complete.*

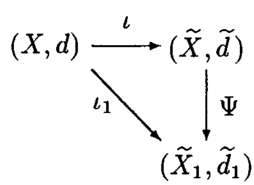
◁ The claim is immediate from 4.5.8 and 4.5.4. ▷

4.5.10. *Let X_0 be dense in X (i.e.,* $\operatorname{cl} X_0 = X$*). Assume further that $f_0 : X_0 \to Y$ is a uniformly continuous mapping from X_0 into some complete space Y. Then there is a unique uniformly continuous mapping $f : X \to Y$ extending f_0; i.e., $f|_{X_0} = f_0$.*

◁ For $x \in X$, the filter $\mathscr{F}_x := \{U \cap X_0 : U \in \tau_X(x)\}$ is a Cauchy filter on X_0. Therefore, 4.5.8 implies that $f_0(\mathscr{F}_X)$ is a Cauchy filter on Y. By the completeness of Y, there is a limit $y \in Y$; that is, $f_0(\mathscr{F}_x) \to y$. Moreover, this limit is unique (cf. 4.1.18). Define $f(x) := y$. Checking uniform continuity for f readily completes the proof. ▷

4.5.11. Definition. An *isometry* or *isometric embedding* or *isometric mapping* of X into $\widetilde{X}$ is a mapping $f : (X, d) \to (\widetilde{X}, \widetilde{d})$ such that $d = \widetilde{d} \circ f^{\times}$. A mapping f is an *isometry onto* $\widetilde{X}$ if f is an isometry of X into $\widetilde{X}$ and, moreover, $\operatorname{im} f = \widetilde{X}$. The expressions, "an isometry of X and $\widetilde{X}$" or "an isometry between X and $\widetilde{X}$" and the like, are also in common parlance.

4.5.12. Hausdorff Completion Theorem. *If (X, d) is a metric space then there are a complete metric space $(\widetilde{X}, \widetilde{d})$ and an isometry $\iota : (X, d) \to (\widetilde{X}, \widetilde{d})$ onto a dense subspace of $(\widetilde{X}, \widetilde{d})$. The space $(\widetilde{X}, \widetilde{d})$ is unique to within isometry in the following sense: The diagram*



commutes for some isometry $\Psi : (\widetilde{X}, \widetilde{d}) \to (\widetilde{X}_1, \widetilde{d}_1)$, *where* $\iota_1 : (X, d) \to (\widetilde{X}_1, \widetilde{d}_1)$ *is an isometry of* X *onto a dense subspace of a complete space* $(\widetilde{X}_1, \widetilde{d}_1)$.

◁ Uniqueness up to isometry follows from 4.5.10. For, if $\Psi_0 := \iota_1 \circ \iota^{-1}$ then Ψ_0 is an isometry of the dense subspace $\iota(X)$ of $\widetilde{X}$ onto the dense subspace $\iota_1(X)$ of $\widetilde{X}_1$. Let Ψ be the unique extension of Ψ_0 to the whole of $\widetilde{X}$. It is sufficient to show that Ψ acts onto $\widetilde{X}_1$. Take $\widetilde{x}_1$ in $\widetilde{X}_1$. This element is the limit of some sequence $(\iota_1(x_n))$, with $x_n \in X$. Clearly, the sequence (x_n) is fundamental. Thus, $(\iota(x_n))$ is a fundamental sequence in $\widetilde{X}$. Let $\widetilde{x} := \lim \iota(x_n)$, $\widetilde{x} \in \widetilde{X}$. Proceed as follows: $\Psi(\widetilde{x}) = \lim \Psi_0(\iota(x_n)) = \lim \iota_1 \circ \iota^{-1}(\iota(x_n)) = \lim \iota_1(x_n) = \widetilde{x}_1$.

We now sketch out the proof that $\widetilde{X}$ exists. Consider the set $\mathscr{X}$ of all fundamental sequences in X. Define some equivalence relation in X by putting $\overline{x}_1 \sim \overline{x}_2 \Leftrightarrow d(\overline{x}_1(n), \overline{x}_2(n)) \to 0$. Assign $\widetilde{X} := \mathscr{X}/\!\sim$ and $\widetilde{d}(\varphi(\overline{x}_1), \varphi(\overline{x}_2)) := \lim d(\overline{x}_1(n), \overline{x}_2(n))$, where $\varphi : \mathscr{X} \to \widetilde{X}$ is the coset mapping. An isometry $\iota : (X, d) \to (\widetilde{X}, \widetilde{d})$ is immediate: $\iota(x) := \varphi(n \mapsto x\ (n \in \mathbb{N}))$. ▷

4.5.13. DEFINITION. The space $(\widetilde{X}, \widetilde{d})$ introduced in Theorem 4.5.12, as well as each space isomorphic to it, is called a *completion* of (X, d).

4.5.14. DEFINITION. A set X_0 in a metric space (X, d) is said to be *complete* if the metric space $(X_0, d|_{X_0^2})$, a *subspace* of (X, d), is complete.

4.5.15. *Every closed subset of a complete space is complete. Every complete set is closed.* ◁▷

4.5.16. *If* X_0 *is a subspace of a complete metric space* X *then a completion of* X_0 *is isometric to the closure of* X_0 *in* X.

◁ Let $\widetilde{X} := \operatorname{cl} X_0$ and let $\iota : X_0 \to \widetilde{X}$ be the identical embedding. It is evident that ι is an isometry onto a dense subspace. Moreover, by 4.5.15 $\widetilde{X}$ is complete. Appealing to 4.5.12 ends the proof. ▷

4.6. Compactness and Completeness

4.6.1. *A compact space is complete.* ◁▷

4.6.2. DEFINITION. Let U be a subset of X and $V \in \mathscr{U}_X$. A set E in X is a *V-net* for U if $U \subset V(E)$.

4.6.3. DEFINITION. A subset U of X is a *totally bounded* set in X if for every V in $\mathscr{U}_X$ there is a finite V-net for U.

4.6.4. *If for every* V *in* $\mathscr{U}$ *a set* U *in* X *has a totally bounded* V*-net then* U *is totally bounded.*

◁ Let $V \in \mathscr{U}_X$ and $W \in \mathscr{U}_X$ be such that $W \circ W \subset V$. Take a totally bounded W-net F for U; i.e., $U \subset W(F)$. Since F is totally bounded, there is a finite W-net

E for F; that is, $F \subset W(E)$. Finally,

$$U \subset W(F) \subset W(W(E)) = W \circ W(E) \subset V(E);$$

i.e., E is a finite V-net for U. ▷

4.6.5. *A subset U of X is totally bounded if and only if for every V in $\mathscr{U}_X$ there is a family $U_1, \dots, U_n$ of subsets of U such that $U = U_1 \cup \dots \cup U_n$ and each of the sets $U_1, \dots, U_n$ is V-small.* ◁▷

4.6.6. REMARK. The claim of 4.6.5 is verbalized as follows: "A set is totally bounded if and only if it has finite covers consisting of arbitrarily small sets."

4.6.7. ***Hausdorff Criterion.*** *A set is compact if and only if it is complete and totally bounded.* ◁▷

4.6.8. *Let $C(Q,\ \mathbb{F})$ be the space of continuous functions with domain a compact set Q and range a subset of $\mathbb{F}$. Furnish this space with the Chebyshëv metric*

$$d(f,\ g) := \sup_{x \in Q} d_{\mathbb{F}}(f(x),\ g(x)) := \sup_{x \in Q} |f(x) - g(x)| \quad (f,\ g \in C(Q,\ \mathbb{F}));$$

and, given $\theta \in \mathscr{U}_{\mathbb{F}}$, put

$$U_\theta := \left\{(f,\ g) \in C(Q,\ \mathbb{F})^2 :\ g \circ f^{-1} \subset \theta\right\}.$$

Then $\mathscr{U}_d = \mathrm{fil}\ \{U_\theta :\ \theta \in \mathscr{U}_{\mathbb{F}}\}$. ◁▷

4.6.9. *The space $C(Q,\ \mathbb{F})$ is complete.* ◁▷

4.6.10. ***Ascoli–Arzelà Theorem.*** *A subset $\mathscr{E}$ of $C(Q,\ \mathbb{F})$ is relatively compact if and only if $\mathscr{E}$ is equicontinuous and the set $\cup\{g(Q) :\ g \in \mathscr{E}\}$ is totally bounded in $\mathbb{F}$.*

◁ $\Rightarrow$: It is beyond a doubt that $\cup\{g(Q) :\ g \in \mathscr{E}\}$ is totally bounded. To show equicontinuity for $\mathscr{E}$ take $\theta \in \mathscr{U}_{\mathbb{F}}$ and choose a symmetric entourage θ' such that $\theta' \circ \theta' \circ \theta' \subset \theta$. By the Hausdorff Criterion, there is a finite $U_{\theta'}$-net $\mathscr{E}'$ for $\mathscr{E}$. Consider the entourage $U \in \mathscr{U}_Q$ defined as

$$U := \bigcap_{f \in \mathscr{E}'} f^{-1} \circ \theta' \circ f$$

(cf. 4.2.9). Given $g \in \mathscr{E}$ and $f \in \mathscr{E}'$ such that $g \circ f^{-1} \subset \theta'$, observe that

$$\theta' = \theta'^{-1} \supset (g \circ f^{-1})^{-1} = (f^{-1})^{-1} \circ g^{-1} = f \circ g^{-1}.$$

Moreover, the composition rules for correspondences and 4.6.8 imply

$$g^{\times}(U) = g \circ U \circ g^{-1} \subset g \circ (f^{-1} \circ \theta' \circ f) \circ g^{-1}$$
$$\subset (g \circ f^{-1}) \circ \theta' \circ (f \circ g^{-1}) \subset \theta' \circ \theta' \circ \theta' \subset \theta.$$

Since g is arbitrary, the resulting inclusion guarantees that $\mathscr{E}$ is equicontinuous.

$\Leftarrow$: By 4.5.15, 4.6.7, 4.6.8, and 4.6.9, it is sufficient to construct a finite U_θ-net for $\mathscr{E}$ given $\theta \in \mathscr{U}_{\mathbb{F}}$. Choose $\theta' \in \mathscr{U}_{\mathbb{F}}$ such that $\theta' \circ \theta' \circ \theta' \subset \theta$ and find an open symmetric entourage $U \in \mathscr{U}_Q$ from the condition

$$U \subset \bigcap_{g \in \mathscr{E}} g^{-1} \circ \theta' \circ g$$

(by the equicontinuity property of $\mathscr{E}$, such an U is available). The family $\{U(x) : x \in Q\}$ clearly forms an open cover of Q. By the compactness of Q, refine a finite subcover $\{U(x_0) : x_0 \in Q_0\}$. In particular, from 1.1.10 derive

$$I_Q \subset \bigcup_{x_0 \in Q_0} U(x_0) \times U(x_0)$$
$$= \bigcup_{(x_0, x_0) \in I_{Q_0}} U^{-1}(x_0) \times U(x_0) = U \circ I_{Q_0} \circ U.$$

The set $\{g|_{Q_0} : g \in \mathscr{E}\}$ is totally bounded in $\mathbb{F}^{Q_0}$. Consequently, there is a finite θ'-net for this set. Speaking more precisely, there is a finite subset $\mathscr{E}'$ of $\mathscr{E}$ with the following property:

$$g \circ I_{Q_0} \circ f^{-1} \subset \theta'$$

for every g in $\mathscr{E}$ and some f in $\mathscr{E}'$. Using the estimates, successively infer that

$$\begin{aligned} g \circ f^{-1} &= g \circ I_Q \circ f^{-1} \subset g \circ (U \circ I_{Q_0} \circ U) \circ f^{-1} \\ &\subset g \circ (g^{-1} \circ \theta' \circ g) \circ I_{Q_0} \circ (f^{-1} \circ \theta' \circ f) \circ f^{-1} \\ &= (g \circ g^{-1}) \circ \theta' \circ (g \circ I_{Q_0} \circ f^{-1}) \circ \theta' \circ (f \circ f^{-1}) \\ &= I_{\operatorname{im} g} \circ \theta' \circ (g \circ I_{Q_0} \circ f^{-1}) \circ \theta' \circ I_{\operatorname{im} f} \\ &\subset \theta' \circ \theta' \circ \theta' \subset \theta. \end{aligned}$$

Thus, by 4.6.8, $\mathscr{E}'$ is a finite U_θ-net for $\mathscr{E}$. $\triangleright$

4.6.11. Remark. It is an enlightening exercise to translate the proof of the Ascoli–Arzelà Theorem into the "ε-δ" language. The necessary vocabulary is in hand: "θ and U_θ stand for ε," "θ' is ${}^{\varepsilon}/_{3}$," and "δ is U." It is also rewarding and instructive to find generalizations of the Ascoli–Arzelà Theorem for mappings acting into more abstract spaces.

4.7. Baire Spaces

4.7.1. DEFINITION. A set U is said to be *nowhere dense* or *rare* whenever its closure lacks interior points; i.e., $\operatorname{int}\operatorname{cl} U = \varnothing$. A set U is *meager* or a set of *first category* if U is included into a countable union of rare sets; i.e., $U \subset \bigcup_{n\in\mathbb{N}} U_n$ with $\operatorname{int}\operatorname{cl} U_n = \varnothing$. A *nonmeager* set (which is not of first category by common parlance) is also referred to as a *set of second category.*

4.7.2. DEFINITION. A space is a *Baire space* if its every nonempty open set is nonmeager.

4.7.3. *The following statements are equivalent:*

(1) *X is a Baire space;*

(2) *every countable union of closed rare sets in X lacks interior points;*

(3) *the intersection of a countable family of open everywhere dense sets (i.e., dense in X) is everywhere dense;*

(4) *the complement of each meager set to X is everywhere dense.*

◁ (1) ⇒ (2): If $U := \bigcup_{n\in\mathbb{N}} U_n$, $U_n = \operatorname{cl} U_n$, and $\operatorname{int} U_n = \varnothing$ then U is meager. Observe that $\operatorname{int} U \subset U$ and $\operatorname{int} U$ is open; hence, $\operatorname{int} U$ as a meager set is necessarily empty, for X is a Baire space.

(2) ⇒ (3): Let $U := \bigcap_{n\in\mathbb{N}} G_n$, where G_n's are open and $\operatorname{cl} G_n = X$. Then $X \setminus U = X \setminus \bigcap_{n\in\mathbb{N}} G_n = \bigcup_{n\in\mathbb{N}}(X \setminus G_n)$. Moreover, $X \setminus G_n$ is closed and $\operatorname{int}(X \setminus G_n) = \varnothing$ (since $\operatorname{cl} G_n = X$). Therefore, $\operatorname{int}(X \setminus U) = \varnothing$, which implies that the exterior of U is empty; i.e., U is everywhere dense.

(3) ⇒ (4): Let U be a meager set in X; i.e., $U \subset \bigcup_{n\in\mathbb{N}} U_n$ and $\operatorname{int}\operatorname{cl} U_n = \varnothing$. It may be assumed that $U_n = \operatorname{cl} U_n$. Then $G_n := X \setminus U_n$ is open and everywhere dense. By hypothesis, $\bigcap_{n\in\mathbb{N}} G_n = X \setminus \bigcup_{n\in\mathbb{N}} U_n$ is everywhere dense. Moreover, the set is included into $X \setminus U$, and so $X \setminus U$ is everywhere dense.

(4) ⇒ (1): If U is nonempty open set in X then $X \setminus U$ is not everywhere dense. Consequently, U is nonmeager. ▷

4.7.4. REMARK. In connection with 4.7.3 (4), observe that the complement of a meager set is (sometimes) termed a *residual* or *comeager set.* A residual set in a Baire space is nonmeager.

4.7.5. Osgood Theorem. *Let X be a Baire space and let $(f_\xi : X \to \mathbb{R})_{\xi\in\Xi}$ be a family of lower semicontinuous functions such that $\sup\{f_\xi(x) : \xi \in \Xi\} < +\infty$ for all $x \in X$. Then each nonempty open set G in X includes a nonempty open subset G_0 on which $(f_\xi)_{\xi\in\Xi}$ is uniformly bounded above; i.e., $\sup_{x\in G_0}\sup\{f_\xi(x) : \xi \in \Xi\} \leq +\infty$.* ◁▷

4.7.6. Baire Category Theorem. *A complete metric space is a Baire space.*

◁ Let G be a nonempty open set and $x_0 \in G$. Suppose by way of contradiction that G is meager; i.e., $G \subset \bigcup_{n\in\mathbb{N}} U_n$, where $\operatorname{int} U_n = \varnothing$ and $U_n = \operatorname{cl} U_n$. Take $\varepsilon_0 > 0$ satisfying $B_{\varepsilon_0}(x_0) \subset G$. It is obvious that U_1 is not included into $B_{\varepsilon_0/2}(x_0)$.

Consequently, there is an element x_1 in $B_{\varepsilon_0/2}(x_0) \setminus U_1$. Since U_1 is closed, find ε_1 satisfying $0 < \varepsilon_1 \le {}^{\varepsilon_0}/_2$ and $B_{\varepsilon_1}(x_1) \cap U_1 = \varnothing$. Check that $B_{\varepsilon_1}(x_1) \subset B_{\varepsilon_0}(x_0)$. For, if $d(x_1,\ y_1) \le \varepsilon_1$, then $d(y_1,\ x_0) \le d(y_1,\ x_1) + d(x_1,\ x_0) \le \varepsilon_1 + {}^{\varepsilon_0}/_2$, because $d(x_1,\ x_0) \le {}^{\varepsilon_0}/_2$. The ball $B_{\varepsilon_1/2}(x_1)$ does not lie in U_2 entirely. It is thus possible to find $x_2 \in B_{\varepsilon_1/2}(x_1) \setminus U_2$ and $0 < \varepsilon_2 \le {}^{\varepsilon_1}/_2$ satisfying $B_{\varepsilon_2}(x_2) \cap U_2 = \varnothing$. It is easy that again $B_{\varepsilon_2}(x_2) \subset B_{\varepsilon_1}(x_1)$. Proceeding by induction, obtain the sequence of nested balls $B_{\varepsilon_0}(x_0) \supset B_{\varepsilon_1}(x_1) \supset B_{\varepsilon_2}(x_2) \supset \dots$; moreover, $\varepsilon_{n+1} \le {}^{\varepsilon_n}/_2$ and $B_{\varepsilon_n}(x_n) \cap U_n = \varnothing$. By the Nested Ball Theorem, the balls have a common point, $x := \lim x_n$. Further, $x \ne \cup_{n \in \mathbb{N}} U_n$; and, hence, $x \notin G$. On the other hand, $x \in B_{\varepsilon_0}(x_0) \subset G$, a contradiction. ▷

4.7.7. REMARK. The Baire Category Theorem is often used as a "pure existence theorem." As a classical example, consider the existence problem for continuous nowhere differentiable functions.

Given $f : [0, 1] \to \mathbb{R}$ and $x \in [0, 1)$, put

$$D_+ f(x) := \liminf_{h \downarrow 0} \frac{f(x+h) - f(x)}{h};$$

$$D^+ f(x) := \limsup_{h \downarrow 0} \frac{f(x+h) - f(x)}{h}.$$

The elements $D_+ f(x)$ and $D^+ f(x)$ of the extended axis $\overline{\mathbb{R}}$ are the *lower right* and *upper right Dini derivatives* of f at x. Further, let D stand for the set of functions f in $C([0,\ 1],\ \mathbb{R})$ such that for some $x \in [0,\ 1)$ the elements $D_+ f(x)$ and $D^+ f(x)$ belong to $\mathbb{R}$; i.e., they are finite. Then D is a meager set. Hence, the functions lacking derivatives at every point of $(0,\ 1)$ are everywhere dense in $C([0,\ 1],\ \mathbb{R})$. However, the explicit examples of such functions are not at all easy to find and grasp. Below a few of the most popular are listed:

$$\textit{van der Waerden's function:}\ \sum_{n=0}^{\infty} \frac{\langle\langle 4^n x \rangle\rangle}{4^n},$$

with $\langle\langle x \rangle\rangle := (x - [x]) \wedge (1 + [x] - x)$, the distance from x to the whole number nearest to x;

$$\textit{Riemann's function:}\ \sum_{n=0}^{+\infty} \frac{1}{n^2} \sin\,(n^2 \pi x);$$

and, finally, the historically first

$$\textit{Weierstrass's function:}\ \sum_{n=0}^{\infty} b^n \cos\,(a^n \pi x),$$

with a an odd positive integer, $0 < b < 1$ and $ab > 1 + {}^{3\pi}/_2$.

4.8. The Jordan Curve Theorem and Rough Drafts

4.8.1. REMARK. In topology, in particular, many significant and curious facts of the metric space $\mathbb{R}^2$ are scrutinized. Here we recall those of them which are of use in the sequel and whose role is known from complex analysis.

4.8.2. DEFINITION. A (*Jordan*) *arc* is a homeomorphic image of a (nondegenerate) interval of the real axis. Recall that a *homeomorphism* or a *topological mapping* is by definition a one-to-one continuous mapping whose inverse is also continuous. A (*simple Jordan*) *loop* is a homeomorphic image of a circle. Concepts like "smooth loop" are understood naturally.

4.8.3. ***Jordan Curve Theorem.*** *If γ is a simple loop in $\mathbb{R}^2$ then there are open sets G_1 and G_2 such that*

$$G_1 \cup G_2 = \mathbb{R}^2 \setminus \gamma;\ \gamma = \partial G_1 = \partial G_2. \ \triangleleft\triangleright$$

4.8.4. REMARK. Either G_1 or G_2 in 4.8.3 is bounded. Moreover, each of the two sets is *connected*; i.e., it cannot be presented as the union of two nonempty disjoint open sets. In this regard the Jordan Curve Theorem is often read as follows: "A simple loop divides the plane into two domains and serves as their mutual boundary."

4.8.5. DEFINITION. Let $D_1, \dots, D_n$ and D be closed disks (= closed balls) in the plane which satisfy $D_1, \dots, D_n \subset \operatorname{int} D$ and $D_m \cap D_k = \varnothing$ as $m \neq k$. The set

$$D \setminus \bigcup_{k=1}^{n} \operatorname{int} D_k$$

is a *holey disk* or, more formally, a *perforated disk*. A subset of the plane which is diffeomorphic (= "smoothly homeomorphic") to a holey disk is called a *connected elementary compactum*. The union of a finite family of pairwise disjoint connected elementary compacta is an *elementary compactum*.

4.8.6. REMARK. The boundary ∂F of an elementary compactum F comprises finitely many disjoint smooth loops. Furthermore, the embedding of F into the (oriented) plane $\mathbb{R}^2$ induces on F the structure of an (oriented) manifold with (oriented) boundary ∂F. Observe also that, by 4.8.3, it makes sense to specify the positive orientation of a smooth loop. This is done by indicating the orientation induced on the boundary of the compact set surrounded by the loop.

4.8.7. *If K is a compact subset of the plane and G is a nonempty open set that includes K then there is a nonempty elementary compactum F such that*

$$K \subset \operatorname{int} F \subset F \subset G. \ \triangleleft\triangleright$$

4.8.8. DEFINITION. Every set F appearing in 4.8.7 is referred to as a *rough draft* for the pair $(K,\ G)$ or an *oriented envelope* of K in G.

Exercises

4.1. Give examples of metric spaces. Find methods for producing new metric spaces.

4.2. Which filter on X^2 coincides with some metric uniformity on X?

4.3. Let S be the space of measurable functions on $[0, 1]$ endowed with the metric

$$d(f, g) := \int_0^1 \frac{|f(t) - g(t)|}{1 + |f(t) - g(t)|} dt \quad (f, g \in S)$$

with some natural identification implied (specify it!). Find the meaning of convergence in the space.

4.4. Given $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$, put

$$d(\alpha, \beta) := 1 / \min \{k \in \mathbb{N} : \alpha_k \neq \beta_k\}.$$

Check that d is a metric and the space $\mathbb{N}^{\mathbb{N}}$ is homeomorphic with the set of irrational numbers.

4.5. Is it possible to metrize pointwise convergence in the sequence space? In the function space $\mathbb{F}^{[0,1]}$?

4.6. How to introduce a reasonable metric into the countable product of metric spaces? Into an arbitrary product of metric spaces?

4.7. Describe the function classes distinguishable by erroneous definitions of continuity and uniform continuity.

4.8. Given nonempty compact subsets A and B of the spaces $\mathbb{R}^N$, define

$$d(A, B) := \left(\sup_{x \in A} \inf_{y \in B} |x - y| \right) \vee \left(\sup_{y \in B} \inf_{x \in A} |x - y| \right).$$

Show that d is a metric. The metric is called the *Hausdorff metric*. What is the meaning of convergence in this metric?

4.9. Prove that nonempty compact convex subsets of a compact convex set in $\mathbb{R}^N$ constitute a compact set with respect to the Hausdorff metric. How does this claim relate to the Ascoli–Arzelà Theorem?

4.10. Prove that each lower semicontinuous function on $\mathbb{R}^N$ is the upper envelope of some family of continuous functions.

4.11. Explicate the interplay between continuous and closed mappings (in the product topology) of metric spaces.

4.12. Find out when a continuous mapping of a metric space into a complete metric spaces is extendible onto a completion of the initial space.

4.13. Describe compact sets in the product of metric spaces.

4.14. Let (Y, d) be a complete metric spaces. A mapping $F : Y \to Y$ is called ***expanding*** whenever $d(F(x), F(y)) \geq \beta d(x, y)$ for some $\beta > 1$ and all $x, y \in Y$. Assume that an expanding mapping $F : Y \to Y$ acts onto Y. Prove that F is one-to-one and possesses a sole fixed point.

4.15. Prove that no compact set can be mapped isometrically onto a proper part of it.

4.16. Show normality of an arbitrary metric space (see 9.3.11).

4.17. Under what conditions a countable subset of a complete metric spaces is nonmeager?

4.18. Is it possible to characterize uniform continuity in terms of convergent sequences?

4.19. In which metric spaces does each continuous real-valued function attain the supremum and infimum of its range? When is it bounded?

Chapter 5
Multinormed and Banach Spaces

5.1. Seminorms and Multinorms

5.1.1. *Let X be a vector space over a basic field $\mathbb{F}$ and let $p : X \to \mathbb{R}^{\cdot}$ be a seminorm. Then*

(1) $\operatorname{dom} p$ *is a subspace of* X;

(2) $p(x) \geq 0$ *for all* $x \in X$;

(3) *the kernel* $\ker p := \{p = 0\}$ *is a subspace in* X;

(4) *the sets* $\overset{\circ}{B}_p := \{p < 1\}$ *and* $B_p := \{p \leq 1\}$ *are absolutely convex; moreover,* p *is the Minkowski functional of every set* B *such that* $\overset{\circ}{B}_p \subset B \subset B_p$;

(5) $X = \operatorname{dom} p$ *if and only if* $\overset{\circ}{B}_p$ *is absorbing.*

◁ If $x_1,\ x_2 \in \operatorname{dom} p$ and $\alpha_1,\ \alpha_2 \in \mathbb{F}$ then by 3.7.6

$$p(\alpha_1 x_1 + \alpha_2 x_2) \leq |\alpha_1| p(x_1) + |\alpha_2| p(x_2) < +\infty + (+\infty) = +\infty.$$

Hence, (1) holds. Suppose to the contrary that (2) is false; i.e., $p(x) < 0$ for some $x \in X$. Observe that $0 \leq p(x) + p(-x) < p(-x) = p(x) < 0$, a contradiction. The claim of (3) is immediate from (2) and the subadditivity of p. The validity of (4) and (5) has been examined in part (cf. 3.8.8). What was left unproven follows from the Gauge Theorem. ▷

5.1.2. *If $p,\ q : X \to \mathbb{R}^{\cdot}$ are two seminorms then the inequality $p \leq q$ (in $(\mathbb{R}^{\cdot})^X$) holds if and only if $B_p \supset B_q$.*

◁ $\Rightarrow$: Evidently, $\{q \leq 1\} \subset \{p \leq 1\}$.

$\Leftarrow$: In view of 5.1.1 (4), observe that $p = p_{B_p}$ and $q = p_{B_q}$. Take $t_1,\ t_2 \in \mathbb{R}$ such that $t_1 < t_2$. If $t_1 < 0$ then $\{q \leq t_1\} = \varnothing$, and so $\{q \leq t_1\} \subset \{p \leq t_2\}$. If $t_1 \geq 0$ then $t_1 B_q \subset t_1 B_p \subset t_2 B_p$. Thus, by 3.8.3, $p \leq q$. ▷

5.1.3. *Let X and Y be vector spaces, let $T \subset X \times Y$ be a linear correspondence, and let $p : Y \to \mathbb{R}^{\cdot}$ be a seminorm. If $p_T(x) := \inf p \circ T(x)$ for $x \in X$ then*

$p_T : X \to \mathbb{R}^{\cdot}$ is a seminorm and the set $B_T := T^{-1}(B_p)$ is absolutely convex. In addition, $p_T = p_{B_T}$.

◁ Given $x_1,\ x_2 \in X$ and $\alpha_1,\ \alpha_2 \in \mathbb{F}$, infer that

$$p_T(\alpha_1 x_1 + \alpha_2 x_2) = \inf p(T(\alpha_1 x_1 + \alpha_2 x_2))$$

$$\le \inf p(\alpha_1 T(x_1) + \alpha_2 T(x_2)) \le \inf(|\alpha_1| p(T(x_1)) + |\alpha_2| p(T(x_2)))$$

$$= |\alpha_1| p_T(x_1) + |\alpha_2| p_T(x_2);$$

i.e., p_T is a seminorm.

The absolute convexity of B_T is a consequence of 5.1.1 (4) and 3.1.8. If $x \in B_T$ then $(x,\ y) \in T$ for some $y \in B_p$. Hence, $p_T(x) \le p(y) \le 1$; that is, $B_T \subset B_{p_T}$. If in turn $x \in \overset{\circ}{B}_{p_T}$ then $p_T(x) = \inf\{p(y) :\ (x,\ y) \in T\} < 1$. Thus, there is some $y \in T(x)$ such that $p(y) < 1$. Therefore, $x \in T^{-1}(\overset{\circ}{B}_p) \subset T^{-1}(B_p) = B_T$. Finally, $\overset{\circ}{B}_{p_T} \subset B_T \subset B_{p_T}$. Referring to 5.1.1 (4), conclude that $p_{B_T} = p_T$. ▷

5.1.4. DEFINITION. The seminorm p_T, constructed in 5.1.3, is the *inverse image* or *preimage* of p under T.

5.1.5. DEFINITION. Let $p : X \to \mathbb{R}$ be a seminorm (by 3.4.3, this implies that $\operatorname{dom} p = X$). A pair $(X,\ p)$ is referred to as a *seminormed space*. It is convenient to take the liberty of calling X itself a seminormed space.

5.1.6. DEFINITION. A *multinorm* on X is a nonempty set (a subset of $\mathbb{R}^X$) of everywhere-defined seminorms. Such a multinorm is denoted by $\mathfrak{M}_X$ or simply by $\mathfrak{M}$, if the underlying vector space is clear from the context. A pair $(X,\ \mathfrak{M}_X)$, as well as X itself, is called a *multinormed space*.

5.1.7. *A set $\mathfrak{M}$ in $(\mathbb{R}^{\cdot})^X$ is a multinorm if and only if $(X,\ p)$ is a seminormed space for every $p \in \mathfrak{M}$.* ◁▷

5.1.8. DEFINITION. A multinorm $\mathfrak{M}_X$ is a *Hausdorff* or *separated multinorm* whenever for all $x \in X,\ x \ne 0$, there is a seminorm $p \in \mathfrak{M}_X$ such that $p(x) \ne 0$. In this case X is called a *Hausdorff* or *separated multinormed space*.

5.1.9. DEFINITION. A *norm* is a Hausdorff multinorm presenting a singleton. The sole element of a norm on a vector space X is also referred to as the *norm* on X and is denoted by $\|\cdot\|$ or (rarely) by $\|\cdot\|_X$ or even $\|\cdot \mid X\|$ if it is necessary to indicate the space X. A pair $(X,\ \|\cdot\|)$ is called a *normed space*; as a rule, the same term applies to X.

5.1.10. EXAMPLES.

(1) A seminormed space $(X,\ p)$ can be treated as a multinormed space $(X,\ \{p\})$. The same relates to a normed space.

(2) If $\mathfrak{M}$ is the set of all (everywhere-defined) seminorms on a space X then $\mathfrak{M}$ is a Hausdorff multinorm called the *finest multinorm* on X.

(3) Let $(Y,\ \mathfrak{N})$ be a multinormed space, and let $T \subset X \times Y$ be a linear correspondence such that dom $T = X$. By 3.4.10 and 5.1.1 (5), for every p in $\mathfrak{N}$ the seminorm p_T is defined everywhere, and hence $\mathfrak{M} := \{p_T :\ p \in \mathfrak{N}\}$ is a multinorm on X. The multinorm $\mathfrak{N}$ is called the *inverse image* or *preimage* of $\mathfrak{N}$ under the correspondence T and is (sometimes) denoted by $\mathfrak{N}_T$. Given $T \in \mathscr{L}(X,\ Y)$, set $\mathfrak{M} := \{p \circ T :\ p \in \mathfrak{N}\}$ and use the natural notation $\mathfrak{N} \circ T := \mathfrak{M}$. Observe in particular the case in which X is a subspace Y_0 of Y and T is the identical embedding $\iota : Y_0 \to Y$. At this juncture Y_0 is treated as a multinormed space with multinorm $\mathfrak{N} \circ \iota$. Moreover, the abuse of the phrase "$\mathfrak{N}$ is a multinorm on Y_0" is very convenient.

(4) Each basic field $\mathbb{F}$ is endowed, as is well known, with the standard norm $|\cdot| : \mathbb{F} \to \mathbb{R}$, the *modulus of a scalar*. Consider a vector space X and $f \in X^{\#}$. Since $f : X \to \mathbb{F}$, it is possible to define the inverse image of the norm on $\mathbb{F}$ as $p_f(x) := |f(x)|\ (x \in X)$. If $\mathscr{X}$ is some subspace of $X^{\#}$ then the multinorm $\sigma(X,\ \mathscr{X}) := \{p_f :\ f \in \mathscr{X}\}$ is the *weak multinorm* on X induced by $\mathscr{X}$.

(5) Let $(X,\ p)$ be a seminormed space. Assume further that X_0 is a subspace of X and $\varphi : X \to X/X_0$ is the coset mapping. The linear correspondence φ^{-1} is defined on the whole of X/X_0. Hence, the seminorm $p_{\varphi^{-1}}$ appears, called the *quotient seminorm* of p by X_0 and denoted by p_{X/X_0}. The space $(X/X_0,\ p_{X/X_0})$ is called the *quotient space* of $(X,\ p)$ by X_0. The definition of quotient space for an arbitrary multinormed space requires some subtlety and is introduced in 5.3.11.

(6) Let X be a vector space and let $\mathfrak{M} \subset (\mathbb{R}^{\cdot})^X$ be a set of seminorms on X. In this case $\mathfrak{M}$ can be treated as a multinorm on the space $X_0 := \cap\{\text{dom}\ p :\ p \in \mathfrak{M}\}$. More precisely, thinking of the multinormed space $(X_0,\ \{p_\iota : p \in \mathfrak{M}\})$, where ι is the identical embedding of X_0 into X, we say: "$\mathfrak{M}$ is a multinorm," or "Consider the (multinormed) space specified by $\mathfrak{M}$." The next example is typical: The family of seminorms

$$\left\{ p_{\alpha,\beta}(f) := \sup_{x \in \mathbb{R}^N} |x^\alpha \partial^\beta f(x)| :\ \alpha \text{ and } \beta \text{ are multi-indices} \right\}$$

specifies the (multinormed) space of infinitely differentiable functions on $\mathbb{R}^N$ decreasing rapidly at infinity (such functions are often called *tempered*, cf. 10.11.6).

(7) Let $(X,\ \|\cdot\|)$ and $(Y,\ \|\cdot\|)$ be normed spaces (over the same ground field $\mathbb{F}$). Given $T \in \mathscr{L}(X,\ Y)$, consider the *operator norm* of T, i.e. the quantity

$$\|T\| := \sup\{\|Tx\| :\ x \in X,\ \|x\| \le 1\} = \sup_{x \in X} \frac{\|Tx\|}{\|x\|}.$$

(From now on, in analogous situations we presume ${}^0/_0 := 0$.)

It is easily seen that $\|\cdot\| : \mathscr{L}(X, Y) \to \mathbb{R}^{\cdot}$ is a seminorm. Indeed, putting $B_X := \{\|\cdot\|_X \leq 1\}$ for $T_1, T_2 \in \mathscr{L}(X, Y)$ and $\alpha_1, \alpha_2 \in \mathbb{F}$, deduce that

$$\|\alpha_1 T_1 + \alpha_2 T_2\| = \sup \|\cdot\|_{\alpha_1 T_1 + \alpha_2 T_2}(B_X)$$
$$= \sup \|\cdot\|((\alpha_1 T_1 + \alpha_2 T_2)(B_X)) \leq \sup \|\alpha_1 T_1(B_X) + \alpha_2 T_2(B_X)\|$$
$$\leq |\alpha_1| \sup \|\cdot\|_{T_1}(B_X) + |\alpha_2| \sup \|\cdot\|_{T_2}(B_X) = |\alpha_1|\ \|T_1\| + |\alpha_2|\ \|T_2\|.$$

The subspace $B(X, Y)$, the effective domain of definition of the above seminorm, is the *space of bounded operators*; and an element of $B(X, Y)$ is a *bounded operator*. Observe that a shorter term "operator" customarily implies a bounded operator. It is clear that $B(X, Y)$ is a normed space (under the operator norm). Note also that T in $\mathscr{L}(X, Y)$ is bounded if and only if T maintains the *normative inequality*; i.e., there is a strictly positive number K such that

$$\|Tx\|_Y \leq K\ \|x\|_X \quad (x \in X).$$

Moreover, $\|T\|$ is the greatest lower bound of the set of Ks appearing in the normative inequality. ◁▷

(8) Let X be a vector space over $\mathbb{F}$ and let $\|\cdot\|$ be a norm on X. Assume further that $X' := B(X, \mathbb{F})$ is the (*normed*) *dual* of X, i.e. the space of bounded functionals fs with the *dual norm*

$$\|f\| = \sup\{|f(x)| : \ \|x\| \leq 1\} = \sup_{x \in X} \frac{|f(x)|}{\|x\|}.$$

Consider $X'' := (X')' := B(X', \mathbb{F})$, the *second dual* of X. Given $x \in X$ and $f \in X'$, put $x'' := \iota(x) : f \mapsto f(x)$. Undoubtedly, $\iota(x) \in (X')^{\#} = \mathscr{L}(X', \mathbb{F})$. In addition,

$$\|x''\| = \|\iota(x)\| = \sup\{|\iota(x)(f)| : \ \|f\|_{X'} \leq 1\}$$
$$= \sup\{|f(x)| : \ (\forall x \in X)\, |f(x)| \leq \|x\|_X\} = \sup\{|f(x)| : \ f \in |\partial|(\|\cdot\|_X)\} = \|x\|_X .$$

The final equality follows for instance from Theorem 3.6.5 and Lemma 3.7.9. Thus, $\iota(x) \in X''$ for every x in X. It is plain that the operator $\iota : X \to X''$, defined as $\iota : x \mapsto \iota(x)$, is linear and bounded; moreover, ι is a monomorphism and $\|\iota x\| = \|x\|$ for all $x \in X$. The operator ι is referred to as the *canonical embedding* of X into the second dual or more suggestively the *double prime mapping*. As a rule, it is convenient to treat x and $x'' := \iota x$ as the same element; i.e., to consider X as a subspace of X''. A normed space X is *reflexive* if X and X'' coincide (under the indicated embedding!). Reflexive spaces possess many advantages. Evidently, not all spaces are reflexive. Unfortunately, such is $C([0, 1], \mathbb{F})$ which is *irreflexive* (the term "nonreflexive" is also is common parlance). ◁▷

5.1.11. REMARK. The constructions, carried out in 5.1.10 (8), show some symmetry or duality between X and X'. In this regard, the notation $(x,\ f) := \langle x \mid f \rangle := f(x)$ symbolizes the action of $x \in X$ on $f \in X'$ (or the action of f on x). To achieve and ensure the utmost conformity, it is a common practice to denote elements of X' by symbols like x'; for instance, $\langle x \mid x' \rangle = (x,\ x') = x'(x)$.

5.2. The Uniformity and Topology of a Multinormed Space

5.2.1. *Let $(X,\ p)$ be a seminormed space. Given $x_1,\ x_2 \in X$, put $d_p(x_1,\ x_2) := p(x_1 - x_2)$. Then*

(1) $d_p(X^2) \subset \mathbb{R}_+$ *and* $\{d \le 0\} \supset I_X$;

(2) $\{d_p \le t\} = \{d_p \le t\}^{-1}$ *and* $\{d_p \le t\} = t\{d_p \le 1\}$ $(t \in \mathbb{R}_+ \setminus 0)$;

(3) $\{d_p \le t_1\} \circ \{d_p \le t_2\} \subset \{d_p \le t_1 + t_2\}$ $(t_1,\ t_2 \in \mathbb{R}_+)$;

(4) $\{d_p \le t_1\} \cap \{d_p \le t_2\} \supset \{d_p \le t_1 \wedge t_2\}$ $(t_1,\ t_2 \in \mathbb{R}_+)$;

(5) *p is a norm $\Leftrightarrow$ d_p is a metric.* ◁▷

5.2.2. DEFINITION. The *uniformity* of a seminormed space $(X,\ p)$ is the filter $\mathscr{U}_p := \operatorname{fil}\ \{\{d_p \le t\} :\ t \in \mathbb{R}_+ \setminus 0\}$.

5.2.3. *If $\mathscr{U}_p$ is the uniformity of a seminormed space $(X,\ p)$ then*

(1) $\mathscr{U}_p \subset \operatorname{fil}\ \{I_X\}$;

(2) $U \in \mathscr{U}_p \Rightarrow U^{-1} \in \mathscr{U}_p$;

(3) $(\forall U \in \mathscr{U}_p)\ (\exists\ V \in \mathscr{U}_p)\ V \circ V \subset U$. ◁▷

5.2.4. DEFINITION. Let $(X,\ \mathfrak{M})$ be a multinormed space. The filter $\mathscr{U} := \sup\{\mathscr{U}_p\ :\ p \in \mathfrak{M}\}$ is called the *uniformity* of X (the other designations are $\mathscr{U}_{\mathfrak{M}}$, $\mathscr{U}_X$, etc.). (By virtue of 5.2.3 (1) and 1.3.13, the definition is sound.)

5.2.5. *If $(X,\ \mathfrak{M})$ is a multinormed space and $\mathscr{U}$ is the uniformity of X then*

(1) $\mathscr{U} \subset \operatorname{fil}\ \{I_X\}$;

(2) $U \in \mathscr{U} \Rightarrow U^{-1} \in \mathscr{U}$;

(3) $(\forall U \in \mathscr{U})\ (\exists V \in \mathscr{U})\ V \circ V \subset U$.

◁ Examine (3). Given $U \in \mathscr{U}$, by 1.2.18 and 1.3.8 there are seminorms $p_1, \dots, p_n \in \mathfrak{M}$ such that $U = \mathscr{U}_{\{p_1, \dots, p_n\}} = \mathscr{U}_{p_1} \vee \dots \vee \mathscr{U}_{p_n}$. Using 1.3.13, find sets $U_k \in \mathscr{U}_{p_k}$ satisfying $U \supset U_1 \cap \dots \cap U_n$. Applying 5.2.3 (3), choose $V_k \in \mathscr{U}_{p_k}$ so as to have $V_k \circ V_k \subset U_k$. It is clear that

$$(V_1 \cap \dots \cap V_n) \circ (V_1 \cap \dots \cap V_n) \subset V_1 \circ V_1 \cap \dots \cap V_n \circ V_n \subset U_1 \cap \dots \cap U_n.$$

Moreover, $V_1 \cap \dots \cap V_n \in \mathscr{U}_{p_1} \vee \dots \vee \mathscr{U}_{p_n} \subset \mathscr{U}$. ▷

5.2.6. *A multinorm $\mathfrak{M}$ on X is separated if and only if so is the uniformity $\mathscr{U}_{\mathfrak{M}}$; i.e., $\cap\{V :\ V \in \mathscr{U}_{\mathfrak{M}}\} = I_X$.*

◁ $\Rightarrow$: Let $(x,\ y) \notin I_X$; i.e., $x \neq y$. Then $p(x-y) > 0$ for some seminorm p in $\mathfrak{M}$. Hence, $(x,\ y) \notin \{d_p \leq {}^1/_2\, p(x-y)\}$. But the last set is included in $\mathscr{U}_p$, and thus in $\mathscr{U}_{\mathfrak{M}}$. Consequently, $X^2 \setminus I_X \subset X^2 \setminus \cap\{V :\ V \in \mathscr{U}_{\mathfrak{M}}\}$. Furthermore, $I_X \subset \cap\{V :\ V \in \mathscr{U}_{\mathfrak{M}}\}$.

$\Leftarrow$: If $p(x) = 0$ for all $p \in \mathfrak{M}$ then $(x,\ 0) \in V$ for every V in $\mathscr{U}_{\mathfrak{M}}$. Hence, $(x,\ 0) \in I_X$ by hypothesis. Therefore, $x = 0$. ▷

5.2.7. *Given a space X with uniformity $\mathscr{U}_X$, define*

$$\tau(x) := \{U(x) :\ U \in \mathscr{U}_X\} \quad (x \in X).$$

Then $\tau(x)$ is a filter for every $x \in X$. Moreover,

(1) $\tau(x) \subset \mathrm{fil}\ \{x\}$;

(2) $(\forall U \in \tau(x))\ (\exists V \in \tau(x)\ \&\ V \subset U)\ (\forall y \in V)\ V \in \tau(y)$.

◁ All is evident (cf. 4.1.8). ▷

5.2.8. Definition. The mapping $\tau\ :\ x \mapsto \tau(x)$ is called the ***topology*** of a multinormed space $(X,\ \mathfrak{M})$; a member of $\tau(x)$ is a ***neighborhood*** about x. The designation for the topology can be more detailed: τ_X, $\tau_{\mathfrak{M}}$, $\tau(\mathscr{U}_{\mathfrak{M}})$, etc.

5.2.9. *The following presentation holds:*

$$\tau_X(x) = \sup\{\tau_p(x) :\ p \in \mathfrak{M}_X\}$$

for all $x \in X$. ◁▷

5.2.10. *If X is a multinormed space then*

$$U \in \tau(x) \Leftrightarrow U - x \in \tau_X(0)$$

for all $x \in X$.

◁ By 5.2.9 and 1.3.13, it suffices to consider a seminormed space $(X,\ p)$. In this case for every $\varepsilon > 0$ the equality holds: $\{d_p \leq \varepsilon\}(x) = \varepsilon B_p + x$, where $B_p := \{p \leq 1\}$. Indeed, if $p(y-x) \leq \varepsilon$ then $y = \varepsilon(\varepsilon^{-1}(y-x)) + x$ and $\varepsilon^{-1}(y-x) \in B_p$. In turn, if $y \in \varepsilon B_p + x$, then $p(y-x) = \inf\{t > 0 :\ y - x \in tB_p\} \leq \varepsilon$. ▷

5.2.11. Remark. The proof of 5.2.10 demonstrates that in a seminormed space $(X,\ p)$ a key role is performed by the ball with radius 1 and centered at zero. The ball bears the name of the ***unit ball*** of X and is denoted by B_p, B_X, etc.

5.2.12. *A multinorm $\mathfrak{M}_X$ is separated if and only if so is the topology τ_X; i.e., given distinct x_1 and x_2 in X, there are neighborhoods U_1 in $\tau_X(x_1)$ and U_2 in $\tau_X(x_2)$ such that $U_1 \cap U_2 = \varnothing$.*

◁ ⇒: Let $x_1 \neq x_2$ and let $\varepsilon := p(x_1 - x_2) > 0$ for $p \in \mathfrak{M}_X$. Put $U_1 := x_1 + {}^{\varepsilon}/_3 B_p$ and $U_2 := x_2 + {}^{\varepsilon}/_3 B_p$. By 5.2.10, $U_k \in \tau_X(x_k)$. Verify that $U_1 \cap U_2 = \varnothing$. Indeed, if $y \in U_1 \cap U_2$ then $p(x_1 - y) \le {}^{\varepsilon}/_3$ and $p(x_2 - y) \le {}^{\varepsilon}/_3$. Therefore, $p(x_1 - x_2) \le {}^2/_3\, \varepsilon < \varepsilon = p(x_1 - x_2)$, which is impossible.

⇐: If $(x_1,\ x_2) \in \cap\{V : V \in \mathscr{U}_X\}$ then $x_2 \in \cap\{V(x_1) : V \in \mathscr{U}_X\}$. Thus, $x_1 = x_2$ and, consequently, $\mathfrak{M}_X$ is separated by 5.2.6. ▷

5.2.13. REMARK. The presence of a uniformity and the corresponding topology in a multinormed space readily justifies using uniform and topological concepts such as uniform continuity, completeness, continuity, openness, closure, etc.

5.2.14. *Let $(X,\ p)$ be a seminormed space and let X_0 be a subspace of X. The quotient space $(X/X_0,\ p_{X/X_0})$ is separated if and only if X_0 is closed.*

◁ ⇒: If $x \notin X_0$ then $\varphi(x) \neq 0$ where, as usual, $\varphi : X \to X/X_0$ is the coset mapping. By hypothesis, $0 \neq \varepsilon := p_{X/X_0}(\varphi(x)) = p_{\varphi^{-1}}(\varphi(x)) = \inf\{p(x + x_0) : x_0 \in X_0\}$. Hence, the ball $x + {}^{\varepsilon}/_2 B_p$ does not meet X_0 and x is an exterior point of X_0. Thus, X_0 is closed.

⇐: Suppose that $\overline{x}$ is a nonzero point of X/X_0 and $\overline{x} = \varphi(x)$ for some x in X. If $p_{X/X_0}(\overline{x}) = 0$ then $0 = \inf\{p(x - x_0) : x_0 \in X_0\}$. In other words, there is a sequence (x_n) in X_0 converging to x. Consequently, by 4.1.19, $x \in X_0$ and $\overline{x} = 0$, a contradiction. ▷

5.2.15. *The closure of a Γ-set is a Γ-set.*

◁ Given $U \in (\Gamma)$, suppose that $U \neq \varnothing$ (otherwise there is nothing worthy of proving). By 4.1.9, for $x,\ y \in \mathrm{cl}\, U$ there are nets (x_γ) and (y_γ) in U such that $x_\gamma \to x$ and $y_\gamma \to y$. If $(\alpha,\ \beta) \in \Gamma$ then $\alpha x_\gamma + \beta y_\gamma \in U$. Appealing to 4.1.19 again, infer $\alpha x + \beta y = \lim(\alpha x_\gamma + \beta y_\gamma) \in \mathrm{cl}\, U$. ▷

5.3. Comparison Between Topologies

5.3.1. DEFINITION. If $\mathfrak{M}$ and $\mathfrak{N}$ are two multinorms on a vector space then $\mathfrak{M}$ is said to be *finer* or *stronger* than $\mathfrak{N}$ (in symbols, $\mathfrak{M} \succ \mathfrak{N}$) if $\mathscr{U}_{\mathfrak{M}} \supset \mathscr{U}_{\mathfrak{N}}$. If $\mathfrak{M} \succ \mathfrak{N}$ and $\mathfrak{N} \succ \mathfrak{M}$ simultaneously, then $\mathfrak{M}$ and $\mathfrak{N}$ are said to be *equivalent* (in symbols, $\mathfrak{M} \sim \mathfrak{N}$).

5.3.2. ***Multinorm Comparison Theorem.*** *For multinorms $\mathfrak{M}$ and $\mathfrak{N}$ on a vector space X the following statements are equivalent:*

(1) $\mathfrak{M} \succ \mathfrak{N}$;

(2) *the inclusion $\tau_{\mathfrak{M}}(x) \supset \tau_{\mathfrak{N}}(x)$ holds for all $x \in X$;*

(3) $\tau_{\mathfrak{M}}(0) \supset \tau_{\mathfrak{N}}(0)$;

(4) $(\forall q \in \mathfrak{N})\ (\exists p_1, \dots, p_n \in \mathfrak{M})\ (\exists \varepsilon_1, \dots, \varepsilon_n \in \mathbb{R}_+ \setminus 0)$ $B_q \supset \varepsilon_1 B_{p_1} \cap \dots \cap \varepsilon_n B_{p_n}$;

(5) $(\forall q \in \mathfrak{N})\ (\exists p_1, \dots, p_n \in \mathfrak{M})\ (\exists t > 0)\ q \le t(p_1 \vee \dots \vee p_n)$ (*with respect to the order of the K-space $\mathbb{R}^X$*).

◁ The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are evident.

(4) ⇒ (5): Applying the Gauge Theorem (cf. 5.1.2), infer that

$$q \le p_{B_{p_1/\varepsilon_1}} \vee \dots \vee p_{B_{p_n/\varepsilon_n}} = \left({}^1/_{\varepsilon_1} p_1\right) \vee \dots \vee \left({}^1/_{\varepsilon_n} p_n\right)$$
$$\le \left({}^1/_{\varepsilon_1}\right) \vee \dots \vee \left({}^1/_{\varepsilon_n}\right) p_1 \vee \dots \vee p_n.$$

(5) ⇒ (1): It is sufficient to check that $\mathfrak{M} \succ \{q\}$ for a seminorm q in $\mathfrak{N}$. If $V \in \mathscr{U}_q$ then $V \supset \{d_q \le \varepsilon\}$ for some $\varepsilon > 0$. By hypothesis

$$\{d_q \le \varepsilon\} \supset \{d_{p_1} \le {}^\varepsilon/_t\} \cap \dots \cap \{d_{p_n} \le {}^\varepsilon/_t\}$$

with suitable $p_1, \dots, p_n \in \mathfrak{M}$ and $t > 0$. The right side of this inclusion is an element of $\mathscr{U}_{p_1} \vee \dots \vee \mathscr{U}_{p_n} = \mathscr{U}_{\{p_1,\dots,p_n\}} \subset \mathscr{U}_{\mathfrak{M}}$. Hence, V is also a member of $\mathscr{U}_{\mathfrak{M}}$. ▷

5.3.3. DEFINITION. Let $p,\ q : X \to \mathbb{R}$ be two seminorms on X. Say that p is *finer* or *stronger* than q and write $p \succ q$ whenever $\{p\} \succ \{q\}$. The *equivalence* of seminorms $p \sim q$ is understood in a routine fashion.

5.3.4. $p \succ q \Leftrightarrow (\exists t > 0)\ q \le tp \Leftrightarrow (\exists t \ge 0)\ B_q \supset tB_p$;

$$p \sim q \Leftrightarrow (\exists t_1,\ t_2 > 0)\ t_2 p \le q \le t_1 p \Leftrightarrow (\exists t_1,\ t_2 > 0)\ t_1 B_p \subset B_q \subset t_2 B_p.$$

◁ Everything follows from 5.3.2 and 5.1.2. ▷

5.3.5. *Riesz Theorem.* *If* $p,\ q : \mathbb{F}^N \to \mathbb{R}$ *are seminorms on the finite-dimensional space* $\mathbb{F}^N$ *then* $p \succ q \Leftrightarrow \ker p \subset \ker q$. ◁▷

5.3.6. *Corollary.* *All norms in finite dimensions are equivalent.* ◁▷

5.3.7. *Let* $(X,\ \mathfrak{M})$ *and* $(Y,\ \mathfrak{N})$ *be multinormed spaces, and let* T *be a linear operator, a member of* $\mathscr{L}(X,\ Y)$. *The following statements are equivalent:*

(1) $\mathfrak{N} \circ T \prec \mathfrak{M}$;

(2) $T^\times(\mathscr{U}_X) \supset \mathscr{U}_Y$ *and* $T^{\times -1}(\mathscr{U}_Y) \subset \mathscr{U}_X$;

(3) $x \in X \Rightarrow T(\tau_X(x)) \supset \tau_Y(Tx)$;

(4) $T(\tau_X(0)) \supset \tau_Y(0)$ *and* $\tau_X(0) \supset T^{-1}(\tau_Y(0))$;

(5) $(\forall q \in \mathfrak{N})\ (\exists p_1, \dots, p_n \in \mathfrak{M})\ q \circ T \prec p_1 \vee \dots \vee p_n$. ◁▷

5.3.8. *Let* $(X,\ \|\cdot\|_X)$ *and* $(Y,\ \|\cdot\|_Y)$ *be normed spaces and let* T *be a linear operator, a member of* $\mathscr{L}(X,\ Y)$. *The following statements are equivalent:*

(1) *T is bounded (that is,* $T \in B(X,\ Y)$*);*

(2) $\|\cdot\|_X \succ \|\cdot\|_Y \circ T$;

(3) *T is uniformly continuous;*

(4) *T is continuous;*

(5) *T is continuous at zero.*

◁ Each of the claims is a particular case of 5.3.7. ▷

5.3.9. REMARK. The message of 5.3.7 shows that it is sometimes convenient to substitute for $\mathfrak{M}$ a multinorm equivalent to $\mathfrak{M}$ but filtered upward (with respect to the relation $\geq$ or $\succ$). For example, we may take the multinorm $\overline{\mathfrak{M}} := \{\sup \mathfrak{M}_0 : \mathfrak{M}_0$ is a nonempty finite subset of $\mathfrak{M}\}$. Observe that unfiltered multinorms should be treated with due precaution.

5.3.10. COUNTEREXAMPLE. Let $X := \mathbb{F}^{\Xi}$; and let X_0 comprise all *constant functions*; i.e., $X_0 := \mathbb{F}\mathbf{1}$, where $\mathbf{1} : \xi \mapsto 1$ $(\xi \in \Xi)$. Set $\mathfrak{M} := \{p_\xi : \xi \in \Xi\}$, with $p_\xi(x) := |x(\xi)|$ $(x \in \mathbb{F}^{\Xi})$. It is clear that $\mathfrak{M}$ is a multinorm on X. Now let $\varphi : X \to X/X_0$ stand for the coset mapping. Undoubtedly, $\mathfrak{M}_{\varphi^{-1}}$ consists of the sole element, zero. At the same time $\overline{\mathfrak{M}}_{\varphi^{-1}}$ is separated.

5.3.11. DEFINITION. Let $(X, \mathfrak{M})$ be a multinormed space and let X_0 be a subspace of X. The multinorm $\overline{\mathfrak{M}}_{\varphi^{-1}}$, with $\varphi : X \to X/X_0$ the coset mapping, is referred to as the *quotient multinorm* and is denoted by $\mathfrak{M}_{X/X_0}$. The space $(X/X_0, \mathfrak{M}_{X/X_0})$ is called the *quotient space* of X by X_0.

5.3.12. *The quotient space X/X_0 is separated if and only if X_0 is closed.* ◁▷

5.4. Metrizable and Normable Spaces

5.4.1. DEFINITION. A multinormed space $(X, \mathfrak{M})$ is *metrizable* if there is a metric d on X such that $\mathscr{U}_{\mathfrak{M}} = \mathscr{U}_d$. Say that X is *normable* if there is a norm on X equivalent to the initial multinorm $\mathfrak{M}$. Say that X is *countably normable* if there is a countable multinorm on X equivalent to the initial.

5.4.2. ***Metrizability Criterion.*** *A multinormed space is metrizable if and only if it is countably normable and separated.*

◁ $\Rightarrow$: Let $\mathscr{U}_{\mathfrak{M}} = \mathscr{U}_d$. Passing if necessary to the multinorm $\overline{\mathfrak{M}}$, assume that for every n in $\mathbb{N}$ it is possible to find a seminorm p_n in $\mathfrak{M}$ and $t_n > 0$ such that $\{d \leq {}^1/_n\} \supset \{d_{p_n} \leq t_n\}$. Put $\mathfrak{N} := \{p_n : n \in \mathbb{N}\}$. Clearly, $\mathfrak{M} \succ \mathfrak{N}$. If $V \in \mathscr{U}_{\mathfrak{M}}$ then $V \supset \{d \leq {}^1/_n\}$ for some $n \in \mathbb{N}$ by the definition of metric uniformity. Hence, by construction, $V \in \mathscr{U}_{p_n} \subset \mathscr{U}_{\mathfrak{M}}$, i.e., $\mathfrak{M} \prec \mathfrak{N}$. Thus, $\mathfrak{M} \sim \mathfrak{N}$. The uniformity $\mathscr{U}_d$ is separated, as indicated in 4.1.7. Applying 5.2.6, observe that $\mathscr{U}_{\mathfrak{M}}$ and $\mathscr{U}_{\mathfrak{N}}$ are both separated.

$\Leftarrow$: Passing if necessary to an equivalent multinorm, suppose that X, the space in question, is countably normed and separated; that is, $\mathfrak{M} := \{p_n : n \in \mathbb{N}\}$ and $\mathfrak{M}$ is a Hausdorff multinorm on X. Given $x_1, x_2 \in X$, define

$$d(x_1, x_2) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(x_1 - x_2)}{1 + p_k(x_1 - x_2)}$$

(the series on the right side of the above formula is dominated by the convergent series $\sum_{k=1}^{\infty} {}^1/_{2^k}$, and so d is defined soundly).

Check that d is a metric. It suffices to validate the triangle inequality. For a start, put $\alpha(t) := t(1+t)^{-1}$ $(t \in \mathbb{R}_+)$. It is evident that $\alpha'(t) = (1+t)^{-2} > 0$. Therefore, α increases. Furthermore, α is subadditive:

$$\alpha(t_1 + t_2) = (t_1 + t_2)(1 + t_1 + t_2)^{-1}$$

$$= t_1(1 + t_1 + t_2)^{-1} + t_2(1 + t_1 + t_2)^{-1} \leq t_1(1 + t_1)^{-1} + t_2(1 + t_2)^{-1}$$

$$= \alpha(t_1) + \alpha(t_2).$$

Thus, given $x,\ y,\ z \in X$, infer that

$$d(x,\ y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \alpha(p_k(x-y)) \leq \sum_{k=1}^{\infty} \frac{1}{2^k} \alpha(p_k(x-z) + p_k(z-y))$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{2^k} (\alpha(p_k(x-z)) + \alpha(p_k(z-y))) = d(x,\ z) + d(z,\ y).$$

It remains to established that $\mathscr{U}_d$ and $\mathscr{U}_{\mathfrak{M}}$ coincide.

First, show that $\mathscr{U}_d \subset \mathscr{U}_{\mathfrak{M}}$. Take a cylinder, say, $\{d \leq \varepsilon\}$; and let $(x,\ y) \in \{d_{p_1} \leq t\} \cap \ldots \cap \{d_{p_n} \leq t\}$. Since α is an increasing function, deduce that

$$d(x,\ y) = \sum_{k=1}^{n} \frac{1}{2^k} \frac{p_k(x-y)}{1 + p_k(x-y)} + \sum_{k=n+1}^{\infty} \frac{1}{2^k} \frac{p_k(x-y)}{1 + p_k(x-y)}$$

$$\leq \frac{t}{1+t} \sum_{k=1}^{n} \frac{1}{2^k} + \sum_{k=n+1}^{\infty} \frac{1}{2^k} \leq \frac{t}{1+t} + \frac{1}{2^n}.$$

Since $t(1+t)^{-1} + 2^{-n}$ tends to zero as $n \to \infty$ and $t \to 0$, for appropriate t and n observe that $(x,\ y) \in \{d \leq \varepsilon\}$. Hence, $\{d \leq \varepsilon\} \in \mathscr{U}_{\mathfrak{M}}$, which is required.

Now establish that $\mathscr{U}_{\mathfrak{M}} \subset \mathscr{U}_d$. To demonstrate the inclusion, given $p_n \in \mathfrak{M}$ and $t > 0$, find $\varepsilon > 0$ such that $\{d_{p_n} \leq t\} \supset \{d \leq \varepsilon\}$. For this purpose, take

$$\varepsilon := \frac{1}{2^n} \frac{t}{1+t},$$

which suffices, since from the relations

$$\frac{1}{2^n} \frac{p_n(x-y)}{1 + p_n(x-y)} \leq d(x,\ y) \leq \varepsilon = \frac{1}{2^n} \frac{t}{1+t}$$

holding for all $x,\ y \in X$ it follows that $p_n(x-y) \leq t$. ▷

5.4.3. DEFINITION. A subset V of a multinormed space $(X, \mathfrak{M})$ is a ***bounded set*** in X if $\sup p(V) < +\infty$ for all $p \in \mathfrak{M}$ which means that the set $p(V)$ is bounded above in $\mathbb{R}$ for every seminorm p in $\mathfrak{M}$.

5.4.4. *For a set V in $(X, \mathfrak{M})$ the following statements are equivalent:*

(1) *V is bounded;*

(2) *for every sequence $(x_n)_{n\in\mathbb{N}}$ in V and every sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ such that $\lambda_n \to 0$, the sequence $(\lambda_n x_n)$ vanishes: $\lambda_n x_n \to 0$ (i.e., $p(\lambda_n x_n) \to 0$ for each seminorm p in $\mathfrak{M}$);*

(3) *every neighborhood of zero absorbs V.*

◁ (1) ⇒ (2): $p(\lambda_n x_n) \le |\lambda_n| p(x_n) \le |\lambda_n| \sup p(V) \to 0$.

(2) ⇒ (3): Let $U \in \tau_X(0)$ and suppose that U fails to absorb V. From Definition 3.4.9, it follows that $(\forall n \in \mathbb{N})\ (\exists x_n \in V)\ x_n \notin nU$. Thus, ${}^1/_n\, x_n \notin U$ for all $n \in \mathbb{N}$; i.e., $({}^1/_n\, x_n)$ does not converge to zero.

(3) ⇒ (1): Given $p \in \mathfrak{M}$, find $n \in \mathbb{N}$ satisfying $V \subset nB_p$. Obviously, $\sup p(V) \le \sup p(nB_p) = n < +\infty$. ▷

5.4.5. *Kolmogorov Normability Criterion.* *A multinormed space is normable if and only if it is separated and has a bounded neighborhood about zero.*

◁ ⇒: It is clear.

⇐: Let V be a bounded neighborhood of zero. Without loss of generality, it may be assumed that $V = B_p$ for some seminorm p in the given multinorm $\mathfrak{M}$. Undoubtedly, $p \prec \mathfrak{M}$. Now if $U \in \tau_{\mathfrak{M}}(0)$ then $nU \supset V$ for some $n \in \mathbb{N}$. Hence, $U \in \tau_p(0)$. Using Theorem 5.3.2, observe that $p \succ \mathfrak{M}$. Thus, $p \sim \mathfrak{M}$; and, therefore, p is also separated by 5.2.12. This means that p is a norm. ▷

5.4.6. REMARK. Incidentally, 5.4.5 shows that the presence of a bounded neighborhood of zero in a multinormed space X amounts to the "seminormability" of X.

5.5. Banach Spaces

5.5.1. DEFINITION. A ***Banach space*** is a complete normed space.

5.5.2. REMARK. The concept of ***Fréchet space***, complete metrizable multinormed space, serves as a natural abstraction of Banach space. It may be shown that the class of Fréchet spaces is the least among those containing all Banach spaces and closed under the taking of countable products. ◁▷

5.5.3. *A normed space X is a Banach space if and only if every norm convergent (= absolutely convergent) series in X converges.*

◁ ⇒: Let $\sum_{n=1}^{\infty} \|x_n\| < +\infty$ for some (countable) sequence (x_n). Then the sequence of partial sums $s_n := x_1 + \dots + x_n$ is fundamental because

$$\|s_m - s_k\| = \left\| \sum_{n=k+1}^{m} x_n \right\| \le \sum_{n=k+1}^{m} \|x_n\| \to 0$$

for $m > k$.

⇐: Given a fundamental sequence (x_n), choose an increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that $\|x_n - x_m\| \le 2^{-k}$ as $n,\ m \ge n_k$. Then the series $x_{n_1} + (x_{n_2} - x_{n_1}) + (x_{n_3} - x_{n_2}) + \cdots$ converges in norm to some x; i.e., $x_{n_k} \to x$. Observe simultaneously that $x_n \to x$. ▷

5.5.4. *If X is a Banach space and X_0 is a closed subspace of X then the quotient space X/X_0 is also a Banach space.*

◁ Let $\varphi : X \to \mathscr{X} := X/X_0$ be the coset mapping. Undoubtedly, for every $\overline{x} \in \mathscr{X}$ there is some $x \in \varphi^{-1}(\overline{x})$ such that $2\|\overline{x}\| \ge \|x\| \ge \|\overline{x}\|$. Hence, given $\sum_{n=1}^{\infty} \overline{x}_n$, a norm convergent series in $\mathscr{X}$, it is possible to choose $x_n \in \varphi^{-1}(\overline{x}_n)$ so that the norm series $\sum_{n=1}^{\infty} \|x_n\|$ be convergent. According to 5.5.3, the sum $x := \sum_{n=1}^{\infty} x_n$ is available. If $\overline{x} := \varphi(x)$ then

$$\left\| \overline{x} - \sum_{k=1}^{n} \overline{x}_k \right\| \le \left\| x - \sum_{k=1}^{n} x_k \right\| \to 0.$$

Appealing to 5.5.3 again, conclude that $\mathscr{X}$ is a Banach space. ▷

5.5.5. REMARK. The claim of 5.5.3 may be naturally translated to seminormed spaces. In particular, if $(X,\ p)$ is a complete seminormed space then the quotient space $X/\ker p$ is a Banach space. ◁▷

5.5.6. Theorem. *If X and Y are normed spaces and $X \ne 0$ then $B(X,\ Y)$, the space of bounded operators, is a Banach space if and only if so is Y.*

◁ ⇐: Consider a Cauchy sequence (T_n) in $B(X,\ Y)$. By the normative inequality, $\|T_m x - T_k x\| \le \|T_m - T_k\|\ \|x\| \to 0$ for all $x \in X$; i.e., $(T_n x)$ is fundamental in Y. Thus, there is a limit $Tx := \lim T_n x$. Plainly, the so-defined operator T is linear. By virtue of the estimate $|\|T_m\| - \|T_k\|| \le \|T_m - T_k\|$ the sequence $(\|T_n\|)$ is fundamental in $\mathbb{R}$ and, hence, bounded; that is, $\sup_n \|T_n\| < +\infty$. Therefore, passing to the limit in $\|T_n x\| \le \sup_n \|T_n\|\ \|x\|$, obtain $\|T\| < +\infty$. It remains to check that $\|T_n - T\| \to 0$. Given $\varepsilon > 0$, choose a number n_0 such that $\|T_m - T_n\| \le {}^{\varepsilon}/_2$ as $m,\ n \ge n_0$. Further, for $x \in B_X$ find $m \ge n_0$ satisfying $\|T_m x - Tx\| \le {}^{\varepsilon}/_2$. Then $\|T_n x - Tx\| \le \|T_n x - T_m x\| + \|T_m x - Tx\| \le \|T_n - T_m\| + \|T_m x - Tx\| \le \varepsilon$ as $n \ge n_0$. In other words, $\|T_n - T\| = \sup\{\|T_n x - Tx\| :\ x \in B_X\} \le \varepsilon$ for all sufficiently large n.

$\Rightarrow$: Let (y_n) be a Cauchy sequence in Y. By hypothesis, there is a *norm-one element* x in X; i.e., x has norm 1: $\|x\| = 1$. Applying 3.5.6 and 3.5.2 (1), find an element $x' \in |\partial|(\|\cdot\|)$ satisfying $(x, x') = \|x\| = 1$. Obviously, the rank-one operator (with range of dimension 1) $T_n := x' \otimes y_n : x \mapsto (x, x')y_n$ belongs to $B(X, Y)$, since $\|T_n\| = \|x'\| \, \|y_n\|$. Hence, $\|T_m - T_k\| = \|x' \otimes (y_m - y_k)\| = \|x'\| \, \|y_m - y_k\| = \|y_m - y_k\|$, i.e., (T_n) is fundamental in $B(X, Y)$. Assign $T := \lim T_n$. Then $\|Tx - T_n x\| = \|Tx - y_n\| \le \|T - T_n\| \, \|x\| \to 0$. In other words, Tx is the limit of (y_n) in Y. ▷

5.5.7. Corollary. *The dual of a normed space (furnished with the dual norm) is a Banach space.* ◁▷

5.5.8. Corollary. *Let X be a normed space; and let $\iota : X \to X''$, the double prime mapping, be the canonical embedding of X into the second dual X''. Then the closure* cl $\iota(X)$ *is a completion of X.*

◁ By virtue of 5.5.7, X'' is a Banach space. By 5.1.10 (8), ι is an isometry from X into X''. Appealing to 4.5.16 ends the proof. ▷

5.5.9. EXAMPLES.

(1) "Abstract" examples: a basic field, a closed subspace of a Banach space, the product of Banach spaces, and 5.5.4–5.5.8.

(2) Let $\mathscr{E}$ be a nonempty set. Given $x \in \mathbb{F}^{\mathscr{E}}$, put $\|x\|_\infty := \sup |x(\mathscr{E})|$. The space $l_\infty(\mathscr{E}) := l_\infty(\mathscr{E}, \mathbb{F}) := \operatorname{dom} \|\cdot\|_\infty$ is called the *space of bounded functions* on $\mathscr{E}$. The designations $B(\mathscr{E})$ and $B(\mathscr{E}, \mathbb{F})$ are also used. For $\mathscr{E} := \mathbb{N}$, it is customary to put $m := l_\infty := l_\infty(\mathscr{E})$.

(3) Let a set $\mathscr{E}$ be *infinite*, i.e. not finite, and let $\mathscr{F}$ stand for a filter on $\mathscr{E}$. By definition, $x \in c(\mathscr{E}, \mathscr{F}) \Leftrightarrow (x \in l_\infty(\mathscr{E})$ and $x(\mathscr{F})$ is a Cauchy filter on $\mathbb{F})$. In the case $\mathscr{E} := \mathbb{N}$ and $\mathscr{F}$ is the *finite complement filter* (comprising all cofinite sets each of which is the complement of a finite subset) of $\mathbb{N}$, the notation $c := c(\mathscr{E}, \mathscr{F})$ is employed, and c is called the *space of convergent sequences*. In $c(\mathscr{E}, \mathscr{F})$ the subspace $c_0(\mathscr{E}, \mathscr{F}) := \{x \in c(\mathscr{E}, \mathscr{F}) : x(\mathscr{F}) \to 0\}$ is distinguished. If $\mathscr{F}$ is the finite complement filter then the shorter notation $c_0(\mathscr{E})$ is used and we speak of the *space of functions vanishing at infinity*. Given $\mathscr{E} := \mathbb{N}$, write $c_0 := c_0(\mathscr{E})$. The space c_0 is referred to as the *space of vanishing sequences*. It is worth keeping in mind that each of these spaces without further specification is endowed with the norm taken from the corresponding space $l_\infty(\mathscr{E}, \mathscr{F})$.

(4) Let $S := (\mathscr{E}, X, \int)$ be a *system with integration*. This means that X is a vector sublattice of $\mathbb{R}^{\mathscr{E}}$, with the lattice operations in X coincident with those in $\mathbb{R}^{\mathscr{E}}$, and $\int : X \to \mathbb{R}$ is a *(pre)integral*; i.e., $\int \in X_+^{\#}$ and $\int x_n \downarrow 0$ whenever $x_n \in X$ and $x_n(e) \downarrow 0$ for $e \in \mathscr{E}$. Moreover, let $f \in \mathbb{F}^{\mathscr{E}}$ be a measurable mapping (with respect to S) (as usual, we may speak of almost everywhere finite and almost everywhere defined measurable functions).

Denote $\mathscr{N}_p(f) := \left(\int |f|^p\right)^{1/p}$ for $p \ge 1$, where $\int$ is the corresponding Lebesgue extension of the initial integral $\int$. (The traditional liberty is taken of using the same symbol for the original and its successor.)

An element of dom $\mathscr{N}_1$ is an *integrable* or *summable function*. The integrability of $f \in \mathbb{F}^{\mathscr{E}}$ is equivalent to the integrability of its *real part* $\operatorname{Re} f$ and *imaginary part* $\operatorname{Im} f$, both members of $\mathbb{R}^{\mathscr{E}}$. For the sake of completeness, recall the definition

$$N(g) := \inf\left\{\sup \int x_n : (x_n) \subset X,\ x_n \le x_{n+1},\ (\forall\, e \in \mathscr{E})\ |g(e)| = \lim_n\ x_n(e)\right\}$$

for an arbitrary g in $\mathbb{F}^{\mathscr{E}}$. If $\mathbb{F} = \mathbb{R}$ then dom $\mathscr{N}_1$ obviously presents the closure of X in the normed space (dom N, N).

The Hölder inequality is valid:

$$\mathscr{N}_1(fg) \le \mathscr{N}_p(f)\mathscr{N}_{p'}(g),$$

with p' the conjugate exponent of p, i.e. ${}^1/_p + {}^1/_{p'} = 1$ for $p > 1$.

◁ This is a consequence of the *Young inequality*, $xy - {}^{x^p}/_p \le {}^{y^{p'}}/_{p'}$ for all $x,\ y \in \mathbb{R}_+$, applied to $|f|/\mathscr{N}_p(f)$ and $|g|/\mathscr{N}_{p'}(g)$ when $\mathscr{N}_p(f)$ and $\mathscr{N}_{p'}(g)$ are both nonzero. If $\mathscr{N}_p(f)\mathscr{N}_{p'}(g) = 0$ then the Hölder inequality is beyond a doubt. ▷

The set $\mathscr{L}_p :=$ dom $\mathscr{N}_p$ is a vector space.

◁ $|f+g|^p \le (|f|+|g|)^p \le 2^p(|f| \vee |g|)^p = 2^p(|f|^p \vee |g|^p) \le 2^p(|f|^p + |g|^p)$ ▷

The function $\mathscr{N}_p$ is a seminorm, satisfying the Minkowski inequality:

$$\mathscr{N}_p(f+g) \le \mathscr{N}_p(f) + \mathscr{N}_p(g).$$

◁ For $p = 1$, this is trivial. For $p > 1$ the Minkowski inequality follows from the presentation

$$\mathscr{N}_p(f) = \sup\{\mathscr{N}_1(fg)/\mathscr{N}_{p'}(g) :\ 0 < \mathscr{N}_{p'}(g) < +\infty\} \quad (f \in \mathscr{L}_p)$$

whose right side is the upper envelope of a family of seminorms. To prove the above presentation, using the Hölder inequality, note that $g := |f|^{{}^p/_{p'}}$ lies in $\mathscr{L}_q$ when $\mathscr{N}_p(f) > 0$; furthermore, $\mathscr{N}_p(f) = \mathscr{N}_1(fg)/\mathscr{N}_{p'}(g)$. Indeed, $\mathscr{N}_1(fg) = \int |f|^{{}^p/_{p'}+1} = \mathscr{N}_p(f)^p$, because ${}^p/_{p'} + 1 = p\left(1 - {}^1/_p\right) + 1 = p$. Continue arguing to find $\mathscr{N}_{p'}(g)^{p'} = \int |g|^{p'} = \int |f|^p = \mathscr{N}_p(f)^p$, and so $\mathscr{N}_{p'}(g) = \mathscr{N}_p(f)^{{}^p/_{p'}}$. Finally,

$$\mathscr{N}_1(fg)/\mathscr{N}_{p'}(g) = \mathscr{N}_p(f)^p/\mathscr{N}_p(f)^{{}^p/_{p'}} = \mathscr{N}_p(f)^{p-{}^p/_{p'}} = \mathscr{N}_p(f)^{p(1-{}^1/_{p'})} = \mathscr{N}_p(f). \ ▷$$

The quotient space $\mathscr{L}_p/\ker \mathscr{N}_p$, together with the corresponding quotient norm $\|\cdot\|_p$, is called the *space of p-summable functions* or *L_p space* with more complete designations $L_p(S)$, $L_p(\mathscr{E},\ X,\ \int)$, etc.

Finally, if a system with integration S arises from inspection of measurable step functions on a *measure space* $(\Omega,\ \mathscr{A},\ \mu)$ then it is customary to write $L_p(\Omega,\ \mathscr{A},\ \mu)$, $L_p(\Omega,\ \mu)$ and even $L_p(\mu)$, with the unspecified parameters clear from the context.

Riesz–Fisher Completeness Theorem. *Each L_p space is a Banach space.*

◁ We sketch out the proof. Consider $t := \sum_{k=1}^{\infty} \mathscr{N}_p(f_k)$, where $f_k \in \mathscr{L}_p$. Put $\sigma_n := \sum_{k=1}^{n} f_k$ and $s_n := \sum_{k=1}^{n} |f_k|$. It is seen that (s_n) has positive entries and increases. The same is true of (s_n^p). Furthermore, $\int s_n^p \le t^p < +\infty$. Hence, by the Levy Monotone Convergence Theorem, for almost all $e \in \mathscr{E}$ there exists a limit $g(e) := \lim s_n^p(e)$, with the resulting function g a member of $\mathscr{L}_1$. Putting $h(e) := g^{1/p}(e)$, observe that $h \in \mathscr{L}_p$ and $s_n(e) \to h(e)$ for almost all $e \in \mathscr{E}$. The inequalities $|\sigma_n| \le s_n \le h$ imply that for almost all $e \in \mathscr{E}$ the series $\sum_{k=1}^{\infty} f_k(e)$ converges. For the sum $f_0(e)$ the estimate holds: $|f_0(e)| \le h(e)$. Hence, it may be assumed that $f_0 \in \mathscr{L}_p$. Appealing to the Lebesgue Dominated Convergence Theorem, conclude that $\mathscr{N}_p(\sigma_n - f_0) = \left(\int |\sigma_n - f_0|^p\right)^{1/p} \to 0$. Thus, in the seminormed space under consideration each seminorm convergent series converges. To complete the proof, apply 5.5.3–5.5.5. ▷

If S is the system of *conventional summation* on $\mathscr{E}$; i.e., $X := \sum_{e\in\mathscr{E}} \mathbb{R}$ is the direct sum of suitably many copies of the ground field $\mathbb{R}$ and $\int x := \sum_{e\in\mathscr{E}} x(e)$, then L_p comprises all *p-summable families*. This space is denoted by $l_p(\mathscr{E})$. Further, $\|x\|_p := \left(\sum_{e\in\mathscr{E}} |x(e)|^p\right)^{1/p}$. In the case $\mathscr{E} := \mathbb{N}$ the notation l_p is used and l_p is referred to as the *space of p-summable sequences.*

(5) Define L_∞ as follows: Let X be an ordered vector space and let $e \in X_+$ be a positive element. The seminorm p_e *associated* with e is the Minkowski functional of the order interval $[-e,\ e]$, i.e.,

$$p_e(x) := \inf\{t > 0 :\ -te \le x \le te\}.$$

The effective domain of definition of p_e is the *space of bounded elements* (with respect to e); the element e itself is referred to as the *strong order-unit* in X_e. An element of $\ker p_e$ is said to be *nonarchimedean* (with respect to e). The quotient space $X_e/\ker p_e$ furnished with the corresponding quotient seminorm is called the *normed space of bounded elements* (generated by e in X). For example, $C(Q,\ \mathbb{R})$, the space of continuous real-valued functions on a nonempty compact set Q, presents the normed space of bounded elements with respect to $\mathbf{1} := \mathbf{1}_Q :\ q \mapsto 1\ (q \in Q)$ (in itself). In $\mathbb{R}^{\mathscr{E}}$ the same element $\mathbf{1}$ generates the space $l_\infty(\mathscr{E})$.

Given a system with integration $S := (\mathscr{E},\ X,\ \int)$, assume that $\mathbf{1}$ is measurable and consider the space of functions acting from $\mathscr{E}$ into $\mathbb{F}$ and satisfying

$$\mathscr{N}_\infty(f) := \inf\{t > 0 :\ |f| \le t\mathbf{1}\} < +\infty,$$

where $\le$ means "less almost everywhere than." This space is called the *space of essentially bounded functions* and is labelled with $\mathscr{L}_\infty$. To denote the quotient space $\mathscr{L}_\infty/\ker \mathscr{N}_\infty$ and its norm the symbols L_∞ and $\|\cdot\|_\infty$ are in use.

It is in common parlance to call the elements of L_∞ (like the elements of $\mathscr{L}_\infty$) essentially bounded functions. The space L_∞ presents a Banach space. ◁▷

The space L_∞, as well as the spaces $C(Q,\ \mathbb{F})$, $l_p(\mathscr{E})$, $c_0(\mathscr{E})$, c, l_p, and L_p $(p \geq 1)$, also bears the unifying title "classical Banach space." Nowadays a *Lindenstrauss space* which is a space whose dual is isometric to L_1 (with respect to some system with integration) is also regarded as classical. It can be shown that a Banach space X is classical if and only if the dual X' is isomorphic to one of the L_p spaces with $p \geq 1$.

(6) Consider a system with integration $S := \big(\mathscr{E},\ X,\ \int\big)$ and let $p \geq 1$. Suppose that for every e in $\mathscr{E}$ there is a Banach space $(Y_e,\ \|\cdot\|_{Y_e})$. Given an arbitrary element f in $\prod_{e\in\mathscr{E}} Y_e$, define $|||f||| : e \mapsto \|f(e)\|_{Y_e}$. Put $N_p(f) := \inf\{\mathscr{N}_p(g) : g \in \mathscr{L}_p,\ g \geq |||f|||\}$. It is clear that dom N_p is a vector space equipped with the seminorm N_p. The *sum of the family in the sense of* L_p or simply the *p-sum* of $(Y_e)_{e\in\mathscr{E}}$ (with respect to the system with integration S) is the quotient space dom $N_p/\ker N_p$ under the corresponding (quotient) norm $|||\cdot|||_p$.

The p-sum of a family of Banach spaces is a Banach space.

◁ If $\sum_{k=1}^\infty N_p(f_k) < +\infty$ then the sequence $(s_n := \sum_{k=1}^n |||f_k|||)$ tends to some almost everywhere finite positive function g and $N_p(g) < +\infty$. It follows that for almost all $e \in \mathscr{E}$ the sequence $(s_n(e))$ (i.e., the series $\sum_{k=1}^\infty \|f_k(e)\|_{Y_e}$) converges. By the completeness of Y_e, the series $\sum_{k=1}^\infty f_k(e)$ converges to some sum $f_0(e)$ in Y_e with $\|f_0(e)\|_{Y_e} \leq g(e)$ for almost every $e \in \mathscr{E}$. Therefore, it may be assumed that $f_0 \in$ dom N_p. Finally, $N_p\left(\sum_{k=1}^n f_k - f_0\right) \leq \sum_{k=n+1}^\infty N_p(f_k) \to 0$. ▷

In the case when $\mathscr{E} := \mathbb{N}$ with conventional summation, for the sum $\mathfrak{Y}$ of a sequence of Banach spaces $(Y_n)_{n\in\mathbb{N}}$ (in the sense of L_p) the following notation is often employed:

$$\mathfrak{Y} := (Y_1 \oplus Y_2 \oplus \cdots)_p,$$

with p the type of summation. An element $\overline{y}$ in $\mathfrak{Y}$ presents a sequence $(y_n)_{n\in\mathbb{N}}$ such that $y_n \in Y_n$ and

$$|||\overline{y}|||_p := \left(\sum_{k=1}^\infty \|y_n\|_{Y_n}^p\right)^{1/p} < +\infty.$$

In the case $Y_e := \mathfrak{X}$ for all $e \in \mathscr{E}$, where $\mathfrak{X}$ is some Banach space over $\mathbb{F}$, put $\mathscr{F}_p :=$ dom N_p and $F_p := \mathscr{F}_p/\ker N_p$. An element of the so-constructed space is a *vector field* or a *$\mathfrak{X}$-valued function* on $\mathscr{E}$ (having a p-summable norm). Undoubtedly, F_p is a Banach space. At the same time, if the initial system with integration contains a nonmeasurable set then extraordinary elements are plentiful in F_p (in particular, for the usual Lebesgue system with integration $F_p \neq L_p$). In this connection the functions with finite range, assuming each value on a measurable set, are selected in $\mathscr{F}_p$. Such a function, as well as the corresponding coset in F_p, is a *simple, finite-valued* or *step function.* The closure in F_p of the set of simple functions is denoted by L_p (or more completely $L_p(\mathfrak{X})$, $L_p(S,\ \mathfrak{X})$, $L_p(\Omega,\ \mathscr{A},\ \mu)$, $L_p(\Omega,\ \mu)$, etc.) and is the *space of $\mathfrak{X}$-valued p-summable functions.* Evidently, $L_p(\mathfrak{X})$ is a Banach space.

It is in order to illustrate one of the advantages of these spaces for $p = 1$. First, notice that a simple function f can be written as a finite combination of *characteristic functions*:

$$f = \sum_{x\in \operatorname{im} f} \chi_{f^{-1}(x)}x,$$

where $f^{-1}(x)$ is a measurable set as $x \in \operatorname{im} f$, with $\chi_E(e) = 1$ for $e \in E$ and $\chi_E(e) = 0$ otherwise. Moreover,

$$\int |||f||| = \int \sum_{x\in \operatorname{im} f} \|\chi_{f^{-1}(x)}x\|$$
$$= \int \sum_{x\in \operatorname{im} f} \chi_{f^{-1}(x)}\,\|x\| = \sum_{x\in \operatorname{im} f} \|x\| \int \chi_{f^{-1}(x)} < +\infty.$$

Next, associate with each simple function f some element in $\mathfrak{X}$ by the rule

$$\int f := \sum_{x\in \operatorname{im} f} \int \chi_{f^{-1}(x)}x.$$

Straightforward calculation shows that the integral $\int$ defined on the subspace of simple functions is linear. Furthermore, it is bounded because

$$\left\|\int f\right\| = \left\|\sum_{x\in \operatorname{im} f} \int \chi_{f^{-1}(x)}x\right\| \le \sum_{x\in \operatorname{im} f} \int \chi_{f^{-1}(x)}\,\|x\|$$
$$= \int \sum_{x\in \operatorname{im} f} \|x\|\,\chi_{f^{-1}(x)} = \int |||f|||\,.$$

By virtue of 4.5.10 and 5.3.8, the operator $\int$ has a unique extension to an element of $B(L_1(\mathfrak{X}),\ \mathfrak{X})$. This element is denoted by the same symbol, $\int$ (or $\int_{\mathscr{E}}$, etc.), and is referred to as the *Bochner integral.*

(7) In the case of conventional summation, the usage of the scalar theory is preserved for the Bochner integral. Namely, the common parlance favours the term "sum of a family" rather than "integral of a summable function," and the symbols pertinent to summation are perfectly welcome. What is more important, infinite dimensions bring about significant complications.

Let (x_n) be a family of elements of a Banach space. Its summability (in the sense of the Bochner integral) means the summability of the numeric family $(\|x_n\|)$, i.e. the norm convergence of (x_n) as a series. Consequently, (x_n) has at most countably many nonzero elements and may thus be treated as a (countable) sequence. Moreover, $\sum_{n=1}^{\infty} \|x_n\| < +\infty$; i.e. the series $x_1 + x_2 + \cdots$ converges in norm. By 5.5.3, for the *series sum* $x = \sum_{n=1}^{\infty} x_n$ observe that $x = \lim_\theta s_\theta$, where $s_\theta := \sum_{n\in\theta} x_n$ is a *partial sum* and θ ranges over the direction of all finite subsets of $\mathbb{N}$. In this case, the resulting x is sometimes called the *unordered sum* of (x_n), whereas the sequence (x_n) is called *unconditionally* or *unorderly summable* to x (in symbols, $x = \sum_{n\in\mathbb{N}} x_n$). Using these terms, observe that summability in norm implies unconditional summability (to the same sum). If $\dim X < +\infty$ then the converse holds which is the *Riemann Theorem on Series.* The general case is explained by the following deep and profound assertion:

Dvoretzky–Rogers Theorem. *In an arbitrary infinite-dimensional Banach space X, for every sequence of positive numbers (t_n) such that $\sum_{n=1}^{\infty} t_n^2 < +\infty$ there is an unconditionally summable sequence of elements (x_n) with $\|x_n\| = t_n$ for all $n \in \mathbb{N}$.*

In this regard for a family of elements $(x_e)_{e\in\mathscr{E}}$ of an arbitrary multinormed space $(X, \mathfrak{M})$ the following terminology is accepted: Say that $(x_e)_{e\in\mathscr{E}}$ is *summable* or *unconditionally summable* (to a sum x) and write $x := \sum_{e\in\mathscr{E}} x_e$ whenever x is the limit in $(X, \mathfrak{M})$ of the corresponding net of partial sums s_θ, with θ a finite subset of $\mathscr{E}$; i.e., $s_\theta \to x$ in $(X, \mathfrak{M})$. If for every p there is a sum $\sum_{e\in\mathscr{E}} p(x_e)$ then the family $(x_e)_{e\in\mathscr{E}}$ is said to be *multinorm summable* (or, what is more exact, *fundamentally summable*, or even *absolutely fundamental*).

In conclusion, consider a Banach space $\mathfrak{Y}$ and $T \in B(\mathfrak{X}, \mathfrak{Y})$. The operator T can be uniquely extended to an operator from $L_1(\mathfrak{X})$ into $L_1(\mathfrak{Y})$ by putting $Tf : e \mapsto Tf(e)$ $(e \in \mathscr{E})$ for an arbitrary simple $\mathfrak{X}$-valued function f. Then, given $f \in L_1(\mathfrak{X})$, observe that $Tf \in L_1(\mathfrak{Y})$ and $\int_{\mathscr{E}} Tf = T \int_{\mathscr{E}} f$. This fact is verbalized as follows: "The Bochner integral commutes with every bounded operator." ◁▷

5.6. The Algebra of Bounded Operators

5.6.1. *Let X, Y, and Z be normed spaces. If $T \in \mathscr{L}(X, Y)$ and $S \in \mathscr{L}(Y, Z)$ are linear operators then $\|ST\| \leq \|S\|\,\|T\|$; i.e., the operator norm is submultiplicative.*

◁ Given $x \in X$ and using the normative inequality twice, infer that

$$\|STx\| \leq \|S\|\ \|Tx\| \leq \|S\|\ \|T\|\ \|x\|. \ ▷$$

5.6.2. Remark. In algebra, in particular, (associative) algebras are studied. An *algebra* (over $\mathbb{F}$) is a vector space A (over $\mathbb{F}$) together with some associative multiplication $\circ : (a,\ b) \mapsto ab\ (a,\ b \in A)$. This multiplication must be distributive with respect to addition (i.e., $(A,\ +,\ \circ)$ is an (associative) ring) and, moreover, the operation $\circ$ must agree with scalar multiplication in the following sense: $\lambda(ab) = (\lambda a)b = a(\lambda b)$ for all $a,\ b \in A$ and $\lambda \in \mathbb{F}$. A displayed notation for an algebra is $(A,\ \mathbb{F},\ +,\ \cdot,\ \circ)$. However, just like on the other occasions it is customary to use the term "algebra" simply for A.

5.6.3. DEFINITION. A *normed algebra* (over a ground field) is an associative algebra (over this field) together with a submultiplicative norm. A *Banach algebra* is a complete normed algebra.

5.6.4. *Let $B(X) := B(X,\ X)$ be the space of bounded endomorphisms of a normed space X. The space $B(X)$ with the operator norm and composition as multiplication presents a normed algebra. If $X \neq 0$ then $B(X)$ has a neutral element (with respect to multiplication), the identity operator I_X; i.e., $B(X)$ is an algebra with unity. Moreover, $\|I_X\| = 1$. The algebra $B(X)$ is a Banach algebra if and only if X is a Banach space.*

$\triangleleft$ If $X = 0$ then there is nothing left to proof. Given $X \neq 0$, apply 5.5.6. $\triangleright$

5.6.5. REMARK. It is usual to refer to $B(X)$ as the *algebra of bounded operators* in X or even as the (*bounded*) *endomorphism algebra* of X. In connection with 5.6.4, given $\lambda \in \mathbb{F}$, it is convenient to retain the same symbol λ for λI_X. (In particular, $1 = I_0 = 0$!) For $X \neq 0$ this procedure may be thought as identification of $\mathbb{F}$ with $\mathbb{F}I_X$.

5.6.6. DEFINITION. If X is a normed space and $T \in B(X)$ then the *spectral radius* of T is the number $r(T) := \inf \left\{ \|T^n\|^{1/n} \ :\ n \in \mathbb{N} \right\}$. (The rationale of this term will transpire later (cf. 8.1.12).)

5.6.7. *The norm of T is greater than the spectral radius of T.*

$\triangleleft$ Indeed, by 5.6.1, the inequality $\|T^n\| \leq \|T\|^n$ is valid. $\triangleright$

5.6.8. *The Gelfand formula holds:*

$$r(T) = \lim \sqrt[n]{\|T^n\|}.$$

$\triangleleft$ Take $\varepsilon > 0$ and let $s \in \mathbb{N}$ satisfy $\|T^s\| \leq r(T) + \varepsilon$. Given $n \in \mathbb{N}$ with $n \geq s$, observe the presentation $n = k(n)s + l(n)$ with $k(n),\ l(n) \in \mathbb{N}$ and $0 \leq l(n) \leq s-1$. Hence,

$$\begin{aligned}\|T^n\| &= \left\|T^{k(n)s}T^{l(n)}\right\| \leq \|T^s\|^{k(n)} \left\|T^{l(n)}\right\| \\ &\leq \left(1 \vee \|T\| \vee \ldots \vee \left\|T^{s-1}\right\|\right) \|T^s\|^{k(n)} = M\ \|T^s\|^{k(n)}.\end{aligned}$$

Consequently,

$$\begin{aligned} r(T) &\le \|T^n\|^{1/n} \le M^{1/n}\,\|T^s\|^{k(n)/n} \\ &\le M^{1/n}(r(T)+\varepsilon)^{k(n)s/n} = M^{1/n}(r(T)+\varepsilon)^{(n-l(n))/n}. \end{aligned}$$

Since $M^{1/n} \to 1$ and ${}^{(n-l(n))}/_n \to 1$, find $r(T) \le \limsup \|T^n\|^{1/n} \le r(T)+\varepsilon$. The inequality $\liminf \|T^n\|^{1/n} \ge r(T)$ is evident. Recall that ε is arbitrary, thus completing the proof. ▷

5.6.9. Neumann Series Expansion Theorem. *With X a Banach space and $T \in B(X)$, the following statements are equivalent:*

(1) *the Neumann series $1+T+T^2+\cdots$ converges in the operator norm of $B(X)$;*

(2) $\|T^k\| < 1$ *for some k and $\mathbb{N}$;*

(3) $r(T) < 1$.

If one of the conditions (1)–(3) *holds then* $\sum_{k=0}^{\infty} T^k = (1-T)^{-1}$.

◁ (1) ⇒ (2): With the Neumann series convergent, the general term (T^k) tends to zero.

(2) ⇒ (3): This is evident.

(3) ⇒ (1): According to 5.6.8, given a suitable $\varepsilon > 0$ and a sufficiently large $k \in \mathbb{N}$, observe that $r(T) \le \|T^k\|^{1/k} \le r(T)+\varepsilon < 1$. In other words, some tail of the series $\sum_{k=0}^{\infty} \|T^k\|$ is dominated by a convergent series. The completeness of $B(X)$ and 5.5.3 imply that $\sum_{k=0}^{\infty} T^k$ converges in $B(X)$.

Now let $S := \sum_{k=0}^{\infty} T^k$ and $S_n := \sum_{k=0}^{n} T^k$. Then

$$S(1-T) = \lim S_n(1-T) = \lim\,(1+T+\cdots+T^n)(1-T) = \lim(1-T^{n+1}) = 1;$$
$$(1-T)S = \lim(1-T)S_n = \lim(1-T)(1+T+\cdots+T^n) = \lim\left(1-T^{n+1}\right) = 1,$$

because $T^n \to 0$. Thus, by 2.2.7 $S = (1-T)^{-1}$. ▷

5.6.10. Corollary. *If $\|T\| < 1$ then $(1-T)$ is invertible* (= *has a bounded inverse*); *i.e., the inverse correspondence $(1-T)^{-1}$is a bounded linear operator. Moreover,* $\|(1-T)^{-1}\| \le (1-\|T\|)^{-1}$.

◁ The Neumann series converges and

$$\|(1-T)^{-1}\| \le \sum_{k=0}^{\infty} \|T^k\| \le \sum_{k=0}^{\infty} \|T\|^k = (1-\|T\|)^{-1}. \ ▷$$

5.6.11. Corollary. *If $\|1-T\| < 1$ then T is invertible and*

$$\|1-T^{-1}\| \le \frac{\|1-T\|}{1-\|1-T\|}.$$

◁ By Theorem 5.6.9,

$$1+\sum_{k=1}^{\infty}(1-T)^k=\sum_{k=0}^{\infty}(1-T)^k=(1-(1-T))^{-1}=T^{-1}.$$

Hence,

$$\|T^{-1}-1\|=\left\|\sum_{k=1}^{\infty}(1-T)^k\right\|\le\sum_{k=1}^{\infty}\|(1-T)^k\|\le\sum_{k=1}^{\infty}\|1-T\|^k. \ \triangleright$$

5.6.12. *Banach Inversion Stability Theorem.* *Let X and Y be Banach spaces. The set of invertible operators* $\operatorname{Inv}(X, Y)$ *is open. Moreover, the inversion $T\mapsto T^{-1}$ acting from* $\operatorname{Inv}(X, Y)$ *to* $\operatorname{Inv}(Y, X)$ *is continuous.*

◁ Take $S, T\in B(X, Y)$ such that $T^{-1}\in B(Y, X)$ and $\|T^{-1}\|\,\|S-T\|\le {}^1/_2$. Consider the operator $T^{-1}S$ in $B(X)$. Observe that

$$\|1-T^{-1}S\|=\|T^{-1}T-T^{-1}S\|\le\|T^{-1}\|\,\|T-S\|\le {}^1/_2<1.$$

Hence, by 5.6.11, $(T^{-1}S)^{-1}$ belongs to $B(X)$. Put $R:=(T^{-1}S)^{-1}T^{-1}$. Clearly, $R\in B(Y, X)$ and, moreover, $R=S^{-1}(T^{-1})^{-1}T^{-1}=S^{-1}$. Further,

$$\|S^{-1}\|-\|T^{-1}\|\le\|S^{-1}-T^{-1}\|$$
$$=\|S^{-1}(T-S)T^{-1}\|\le\|S^{-1}\|\,\|T-S\|\,\|T^{-1}\|\le {}^1/_2\,\|S^{-1}\|.$$

This implies $\|S^{-1}\|\le 2\,\|T^{-1}\|$, yielding the inequalities

$$\|S^{-1}-T^{-1}\|\le\|S^{-1}\|\,\|T-S\|\,\|T^{-1}\|\le 2\,\|T^{-1}\|^2\,\|T-S\|. \ \triangleright$$

5.6.13. Definition. If X is a Banach space over $\mathbb{F}$ and $T\in B(X)$ then a scalar $\lambda\in\mathbb{F}$ is a *regular* or *resolvent value* of T whenever $(\lambda-T)^{-1}\in B(X)$. In this case put $R(T, \lambda):=(\lambda-T)^{-1}$ and say that $R(T, \lambda)$ is the *resolvent* of T at λ. The set of the resolvent values of T is denoted by $\operatorname{res}(T)$ and called the *resolvent set* of T. The mapping $\lambda\mapsto R(T, \lambda)$ from $\operatorname{res}(T)$ into $B(X)$ is naturally called the *resolvent* of T. The set $\mathbb{F}\setminus\operatorname{res}(T)$ is referred to as the *spectrum* of T and is denoted by $\operatorname{Sp}(T)$ or $\sigma(T)$. A member of $\operatorname{Sp}(T)$ is said to be a *spectral value* of T (which is enigmatic for the time being).

5.6.14. Remark. If $X=0$ then the spectrum of the only operator $T=0$ in $B(X)$ is the empty set. In this regard, in spectral analysis it is silently presumed that $X\ne 0$. In the case $X\ne 0$ for $\mathbb{F}:=\mathbb{R}$ the spectra of some operators can also be void, whereas for $\mathbb{F}:=\mathbb{C}$ it is impossible (cf. 8.1.11). ◁▷

5.6.15. *The set* res (T) *is open. If* $\lambda_0 \in$ res (T) *then*

$$R(T,\ \lambda) = \sum_{k=0}^{\infty} (-1)^k (\lambda - \lambda_0)^k R(T,\ \lambda_0)^{k+1}$$

in some neighborhood of λ_0. *If* $|\lambda| > \|T\|$ *then* $\lambda \in$ res (T) *and the expansion*

$$R(T,\ \lambda) = \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{T^k}{\lambda^k}$$

holds. Moreover, $\|R(T,\ \lambda)\| \to 0$ *as* $|\lambda| \to +\infty$.

◁ Since $\|(\lambda - T) - (\lambda_0 - T)\| = |\lambda - \lambda_0|$, the openness property of res (T) follows from 5.6.12. Proceed along the lines

$$\lambda - T = (\lambda - \lambda_0) + (\lambda_0 - T) = (\lambda_0 - T)R(T,\ \lambda_0)(\lambda - \lambda_0) + (\lambda_0 - T)$$
$$= (\lambda_0 - T)((\lambda - \lambda_0)R(T,\ \lambda_0) + 1) = (\lambda_0 - T)(1 - ((-1)(\lambda - \lambda_0)R(T,\ \lambda_0))).$$

In a suitable neighborhood of λ_0, from 5.6.9 derive

$$R(T,\ \lambda) = (\lambda - T)^{-1}$$
$$= (1 - ((-1)(\lambda - \lambda_0)R(T,\ \lambda_0)))^{-1}(\lambda_0 - T)^{-1} = \sum_{k=0}^{\infty} (-1)^k (\lambda - \lambda_0)^k R(T,\ \lambda_0)^{k+1}.$$

According to 5.6.9, for $|\lambda| > \|T\|$ there is an operator $\left(1 - {}^T/_\lambda\right)^{-1}$ presenting the sum of the Neumann series; i.e.,

$$R(T,\ \lambda) = \frac{1}{\lambda}\left(1 - {}^T/_\lambda\right)^{-1} = \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{T^k}{\lambda^k}.$$

It is clear that

$$\|R(T,\ \lambda)\| \le \frac{1}{|\lambda|} \cdot \frac{1}{1 - {}^{\|T\|}/_{|\lambda|}}. \ ▷$$

5.6.16. *The spectrum of every operator is compact.* ◁▷

5.6.17. Remark. It is worth keeping in mind that the inequality $|\lambda| > r(T)$ is a necessary and sufficient condition for the convergence of the Laurent series, $R(T,\ \lambda) = \sum_{k=0}^{\infty} T^k/\lambda^{k+1}$, which expands the resolvent of T at infinity.

5.6.18. *An operator* S *commutes with an operator* T *if and only if* S *commutes with the resolvent of* T.

◁ ⇒: $ST = TS \Rightarrow S(\lambda - T) = \lambda S - ST = \lambda S - TS = (\lambda - T)S \Rightarrow R(T,\ \lambda)S(\lambda - T) = S \Rightarrow R(T,\ \lambda)S = S\,R(T,\ \lambda)$ ($\lambda \in \operatorname{res}(T)$).

⇐: $SR(T,\ \lambda_0) = R(T,\ \lambda_0)S \Rightarrow S = R(T,\ \lambda_0)S(\lambda_0 - T) \Rightarrow (\lambda_0 - T)S = S(\lambda_0 - T) \Rightarrow TS = ST$. ▷

5.6.19. *If* $\lambda,\ \mu \in \operatorname{res}(T)$ *then the first resolvent equation, the Hilbert identity, holds*:

$$R(T,\ \lambda) - R(T,\ \mu) = (\mu - \lambda)R(T,\ \mu)R(T,\ \lambda).$$

◁ "Multiplying the equality $\mu - \lambda = (\mu - T) - (\lambda - T)$, first, by $R(T,\ \lambda)$ from the right and, second, by $R(T,\ \mu)$ from the left," successively infer the sought identity. ▷

5.6.20. *If* $\lambda,\ \mu \in \operatorname{res}(T)$ *then* $R(T,\ \lambda)R(T,\ \mu) = R(T,\ \mu)R(T,\ \lambda)$. ◁▷

5.6.21. *For* $\lambda \in \operatorname{res}(T)$ *the equality holds*:

$$\frac{d^k}{d\lambda^k}R(T,\ \lambda) = (-1)^k k!\,R(T,\ \lambda)^{k+1}.$$ ◁▷

5.6.22. Composition Spectrum Theorem. *The spectra* $\operatorname{Sp}(ST)$ *and* $\operatorname{Sp}(TS)$ *may differ only by zero.*

◁ It suffices to establish that $1 \notin \operatorname{Sp}(ST) \Rightarrow 1 \notin \operatorname{Sp}(TS)$. Indeed, from $\lambda \notin \operatorname{Sp}(ST)$ and $\lambda \neq 0$ it will follow that

$$1 \notin {}^1/_\lambda \operatorname{Sp}(ST) \Rightarrow 1 \notin \operatorname{Sp}\left({}^1/_\lambda ST\right) \Rightarrow 1 \notin \operatorname{Sp}\left({}^1/_\lambda TS\right) \Rightarrow \lambda \notin \operatorname{Sp}(TS).$$

Therefore, consider the case $1 \notin \operatorname{Sp}(ST)$. The formal Neumann series expansions

$$(1 - ST)^{-1} \sim 1 + ST + (ST)(ST) + (ST)(ST)(ST) + \cdots,$$
$$T(1 - ST)^{-1}S \sim TS + TSTS + TSTSTS + \cdots \sim (1 - TS)^{-1} - 1$$

lead to conjecturing that the presentation is valid:

$$(1 - TS)^{-1} = 1 + T(1 - ST)^{-1}S$$

(which in turn means $1 \notin \operatorname{Sp}(TS)$). Straightforward calculation demonstrates the above presentation, thus completing the entire proof:

$$\begin{aligned}(1 + T(1 - ST)^{-1}S)(1 - TS) &= 1 + T(1 - ST)^{-1}S - TS + T(1 - ST)^{-1}(-ST)S\\ &= 1 + T(1 - ST)^{-1}S - TS + T(1 - ST)^{-1}(1 - ST - 1)S\\ &= 1 + T(1 - ST)^{-1}S - TS + TS - T(1 - ST)^{-1}S = 1;\\ (1 - TS)(1 + T(1 - ST)^{-1}S) &= 1 - TS + T(1 - ST)^{-1}S + T(-ST)(1 - ST)^{-1}S\\ &= 1 - TS + T(1 - ST)^{-1}S + T(1 - ST - 1)(1 - ST)^{-1}S\\ &= 1 - TS + T(1 - ST)^{-1}S + TS - T(1 - ST)^{-1}S = 1.\end{aligned}$$ ▷

Exercises

5.1. Prove that a normed space is finite-dimensional if and only if every linear functional on the space is bounded.

5.2. Demonstrate that it is possible to introduce a norm into each vector space.

5.3. Show that a vector space X is finite-dimensional if and only if all norms on X are equivalent to each other.

5.4. Demonstrate that all separated multinorms introduce the same topology in a finite-dimensional space.

5.5. Each norm on $\mathbb{R}^N$ is appropriate for norming the product of finitely many normed spaces, isn't it?

5.6. Find conditions for continuity of an operator acting between multinormed spaces and having finite-dimensional range.

5.7. Describe the operator norms in the space of square matrices. When are such norms comparable?

5.8. Calculate the distance between hyperplanes in a normed space.

5.9. Find the general form of a continuous linear functional on a classical Banach space.

5.10. Study the question of reflexivity for classical Banach spaces.

5.11. Find the mutual disposition of the spaces l_p and $l_{p'}$ as well as L_p and $L_{p'}$. When is the complement of one element of every pair is dense in the other?

5.12. Find the spectrum and resolvent of the *Volterra operator* (the taking of a primitive), a projection, and a rank-one operator.

5.13. Construct an operator whose spectrum is a prescribed nonempty compact set in $\mathbb{C}$.

5.14. Prove that the identity operator (in a nonzero space) is never the commutator of any pair of operators.

5.15. Is it possible to define some reasonable spectrum for an operator in a multinormed space?

5.16. Does every Banach space over $\mathbb{F}$ admit an isometric embedding into the space $C(Q, \mathbb{F})$, with Q a compact space?

5.17. Find out when $L_p(X)' = L_{p'}(X')$, with X a Banach space.

5.18. Let (X_n) be a sequence of normed spaces and let

$$X_0 = \left\{ x \in \prod_{n \in \mathbb{N}} X_n : \|x_n\| \to 0 \right\}$$

be their c_0-sum (with the norm $\|x\| = \sup\{\|x_n\| : n \in \mathbb{N}\}$ induced from the l_∞-sum). Prove that X_0 is separable if and only if so is each of the spaces X_n.

5.19. Prove that the space $C^{(p)}[0, 1]$ presents the sum of a finite-dimensional subspace and a space isomorphic to $C[0, 1]$.

Chapter 6
Hilbert Spaces

6.1. Hermitian Forms and Inner Products

6.1.1. DEFINITION. Let H be a vector space over a basic field $\mathbb{F}$. A mapping $f : H^2 \to \mathbb{F}$ is a *hermitian form* on H provided that

(1) the mapping $f(\cdot,\ y) : x \mapsto f(x,\ y)$ belongs to $H^{\#}$ for every y in Y;

(2) $f(x,\ y) = f(y,\ x)^*$ for all $x,\ y \in H$, where $\lambda \mapsto \lambda^*$ is the natural involution in $\mathbb{F}$; that is, the taking of the complex conjugate of a complex number.

6.1.2. REMARK. It is easy to see that, for a hermitian form f and each x in H the mapping $f(x,\ \cdot) : y \mapsto (x,\ y)$ lies in $H_*^{\#}$, where H_* is the twin of H (see 2.1.4 (2)). Consequently, in case $\mathbb{F} := \mathbb{R}$ every hermitian form is *bilinear*, i.e., linear in each argument; and in case $\mathbb{F} := \mathbb{C}$, *sesquilinear*, i.e., linear in the first argument and $*$-linear in the second.

6.1.3. *Every hermitian form f satisfies the polarization identity:*

$$f(x+y,\ x+y) - f(x-y,\ x-y) = 4\mathrm{Re}\, f(x,\ y) \quad (x,\ y \in H).$$

$$\triangleleft \quad -\ \begin{array}{r} f(x+y,\ x+y) = f(x,\ x) + f(x,\ y) + f(y,\ x) + f(y,\ y) \\ f(x-y,\ x-y) = f(x,\ x) - f(x,\ y) - f(y,\ x) + f(y,\ y) \\ \hline 2(f(x,\ y) + f(y,\ x)) \end{array} \quad \triangleright$$

6.1.4. DEFINITION. A hermitian form f is *positive* or *positive semidefinite* provided that $f(x,\ x) \geq 0$ for all $x \in H$. In this event, write $(x,\ y) := \langle x \mid y \rangle := f(x,\ y)$ $(x,\ y \in H)$. A positive hermitian form is usually referred to as a *semi-inner product* on H. A semi-inner product on H is an *inner product* or a (*positive definite*) *scalar product* whenever $(x,\ x) = 0 \Rightarrow x = 0$ with $x \in H$.

6.1.5. *The Cauchy–Bunyakovskiĭ–Schwarz inequality holds:*

$$|(x,\ y)|^2 \leq (x,\ x)(y,\ y) \quad (x,\ y \in H).$$

◁ If $(x, x) = (y, y) = 0$ then $0 \le (x+ty, x+ty) = t(x, y)^* + t^*(x, y)$. Letting $t := -(x, y)$, find that $-2|(x, y)|^2 \ge 0$; i.e., in this case the claim is established.

If, for definiteness, $(y, y) \ne 0$; then in view of the estimate

$$0 \le (x + ty, x + ty) = (x, x) + 2t\mathrm{Re}\,(x, y) + t^2(y, y) \quad (t \in \mathbb{R})$$

conclude that $\mathrm{Re}\,(x, y)^2 \le (x, x)(y, y)$.

If $(x, y) = 0$ then nothing is left to proof. If $(x, y) \ne 0$ then let $\theta := |(x, y)|\,(x, y)^{-1}$ and $\overline{x} := \theta x$. Now $|\theta| = 1$ and, furthermore,

$$(\overline{x}, \overline{x}) = (\theta x, \theta x) = \theta\theta^*(x, x) = |\theta|^2(x, x) = (x, x);$$
$$|(x, y)| = \theta(x, y) = (\theta x, y) = (\overline{x}, y) = \mathrm{Re}\,(\overline{x}, y).$$

Consequently, $|(x, y)|^2 = \mathrm{Re}\,(\overline{x}, y)^2 \le (x, x)(y, y)$. ▷

6.1.6. *If $(\cdot, \cdot)$ is a semi-inner product on H then the mapping $\|\cdot\| : x \mapsto (x, x)^{1/2}$ is a seminorm on H.*

◁ It suffices to prove the triangle inequality. Applying the Cauchy–Bunyakovskiĭ–Schwarz inequality, observe that

$$\|x + y\|^2 = (x, x) + (y, y) + 2\mathrm{Re}\,(x, y)$$
$$\le (x, x) + (y, y) + 2\|x\|\,\|y\| = (\|x\| + \|y\|)^2. \ ▷$$

6.1.7. Definition. A space H endowed with a semi-inner product $(\cdot, \cdot)$ and the *associate seminorm* $\|\cdot\|$ is a *pre-Hilbert space.* A pre-Hilbert space H is a *Hilbert space* provided that the seminormed space $(H, \|\cdot\|)$ is a Banach space.

6.1.8. *In a pre-Hilbert space H, the Parallelogram Law is effective*

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad (x, y \in H)$$

which reads: the sum of the squares of the lengths of the diagonals equals the sum of the squares of the lengths of all sides.

$$◁\ \|x + y\|^2 = (x + y, x + y) = \|x\|^2 + 2\mathrm{Re}\,(x, y) + \|y\|^2;$$
$$\|x - y\|^2 = (x - y, x - y) = \|x\|^2 - 2\mathrm{Re}\,(x, y) + \|y\|^2 \ ▷$$

6.1.9. Von Neumann–Jordan Theorem. *If a seminormed space $(H, \|\cdot\|)$ obeys the Parallelogram Law then H is a pre-Hilbert space; i.e., there is a unique semi-inner product $(\cdot, \cdot)$ on H such that $\|x\| = (x, x)^{1/2}$ for all $x \in H$.*

◁ Considering the real carrier $H_{\mathbb{R}}$ of H and $x,\ y \in H_{\mathbb{R}}$, put

$$(x,\ y)_{\mathbb{R}} := {}^1/_4 \left(\|x+y\|^2 - \|x-y\|^2\right).$$

Given the mapping $(\,\cdot\,,\ y)_{\mathbb{R}}$, from the Parallelogram Law successively derive

$$\begin{aligned}
&(x_1,\ y)_{\mathbb{R}} + (x_2,\ y)_{\mathbb{R}} \\
&= {}^1/_4 \left(\|x_1+y\|^2 - \|x_1-y\|^2 + \|x_2+y\|^2 - \|x_2-y\|^2\right) \\
&= {}^1/_4 \left(\left(\|x_1+y\|^2 + \|x_2+y\|^2\right) - \left(\|x_1-y\|^2 + \|x_2-y\|^2\right)\right) \\
&= {}^1/_4 \left({}^1/_2(\|(x_1+y)+(x_2+y)\|^2 + \|x_1-x_2\|^2\right) \\
&\quad -{}^1/_2 \left(\|(x_1-y)+(x_2-y)\|^2 + \|x_1-x_2\|^2)\right) \\
&= {}^1/_4 \left({}^1/_2\|x_1+x_2+2y\|^2 - {}^1/_2\|x_1+x_2-2y\|^2\right) \\
&= {}^1/_2 \left(\left\|{}^{(x_1+x_2)}/_2 + y\right\|^2 - \left\|{}^{(x_1-x_2)}/_2 - y\right\|^2\right) \\
&= 2\left({}^{(x_1+x_2)}/_2,\ y\right)_{\mathbb{R}}.
\end{aligned}$$

In particular, $(x_2,\ y)_{\mathbb{R}} = 0$ in case $x_2 := 0$, i.e. ${}^1/_2(x_1,\ y)_{\mathbb{R}} = \left({}^1/_2 x_1,\ y\right)_{\mathbb{R}}$. Analogously, given $x_1 := 2x_1$ and $x_2 := 2x_2$, infer that

$$(x_1 + x_2,\ y)_{\mathbb{R}} = (x_1,\ y)_{\mathbb{R}} + (x_2,\ y)_{\mathbb{R}}.$$

Since the mapping $(\,\cdot\,,\ y)_{\mathbb{R}}$ is continuous for obvious reasons, conclude that $(\,\cdot\,,\ y)_{\mathbb{R}} \in (H_{\mathbb{R}})^{\#}$. Put

$$(x,\ y) := (\mathbb{R}\mathrm{e}^{-1}(\,\cdot\,,\ y)_{\mathbb{R}})(x),$$

where $\mathbb{R}\mathrm{e}^{-1}$ is the complexifier (see 3.7.5).

In case $\mathbb{F} := \mathbb{R}$ it is clear that $(x,\ y) = (x,\ y)_{\mathbb{R}} = (y,\ x)$ and $(x,\ x) = \|x\|^2$; i.e., nothing is to be proven. On the other hand, if $\mathbb{F} := \mathbb{C}$ then

$$(x,\ y) = (x,\ y)_{\mathbb{R}} - i(ix,\ y)_{\mathbb{R}}.$$

This entails sesquilinearity:

$$\begin{aligned}
(y,\ x) &= (y,\ x)_{\mathbb{R}} - i(iy,\ x)_{\mathbb{R}} = (x,\ y)_{\mathbb{R}} - i(x,\ iy)_{\mathbb{R}} \\
&= (x,\ y)_{\mathbb{R}} + i(ix,\ y)_{\mathbb{R}} = (x,\ y)^*,
\end{aligned}$$

since

$$\begin{aligned}
&(x,\ iy)_{\mathbb{R}} = {}^1/_4 \left(\|x+iy\|^2 - \|x-iy\|^2\right) \\
&= {}^1/_4 \left(|i|\ \|y-ix\|^2 - |-i|\ \|ix+y\|^2\right) = -(ix,\ y)_{\mathbb{R}}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}(x,\ x) &= (x,\ x)_{\mathbb{R}} - i(ix,\ x)_{\mathbb{R}} \\ &= \|x^2\| - {}^i/_4\left(\|ix+x\|^2 - \|ix-x\|^2\right) \\ &= \|x\|^2\left(1 - {}^i/_4\left(|1+i|^2 - |1-i|^2\right)\right) = \|x\|^2.\end{aligned}$$

The claim of uniqueness follows from 6.1.3. ▷

6.1.10. EXAMPLES.

(1) A Hilbert space is exemplified by the L_2 space (over some system with integration), the inner product introduced as follows $(f,\ g) := \int fg^*$ for $f,\ g \in L^2$. In particular, $(x,\ y) := \sum_{e\in\mathscr{E}} x_e y_e^*$ for $x,\ y \in l_2(\mathscr{E})$.

(2) Assume that H is a pre-Hilbert space and $(\cdot,\ \cdot) : H^2 \to \mathbb{F}$ is a semi-inner product on H. It is clear that the real carrier $H_{\mathbb{R}}$ with the semi-inner product $(\cdot,\ \cdot)_{\mathbb{R}} : (x,\ y) \mapsto \operatorname{Re}(x,\ y)$ presents a pre-Hilbert space with the norm of an element of H independent of whether it is calculated in H or in $H_{\mathbb{R}}$. The pre-Hilbert space $(H_{\mathbb{R}},\ (\cdot,\ \cdot)_{\mathbb{R}})$ is the *realification* or *decomplexification* of $(H,\ (\cdot,\ \cdot))$. In turn, if the real carrier of a seminormed space is a pre-Hilbert space then the process of complexification leads to some natural pre-Hilbert structure of the original space.

(3) Assume that H is a pre-Hilbert space and H_* is the twin vector space of H. Given $x,\ y \in H_*$, put $(x,\ y)_* := (x,\ y)^*$. Clearly, $(\cdot,\ \cdot)_*$ is a semi-inner product on H_*. The resulting pre-Hilbert space is the *twin* of H, with the denotation H_* preserved.

(4) Let H be a pre-Hilbert space and let $H_0 := \ker \|\cdot\|$ be the kernel of the seminorm $\|\cdot\|$ on H. Using the Cauchy–Bunyakovskiĭ–Schwarz inequality, Theorem 2.3.8 and 6.1.10 (3), observe that there is a natural inner product on the quotient space H/H_0: If $\overline{x}_1 := \varphi(x_1)$ and $\overline{x}_2 := \varphi(x_2)$, with $x_1,\ x_2 \in H$ and $\varphi : H \to H/H_0$ the coset mapping, then $(\overline{x}_1,\ \overline{x}_2) := (x_1,\ x_2)$. Moreover, the pre-Hilbert space H/H_0 may be considered as the quotient space of the seminormed space $(H,\ \|\cdot\|)$ by the kernel of the seminorm $\|\cdot\|$. Thus, H/H_0 is a Hausdorff space referred to as the Hausdorff pre-Hilbert space *associated* with H. Completing the normed space H/H_0, obtain a Hilbert space (for instance, by the von Neumann–Jordan Theorem). The so-constructed Hilbert space is called *associated* with the original pre-Hilbert space.

(5) Assume that $(H_e)_{e\in\mathscr{E}}$ is a family of Hilbert spaces and H is the 2-sum of the family; i.e., $h \in H$ if and only if $h := (h_e)_{e\in\mathscr{E}}$, where $h_e \in H_e$ for $e \in \mathscr{E}$ and

$$\|h\| := \left(\sum_{e\in\mathscr{E}} \|h_e\|^2\right)^{1/2} < +\infty.$$

By 5.5.9 (6), H is a Banach space. Given $f,\ g \in H$, on successively applying the Parallelogram Law, deduce that

$$\begin{aligned} &{}^1/_2 \left(\|f+g\|^2 + \|f-g\|^2\right) \\ &= {}^1/_2 \left(\sum_{e\in\mathscr{E}} \|f_e + g_e\|^2 + + \sum_{e\in\mathscr{E}} \|f_e - g_e\|^2\right) \\ &= \sum_{e\in\mathscr{E}} {}^1/_2 \left(\|f_e+g_e\|^2 + \|f_e - g_e\|^2\right) = \sum_{e\in\mathscr{E}} \left(\|f_e\|^2 + \|g_e\|^2\right) = \|f\|^2 + \|g\|^2. \end{aligned}$$

Consequently, H is a Hilbert space by the von Neumann–Jordan Theorem. The space H, the *Hilbert sum* of the family $(H_e)_{e\in\mathscr{E}}$, is denoted by $\oplus_{e\in\mathscr{E}} H_e$. With $\mathscr{E} := \mathbb{N}$, it is customary to use the symbol $H_1 \oplus H_2 \oplus \dots$ for H.

(6) *Let H be a Hilbert space and let S be a system with integration. The space $L_2(S,\ H)$ comprising all H-valued square-integrable functions is also a Hilbert space.* ◁▷

6.2. Orthoprojections

6.2.1. *Let U_ε be a convex subset of the spherical layer $(r+\varepsilon)B_H \setminus rB_H$ with $r,\ \varepsilon > 0$ in a Hilbert space H. Then the diameter of U_ε vanishes as ε tends to 0.*

◁ Given $x,\ y \in U_\varepsilon$, on considering that ${}^1/_2(x+y) \in U_\varepsilon$ and applying the Parallelogram Law, for $\varepsilon \le r$ infer that

$$\begin{aligned} \|x-y\|^2 &= 2\left(\|x\|^2 + \|y\|^2\right) - 4\left\|{}^{(x+y)}/_2\right\|^2 \\ &\le 4(r+\varepsilon)^2 - 4r^2 = 8r\varepsilon + 4\varepsilon^2 \le 12r\varepsilon. \ \triangleright \end{aligned}$$

6.2.2. ***Levy Projection Theorem.*** *Let U be a nonempty closed convex set in a Hilbert space H and $x \in H \setminus U$. Then there is a unique element u_0 of U such that*

$$\|x-u_0\| = \inf\{\|x-u\| :\ u \in U\}.$$

◁ Put $U_\varepsilon := \{u \in U :\ \|x-u\| \le \inf\|U-x\| + \varepsilon\}$. By 6.2.1, the family $(U_\varepsilon)_{\varepsilon>0}$ constitutes a base for a Cauchy filter in U. ▷

6.2.3. DEFINITION. The element u_0 appearing in 6.2.2 is the *best approximation* to x in U or the *projection* of x to U.

6.2.4. *Let H_0 be a closed subspace of a Hilbert space H and $x \in H \setminus H_0$. An element x_0 of H_0 is the projection of x to H_0 if and only if $(x-x_0,\ h_0) = 0$ for every h_0 in H_0.*

◁ It suffices to consider the real carrier $(H_0)_{\mathbb{R}}$ of H_0. The convex function $f(h_0) := (h_0 - x,\ h_0 - x)$ is defined on $(H_0)_{\mathbb{R}}$. Further, x_0 in H_0 serves as the projection of x to H_0 if and only if $0 \in \partial_{x_0}(f)$. In view of 3.5.2 (4) this containment means that $(x - x_0,\ h_0) = 0$ for every h_0 in H_0, because $f'(x_0) = 2(x_0 - x,\ \cdot)$. ▷

6.2.5. DEFINITION. Elements x and y of H are *orthogonal*, in symbols $x \perp y$, if $(x,\ y) = 0$. By $U^{\perp}$ we denote the subset of H that comprises all elements orthogonal to every point of a given subset U; i.e.,

$$U^{\perp} := \{y \in H : (\forall\, x \in U)\ x \perp y\}.$$

The set $U^{\perp}$ is the *orthogonal complement* or *orthocomplement* of U (to H).

6.2.6. *Let H_0 be a closed subspace of a Hilbert space H. The orthogonal complement of H_0, the set $H_0^{\perp}$, is a closed subspace and $H = H_0 \oplus H_0^{\perp}$.*

◁ The closure property of $H_0^{\perp}$ in H is evident. It is also clear that $H_0 \wedge H_0^{\perp} = H_0 \cap H_0^{\perp} = 0$. We are left with showing only that $H_0 \vee H_0^{\perp} = H_0 + H_0^{\perp} = H$. Take an element h of $H \setminus H_0$. In virtue of 6.2.2 the projection h_0 of H to H_0 is available and, by 6.2.4, $h - h_0 \in H_0^{\perp}$. Finally, $h = h_0 + (h - h_0) \in H_0 + H_0^{\perp}$. ▷

6.2.7. DEFINITION. The projection onto a (closed) subspace H_0 along $H_0^{\perp}$ is the *orthoprojection* onto H_0, denoted by P_{H_0}.

6.2.8. Pythagoras Lemma. $x \perp y \Rightarrow \|x + y\|^2 = \|x\|^2 + \|y\|^2$. ◁▷

6.2.9. Corollary. *The norm of an orthoprojection is at most one:* $(H \neq 0\ \&\ H_0 \neq 0) \Rightarrow \|P_{H_0}\| = 1$. ◁▷

6.2.10. Orthoprojection Theorem. *For an operator P in $\mathscr{L}(H)$ such that $P^2 = P$, the following statements are equivalent:*

(1) *P is the orthoprojection onto $H_0 :=$ im P;*

(2) $\|h\| \leq 1 \Rightarrow \|Ph\| \leq 1$;

(3) *$(Px,\ P^d y) = 0$ for all $x, y \in H$, with P^d the complement of P, i.e. $P^d = I_H - P$;*

(4) *$(Px,\ y) = (x,\ Py)$ for $x,\ y \in H$.*

◁ (1) ⇒ (2): This is observed in 6.2.9.

(2) ⇒ (3): Let $H_1 :=$ ker $P =$ im P^d. Take $x \in H_1^{\perp}$. Since $x = Px + P^d x$ and $x \perp P^d x$; therefore, $\|x\|^2 \geq \|Px\|^2 = (x - P^d x,\ x - P^d x) = (x,\ x) - 2\mathrm{Re}\,(x,\ P^d x) + (P^d x,\ P^d x) = \|x\|^2 + \|P^d x\|^2$. Whence $P^d x = 0$; i.e., $x \in$ im P. Considering 6.2.6, from $H_1 =$ ker P and $H_1^{\perp} \subset$ im P deduce the equalities $H_1^{\perp} =$ im $P = H_0$. Thus $(Px,\ P^d y) = 0$ for all $x,\ y \in H$, since $Px \in H_0$ and $P^d y \in H_1$.

(3) ⇒ (4): $(Px,\ y) = (Px,\ Py + P^d y) = (Px,\ Py) = (Px,\ Py) + (P^d x,\ Py) = (x,\ Py)$.

(4) ⇒ (1): Show first that H_0 is a closed subspace. Let $h_0 := \lim h_n$ with $h_n \in H_0$, i.e. $Ph_n = h_n$. For every x in H, from the continuity property of the

functionals $(\cdot,\ x)$ and $(\cdot,\ Px)$ successively derive

$$(h_0,\ x) = \lim(h_n,\ x) = \lim(Ph_n,\ x) = \lim(h_n,\ Px) = (Ph_0,\ x).$$

Whence $(h_0 - Ph_0,\ h_0 - Ph_0) = 0$; i.e., $h_0 \in \operatorname{im} P$.

Given $x \in H$ and $h_0 \in H_0$, now infer that $(x - Px,\ h_0) = (x - Px,\ Ph_0) = (P(x - Px),\ h_0) = (Px - P^2x,\ h_0) = (Px - Px,\ h_0) = 0$. Therefore, from 6.2.4 obtain $Px = P_{H_0}x$. ▷

6.2.11. *Let P_1, and P_2 be orthoprojections with $P_1P_2 = 0$. Then $P_2P_1 = 0$.*

◁ $P_1P_2 = 0 \Rightarrow \operatorname{im} P_2 \subset \ker P_1 \Rightarrow \operatorname{im} P_1 = (\ker P_1)^\perp \subset (\operatorname{im} P_2)^\perp = \ker P_2 \Rightarrow P_2P_1 = 0$ ▷

6.2.12. DEFINITION. Orthoprojections P_1 and P_2 are *orthogonal*, in symbols, $P_1 \perp P_2$ or $P_2 \perp P_1$, provided that $P_1P_2 = 0$.

6.2.13. Theorem. *Let $P_1, \dots, P_n$ be orthoprojections. The operator $P := P_1 + \dots + P_n$ is an orthoprojection if and only if $P_l \perp P_m$ for $l \neq m$.*

◁ $\Rightarrow$: First, given an orthoprojection P_0, observe that $\|P_0x\|^2 = (P_0x,\ P_0x) = (P_0^2x,\ x) = (P_0x,\ x)$ by Theorem 6.2.10. Consequently,

$$\begin{aligned}&\|P_lx\|^2 + \|P_mx\|^2 \\ \le \sum_{k=1}^{n} \|P_kx\|^2 = \sum_{k=1}^{n} (P_kx,\ x) &= (Px,\ x) = \|Px\|^2 \le \|x\|^2\end{aligned}$$

for $x \in H$ and $l \neq m$.

In particular, putting $x := P_lx$, observe that

$$\|P_lx\|^2 + \|P_mP_lx\|^2\|P_lx\|^2 \Rightarrow \|P_mP_l\| = 0.$$

$\Leftarrow$: Straightforward calculation shows that P is an idempotent operator. Indeed,

$$P^2 = \left(\sum_{k=1}^{n} P_k\right)^2 = \sum_{l=1}^{n}\sum_{m=1}^{n} P_lP_m = \sum_{k=1}^{n} P_k^2 = P.$$

Furthermore, in virtue of 6.2.10 (4), $(P_kx,\ y) = (x,\ P_ky)$ and so $(Px,\ y) = (x,\ Py)$. It suffices to appeal to 6.2.10 (4) once again. ▷

6.2.14. REMARK. Theorem 6.2.13 is usually referred to as the *pairwise orthogonality criterion for finitely many orthoprojections.*

6.3. A Hilbert Basis

6.3.1. DEFINITION. A family $(x_e)_{e\in\mathscr{E}}$ of elements of a Hilbert space H is *orthogonal*, if $e_1 \neq e_2 \Rightarrow x_{e_1} \perp x_{e_2}$. By specification, a subset $\mathscr{E}$ of a Hilbert space H is *orthogonal* if so is the family $(e)_{e\in\mathscr{E}}$.

6.3.2. *Pythagoras Theorem.* *An orthogonal family $(x_e)_{e\in\mathscr{E}}$ of elements of a Hilbert space is (unconditionally) summable if and only if the numeric family $(\|x_e\|^2)_{e\in\mathscr{E}}$ is summable. Moreover,*

$$\left\|\sum_{e\in\mathscr{E}} x_e\right\|^2 = \sum_{e\in\mathscr{E}} \|x_e\|^2.$$

◁ Let $s_\theta := \sum_{e\in\mathscr{E}} x_e$, where θ is a finite subset of $\mathscr{E}$. By 6.2.8, $\|s_\theta\|^2 = \sum_{e\in\theta}\|x_e\|^2$. Given a finite subset θ' of $\mathscr{E}$ which includes θ, thus observe that

$$\|s_{\theta'} - s_\theta\|^2 = \|s_{\theta'\setminus\theta}\|^2 = \sum_{e\in\theta'\setminus\theta} \|x_e\|^2.$$

In other words, the fundamentalness of (s_θ) amounts to the fundamentalness of the net of partial sums of the family $(\|x_e\|^2)_{e\in\mathscr{E}}$. On using 4.5.4, complete the proof. ▷

6.3.3. *Orthoprojection Summation Theorem.* *Let $(P_e)_{e\in\mathscr{E}}$ be a family of pairwise orthogonal orthoprojections in a Hilbert space H. Then for every x in H the family $(P_e x)_{e\in\mathscr{E}}$ is (unconditionally) summable. Moreover, the operator $Px := \sum_{e\in\mathscr{E}} P_e x$ is the orthoprojection onto the subspace*

$$\mathscr{H} := \left\{\sum_{e\in\mathscr{E}} x_e \,:\, x_e \in H_e := \operatorname{im} P_e,\ \sum_{e\in\mathscr{E}} \|x_e\|^2 < +\infty\right\}.$$

◁ Given a finite subset θ of $\mathscr{E}$, put $s_\theta := \sum_{e\in\theta} P_e$. By Theorem 6.2.13, s_θ is an orthoprojection. Hence, in view of 6.2.8, $\|s_\theta x\|^2 = \sum_{e\in\theta} \|P_e x\|^2 \leq \|x\|^2$ for every x in H. Consequently, the family $(\|P_e x\|^2)_{e\in\mathscr{E}}$ is summable (the net of partial sums is increasing and bounded). By the Pythagoras Theorem, there is a sum $Px := \sum_{e\in\mathscr{E}} P_e x$; i.e., $Px = \lim_\theta s_\theta x$. Whence $P^2 x = \lim_\theta s_\theta P x = \lim_\theta s_\theta \lim_{\theta'} s_{\theta'} x = \lim_\theta \lim_{\theta'} s_\theta s_{\theta'} x = \lim_\theta \lim_{\theta'} s_{\theta\cap\theta'} x = \lim_\theta s_\theta x = Px$. Finally, $\|Px\| = \|\lim_\theta s_\theta x\| = \lim_\theta \|s_\theta x\| \leq \|x\|$ and, moreover, $P^2 = P$. Appealing to 6.2.10, conclude that P is the orthoprojection onto $\operatorname{im} P$.

If $x \in \operatorname{im} P$, i.e., $Px = x$; then $x = \sum_{e\in\mathscr{E}} P_e x$ and by the Pythagoras Theorem $\sum_{e\in\mathscr{E}} \|P_e x\|^2 = \|x\|^2 = \|Px\|^2 < +\infty$. Since $P_e x \in H_e$ $(e \in \mathscr{E})$; therefore, $x \in \mathscr{H}$. If $x_e \in H_e$ and $\sum_{e\in\mathscr{E}} \|x_e\|^2 < +\infty$ then for $x := \sum_{e\in\mathscr{E}} x_e$ (existence follows from the same Pythagoras Theorem) observe that $x = \sum_{e\in\mathscr{E}} x_e = \sum_{e\in\mathscr{E}} P_e x_e = Px$; i.e., $x \in \operatorname{im} P$. Thus, $\operatorname{im} P = \mathscr{H}$. ▷

6.3.4. REMARK. This theorem may be treated as asserting that the space $\mathscr{H}$ and the Hilbert sum of the family $(H_e)_{e\in\mathscr{E}}$ are isomorphic. The identification is clearly accomplished by the Bochner integral presenting the process of summation in this case.

6.3.5. REMARK. Let h in H be a *normalized* or *unit* or *norm-one element*; i.e., $\|h\| = 1$. Assume further that $H_0 := \mathbb{F}h$ is a one-dimensional subspace of H spanned over h_0. For every element x of H and every scalar λ, a member of $\mathbb{F}$, observe that

$$(x - (x,\ h)h,\ \lambda h) = \lambda^*((x,\ h) - (x,\ h))(h,\ h) = 0.$$

Therefore, by 6.2.4, $P_{H_0} = (\cdot,\ h) \otimes h$. To denote this orthoprojection, it is convenient to use the symbol $\langle h\rangle$. Thus, $\langle h\rangle : x \mapsto (x,\ h)h\ (x \in H)$.

6.3.6. DEFINITION. A family of elements of a Hilbert space is called *orthonormal* (or *orthonormalized*) if, first, the family is orthogonal and, second, the norm of each member of the family equals one. *Orthonormal sets* are defined by specification.

6.3.7. *For every orthonormal subset $\mathscr{E}$ of H and every element x of H, the family $(\langle e\rangle x)_{e\in\mathscr{E}}$ is (unconditionally) summable. Moreover, the Bessel inequality holds*:

$$\|x\|^2 \geq \sum_{e\in\mathscr{E}} |(x,\ e)|^2.$$

◁ It suffices to refer to the Orthoprojection Summation Theorem, for

$$\|x\|^2 \geq \left\|\sum_{e\in\mathscr{E}} \langle e\rangle x\right\|^2 = \left\|\sum_{e\in\mathscr{E}} (x,\ e)e\right\|^2 = \sum_{e\in\mathscr{E}} \|(x,\ e)e\|^2 . \ ▷$$

6.3.8. DEFINITION. An orthonormal set $\mathscr{E}$ in a Hilbert space H is a *Hilbert basis* (for H) if $x = \sum_{e\in\mathscr{E}}\langle e\rangle x$ for every x in H. An orthonormal family of elements of a Hilbert space is a Hilbert basis if the range of the family is a Hilbert basis.

6.3.9. *An orthonormal set $\mathscr{E}$ is a Hilbert basis for H if and only if* $\operatorname{lin}(\mathscr{E})$, *the linear span of $\mathscr{E}$, is dense in H.* ◁▷

6.3.10. DEFINITION. A subset $\mathscr{E}$ of a Hilbert space is said to meet the *Steklov condition* if $\mathscr{E}^\perp = 0$.

6.3.11. ***Steklov Theorem.*** *An orthonormal set $\mathscr{E}$ is a Hilbert basis if and only if $\mathscr{E}$ meets the Steklov condition.*

◁ $\Rightarrow$: Let $h \in \mathscr{E}^\perp$. Then $h = \sum_{e\in\mathscr{E}}\langle e\rangle h = \sum_{e\in\mathscr{E}}(h,\ e)e = \sum_{e\in\mathscr{E}} 0 = 0$.

$\Leftarrow$: For $x \in H$, in virtue of 6.3.3 and 6.2.4, $x - \sum_{e\in\mathscr{E}}\langle e\rangle x \in \mathscr{E}^\perp$. ▷

6.3.12. ***Theorem.*** *Each Hilbert space has a Hilbert basis.*

◁ By the Kuratowski–Zorn Lemma, each Hilbert space H has an orthonormal set $\mathscr{E}$ maximal by inclusion. If there were some h in $H \setminus H_0$, with $H_0 := \mathrm{cl}\,\mathrm{lin}\,(\mathscr{E})$; then the element $h_1 := h - P_{H_0}h$ would be orthogonal to every element in $\mathscr{E}$. Thus, for $H \neq 0$ we would have $\mathscr{E} \cup \{\|h_1\|^{-1}h_1\} = \mathscr{E}$. A contradiction. In case $H = 0$ there is nothing left to proof. ▷

6.3.13. REMARK. It is possible to show that two Hilbert bases for a Hilbert space H have the same cardinality. This cardinality is the *Hilbert dimension* of H.

6.3.14. REMARK. Let $(x_n)_{n\in\mathbb{N}}$ be a countable sequence of linearly independent elements of a Hilbert space H. Put $x_0 := 0$, $e_0 := 0$ and

$$y_n := x_n - \sum_{k=0}^{n-1} \langle e_k \rangle x_n, \quad e_n := \frac{y_n}{\|y_n\|} \quad (n \in \mathbb{N}).$$

Evidently, $(y_n,\ e_k) = 0$ for $0 \leq k \leq n-1$ (for instance, by 6.2.13). Also, $y_n \neq 0$, since H is infinite-dimensional. Say that the orthonormal sequence $(e_n)_{n\in\mathbb{N}}$ results from the sequence $(x_n)_{n\in\mathbb{N}}$ by the *Gram–Schmidt orthogonalization process.* Using the process, it is easy to prove that a Hilbert space has a countable Hilbert basis if and only if the space has a countable dense subset; i.e., whenever the space is *separable.* ◁▷

6.3.15. DEFINITION. Let $\mathscr{E}$ be a Hilbert basis for a space H and $x \in H$. The numeric family $\widehat{x} := (\widehat{x}_e)_{e\in\mathscr{E}}$ in $\mathbb{F}^{\mathscr{E}}$, given by the identity $\widehat{x}_e := (x,\ e)$, is the *Fourier coefficient family* of x with respect to $\mathscr{E}$ or the *Fourier transform* of X (relative to $\mathscr{E}$).

6.3.16. *Riesz–Fisher Isomorphism Theorem.* *Let $\mathscr{E}$ be a Hilbert basis for H. The Fourier transform $\mathscr{F} : x \mapsto \widehat{x}$ (relative to $\mathscr{E}$) is an isometric isomorphism of H onto $l_2(\mathscr{E})$. The inverse Fourier transform, the Fourier summation $\mathscr{F}^{-1} : l_2(\mathscr{E}) \to H$, acts by the rule $\mathscr{F}^{-1}(x) := \sum_{e\in\mathscr{E}} x_e e$ for $x := (x_e)_{e\in\mathscr{E}} \in l_2(\mathscr{E})$. Moreover, for all $x,\ y \in H$ the Parseval identity holds:*

$$(x,\ y) = \sum_{e\in\mathscr{E}} \widehat{x}_e \widehat{y}_e^{\,*}.$$

◁ By the Pythagoras Theorem, the Fourier transform acts in $l_2(\mathscr{E})$. By Theorem 6.3.3, $\widehat{\ }$ is an epimorphism. By the Steklov Theorem, $\widehat{\ }$ is a monomorphism. It is beyond a doubt that $\mathscr{F}^{-1}\widehat{x} = x$ for $x \in H$ and $\widehat{\mathscr{F}^{-1}(x)} = x$ for $x \in l_2(\mathscr{E})$. The equality

$$\|x\|^2 = \sum_{e\in\mathscr{E}} \|\widehat{x}\|^2 = \|\widehat{x}_e\|_2^2 \quad (x \in H)$$

follows from the Pythagoras Theorem. At the same time

$$(x,\ y) = \left(\sum_{e\in\mathscr{E}} \widehat{x}_e e,\ \sum_{e\in\mathscr{E}} \widehat{y}_e e\right) = \sum_{e,e'\in\mathscr{E}} \widehat{x}_e \widehat{y}^*_{e'}(e,\ e') = \sum_{e\in\mathscr{E}} \widehat{x}_e \widehat{y}^*_e. \ \triangleright$$

6.3.17. REMARK. The Parseval identity shows that the Fourier transform preserves inner products. Therefore, the Fourier transform is a *unitary operator* or a *Hilbert-space isomorphism*; i.e., an isomorphism preserving inner products. This is why the Riesz–Fisher Theorem is sometimes referred to as the *theorem on Hilbert isomorphy* between Hilbert spaces (of the same Hilbert dimension).

6.4. The Adjoint of an Operator

6.4.1. ***Riesz Prime Theorem.*** *Let H be a Hilbert space. Given $x \in H$, put $x' := (\cdot,\ x)$. Then the prime mapping $x \mapsto x'$ presents an isometric isomorphism of H_* onto H'.*

$\triangleleft$ It is clear that $x = 0 \Rightarrow x' = 0$. If $x \neq 0$ then

$$\|y'\|_{H'} = \sup_{\|y\|\le 1} |(y,\ x)| \le \sup_{\|y\|\le 1} \|y\|\ \|x\| \le \|x\|;$$

$$\|x'\|_{H'} = \sup_{\|y\|\le 1} |(y,\ x)| \ge \left|\left({}^{x}/_{\|x\|},\ x\right)\right| = \|x\|.$$

Therefore, $x \mapsto x'$ is an isometry of H_* into H'. Check that this mapping is an epimorphism.

Let $l \in H'$ and $H_0 := \ker l \neq H$ (if there no such l then nothing is to be proven). Choose a norm-one element e in $H_0^\perp$ and put $\operatorname{grad} l := l(e)^* e$. If $x \in H_0$ then

$$(\operatorname{grad} l)'(x) = (x,\ \operatorname{grad} l) = (x,\ l(e)^* e) = l(e)^{**}(x,\ e) = 0.$$

Consequently, for some α in $\mathbb{F}$ and all $x \in H$ in virtue of 2.3.12 $(\operatorname{grad} l)'(x) = \alpha l(x)$. In particular, letting $x := e$, find

$$(\operatorname{grad} l)'(e) = (e,\ \operatorname{grad} l) = l(e)(e,\ e) = \alpha l(e);$$

i.e., $\alpha = 1$. $\triangleright$

6.4.2. REMARK. From the Riesz Prime Theorem it follows that the dual space H' possesses a natural structure of a Hilbert space and the prime mapping $x \mapsto x'$ implements a Hilbert space isomorphism from H_* onto H'. The inverse mapping now coincides with the *gradient mapping* $l \mapsto \operatorname{grad} l$ constructed in the proof of the theorem. Implying this, the claim of 6.4.1 is referred to as the *theorem on the general form of a linear functional in a Hilbert space.*

6.4.3. *Each Hilbert space is reflexive.*

◁ Let $\iota : H \to H''$ be the double prime mapping; i.e. the canonical embedding of H into the second dual H'' which is determined by the rule $x''(l) = \iota(x)(l) = l(x)$, where $x \in H$ and $l \in H'$ (see 5.1.10 (8)). Check that ι is an epimorphism. Let $f \in H''$. Consider the mapping $y \mapsto f(y')$ for $y \in H$. It is clear that this mapping is a linear functional over H_* and so by the Riesz Prime Theorem there is an element $x \in H = H_{**}$ such that $(y,\ x)_* = (x,\ y) = f(y')$ for every y in H. Observe that $\iota(x)(y') = y'(x) = (x,\ y) = f(y')$ for all $y \in H$. Since by the Riesz Prime Theorem $y \mapsto y'$ is a mapping onto H', conclude that $\iota(x) = f$. ▷

6.4.4. *Let H_1 and H_2 be Hilbert spaces and $T \in B(H_1,\ H_2)$. Then there is a unique mapping $T^* : H_2 \to H_1$ such that*

$$(Tx,\ y) = (x,\ T^*y)$$

for all $x \in H_1$ and $y \in H_2$. Moreover, $T^* \in B(H_2,\ H_1)$ and $\|T^*\| = \|T\|$.

◁ Let $y \in H_2$. The mapping $x \mapsto (Tx,\ y)$ is the composition $y' \circ T$; i.e., it presents a continuous linear functional over H_1. By the Riesz Prime Theorem there is precisely one element x of H_1 for which $x' = y' \circ T$. Put $T^*y := x$. It is clear that $T^* \in \mathscr{L}(H_2,\ H_1)$. Furthermore, using the Cauchy–Bunyakovskiĭ–Schwarz inequality and the normative inequality, infer that

$$|(T^*y,\ T^*y)| = |(TT^*y,\ y)| \leq \|TT^*y\|\ \|y\| \leq \|T\|\ \|T^*y\|\ \|y\|.$$

Hence, $\|T^*y\| \leq \|T\|\ \|y\|$ for all $y \in H_2$; i.e., $\|T^*\| \leq \|T\|$. At the same time $T = T^{**} := (T^*)^*$; i.e., $\|T\| = \|T^{**}\| \leq \|T^*\|$. ▷

6.4.5. Definition. For $T \in B(H_1,\ H_2)$, the operator T^*, the member of $B(H_2,\ H_1)$ constructed in 6.4.4, is the *adjoint* of T. The terms like "hermitian-conjugate" and "Hilbert-space adjoint" are also in current usage.

6.4.6. *Let H_1 and H_2 be Hilbert spaces. Assume further that $S,\ T \in B(H_1,\ H_2)$ and $\lambda \in \mathbb{F}$. Then*

(1) $T^{**} = T$;

(2) $(S + T)^* = S^* + T^*$;

(3) $(\lambda T)^* = \lambda^* T^*$;

(4) $\|T^*T\| = \|T\|^2$.

◁ (1)–(3) are obvious properties. If $\|x\| \leq 1$ then

$$\|Tx\|^2 = (Tx,\ Tx) = |(Tx,\ Tx)| = |(T^*Tx,\ x)| \leq \|T^*Tx\|\ \|x\| \leq \|T^*T\|.$$

Furthermore, using the submultiplicativity of the operator norm and 6.4.4, infer $\|T^*T\| \leq \|T^*\|\ \|T\| = \|T\|^2$, which proves (4). ▷

6.4.7. *Let H_1, H_2, and H_3 be three Hilbert spaces. Assume further that $T \in B(H_1, H_2)$ and $S \in B(H_2, H_3)$. Then $(ST)^* = T^*S^*$.*

◁ $(STx, z) = (Tx, S^*z) = (x, T^*S^*z) \quad (x \in H_1, z \in H_3)$ ▷

6.4.8. DEFINITION. Consider an elementary diagram $H_1 \xrightarrow{T} H_2$. The diagram $H_1 \xleftarrow{T^*} H_2$ is the *adjoint* of the initial elementary diagram. Given an arbitrary diagram composed of bounded linear mappings between Hilbert spaces, assume that each elementary subdiagram is replaced with its adjoint. Then the resulting diagram is the *adjoint* or, for suggestiveness, the *diagram star* of the initial diagram.

6.4.9. *Diagram Star Principle.* *A diagram is commutative if and only if so is its adjoint diagram.*

◁ Follows from 6.4.7 and 6.4.6 (1). ▷

6.4.10. *Corollary.* *An operator T is invertible if and only if T^* is invertible. Moreover, $T^{*-1} = T^{-1*}$.* ◁▷

6.4.11. *Corollary.* *If $T \in B(H)$ then $\lambda \in \operatorname{Sp}(T) \Leftrightarrow \lambda^* \in \operatorname{Sp}(T^*)$.* ◁▷

6.4.12. *Sequence Star Principle* (cf. 7.6.13). *A sequence*

$$\ldots \to H_{k-1} \xrightarrow{T_k} H_k \xrightarrow{T_{k+1}} H_{k+1} \to \ldots$$

is exact if and only if so is the sequence star

$$\ldots \leftarrow H_{k-1} \xleftarrow{T_k^*} H_k \xleftarrow{T_{k+1}^*} H_{k+1} \leftarrow \ldots .$$ ◁▷

6.4.13. DEFINITION. An *involutive algebra* or *$*$-algebra* A (over a ground field $\mathbb{F}$) is an algebra with an *involution* $*$, i.e. with a mapping $a \mapsto a^*$ from A to A such that

(1) $a^{**} = a \ (a \in A)$;
(2) $(a+b)^* = a^* + b^* \ (a, b \in A)$;
(3) $(\lambda a)^* = \lambda^* a^* \ (\lambda \in \mathbb{F}, a \in A)$;
(4) $(ab)^* = b^* a^* \ (a, b \in A)$.

A Banach algebra A with involution $*$ satisfying $\|a^*a\| = \|a\|^2$ for all $a \in A$ is a *C^*-algebra.*

6.4.14. *The endomorphism space $B(H)$ of a Hilbert space H is a C^*-algebra (with the composition of operators as multiplication and the taking of the adjoint of an endomorphism as involution).* ◁▷

6.5. Hermitian Operators

6.5.1. DEFINITION. Let H be a Hilbert space over a ground field $\mathbb{F}$. An element T of $B(H)$ is a *hermitian operator* or *selfadjoint operator* in H provided that $T = T^*$.

6.5.2. ***Rayleigh Theorem.*** *For a hermitian operator T the equality holds:*

$$\|T\| = \sup_{\|x\| \le 1} |(Tx,\ x)|.$$

$\lhd$ Put $t := \sup\{|(Tx,\ x)| : \ \|x\| \le 1\}$. It is clear that $|(Tx,\ x)| \le \|Tx\|\ \|x\| \le \|T\|$ provided $\|x\| \le 1$. Thus, $t \le \|T\|$.

Since $T = T^*$; therefore, $(Tx,\ y) = (x,\ Ty) = (Ty,\ x)^* = (y,\ Tx)^*$; i.e., $(x,\ y) \mapsto (Tx,\ y)$ is a hermitian form. Consequently, in virtue of 6.1.3 and 6.1.8

$$\begin{aligned} 4\mathrm{Re}\,(Tx,\ y) &= (T(x+y),\ x+y) - (T(x-y),\ x-y) \\ &\le t(\|x+y\|^2 + \|x-y\|^2) = 2t(\|x\|^2 + \|y\|^2). \end{aligned}$$

If $Tx = 0$ then it is plain that $\|Tx\| \le t$. Assume $Tx \ne 0$. Given $\|x\| \le 1$ and putting $y := \|Tx\|^{-1}Tx$, infer that

$$\begin{aligned} \|Tx\| &= \|Tx\| \left(\frac{Tx}{\|Tx\|},\ \frac{Tx}{\|Tx\|} \right) \\ &= (Tx,\ y) = \mathrm{Re}\,(Tx,\ y) \le {}^1\!/_2\, t \left(\|x\|^2 + \left\| {}^{Tx}\!/_{\|Tx\|} \right\|^2 \right) \le t; \end{aligned}$$

i.e., $\|T\| = \sup\{\|Tx\| : \ \|x\| \le 1\} \le t$. $\rhd$

6.5.3. REMARK. As mentioned in the proof of 6.5.2, each hermitian operator T in a Hilbert space H generates the hermitian form $f_T(x,\ y) := (Tx,\ y)$. Conversely, let f be a hermitian form, with the functional $f(\,\cdot\,,\ y)$ continuous for every y in H. Then by the Riesz Prime Theorem there is an element Ty of H such that $f(\,\cdot\,,\ y) = (Ty)'$. Evidently, $T \in \mathscr{L}(H)$ and $(x,\ Ty) = f(x,\ y) = f(y,\ x)^* = (y,\ Tx)^* = (Tx,\ y)$. It is possible to show that in this case $T \in B(H)$ and $T = T^*$. In addition, $f = f_T$. Therefore, the condition $T \in B(H)$ in Definition 6.5.1 can be replaced with the condition $T \in \mathscr{L}(H)$ (the *Hellinger–Toeplitz Theorem*).

6.5.4. ***Weyl Criterion.*** *A scalar λ belongs to the spectrum of a hermitian operator T if and only if*

$$\inf_{\|x\|=1} \|\lambda x - Tx\| = 0.$$

$\lhd$ $\Rightarrow$: Put $t := \inf\{\|\lambda x - Tx\| : \ x \in H,\ \|x\| = 1\} > 0$. Demonstrate that $\lambda \notin \mathrm{Sp}(T)$. Given an x in H, observe that $\|\lambda x - Tx\| \ge t\|x\|$. Thus, first,

$(\lambda - T)$ is a monomorphism; second, $H_0 := \operatorname{im}(\lambda - T)$ is a closed subspace (because $\|(\lambda-T)x_m - (\lambda-T)x_k\| \geq t\|x_m - x_k\|$; i.e., the inverse image of a Cauchy sequence is a Cauchy sequence); and, third, which is final, $(\lambda - T)^{-1} \in B(H)$ whenever $H = H_0$ (in such a situation $\|R(T,\ \lambda)\| \leq t^{-1}$). Suppose to the contrary that $H \neq H_0$. Then there is some y in $H_0^\perp$ satisfying $\|y\| = 1$. For all $x \in H$, note that $0 = (\lambda x - Tx,\ y) = (x,\ \lambda^* y - Ty)$; i.e., $\lambda^* y = Ty$. Further, $\lambda^* = (Ty,\ y)/(y,\ y)$ and the hermiticity of T guarantees $\lambda^* \in \mathbb{R}$. Whence $\lambda^* = \lambda$ and $y \in \ker(\lambda - T)$. We arrive at a contradiction: $1 = \|y\| = \|0\| = 0$.

$\Leftarrow$: If $\lambda \notin \operatorname{Sp}(T)$ then the resolvent of T at λ, the member $R(T,\ \lambda)$ of $B(H)$, is available. Hence, $\inf\{\|\lambda x - Tx\| :\ \|x\| = 1\} \geq \|R(T,\ \lambda)\|^{-1}$. ▷

6.5.5. *Spectral Endpoint Theorem.* *Let T be a hermitian operator in a Hilbert space. Put*

$$m_T := \inf_{\|x\|=1} (Tx,\ x), \quad M_T := \sup_{\|x\|=1} (Tx,\ x).$$

Then $\operatorname{Sp}(T) \subset [m_T,\ M_T]$ *and* $m_T,\ M_T \in \operatorname{Sp}(T)$.

◁ Considering that the operator $T - \operatorname{Re}\lambda$ is hermitian in the space H under study, from the identity

$$\|\lambda x - Tx\|^2 = |\operatorname{Im}\lambda|^2\|x\|^2 + \|Tx - \operatorname{Re}\lambda x\|^2$$

infer the inclusion $\operatorname{Sp}(T) \subset \mathbb{R}$ by 6.5.4. Given a norm-one element x of H and invoking the Cauchy–Bunyakovskiĭ–Schwarz inequality, in case $\lambda < m_T$ deduce that

$$\begin{aligned}\|\lambda x - Tx\| &= \|\lambda x - Tx\|\ \|x\| \geq |(\lambda x - Tx,\ x)| \\ &= |\lambda - (Tx,\ x)| = (Tx,\ x) - \lambda \geq m_T - \lambda > 0.\end{aligned}$$

On appealing to 6.5.4, find $\lambda \in \operatorname{res}(T)$. In case $\lambda > M_T$, similarly infer that

$$\|\lambda x - Tx\| \geq |(\lambda x - Tx,\ x)| = |\lambda - (Tx,\ x)| = \lambda - (Tx,\ x) \geq \lambda - M_T > 0.$$

Once again $\lambda \in \operatorname{res}(T)$. Finally, $\operatorname{Sp}(T) \subset [m_T,\ M_T]$.

Since $(Tx,\ x) \in \mathbb{R}$ for $x \in H$; therefore, in virtue of 6.5.2

$$\begin{aligned}\|T\| &= \sup\{|(Tx,\ x)| :\ \|x\| \leq 1\} \\ &= \sup\{(Tx,\ x) \vee (-(Tx,\ x)) :\ \|x\| \leq 1\} = M_T \vee (-m_T).\end{aligned}$$

Assume first that $\lambda := \|T\| = M_T$. If $\|x\| = 1$ then

$$\|\lambda x - Tx\|^2 = \lambda^2 - 2\lambda(Tx,\ x) + \|Tx\|^2 \leq 2\|T\|^2 - 2\|T\|(Tx,\ x).$$

In other words, the next estimate holds:

$$\inf_{\|x\|=1} \|\lambda x - Tx\|^2 \le 2\|T\| \inf_{\|x\|=1} (\|T\| - (Tx,\ x)) = 0.$$

Using 6.5.4, conclude that $\lambda \in \mathrm{Sp}(T)$.

Now consider the operator $S := T - m_T$. It is clear that $M_S = M_T - m_T \ge 0$ and $m_S = m_T - m_T = 0$. Therefore, $\|S\| = M_S$ and in view of the above $M_S \in \mathrm{Sp}(S)$. Whence it follows that M_T belongs to $\mathrm{Sp}(T)$, since $T = S + m_T$ and $M_T = M_S + m_T$. It suffices to observe that $m_T = -M_{-T}$ and $\mathrm{Sp}(T) = -\mathrm{Sp}(-T)$. ▷

6.5.6. Corollary. *The norm of a hermitian operator equals the radius of its spectrum (and the spectral radius).* ◁▷

6.5.7. Corollary. *A hermitian operator is zero if and only if its spectrum consists of zero.* ◁▷

6.6. Compact Hermitian Operators

6.6.1. DEFINITION. Let X and Y be Banach spaces. An operator T, a member of $\mathscr{L}(X,\ Y)$, is called *compact* (in symbols, $T \in \mathscr{K}(X,\ Y)$) if the image $T(B_X)$ of the unit ball B_X of X is relatively compact in Y.

6.6.2. REMARK. Detailed study of compact operators in Banach spaces is the purpose of the Riesz–Schauder theory to be exposed in Chapter 8.

6.6.3. *Let T be a compact hermitian operator. If $0 \ne \lambda \in \mathrm{Sp}(T)$ then λ is an eigenvalue of T; i.e., $\ker(\lambda - T) \ne 0$.*

◁ By the Weyl Criterion, $\lambda x_n - Tx_n \to 0$ for some sequence (x_n) such that $\|x_n\| = 1$. Without loss of generality, assume that the sequence (Tx_n) converges to $y := \lim Tx_n$. Then from the identity $\lambda x_n = (\lambda x_n - Tx_n) + Tx_n$ obtain that there is a limit (λx_n) and $y = \lim \lambda x_n$. Consequently, $Ty = T(\lim \lambda x_n) = \lambda \lim Tx_n = \lambda y$. Since $\|y\| = |\lambda|$, conclude that y is an *eigenvector* of T, i.e. $y \in \ker(\lambda - T)$. ▷

6.6.4. *Let λ_1 and λ_2 be distinct eigenvalues of a hermitian operator T. Assume further that x_1 and x_2 are eigenvectors with eigenvalues λ_1 and λ_2 (i.e., $x_s \in \ker(\lambda_s - T)$, $s := 1,\ 2$). Then x_1 and x_2 are orthogonal.*

◁ $(x_1,\ x_2) = \frac{1}{\lambda_1}(Tx_1,\ x_2) = \frac{1}{\lambda_1}(x_1,\ Tx_2) = \frac{\lambda_2}{\lambda_1}(x_1,\ x_2)$ ▷

6.6.5. *For whatever strictly positive ε, there are only finitely many eigenvalues of a compact hermitian operator beyond the interval $[-\varepsilon,\ \varepsilon]$.*

◁ Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of pairwise distinct eigenvalues of T satisfying $|\lambda_n| > \varepsilon$. Further, let x_n be an eigenvector corresponding to λ_n and such that $\|x_n\| = 1$. By virtue of 6.6.4 $(x_k,\ x_m) = 0$ for $m \ne k$. Consequently,

$$\|Tx_m - Tx_k\|^2 = \|Tx_m\|^2 + \|Tx_k\|^2 = \lambda_m^2 + \lambda_k^2 \ge 2\varepsilon^2;$$

i.e., the sequence $(Tx_n)_{n\in\mathbb{N}}$ is relatively compact. We arrive at a contradiction to the compactness property of T. ▷

6.6.6. *Spectral Decomposition Lemma.* *Let T be a compact hermitian operator in a Hilbert space H and $0 \neq \lambda \in \mathrm{Sp}(T)$. Put $H_\lambda := \ker(\lambda - T)$. Then H_λ is finite-dimensional and the decomposition $H = H_\lambda \oplus H_\lambda^\perp$ reduces T. Moreover, the matrix presentation holds*

$$T \sim \begin{pmatrix} \lambda & 0 \\ 0 & T_\lambda \end{pmatrix},$$

where the operator T_λ, the part of T in $H_\lambda^\perp$, is hermitian and compact, with $\mathrm{Sp}(T_\lambda) = \mathrm{Sp}(T) \setminus \{\lambda\}$.

◁ The subspace H_λ is finite-dimensional in view of the compactness of T. Furthermore, H_λ is invariant under T. Consequently, the orthogonal complement $H_\lambda^\perp$ of H_λ is an invariant subspace of T^* (coincident with T), since $h \in H_\lambda^\perp \Rightarrow (\forall x \in H_\lambda)\ x \perp h \Rightarrow (\forall x \in H_\lambda)\ 0 = (h,\ Tx) = (T^*h,\ x) \Rightarrow T^*h \in H_\lambda^\perp$.

The part of T in H_λ is clearly λ. The part T_λ of T in $H_\lambda^\perp$ is undoubtedly compact and hermitian. Obviously, for $\mu \neq \lambda$, the operator

$$\mu - T \sim \begin{pmatrix} \mu - \lambda & 0 \\ 0 & \mu - T_\lambda \end{pmatrix}$$

is invertible if and only if so is $\mu - T_\lambda$. It is also clear that λ is not an eigenvalue of T_λ. ▷

6.6.7. *Hilbert–Schmidt Theorem.* *Let H be a Hilbert space and let T be a compact hermitian operator in H. Assume further that P_λ is the orthoprojection onto* $\ker(\lambda - T)$ *for $\lambda \in \mathrm{Sp}(T)$. Then*

$$T = \sum_{\lambda\in\mathrm{Sp}(T)} \lambda P_\lambda.$$

◁ Using 6.5.6 and 6.6.6 as many times as need be, for every finite subset θ of $\mathrm{Sp}(T)$ obtain the equality

$$\left\| T - \sum_{\lambda\in\theta} \lambda P_\lambda \right\| = \sup\{|\lambda| :\ \lambda \in (\mathrm{Sp}(T) \cup 0) \setminus \theta\}.$$

It suffices to refer to 6.6.5. ▷

6.6.8. REMARK. The Hilbert–Schmidt Theorem provides essentially new information, as compared with the case of finite dimensions, only if the operator T has *infinite-rank*, that is, the dimension of its range is infinite or, which is the same, $H_0^\perp$ is an infinite-dimensional space, where $H_0 := \ker T$. In fact, if the operator T has *finite rank* (i.e., its range is finite-dimensional) then, since the subspace $H_0^\perp$ is isomorphic with the range of T, observe that

$$T = \sum_{k=1}^{n} \lambda_k \langle e_k \rangle = \sum_{k=1}^{n} \lambda_k e_k' \otimes e_k,$$

where $\lambda_1, \dots, \lambda_n$ are nonzero points of $\mathrm{Sp}(T)$ counted with multiplicity, and $\{e_1, \dots, e_n\}$ is a properly-chosen orthonormal basis for $H_0^\perp$.

The Hilbert–Schmidt Theorem shows that, to within substitution of series for sum, an infinite-rank compact hermitian operator looks like a finite-rank operator. Indeed, for $\lambda \neq \mu$, where λ and μ are nonzero points of $\mathrm{Sp}(T)$, the eigenspaces H_λ and H_μ are finite-dimensional and orthogonal. Moreover, the Hilbert sum $\oplus_{\lambda \in \mathrm{Sp}(T) \setminus 0} H_\lambda$ equals $H_0^\perp = \mathrm{cl\,im}\, T$, because $H_0 = (\mathrm{im}\, T)^\perp$. Successively selecting a basis for each finite-dimensional space H_λ by enumerating the eigenvalues in decreasing order of magnitudes with multiplicity counted; i.e., putting $\lambda_1 := \lambda_2 := \ldots := \lambda_{\dim H_{\lambda_1}} := \lambda_1$, $\lambda_{\dim H_{\lambda_1+1}} := \ldots := \lambda_{\dim H_{\lambda_1} + \dim H_{\lambda_2}} := \lambda_2$, etc., obtain the decomposition $H = H_0 \oplus H_{\lambda_1} \oplus H_{\lambda_2} \oplus \ldots$ and the presentation

$$T = \sum_{k=1}^{\infty} \lambda_k \langle e_k \rangle = \sum_{k=1}^{\infty} \lambda_k e_k' \otimes e_k,$$

where the series is summed in operator norm. ◁▷

6.6.9. Theorem. *Let T in $\mathscr{K}(H_1, H_2)$ be an infinite-rank compact operator from a Hilbert space H_1 to a Hilbert space H_2. There are orthonormal families $(e_k)_{k \in \mathbb{N}}$ in H_1, $(f_k)_{k \in \mathbb{N}}$ in H_2, and a numeric family $(\mu_k)_{k \in \mathbb{N}}$ in $\mathbb{R}_+ \setminus 0$, $\mu_k \downarrow 0$, such that the following presentation holds*:

$$T = \sum_{k=1}^{\infty} \mu_k e_k' \otimes f_k.$$

◁ Put $S := T^*T$. It is clear that $S \in B(H_1)$ and S is compact. Furthermore, $(Sx, x) = (T^*Tx, x) = (Tx, Tx) = \|Tx\|^2$. Consequently, in virtue of 6.4.6, S is hermitian and $H_0 := \ker S = \ker T$. Observe also that $\mathrm{Sp}(S) \subset \mathbb{R}_+$ by Theorem 6.5.5.

Let $(e_k)_{k \in \mathbb{N}}$ be an orthonormal basis for $H_0^\perp$ comprising all eigenvalues of S and let $(\lambda_k)_{k \in \mathbb{N}}$ be a corresponding decreasing sequence of strictly positive eigenvalues $\lambda_k > 0$, $k \in \mathbb{N}$ (cf. 6.6.8). Then every element $x \in H_1$ expands into the

Fourier series

$$x - P_{H_0}x = \sum_{k=1}^{\infty}(x,\ e_k)e_k.$$

Therefore, considering that $TP_{H_0} = 0$ and assigning $\mu_k := \sqrt{\lambda_k}$ and $f_k := \mu_k^{-1}Te_k$, find

$$Tx = \sum_{k=1}^{\infty}(x,\ e_k)Te_k = \sum_{k=1}^{\infty}(x,\ e_k)\frac{\mu_k}{\mu_k}Te_k = \sum_{k=1}^{\infty}\mu_k(x,\ e_k)f_k.$$

The family $(f_k)_{k\in\mathbb{N}}$ is orthonormal because

$$\begin{aligned}(f_n,\ f_m) &= \left(\frac{Te_n}{\mu_n},\ \frac{Te_m}{\mu_m}\right) = \frac{1}{\mu_n\mu_m}(Te_n,\ Te_m) \\ &= \frac{1}{\mu_n\mu_m}(T^*Te_n,\ e_m) = \frac{1}{\mu_n\mu_m}(Se_n,\ e_m) \\ &= \frac{1}{\mu_n,\ \mu_m}(\lambda_n e_n,\ e_m) = \frac{\mu_n}{\mu_m}(e_n,\ e_m).\end{aligned}$$

Successively using the Pythagoras Theorem and the Bessel inequality, argue as follows:

$$\left\|\left(T - \sum_{k=1}^{n}\mu_k e'_k \otimes f_k\right)x\right\|^2 = \left\|\sum_{k=n+1}^{\infty}\mu_k(x,\ e_k)f_k\right\|^2$$
$$= \sum_{k=n+1}^{\infty}\mu_k^2|(x,\ e_k)|^2 \le \lambda_{n+1}\sum_{k=n+1}^{\infty}|(x,\ e_k)|^2 \le \lambda_{n+1}\|x\|^2.$$

Since $\lambda_k \downarrow 0$, finally deduce that

$$\left\|T - \sum_{k=1}^{n}\mu_k e'_k \otimes f_k\right\| \le \mu_{n+1} \to 0. \ \triangleright$$

6.6.10. REMARK. Theorem 6.6.9 means in particular that a compact operator (and only a compact operator) is an adherent point of the set of finite-rank operators. This fact is also expressed as follows: "Every Hilbert space possesses the approximation property."

Exercises

6.1. Describe the extreme points of the unit ball of a Hilbert space.

6.2. Find out which classical Banach spaces are Hilbert spaces and which are not.

6.3. Is a quotient space of a Hilbert space also a Hilbert space?

6.4. Is it true that each Banach space may be embedded into a Hilbert space?

6.5. Is it possible that the (bounded) endomorphism space of a Hilbert space presents a Hilbert space?

6.6. Describe the second (= repeated) orthogonal complement of a set in a Hilbert space.

6.7. Prove that no Hilbert basis for an infinite-dimensional Hilbert space is a Hamel basis.

6.8. Find the best approximation to a polynomial of degree $n+1$ by polynomials of degree at most n in the L_2 space on an interval.

6.9. Prove that $x \perp y$ if and only if $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ and $\|x+iy\|^2 = \|x\|^2 + \|y\|^2$.

6.10. Given a bounded operator T, prove that

$$(\ker T)^{\perp} = \mathrm{cl}\ \mathrm{im}\ T^*, \quad (\mathrm{im}\ T)^{\perp} = \ker T^*.$$

6.11. Reveal the interplay between hermitian forms and hermitian operators (cf. 6.5.3).

6.12. Find the adjoint of a shift operator, a multiplier, and a finite-rank operator.

6.13. Prove that an operator between Hilbert spaces is compact if and only if so is its adjoint.

6.14. Assume that an operator T is an isometry. Is T^* an isometry too?

6.15. A partial isometry is an operator isometric on the orthogonal complement of its kernel. What is the structure of the adjoint of a partial isometry?

6.16. What are the extreme points of the unit ball of the endomorphism space of a Hilbert space?

6.17. Prove that the weak topology of a separable Hilbert space becomes metrizable if restricted onto the unit ball.

6.18. Show that an idempotent operator P in a Hilbert space is an orthoprojection if and only if P commutes with P^*.

6.19. Let $(a_{kl})_{k,l \in \mathbb{N}}$ be an infinite matrix such that $a_{kl} \geq 0$ for all k and l. Assume further that there are also p_k and $\beta,\ \gamma > 0$ satisfying

$$\sum_{k=1}^{\infty} a_{kl} p_k \leq \beta p_l; \ \sum_{l=1}^{\infty} a_{kl} p_l \leq \gamma p_k \quad (k,\ l \in \mathbb{N}).$$

Then there is some T in $B(l_2)$ such that $(e_k,\ e_l) = a_{kl}$ and $\|T\| = \sqrt{\beta\gamma}$, where e_k is the characteristic function of k, a member of $\mathbb{N}$.

Chapter 7
Principles of Banach Spaces

7.1. Banach's Fundamental Principle

7.1.1. Lemma. *Let U be a convex set with nonempty interior in a multinormed space:* $\operatorname{int} U \neq \varnothing$. *Then*

(1) $0 \le \alpha < 1 \Rightarrow \alpha \operatorname{cl} U + (1-\alpha) \operatorname{int} U \subset \operatorname{int} U$;

(2) $\operatorname{core} U = \operatorname{int} U$;

(3) $\operatorname{cl} U = \operatorname{cl} \operatorname{int} U$;

(4) $\operatorname{int} \operatorname{cl} U = \operatorname{int} U$.

◁ (1): For $u_0 \in \operatorname{int} U$, the set $\operatorname{int} U - u_0$ is an open neighborhood of zero in virtue of 5.2.10. Whence, given $0 \le \alpha < 1$, obtain

$$\begin{aligned} \alpha \operatorname{cl} U = \operatorname{cl} \alpha U &\subset \alpha U + (1-\alpha)(\operatorname{int} U - u_0) \\ &= \alpha U + (1-\alpha) \operatorname{int} U - (1-\alpha) u_0 \\ &\subset \alpha U + (1-\alpha) U - (1-\alpha) u_0 \subset U - (1-\alpha) u_0. \end{aligned}$$

Thus, $(1-\alpha)u_0 + \alpha \operatorname{cl} U \subset U$ and so U includes $(1-\alpha) \operatorname{int} U + \alpha \operatorname{cl} U$. The last set is open as presenting the sum of $\alpha \operatorname{cl} U$ and $(1-\alpha) \operatorname{int} U$, an open set.

(2): Undoubtedly, $\operatorname{int} U \subset \operatorname{core} U$. If $u_0 \in \operatorname{int} U$ and $u \in \operatorname{core} U$ then $u = \alpha u_0 + (1-\alpha) u_1$ for some u_1 in U and $0 < \alpha < 1$. Since $u_1 \in \operatorname{cl} U$, from (1) deduce that $u \in \operatorname{int} U$.

(3): Clearly, $\operatorname{cl} \operatorname{int} U \subset \operatorname{cl} U$ for $\operatorname{int} U \subset U$. If, in turn, $u \in \operatorname{cl} U$; then, choosing u_0 in the set $\operatorname{int} U$ and putting $u_\alpha := \alpha u_0 + (1-\alpha)u$, infer that $u_\alpha \to u$ as $\alpha \to 0$ and $u_\alpha \in \operatorname{int} U$ when $0 < \alpha < 1$. Thus, by construction $u \in \operatorname{cl} \operatorname{int} U$.

(4): From the inclusions $\operatorname{int} U \subset U \subset \operatorname{cl} U$ it follows that $\operatorname{int} U \subset \operatorname{int} \operatorname{cl} U$. If now $u \in \operatorname{int} \operatorname{cl} U$ then, in virtue of (2), $u \in \operatorname{core} \operatorname{cl} U$. Consequently, taking u_0 in the set $\operatorname{int} U$ again, find $u_1 \in \operatorname{cl} U$ and $0 < \alpha < 1$ satisfying $u = \alpha u_0 + (1-\alpha) u_1$. Using (1), finally infer that $u \in \operatorname{int} U$. ▷

7.1.2. Remark. In the case of finite dimensions, the condition $\operatorname{int} U \neq \varnothing$ may be omitted in 7.1.1 (2) and 7.1.1 (4). In the opposite case, the presence of an interior point is an essential requirement, as shown by numerous examples. For instance, take $U := B_{c_0} \cap X$, where c_0 is the space of vanishing sequences and X is the subspace of terminating sequences in c_0, the direct sum of countably many copies of a basic field. Evidently, core $U = \varnothing$ and at the same time cl $U = B_{c_0}$. ◁▷

7.1.3. DEFINITION. A subset U of a (multi)normed space X is an *ideally convex set* in X, if U is closed under the taking of *countable convex combinations.* More precisely, U is ideally convex if for all sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$, with $\alpha_n \in \mathbb{R}_+$, $\sum_{n=1}^{\infty} \alpha_n = 1$ and $u_n \in U$ such that the series $\sum_{n=1}^{\infty} \alpha_n u_n$ converges in X to some u, the containment holds: $u \in U$.

7.1.4. EXAMPLES.

(1) *Translation (by a vector) preserves ideal convexity.* ◁▷

(2) *Every closed convex set is ideally convex.* ◁▷

(3) *Every open convex set is ideally convex.*

◁ Take an open and convex U. If $U = \varnothing$ then nothing is left to proof. If $U \neq \varnothing$ then by 7.1.4 (1) it may be assumed that $0 \in U$ and, consequently, $U = \{p_U < 1\}$, where p_U is the Minkowski functional of U. Let $(u_n)_{n\in\mathbb{N}}$ and $(\alpha_n)_{n\in\mathbb{N}}$ be sequences in U and in $\mathbb{R}_+$ such that $\sum_{n=1}^{\infty} \alpha_n = 1$, with the element $u := \sum_{n=1}^{\infty} \alpha_n u_n$ failing to lie in U. By virtue of 7.1.4 (2), u belongs to cl $U = \{p_U \leq 1\}$ and so $p_U(u) = 1$. On the other hand, it is clear that $p_U(u) \leq \sum_{n=1}^{\infty} \alpha_n p_U(u_n) \leq 1 = \sum_{n=1}^{\infty} \alpha_n$ (cf. 7.2.1). Thus, $0 = \sum_{n=1}^{\infty} (\alpha_n - \alpha_n p_U(u_n)) = \sum_{n=1}^{\infty} \alpha_n (1 - p_U(u_n))$. Whence $\alpha_n = 0$ for all $n \in \mathbb{N}$. We arrive at a contradiction. ▷

(4) *The intersection of a family of ideally convex sets is ideally convex.* ◁▷

(5) *Every convex subset of a finite-dimensional space is ideally convex.* ◁▷

7.1.5. *Banach's Fundamental Principle.* *In a Banach space, each ideally convex set with absorbing closure is a neighborhood of zero.*

◁ Let U be such a set in a Banach space X. By hypothesis $X = \cup_{n\in\mathbb{N}} n \,\mathrm{cl}\, U$. By the Baire Category Theorem, X is nonmeager and so there is some n in $\mathbb{N}$ such that int $n \,\mathrm{cl}\, U \neq \varnothing$. Therefore, int cl $U = {}^1/_n$ int $n \,\mathrm{cl}\, U \neq \varnothing$. By hypothesis $0 \in$ core cl U. Consequently, from 7.1.1 it follows that $0 \in$ int cl U. In other words, there is a $\delta > 0$ such that cl $U \supset \delta B_X$. Consequently,

$$\varepsilon > 0 \Rightarrow \mathrm{cl}\ {}^1/_\varepsilon U \supset {}^\delta/_\varepsilon B_X.$$

Using the above implication, show that $U \supset {}^\delta/_2 B_X$.

Let $x_0 \in {}^\delta/_2 B_X$. Putting $\varepsilon := 2$, choose $y_1 \in {}^1/_\varepsilon U$ from the condition $\|y_1 - x_0\| \leq {}^1/_{2\varepsilon}\delta$. Thus obtain an element u_1 of U such that $\left\|{}^1/_2 u_1 - x_0\right\| \leq {}^1/_{2\varepsilon}\delta = {}^1/_4\delta$.

Now putting $x_0 := -{}^1/_2 u_1 + x_0$ and $\varepsilon := 4$ and applying the argument of the preceding paragraph, find an element u_2 of U satisfying $\|{}^1/_4 u_2 + {}^1/_2 u_1 - x_0\| \le {}^1/_{2\varepsilon}\delta = {}^1/_8\delta$. Proceeding by induction, construct a sequence $(u_n)_{n\in\mathbb{N}}$ of the points of U which possesses the property that the series $\sum_{n=1}^{\infty} {}^1/_{2^n} u_n$ converges to x_0. Since $\sum_{n=1}^{\infty} {}^1/_{2^n} = 1$ and the set U is ideally convex, deduce $x_0 \in U$. ▷

7.1.6. *For every ideally convex set U in a Banach space the following four sets coincide: the core of U, the interior of U, the core of the closure of U and the interior of the closure of U.*

◁ It is clear that $\operatorname{int} U \subset \operatorname{core} U \subset \operatorname{core} \operatorname{cl} U$. If $u \in \operatorname{core} \operatorname{cl} U$ then $\operatorname{cl}(U - u)$ equal to $\operatorname{cl} U - u$ is an absorbing set. An ideally convex set translates into an ideally convex set (cf. 7.1.4 (1)). Consequently, $U - u$ is a neighborhood of zero by Banach's Fundamental Principle. By virtue of 5.2.10, u belongs to $\operatorname{int} U$. Thus, $\operatorname{int} U = \operatorname{core} U = \operatorname{core} \operatorname{cl} U$. Using 7.1.1, conclude that $\operatorname{int} \operatorname{cl} U = \operatorname{int} U$. ▷

7.1.7. *The core and the interior of a closed convex set in a Banach space coincide.*

◁ A closed convex set is ideally convex. ▷

7.1.8. REMARK. Inspection of the proof of 7.1.5 shows that the condition for the ambient space to be a Banach space in 7.1.7 is not utilized to a full extent. There are examples of incomplete normed spaces in which the core and interior of each closed convex set coincide. A space with this property is called *barreled.* The concept of barreledness is seen to make sense also in multinormed spaces. Barreled multinormed spaces are plentiful. In particular, such are all Fréchet spaces.

7.1.9. COUNTEREXAMPLE. *Each infinite-dimensional Banach space contains absolutely convex, absorbing and not ideally convex sets.*

◁ Using, for instance, a Hamel basis, take a discontinuous linear functional f. Then the set $\{|f| \le 1\}$ is what was sought. ▷

7.2. Boundedness Principles

7.2.1. *Let $p : X \to \mathbb{R}$ be a sublinear functional on a normed space $(X, \|\cdot\|)$. The following conditions are equivalent:*

(1) *p is uniformly continuous;*

(2) *p is continuous;*

(3) *p is continuous at zero;*

(4) *$\{p \le 1\}$ is a neighborhood of zero;*

(5) $\|p\| := \sup\{|p(x)| : \|x\| \le 1\} < +\infty$; *i.e., p is bounded.*

◁ The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are immediate.

(4) ⇒ (5): There is some $t > 0$ such that $t^{-1}B_X \subset \{p \le 1\}$. Given $\|x\| \le 1$, thus find $p(x) \le t$. In addition, the inequality $-p(-x) \le p(x)$ implies that $-p(x) \le t$ for $x \in B_X$. Finally, $\|p\| \le t < +\infty$.

(5) ⇒ (1): From the subadditivity of p, given $x,\ y \in X$, observe that

$$p(x) - p(y) \le p(x - y); \quad p(y) - p(x) \le p(y - x).$$

Whence $|p(x) - p(y)| \le p(x - y) \vee p(y - x) \le \|p\|\ \|x - y\|$. ▷

7.2.2. *Gelfand Theorem.* *Every lower semicontinuous sublinear functional with domain a Banach space is continuous.*

◁ Let p be such a functional. Then the set $\{p \le 1\}$ is closed (cf. 4.3.8). Since dom p is the whole space; therefore, by 3.8.8, $\{p \le 1\}$ is an absorbing set. By Banach's Fundamental Principle $\{p \le 1\}$ is a neighborhood of zero. Application to 7.2.1 completes the proof. ▷

7.2.3. Remark. The Gelfand Theorem is stated amply as follows: "If X is a Banach space then each of the equivalent conditions 7.2.1 (1)–7.2.1 (5) amounts to the statement: 'p is lower semicontinuous on X.'" Observe immediately that the requirement dom $p = X$ may be slightly relaxed by assuming dom p to be a nonmeager linear set and withdrawing the condition for X to be a Banach space.

7.2.4. *Equicontinuity Principle.* *Suppose that X is a Banach space and Y is a (semi)normed space. For every nonempty set $\mathscr{E}$ of continuous linear operators from X to Y the following statements are equivalent:*

(1) *$\mathscr{E}$ is pointwise bounded; i.e., for all $x \in X$ the set $\{Tx : T \in \mathscr{E}\}$ is bounded in Y;*

(2) *$\mathscr{E}$ is equicontinuous.*

◁ (1) ⇒ (2): Put $q(x) := \sup\{p(Tx) : T \in \mathscr{E}\}$, with p the (semi)norm of Y. Evidently, q is a lower semicontinuous sublinear functional and so by the Gelfand Theorem $\|q\| < +\infty$; i.e., $p(T(x - y)) \le \|q\|\ \|x - y\|$ for all $T \in \mathscr{E}$. Consequently, $T^{\times -1}(\{d_p \le \varepsilon\}) \subset \{d_{\|\cdot\|} \le \varepsilon / \|q\|\}$ for every T in $\mathscr{E}$, where $\varepsilon > 0$ is taken arbitrarily. This means the equicontinuity property of $\mathscr{E}$.

(2) ⇒ (1): Straightforward. ▷

7.2.5. *Uniform Boundedness Principle.* *Let X be a Banach space and let Y be a normed space. For every nonempty family $(T_\xi)_{\xi \in \Xi}$ of bounded operators the following statements are equivalent:*

(1) $x \in X \Rightarrow \sup_{\xi \in \Xi} \|T_\xi x\| < +\infty$;

(2) $\sup_{\xi \in \Xi} \|T_\xi\| < +\infty$.

◁ It suffices to observe that 7.2.5 (2) is another expression for 7.2.4 (2). ▷

7.2.6. *Let X be a Banach space and let U be a subset of X'. Then the following statements are equivalent:*

(1) *U is bounded in X';*

(2) *for every x in X the numeric set $\{\langle x \mid x' \rangle : x' \in U\}$ is bounded in $\mathbb{F}$.*

◁ This is a particular case of 7.2.5. ▷

7.2.7. *Let X be a normed space and let U be a subset of X. Then the following statements are equivalent:*

(1) *U is bounded in X;*

(2) *for every x' in X' the numeric set $\{\langle x \mid x' \rangle : x \in U\}$ is bounded in $\mathbb{F}$.*

◁ Only (2) ⇒ (1) needs examining. Observe that X' is a Banach space (cf. 5.5.7) and X is isometrically embedded into X'' by the double prime mapping (cf. 5.1.10 (8)). So, the claim follows from 7.2.6. ▷

7.2.8. REMARK. The message of 7.2.7 (2) may be reformulated as "U is bounded in the space $(X, \sigma(X, X'))$" or, in view of 5.1.10 (4), as "U is weakly bounded." The duality between 7.2.6 and 7.2.7 is perfectly revealed in 10.4.6.

7.2.9. ***Banach–Steinhaus Theorem.*** *Let X and Y be Banach spaces. Assume further that $(T_n)_{n\in\mathbb{N}}$, $T_n \in B(X, Y)$, is a sequence of bounded operators. Put $E := \{x \in X : \exists \lim T_n x\}$. The following conditions are equivalent:*

(1) *$E = X$;*

(2) *$\sup_{n\in\mathbb{N}} \|T_n\| < +\infty$ and E is dense in X.*

Under either (and, hence, both) of the conditions (1) and (2) the mapping $T_0 : X \to Y$, defined as $T_0 x := \lim T_n x$, presents a bounded linear operator and $\|T_0\| \le \liminf \|T_n\|$.

◁ If $E = X$ then, of course, $\operatorname{cl} E = X$. In addition, for every x in X the sequence $(T_n x)_{n\in\mathbb{N}}$ is bounded in Y (for, it converges). Consequently, by the Uniform Boundedness Principle $\sup_{n\in\mathbb{N}} \|T_n\| < +\infty$ and (1) ⇒ (2) is proven.

If (2) holds and $x \in X$ then, given $\overline{x} \in E$ and $m,\ k \in \mathbb{N}$, infer that

$$\begin{aligned}
\|T_m x - T_k x\| &= \|T_m x - T_m \overline{x} + T_m \overline{x} - T_k \overline{x} + T_k \overline{x} - T_k x\| \\
&\le \|T_m x - T_m \overline{x}\| + \|T_m \overline{x} - T_k \overline{x}\| + \|T_k \overline{x} - T_k x\| \\
&\le \|T_m\|\ \|x - \overline{x}\| + \|T_m \overline{x} - T_k \overline{x}\| + \|T_k\|\ \|\overline{x} - x\| \\
&\le 2 \sup_{n\in\mathbb{N}} \|T_n\|\ \|x - \overline{x}\| + \|T_m \overline{x} - T_k \overline{x}\|.
\end{aligned}$$

Take $\varepsilon > 0$ and choose, first, $\overline{x} \in E$ such that $2\sup_n \|T_n\|\,\|x - \overline{x}\| \le {}^{\varepsilon}/_2$, and, second, $n \in \mathbb{N}$ such that $\|T_m \overline{x} - T_k \overline{x}\| \le {}^{\varepsilon}/_2$ for $m,\ k \ge n$. By virtue of what was proven $\|T_m x - T_k x\| \le \varepsilon$; i.e., $(T_n x)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. Since Y is a Banach space, conclude that $x \in E$. Thus, (2) ⇒ (1) is proven.

To complete the proof it suffices to observe that

$$\|T_0 x\| = \lim \|T_n x\| \le \liminf \|T_n\|\ \|x\|$$

for all $x \in X$, because every norm is a continuous function. ▷

7.2.10. REMARK. Under the hypotheses of the Banach–Steinhaus Theorem, the validity of either of the equivalent items 7.2.9 (1) and 7.2.9 (2) implies that (T_n) converges to T_0 *compactly* on X (= uniformly on every compact subset of X). In other words,

$$\sup_{x \in Q} \|T_n x - T_0 x\| \to 0$$

for every (nonempty) compact set Q in X.

◁ Indeed, it follows from the Gelfand Theorem that the sublinear functional $p_n(x) := \sup\{\|T_m x - T_0 x\| : m \geq n\}$ is continuous. Moreover, $p_n(x) \geq p_{n+1}(x)$ and $p_n(x) \to 0$ for all $x \in X$. Consequently, the claim follows from the *Dini Theorem*: "Each decreasing sequence of continuous real functions which converges pointwise to a continuous function on a compact set converges uniformly." ▷

7.2.11. ***Singularity Fixation Principle.*** *Let X be a Banach space and let Y be a normed space. If $(T_n)_{n \in \mathbb{N}}$ is a sequence of operators, $T_n \in B(X, Y)$ and $\sup_n \|T_n\| = +\infty$ then there is a point x of X satisfying $\sup_n \|T_n x\| = +\infty$. The set of such points "fixing a singularity" is residual.*

◁ The first part of the assertion is contained in the Uniform Boundedness Principle. The second requires referring to 7.2.3 and 4.7.4. ▷

7.2.12. ***Singularity Condensation Principle.*** *Let X be a Banach space and let Y be a normed space. If $(T_{n,m})_{n,m \in \mathbb{N}}$ is a family of operators, $T_n \in B(X, Y)$, such that $\sup_n \|T_{n,m}\| = +\infty$ for every $m \in \mathbb{N}$ then there is a point x of X satisfying $\sup_n \|T_{n,m} x\| = +\infty$ for all $m \in \mathbb{N}$.* ◁▷

7.3. The Ideal Correspondence Principle

7.3.1. *Let X and Y be vector spaces. A correspondence $F \subset X \times Y$ is convex if and only if for $x_1, x_2 \in X$ and $\alpha_1, \alpha_2 \in \mathbb{R}_+$ such that $\alpha_1 + \alpha_2 = 1$, the inclusion holds:*

$$F(\alpha_1 x_1 + \alpha_2 x_2) \supset \alpha_1 F(x_1) + \alpha_2 F(x_2).$$

◁ ⇐: If $(x_1, y_1), (x_2, y_2) \in F$ and $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$, then $\alpha_1 y + \alpha_2 y_2 \in F(\alpha_1 x_1 + \alpha_2 x_2)$ since $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$.

⇒: If either x_1 or x_2 fails to enter in dom F then there is nothing to prove. If $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$ with $x_1, x_2 \in \operatorname{dom} F$ then $\alpha_1(x_1, y_1) + \alpha_2(x_2, y_2) \in F$ for $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$ (cf. 3.1.2 (8)). ▷

7.3.2. REMARK. Let X and Y be Banach spaces. It is clear that there are many ways for furnishing the space $X \times Y$ with a norm so that the norm topology be coincident with the product of the topologies τ_X and τ_Y. For instance, it is possible to put $\|(x, y)\| := \|x\|_X + \|y\|_Y$; i.e., to define the norm on $X \times Y$ as that of the 1-sum of X and Y. Observe immediately that the concept of ideally convex set has a linear topological character: the class of objects distinguished in a space is independent of the way of introducing the space topology; in particular, the class remains invariant under passage to an equivalent (multi)norm. In this connection the next definition is sound.

7.3.3. DEFINITION. A correspondence $F \subset X \times Y$, with X and Y Banach spaces, is called *ideally convex* or, briefly, *ideal* if F is an ideally convex set.

7.3.4. *Ideal Correspondence Lemma.* *The image of a bounded ideally convex set under an ideal correspondence is an ideally convex set.*

◁ Let $F \subset X \times Y$ be an ideal correspondence and let U be a bounded ideally convex set in X. If $U \cap \operatorname{dom} F = \varnothing$ then $F(U) = \varnothing$ and nothing is left unproven. Let now $(y_n)_{n\in\mathbb{N}} \subset F(U)$; i.e., $y_n \in F(x_n)$, where $x_n \in U$ and $n \in \mathbb{N}$. Let, finally, (α_n) be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \alpha_n = 1$ and, moreover, there is a sum of the series $y := \sum_{n=1}^{\infty} \alpha_n y_n$ in Y. It is beyond a doubt that

$$\sum_{n=1}^{\infty} \|\alpha_n x_n\| = \sum_{n=1}^{\infty} \alpha_n \|x_n\| \le \sum_{n=1}^{\infty} \alpha_n \sup \|U\| = \sup \|U\| < +\infty$$

in view of the boundedness property of U. Since X is complete, from 5.5.3 it follows that X contains the element $x := \sum_{n=1}^{\infty} \alpha_n x_n$. Consequently, $(x,\ y) = \sum_{n=1}^{\infty} \alpha_n (x_n,\ y_n)$ in the space $X \times Y$. Successively using the ideal convexity of F and U, infer that $(x,\ y) \in F$ and $x \in U$. Thus, $y \in F(U)$. ▷

7.3.5. *Ideal Correspondence Principle.* *Let X and Y be Banach spaces. Assume further that $F \subset X \times Y$ is an ideal correspondence and $(x,\ y) \in F$. A correspondence F carries each neighborhood of x onto a neighborhood about y if and only if $y \in \operatorname{core} F(X)$.*

◁ $\Rightarrow$: This is obvious.

$\Leftarrow$: On account of 7.1.4 it may be assumed that $x = 0$ and $y = 0$. Since each neighborhood of zero U includes εB_X for some $\varepsilon > 0$, it suffices to settle the case $U := B_X$. Since U is bounded; by 7.3.4, $F(U)$ is ideally convex. To complete the proof, it suffices to show that $F(U)$ is an absorbing set and to cite 7.1.6.

Take an arbitrary element $\overline{y}$ of Y. Since by hypothesis $0 \in \operatorname{core} F(X)$, there is a real α in $\mathbb{R}_+$ such that $\alpha\overline{y} \in F(X)$. In other words, $\alpha\overline{y} \in F(\overline{x})$ for some $\overline{x}$ in X. If $\|\overline{x}\| \le 1$ then there is nothing to prove. If $\|\overline{x}\| > 1$ then $\lambda := \|\overline{x}\|^{-1} < 1$. Whence, using 7.3.1, infer that

$$\begin{aligned}\alpha\lambda\overline{y} &= (1-\lambda)0 + \lambda\alpha\overline{y} \in (1-\lambda)F(0) + \lambda F(\overline{x}) \\ &\subset F((1-\lambda)0 + \lambda\overline{x}) = F(\lambda\overline{x}) \subset F(B_X) = F(U).\end{aligned}$$

Here use was made of the fact that $\|\lambda\overline{x}\| = 1$; i.e., $\lambda\overline{x} \in B_X$. ▷

7.3.6. REMARK. The property of F, described in 7.3.5, is referred to as the *openness* of F at $(x,\ y)$.

7.3.7. REMARK. The Ideal Correspondence Principle is formally weaker that Banach's Fundamental Principle. Nevertheless, the gap is tiny and can be easily filled in. Namely, the conclusion of 7.3.5 remains valid if we suppose that $y \in \operatorname{core} \operatorname{cl} F(X)$, on additionally requiring ideal convexity from $F(X)$. The requirement is not too stringent and certainly valid provided that the domain of F is bounded in virtue of 7.3.4. As a result of this slight modification, 7.1.5 becomes a particular case of 7.3.5. In this connection the claim of 7.3.5 is often referred to as *Banach's Fundamental Principle for a Correspondence.*

7.3.8. DEFINITION. Let X and Y be Banach spaces and let $F \subset X \times Y$ be a correspondence. Then F is called *closed* if F is a closed set in $X \times Y$.

7.3.9. REMARK. For obvious reasons, a closed correspondence is often referred to as a *closed-graph correspondence.*

7.3.10. *A correspondence F is closed if and only if for all sequences (x_n) in X and (y_n) in Y such that $x_n \in \operatorname{dom} F$, $y_n \in F(x_n)$ and $x_n \to x$, $y_n \to y$, it follows that $x \in \operatorname{dom} F$ and $y \in F(x)$.* ◁▷

7.3.11. *Assume that X and Y are Banach spaces and $F \subset X \times Y$ is a closed convex correspondence. Further, let $(x,\ y) \in F$ and $y \in \operatorname{core} \operatorname{im} F$. Then F carries each neighborhood of x onto a neighborhood about y.*

◁ A closed convex set is ideally convex and so all follows from 7.3.5. ▷

7.3.12. DEFINITION. A correspondence $F \subset X \times Y$ is called *open* if the image of each open set in X is an open set in Y.

7.3.13. **Open Correspondence Principle.** *Let X and Y be Banach spaces and let $F \subset X \times Y$ be an ideal correspondence, with $\operatorname{im} F$ an open set. Then F is an open correspondence.*

◁ Let U be an open set in X. If $y \in F(U)$ then there is some x in U such that $(x,\ y) \in F$. It is clear that $y \in \operatorname{core} \operatorname{im} F$. By the criterion of 7.3.5 $F(U)$ is a neighborhood of y because U is a neighborhood of x. This means that $F(U)$ is an open set. ▷

7.4. Open Mapping and Closed Graph Theorems

7.4.1. DEFINITION. A member T of $\mathscr{L}(X,\ Y)$ is a *homomorphism,* if $T \in B(X,\ Y)$ and T is an open correspondence.

7.4.2. *Assume that X is a Banach space, Y is a normed space and T is a homomorphism from X to Y. Then $\operatorname{im} T = Y$ and Y is a Banach space.*

◁ It is obvious that im $T = Y$. Presuming T to be a monomorphism, observe that $T^{-1} \in \mathscr{L}(Y, X)$. Since T is open, T^{-1} belongs to $B(Y, X)$, which ensures the completeness of Y (the inverse image of a Cauchy sequence in a subset is a Cauchy sequence in the inverse image of the subset). In the general case, consider the coimage coim $T := X/\ker T$ endowed with the quotient norm. In virtue of 5.5.4, coim T is a Banach space. In addition, by 2.3.11 there is a unique quotient $\overline{T}$ of T by coim T, the monoquotient of T. Taking account of the definition of quotient norm and 5.1.3, conclude that $\overline{T}$ is a homomorphism. Furthermore, $\overline{T}$ is a monomorphism by definition. It remains to observe that im $\overline{T} =$ im $T = Y$. ▷

7.4.3. REMARK. As regards the monoquotient $\overline{T}$: coim $T \to Y$ of T, it may be asserted that $\|T\| = \|\overline{T}\|$. ◁▷

7.4.4. *Banach Homomorphism Theorem.* *Every bounded epimorphism from one Banach space onto the other is a homomorphism.*

◁ Let $T \in B(X, Y)$ and im $T = Y$. On applying the Open Correspondence Principle to T, complete the proof. ▷

7.4.5. *Banach Isomorphism Theorem.* *Let X and Y be Banach spaces and $T \in B(X, Y)$. If T is an isomorphism of the vector spaces X and Y, i.e.* ker $T = 0$ *and* im $T = Y$; *then* $T^{-1} \in B(Y, X)$.

◁ A particular case of 7.4.4. ▷

7.4.6. REMARK. The Banach Homomorphism Theorem is often referred to as the *Open Mapping Theorem* for understandable reasons. Theorem 7.4.5 is briefly formulated as follows: "A continuous (algebraic) isomorphism of Banach spaces is a topological isomorphism." It is also worth observing that the theorem is sometimes referred to as the *Well-Posedness Principle* and verbalized as follows: "If an equation $Tx = y$, with $T \in B(X, Y)$ and X and Y Banach spaces, is uniquely solvable given an arbitrary right side; then the solution x depends continuously on the right side y."

7.4.7. *Banach Closed Graph Theorem.* *Let X and Y be Banach spaces and let T in $\mathscr{L}(X, Y)$ be a closed linear operator. Then T is continuous.*

◁ The correspondence T^{-1} is ideal and $T^{-1}(Y) = X$. ▷

7.4.8. Corollary. *Suppose that X and Y are Banach spaces and T is a linear operator from X to Y. The following conditions are equivalent:*

(1) $T \in B(X, Y)$;

(2) *for every sequence $(x_n)_{n\in\mathbb{N}}$ in X, together with some x in X and y in Y satisfying $x_n \to x$ and $Tx_n \to y$, it happens that $y = Tx$.*

◁ (2): This is a reformulation of the closure property of T. ▷

7.4.9. DEFINITION. A subspace X_1 of a Banach space X is *complemented* (rarely, a *topologically complemented*), if X_1 is closed and, moreover, there is a closed subspace X_2 such that $X = X_1 \oplus X_2$ (i.e., $X_1 \wedge X_2 = 0$ and $X_1 \vee X_2 = X$). These X_1 and X_2 are called *complementary* to one another.

7.4.10. Complementation Principle. *For a subspace X_1 of some Banach space X one of the following conditions amounts to the other:*

(1) *X_1 is complemented;*

(2) *X_1 is the range of a bounded projection; i.e., there is a member P of $B(X)$ such that $P^2 = P$ and $\operatorname{im} P = X_1$.*

◁ (1) ⇒ (2): Let P be the projection of X onto X_1 along X_2 (cf. 2.2.9 (4)). Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X with $x_n \to x$ and $Px_n \to y$. It is clear that $Px_n \in X_1$ for $n \in \mathbb{N}$. Since X_1 is closed, by 4.1.19 $y \in X_1$. Similarly, the condition ($x_n - Px_n \in X_2$ for $n \in \mathbb{N}$) implies that $x - y \in X_2$. Consequently, $P(x - y) = 0$. Furthermore, $y = Py$; i.e., $y = Px$. It remains to refer to 7.4.8.

(2) ⇒ (1): It needs showing only that X_1 equal to $\operatorname{im} P$ is closed. Take a sequence $(x_n)_{n\in\mathbb{N}}$ in X_1 such that $x_n \to x$ in X. Then $Px_n \to Px$ in view of the boundedness of P. Obtain $Px_n = x_n$, because $x_n \in \operatorname{im} P$ and P is an idempotent. Finally, $x = Px$, i.e. $x \in X_1$, what was required. ▷

7.4.11. EXAMPLES.

(1) *Every finite-dimensional subspace is complemented.* ◁▷

(2) *The space c_0 is not complemented in l_∞.*

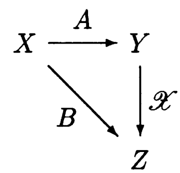
◁ It turns out more convenient to work with $X := l_\infty(\mathbb{Q})$ and $Y := c_0(\mathbb{Q})$, where $\mathbb{Q}$ is the set of rational numbers. Given $t \in \mathbb{R}$, choose a sequence $(\bar{t}_n)$ of pairwise distinct rational numbers other than t and such that $\bar{t}_n \to t$. Let $Q_t := \{\bar{t}_n : n \in \mathbb{N}\}$. Observe that $Q_{t'} \cap Q_{t''}$ is a finite set if $t' \neq t''$.

Let χ_t be the coset containing the characteristic function of Q_t in the quotient space X/Y and $V := \{\chi_t : t \in \mathbb{R}\}$. Since $\chi_{t'} \neq \chi_{t''}$ for $t' \neq t''$, the set V is uncountable. Take $f \in (X/Y)'$ and put $V_f := \{v \in V : f(v) \neq 0\}$. It is evident that $V_f = \cup_{n\in\mathbb{N}} V_f(n)$, where $V_f(n) := \{v \in V : |f(v)| \geq {}^1/_n\}$. Given $m \in \mathbb{N}$ and pairwise distinct $v_1, \dots, v_m$ in $V_f(n)$, put $\alpha_k := |f(v_k)|/f(v_k)$ and $x := \sum_{k=1}^n \alpha_k v_k$ to find $\|x\| \leq 1$ and $\|f\| \geq |f(x)| = |\sum_{k=1}^m \alpha_k f(v_k)| = |\sum_{k=1}^m |f(v_k)|| \geq {}^m/_n$. Hence, $V_f(n)$ is a finite set. Consequently, V_f is countable. Whence it follows that for every countable subset F of $(X/Y)'$ there is an element v of V satisfying $(\forall f \in F)\ f(v) = 0$. At the same time the countable set of the coordinate projections $\delta_q : x \mapsto x(q)$ $(q \in \mathbb{Q})$ is total over $l_\infty(\mathbb{Q})$; i.e., $(\forall q \in \mathbb{Q})\ \delta_q(x) = 0 \Rightarrow x = 0$ for $x \in l_\infty(\mathbb{Q})$. It remains to compare the above observations. ▷

(3) Every closed subspace of a Hilbert space is complemented in virtue of 6.2.6. Conversely, if, in an arbitrary Banach space X with $\dim X \geq 3$, each closed subspace is the range of some projection P with $\|P\| \leq 1$; then X is isometrically isomorphic to a Hilbert space (this is the *Kakutani Theorem*). The next fact is much deeper:

Lindenstrauss–Tzafriri Theorem. *A Banach space having every closed subspace complemented is topologically isomorphic to a Hilbert space.*

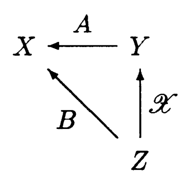
7.4.12. *Sard Theorem.* *Suppose that X, Y, and Z are Banach spaces. Take $A \in B(X, Y)$ and $B \in B(Y, Z)$. Suppose further that* im A *is a complemented subspace in Y. The diagram*



is commutative for some $\mathscr{X}$ in $B(Y, Z)$ if and only if $\ker A \subset \ker B$.

◁ Only ⇐ needs examining. Moreover, in the case im $A = Y$ the sole member $\mathscr{X}_0$ of $\mathscr{L}(Y, Z)$ such that $\mathscr{X}_0 A = B$ is continuous. Indeed, $\mathscr{X}_0^{-1}(U) = A(B^{-1}(U))$ for every open set U in Z. The set $B^{-1}(U)$ is open in virtue of the boundedness property of B, and $A(B^{-1}(U))$ is open by the Banach Homomorphism Theorem. In the general case, construct $\mathscr{X}_0 \in B(\text{im } A, Z)$ and take as $\mathscr{X}$ the operator $\mathscr{X}_0 P$, where P is some continuous projection of Y onto im A. (Such a projection is available by the Complementation Principle.) ▷

7.4.13. *Phillips Theorem.* *Suppose that X, Y, and Z are Banach spaces. Take $A \in B(Y, X)$ and $B \in B(Z, X)$. Suppose further that* $\ker A$ *is a complemented subspace of Y. The diagram*



is commutative for some $\mathscr{X}$ in $B(Z, Y)$ if and only if im $A \supset$ im B.

◁ Once again only ⇐ needs examining. Using the definition of complemented subspace, express Y as the direct sum of $\ker A$ and Y_0, where Y_0 is a closed subspace. By 5.5.9 (1), Y_0 is a Banach space. Consider the part A_0 of A in Y_0. Undoubtedly, im A_0 = im $A \supset$ im B. Consequently, by 2.3.13 and 2.3.14 the equation $A_0 \mathscr{X}_0 = B$ has a unique solution $\mathscr{X}_0 := A_0^{-1}$. It suffices to prove that the operator $\mathscr{X}_0$, treated as a member of $\mathscr{L}(Z, Y_0)$, is bounded.

The operator $\mathscr{X}_0$ is closed. Indeed (cf. 7.4.8), if $z_n \to z$ and $A_0^{-1} B z_n \to y$ then $B z_n \to B z$, since B is bounded. In addition, by the continuity of A_0, the correspondence $A_0^{-1} \subset X \times Y_0$ is closed; and so 7.3.10 yields the equality $y = A_0^{-1} B z$. ▷

7.4.14. REMARK. We use neither the completeness of Z in proving the Sard Theorem nor the completeness of X in proving the Phillips Theorem.

7.4.15. REMARK. The Sard Theorem and the Phillips Theorem are in "formal duality"; i.e., one results from the other by reversing arrows and inclusions and substituting ranges for kernels (cf. 2.3.15).

7.4.16. ***Two Norm Principle.*** *Let a vector space be complete in each of two comparable norms. Then the norms are equivalent.*

$\triangleleft$ For definiteness, assume that $\|\cdot\|_2 \succ \|\cdot\|_1$ in X. Consider the diagram

$$
\begin{array}{ccc}
(X, \|\cdot\|_1) & \xleftarrow{\;I_X\;} & (X, \|\cdot\|_2) \\
 & {\scriptstyle I_X}\nwarrow & \uparrow {\scriptstyle \mathscr{X}} \\
 & & (X, \|\cdot\|_1)
\end{array}
$$

By the Phillips Theorem some continuous operator $\mathscr{X}$ makes the diagram commutative. Such an operator is unique: it is I_X. $\triangleright$

7.4.17. ***Graph Norm Principle.*** *Let X and Y be Banach spaces and let T in $\mathscr{L}(X,\ Y)$ be a closed operator. Given $x \in X$, define the graph norm of x as $\|x\|_{\mathrm{gr}\ T} := \|x\|_X + \|Tx\|_Y$. Then $\|\cdot\|_{\mathrm{gr}\ T} \sim \|\cdot\|_X$.*

$\triangleleft$ Observe that the space $(X,\ \|\cdot\|_{\mathrm{gr}\ T})$ is complete. Further, $\|\cdot\|_{\mathrm{gr}\ T} \geq \|\cdot\|_X$. It remains to refer to the Two Norm Principle. $\triangleright$

7.4.18. DEFINITION. A normed space X is a *Banach range*, if X is the range of some bounded operator given on some Banach space.

7.4.19. ***Kato Criterion.*** *Let X be a Banach space and $X = X_1 \oplus X_2$, where $X_1, X_2 \in \mathrm{Lat}\,(X)$. The subspaces X_1 and X_2 are closed if and only if each of them is a Banach range.*

$\triangleleft \Rightarrow$: A corollary to the Complementation Principle.

$\Leftarrow$: Let Z be a some Banach range, i.e. $Z = T(Y)$ for some Banach space Y and $T \in B(Y,\ Z)$. Passing, if need be, to the monoquotient of T, we may assume that T is an isomorphism. Put $\|z\|_0 := \|T^{-1}z\|_Y$. It is clear that $(Z,\ \|\cdot\|_0)$ is a Banach space and $\|z\| = \|TT^{-1}z\| \leq \|T\|\|T^{-1}z\| = \|T\|\|z\|_0$; i.e., $\|\cdot\|_0 \succ \|\cdot\|_Z$. Applying this construction to X_1 and X_2, obtain Banach spaces $(X_1,\ \|\cdot\|_1)$ and $(X_2,\ \|\cdot\|_2)$. Now $\|\cdot\|_k \succ \|\cdot\|_X$ on X_k for $k := 1, 2$.

Given $x_1 \in X_1$ and $x_2 \in X_2$, put $\|x_1 + x_2\|_0 := \|x_1\|_1 + \|x_2\|_2$. Thereby we introduce in X some norm $\|\cdot\|$ that is stronger than the initial norm $\|\cdot\|_X$. By construction $(X,\ \|\cdot\|_0)$ is a Banach space. It remains to refer to 7.4.16. $\triangleright$

7.5. The Automatic Continuity Principle

7.5.1. *Lemma.* *Let $f : X \to \mathbb{R}^{\cdot}$ be a convex function on a (multi)normed space X. The following statements are equivalent:*

(1) *$U :=$ int dom $f \neq \varnothing$ and $f|_U$ is a continuous function;*

(2) *there is a nonempty open set V such that* $\sup f(V) < +\infty$.

◁ (1) ⇒ (2): This is obvious.

(2) ⇒ (1): It is clear that $U \neq \varnothing$. Using 7.1.1, observe that each point u of U has a neighborhood W in which f is bounded above, i.e. $t := \sup f(W) < +\infty$. Without loss of generality, it may be assumed that $u := 0$, $f(u) := 0$ and W is an absolutely convex set. From the convexity of f, for every $\alpha \in \mathbb{R}_+$ such that $\alpha \leq 1$ and for an arbitrary v in W, obtain

$$f(\alpha v) = f(\alpha v + (1-\alpha)0) \leq \alpha f(v) + (1-\alpha)f(0) = \alpha f(v);$$
$$f(\alpha v) + \alpha f(-v) \geq f(\alpha v) + f(\alpha(-v))$$
$$= 2({}^1/_2 f(\alpha v) + {}^1/_2 f(-\alpha v)) \geq 2f(0) = 0.$$

Therefore, $|f(\alpha W)| \leq \alpha t$, which implies that f is continuous at zero. ▷

7.5.2. *Corollary.* *If $x \in$ int dom f and f is continuous at x then the subdifferential $\partial_x(f)$ contains only continuous functionals.*

◁ If $l \in \partial_x(f)$ then $(\forall \overline{x} \in X)\ l(\overline{x}) \leq l(x) + f(\overline{x}) - f(x)$ and so l is bounded above on some neighborhood about x. Consequently, l is continuous at x by 7.5.1. From 5.3.7 derive that l is continuous. ▷

7.5.3. *Corollary.* *Every convex function on a finite-dimensional space is continuous on the interior of its domain.* ◁▷

7.5.4. Definition. A function $f : X \to \mathbb{R}^{\cdot}$ is called *ideally convex* if epi f is an ideal correspondence.

7.5.5. *Automatic Continuity Principle.* *Every ideally convex function on a Banach space is continuous on the core of its domain.*

◁ Let f be such a function. If core dom $f = \varnothing$ then there is nothing to prove. If $x \in$ core dom f then put $t := f(x)$ and $F := (\text{epi } f)^{-1} \subset \mathbb{R} \times X$. Applying the Ideal Correspondence Principle, find a $\delta > 0$ from the condition $F(t + B_{\mathbb{R}}) \supset x + \delta B_X$. Whence, in particular, infer the estimate $f(x + \delta B_X) \leq t + 1$. In virtue of 7.5.1, f is continuous on int dom f. Since $x \in$ int dom f; therefore, by Lemma 7.1.1, core dom $f =$ int dom f. ▷

7.5.6. Remark. Using 7.3.6, it is possible to prove that an ideally convex function f, defined in a Banach space on a subset with nonempty core, is *locally Lipschitz* on int dom f. In other words, given $x_0 \in$ int dom f, there are a positive number L and a neighborhood U about x_0 such that $\|f(x) - f(x_0)\| \leq L\|x - x_0\|$ whenever $x \in U$. ◁▷

7.5.7. Corollary. *Let $f : X \to \mathbb{R}^{\cdot}$ be an ideally convex function on a Banach space X and $x \in$ core dom f. Then the directional derivative $f'(x)$ is a continuous sublinear functional and $\partial_x(f) \subset X'$.*

◁ Apply the Automatic Continuity Principle twice. ▷

7.5.8. REMARK. In view of 7.5.7, in studying a Banach space X, only continuous functionals on X are usually admitted into the subdifferential of a function $f : X \to \mathbb{R}^{\cdot}$ at a point x; i.e., we agree to define

$$\partial_x(f) := \partial_x(f) \cap X'.$$

Proceed likewise in (multi)normed spaces. If a need is felt to distinguish the "old" (wider) subdifferential, a subset of $X^{\#}$, from the "new" (narrower) subdifferential, a subset of X'; then the first is called *algebraic*, whereas the second is called *topological.* With this in mind, we refer to the facts indicated in 7.5.2 and 7.5.7 as to the *Coincidence Principle for algebraic and topological subdifferentials.* Observe finally that if $f := p$ is a seminorm on X then, for similar reasons, it is customary to put $|\partial|(p) := |\partial|(p) \cap X'$.

7.5.9. Ideal Hahn–Banach Theorem. *Let $f : Y \to \mathbb{R}^{\cdot}$ be an ideally convex function on a Banach space Y. Further, let X be a normed space and $T \in B(X, Y)$. If a point x in X is such that $Tx \in$ core dom f then*

$$\partial_x(f \circ T) = \partial_{Tx}(f) \circ T.$$

◁ The right side of the sought formula is included into its left side for obvious reasons. If l in X' belongs to $\partial_x(f \circ T)$, then by the Hahn–Banach Theorem there is an element l_1 of the algebraic subdifferential of f at Tx for which $l = l_1 \circ T$. It suffices to observe that, in virtue of 7.5.7, l_1 is an element of Y' and so it is a member of the topological subdifferential $\partial_{Tx}(f)$. ▷

7.5.10. Balanced Hahn–Banach Theorem. *Suppose that X and Y are normed spaces. Given $T \in B(X, Y)$, let $p : Y \to \mathbb{R}$ be a continuous seminorm. Then*

$$|\partial|(p \circ T) = |\partial|(p) \circ T.$$

◁ If $l \in |\partial|(p \circ T)$ then $l = l_1 \circ T$ for some l_1 in the algebraic balanced subdifferential of p (cf. 3.7.11). From 7.5.2 it follows that l_1 is continuous. Thus, $|\partial|(p \circ T) \subset |\partial|(p) \circ T$. The reverse inclusion raises no doubts. ▷

7.5.11. Continuous Extension Principle. *Let X_0 be a subspace of X and let l_0 be a continuous linear functional on X_0. Then there is a continuous linear functional l on X extending l_0 and such that $\|l\| = \|l_0\|$.*

◁ Take $p := \|l_0\| \, \|\cdot\|$, and consider the identical embedding $\iota : X_0 \to X$. On account of 7.5.10, $l_0 \in |\partial|(p \circ \iota) = |\partial|(p) \circ \iota = \|l_0\| \, |\partial|(\|\cdot\|) \circ \iota$. It suffices to observe that $|\partial|(\|\cdot\|_X) = B_{X'}$. ▷

7.5.12. *Topological Separation Theorem.* *Let U be a convex set with nonempty interior in a space X. If L is an affine variety in X and $L \cap \operatorname{int} U = \varnothing$ then there is a closed hyperplane H in X such that $H \supset L$ and $H \cap \operatorname{int} U = \varnothing$.* ◁▷

7.5.13. REMARK. When applying Theorem 7.5.12, it is useful to bear in mind that a closed hyperplane is precisely a level set of a nonzero continuous linear functional. ◁▷

7.5.14. *Corollary.* *Let X_0 be a subspace of X. Then*

$$\operatorname{cl} X_0 = \cap \{\ker f : \ f \in X', \ \ker f \supset X_0\}.$$

◁ It is clear that $(f \in X' \ \& \ \ker f \supset X_0) \Rightarrow \ker f \supset \operatorname{cl} X_0$. If $x_0 \notin \operatorname{cl} X_0$ then there is an open convex neighborhood about x_0 disjoint from $\operatorname{cl} X_0$. In virtue of 7.5.12 and 7.5.13 there is a functional f_0, a member of $(X_{\mathbb{R}})'$, such that $\ker f_0 \supset \operatorname{cl} X_0$ and $f_0(x_0) = 1$. From the properties of the complexifier infer that the functional $\mathbb{R}\mathrm{e}^{-1} f_0$ vanishes on X_0 and differs from zero at the point x_0. It is also beyond a doubt that the functional is continuous. ▷

7.6. Prime Principles

7.6.1. *Let X and Y be (multi)normed vector spaces (over the same ground field $\mathbb{F}$). Assume further that X' and Y' are the duals of X and Y respectively. Take a continuous linear operator T from X to Y. Then $y' \circ T \in X'$ for $y' \in Y'$ and the mapping $y' \mapsto y' \circ T$ is a linear operator.* ◁▷

7.6.2. DEFINITION. The operator $T' : Y' \to X'$, constructed in 7.6.1, is the *dual* or *transpose* of $T : X \to Y$.

7.6.3. *Theorem.* *The prime mapping $T \mapsto T'$ implements a linear isometry of the space $B(X,\ Y)$ into the space $B(Y',\ X')$.*

◁ The prime mapping is clearly a linear operator from $B(X,\ Y)$ to $\mathscr{L}(Y',\ X')$. Furthermore, since $\|y\| = \sup\{|l(y) : \ l \in |\partial|(\|\cdot\|)\}$; therefore,

$$\|T'\| = \sup\{\|T'y'\| : \ \|y'\| \le 1\}$$
$$= \sup\{|y'(Tx)| : \ \|y'\| \le 1, \ \|x\| \le 1\} = \sup\{\|Tx\| : \ \|x\| \le 1\} = \|T\|,$$

what was required. ▷

7.6.4. EXAMPLES.

(1) Let X and Y be Hilbert spaces. Take $T \in B(X,\ Y)$. Observe first that, in a plain sense, $T \in B(X,\ Y) \Leftrightarrow T \in B(X_*,\ Y_*)$. Denote the prime mapping of X by $(\cdot)'_X : X_* \to X'$, i.e., $x \mapsto x' := (\,\cdot\,,\ x)$; and denote the prime mapping of Y by $(\cdot)'_Y : Y_* \to Y'$, i.e., $y \mapsto y' := (\,\cdot\,,\ y)$.

The adjoint of T, the member T^* of $B(Y,\ X)$, and the dual of T, the member T' of $B(Y',\ X')$, are related by the commutative diagram:

$$\begin{array}{ccc} X_* & \xleftarrow{T^*} & Y_* \\ (\cdot)'_X \downarrow & & \downarrow (\cdot)'_Y \\ X' & \xleftarrow{T'} & Y' \end{array}$$

◁ Indeed, it is necessary to show the equality $T'y' = (T^*y)'$ for $y \in Y$. Given $x \in X$, by definition observe that

$$T'y'(x) = y'(Tx) = (Tx,\ y) = (x,\ T^*y) = (T^*y)'(x).$$

Since x is arbitrary, the proof is complete. ▷

(2) *Let $\iota : X_0 \to X$ be the identical embedding of X_0 into X. Then $\iota' : X \to X_0'$. Moreover, $\iota'(x')(x_0) = x'(x_0)$ for all $x_0 \in X_0$ and $x' \in X'$ and ι' is an epimorphism; i.e., $X' \xrightarrow{\iota'} X_0' \to 0$ is an exact sequence.* ◁▷

7.6.5. DEFINITION. Let an elementary diagram $X \xrightarrow{T} Y$ be given. The diagram $Y' \xrightarrow{T'} X'$ is referred to as *resulting from setting primes* or as the *diagram prime* of the original diagram or as the *dual diagram.* If primes are set in every elementary subdiagram in an arbitrary diagram composed of bounded linear operators in Banach spaces, then the so-obtained diagram is referred to as *dual* or resulting from setting primes in the original diagram. The term "diagram prime" is used for suggestiveness.

7.6.6. *Double Prime Lemma.* *Let $X'' \xrightarrow{T''} Y''$ be the diagram that results from setting primes in the diagram $X \xrightarrow{T} Y$ twice. Then the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ '' \downarrow & & \downarrow '' \\ X'' & \xrightarrow{T''} & Y'' \end{array}$$

Here $'' : X \to X''$ and $'' : Y \to Y''$ are the respective double prime mappings; i.e. the canonical embeddings of X into X'' and of Y into Y'' (cf. 5.1.10 (8)).

◁ Let $x \in X$. We have to prove that $T''x'' = (Tx)''$. Take $y' \in Y'$. Then

$$T''x''(y') = x''(T'y') = T'y'(x) = y'(Tx) = (Tx)''(y').$$

Since $y' \in Y'$ is arbitrary, the proof is complete. ▷

7.6.7. *Diagram Prime Principle.* *A diagram is commutative if and only if so is its diagram prime.*

◁ It suffices to convince oneself that the triangles

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ {\scriptstyle R}\searrow & & \swarrow{\scriptstyle S} \\ & Z & \end{array} \qquad \begin{array}{ccc} X' & \xrightarrow{T'} & Y' \\ {\scriptstyle R'}\nwarrow & & \nearrow{\scriptstyle S'} \\ & Z' & \end{array}$$

are commutative or not simultaneously. Since $R = ST \Rightarrow R' = (ST)' = T'S'$; therefore, the commutativity of the triangle on the left entails the commutativity of the triangle on the right. If the latter commutes then by what was already proven $R'' = S''T''$. Using 7.6.6, argue as follows: $(Rx)'' = R''x'' = S''T''x'' = S''(T''x'') = S''(Tx)'' = (STx)''$ for all $x \in X$. Consequently, $R = ST$. ▷

7.6.8. DEFINITION. Let X_0 be a subspace of X and let $\mathscr{X}_0$ be a subspace of X'. Put

$$X_0^{\perp} := \{f \in X' : \ \ker f \supset X_0\} = |\partial|(\delta(X_0));$$
$$^{\perp}\mathscr{X}_0 := \{x \in X : \ f \in \mathscr{X}_0 \Rightarrow f(x) = 0\} = \cap\{\ker f : \ f \in \mathscr{X}_0\}.$$

The subspace $X_0^{\perp}$ is the (*direct*) *polar* of X_0, and the subspace $^{\perp}\mathscr{X}_0$ is the *reverse polar* of $\mathscr{X}_0$. A less exact term "annihilator" is also in use.

7.6.9. DEFINITION. Let X and Y be Banach spaces. An arbitrary element T of $B(X,\ Y)$ is a *normally solvable operator*, if $\operatorname{im} T$ is a closed subspace of Y. The natural term "closed range operator" is also in common parlance.

7.6.10. *An operator T, a member of $B(X,\ Y)$, is normally solvable if and only if T is a homomorphism, when regarded as acting from X to* $\operatorname{im} T$.

◁ $\Rightarrow$: The Banach Homomorphism Theorem.

$\Leftarrow$: Refer to 7.4.2. ▷

7.6.11. Polar Lemma. *Let $T \in B(X,\ Y)$. Then*

(1) $(\operatorname{im} T)^{\perp} = \ker(T')$;

(2) *if T is normally solvable then*

$$\operatorname{im} T = {}^{\perp}\ker(T'), \quad (\ker T)^{\perp} = \operatorname{im}(T').$$

◁ (1): $y' \in \ker(T') \Leftrightarrow T'y' = 0$

$\Leftrightarrow (\forall\, x \in X)\ T'y'(x) = 0 \Leftrightarrow (\forall\, x \in X)\ y'(Tx) = 0 \Leftrightarrow y' \in (\operatorname{im} T)^{\perp}$.

(2): The equality $\operatorname{cl} \operatorname{im} T = {}^{\perp}\ker(T')$ follows from 7.5.13. Furthermore, by hypothesis $\operatorname{im} T$ is closed.

If $x' = T'y'$ and $Tx = 0$ then $x'(x) = T'y'(x) = y'(Tx) = 0$, which means that $x' \in (\ker T)^{\perp}$. Consequently, $\operatorname{im}(T') \subset (\ker T)^{\perp}$. Now take $x' \in (\ker T)^{\perp}$. Considering the operator T acting onto $\operatorname{im} T$, apply the Sard Theorem to the left side of the diagram

$$\begin{array}{ccccc} X & \xrightarrow{T} & \operatorname{im} T & \longrightarrow & Y \\ & {\scriptstyle x'}\searrow & \downarrow{\scriptstyle y_0'} & \swarrow{\scriptstyle y'} & \\ & & \mathbb{F} & & \end{array}$$

As a result of this, obtain y'_0 in $(\mathrm{im}\ T)'$ such that $y'_0 \circ T = x'$. By the Continuous Extension Principle there is an element y' of Y' satisfying $y' \supset y'_0$. Thus, $x' = T'y'$, i.e. $x' \in \mathrm{im}\,(T')$. ▷

7.6.12. *Hausdorff Theorem.* *Let X and Y be Banach spaces. Assume further that $T \in B(X,\ Y)$. Then T is normally solvable if and only if T' is normally solvable.*

◁ $\Rightarrow$: In virtue of 7.6.11 (2), $\mathrm{im}\,(T') = (\ker\ T)^\perp$. Evidently, the subspace $(\ker\ T)^\perp$ is closed.

$\Leftarrow$: To begin with, suppose $\mathrm{cl}\ \mathrm{im}\ T = Y$. It is clear that $0 = Y^\perp = (\mathrm{cl}\ \mathrm{im}\ \ T)^\perp = (\mathrm{im}\ T)^\perp = \ker\,(T')$ in virtue of 7.6.11. By the Banach Isomorphism Theorem there is some $S \in B(\mathrm{im}\,(T'),\ Y')$ such that $ST' = I_{Y'}$. The case $r := \|S\| = 0$ is trivial. Therefore, it may be assumed that $\|T'y'\| \ge {}^1/_r\|y'\|$ for all $y' \in Y'$.

Show now that $\mathrm{cl}\ T(B_X) \supset {}^1/_{2r}B_Y$. With this at hand, from the ideal convexity of $T(B_X)$ it is possible to infer that $T(B_X) \supset {}^1/_{4r}B_Y$. The last inclusion implies that T is a homomorphism.

Let $y \notin \mathrm{cl}\ T(B_X)$. Then y does not belong to some open convex set including $T(B_X)$. Passing, if need be, to the real carriers of X and Y, assume that $\mathbb{F} := \mathbb{R}$. Applying the Topological Separation Theorem, find some nonzero y' in Y' satisfying

$$\|y'\|\ \|y\| \ge y'(y) \ge \sup_{\|x\| \le 1} y'(Tx) = \|T'y'\| \ge {}^1/_r\|y'\|.$$

Whence $\|y\| \ge {}^1/_r > {}^1/_{2r}$. Therefore, the sought inclusion is established and the operator T is normally solvable under the above supposition.

Finally, address the general case. Put $Y_0 := \mathrm{cl}\,\mathrm{im}\, T$ and let $\iota : Y_0 \to Y$ be the identical embedding. Then $T = \iota\overline{T}$, where $\overline{T} : X \to Y_0$ is the operator acting by the rule $\overline{T}x = Tx$ for $x \in X$. In addition, $\mathrm{im}\,(T') = \mathrm{im}\,(\overline{T}'\iota') = \overline{T}'(\mathrm{im}\,(\iota')) = \overline{T}'(Y'_0)$, because $\iota'(Y') = Y'_0$ (cf. 7.6.4 (2)). Thus, $\overline{T}'$ is a normally solvable operator. By what was proven, $\overline{T}$ is normally solvable. It remains to observe that $\mathrm{im}\ T = \mathrm{im}\ \overline{T}$. ▷

7.6.13. *Sequence Prime Principle.* *A sequence*

$$\dots \to X_{k-1} \xrightarrow{T_k} X_k \xrightarrow{T_{k+1}} X_{k+1} \to \dots$$

is exact if and only if so is the sequence prime

$$\dots \leftarrow X'_{k-1} \xleftarrow{T'_k} X'_k \xleftarrow{T'_{k+1}} X'_{k+1} \leftarrow \dots .$$

◁ $\Rightarrow$: Since $\mathrm{im}\ T_{k+1} = \ker\ T_{k+2}$; therefore, T_{k+1} is normally solvable. Using the Polar Lemma, conclude that

$$\ker\,(T'_k) = (\mathrm{im}\ T_k)^\perp = (\ker\ T_{k+1})^\perp = \mathrm{im}\,(T'_{k+1}).$$

$\Rightarrow$: By the Hausdorff Theorem T_{k+1} is normally solvable. Once again appealing to 7.6.11 (2), infer that

$$(\mathrm{im}\ T_k)^{\perp} = \ker(T_k') = \mathrm{im}(T_{k+1}') = (\ker\ T_{k+1})^{\perp}.$$

Since T_k is normally solvable by Theorem 7.6.12; therefore, im T_k is a closed subspace. Using 7.5.14, observe that

$$\mathrm{im}\ T^k = {}^{\perp}((\mathrm{im}\ T_k)^{\perp}) = {}^{\perp}((\ker\ T_{k+1})^{\perp}) = \ker\ T_{k+1}.$$

Here account was taken of the fact that ker T_{k+1} itself is a closed subspace. ▷

7.6.14. Corollary. *For a normally solvable operator* T, *the following isomorphisms hold*: $(\ker T)' \simeq \operatorname{coker}(T')$ *and* $(\operatorname{coker} T)' \simeq \ker(T')$.

◁ By virtue of 2.3.5 (6) the sequence

$$0 \to \ker T \to X \xrightarrow{T} Y \to \operatorname{coker} T \to 0$$

is exact. From 7.6.13 obtain that the sequence

$$0 \to (\operatorname{coker} T)' \to Y' \xrightarrow{T'} X' \to (\ker T)' \to 0$$

is exact. ▷

7.6.15. Corollary. *T is an isomorphism $\Leftrightarrow$ T' is an isomorphism.* ◁▷

7.6.16. Corollary. $\mathrm{Sp}(T) = \mathrm{Sp}(T')$. ◁▷

Exercises

7.1. Find out which linear operators are ideal.

7.2. Establish that a separately continuous bilinear form on a Banach space is jointly continuous.

7.3. Is a family of lower semicontinuous sublinear functionals on a Banach space uniformly bounded on the unit ball?

7.4. Let X and Y be Banach spaces and $T : X \to Y$. Prove that $\|Tx\|_Y \geq t\|x\|_X$ for some strictly positive t and all $x \in X$ if and only if $\ker T = 0$ and im T is a complete set.

7.5. Find conditions for normal solvability of the operator of multiplication by a function in the space of continuous functions on a compact set.

7.6. Let T be a bounded epimorphism of a Banach space X onto $l_1(\mathscr{E})$. Show that ker T is complemented.

7.7. Establish that a uniformly closed subspace of $C([a,\ b])$ composed of continuously differentiable functions (i.e., elements of $C^{(1)}([a,\ b])$) is finite-dimensional.

7.8. Let X and Y be different Banach spaces, with X continuously embedded into Y. Establish that X is a meager subset of Y.

7.9. Let X_1 and X_2 be nonzero closed subspaces of a Banach space and $X_1 \cap X_2 = 0$. Prove that the sum $X_1 + X_2$ is closed if and only if the next quantity

$$\inf\{\|x_1 - x_2\|/\|x_1\| : x_1 \neq 0,\ x_1 \in X_1,\ x_2 \in X_2\}$$

is strictly positive.

7.10. Let (a_{mn}) be a double countable sequence such that there is a sequence $(x^{(m)})$ of elements of l_1 for which all the series $\sum_{n=1}^{\infty} a_{mn} x_n^{(m)}$ fail to converge in norm. Prove that there is a sequence x in l_1 such that the series $\sum_{n=1}^{\infty} a_{mn} x_n$ fail to converge in norm for all $m \in \mathbb{N}$.

7.11. Let T be an endomorphism of a Hilbert space H which satisfies $\langle Tx \mid y\rangle = \langle x \mid Ty\rangle$ for all $x,\ y \in H$. Establish that T is bounded.

7.12. Let a closed cone X^+ in a Banach space X be ***reproducing***: $X = X^+ - X^+$. Prove that there is a constant $t > 0$ such that for all $x \in X$ and each presentation $x = x_1 - x_2$ with $x_1,\ x_2 \in X^+$, the estimates hold: $\|x_1\| \leq t\|x\|$ and $\|x_2\| \leq t\|x\|$.

7.13. Let lower semicontinuous sublinear functionals p and q on a Banach space X be such that the cones dom p and dom q are closed and the subspace dom $p -$dom $q =$ dom $q -$dom p is complemented in X. Prove that

$$\partial(p + q) = \partial(p) + \partial(q)$$

in the case of topological subdifferentials (cf. Exercise 3.10).

7.14. Let p be a continuous sublinear functional defined on a normed space X and let T be a continuous endomorphism of X. Assume further that the dual T' of T takes the subdifferential $\partial(p)$ into itself. Establish that $\partial(p)$ contains a fixed point of T'.

7.15. Given a function $f : X \to \mathbb{R}^{\cdot}$ on a (multi)normed space X, put

$$f^*(x') := \sup\{\langle x \mid x'\rangle - f(x) : x \in \operatorname{dom} f\} \quad (x' \in X');$$
$$f^{**}(x) := \sup\{\langle x \mid x'\rangle - f^*(x') : x' \in \operatorname{dom}(f^*)\} \quad (x \in X).$$

Find conditions for f to satisfy $f = f^{**}$.

7.16. Establish that l_∞ is complemented in each ambient Banach space.

7.17. A Banach space X is called ***primary***, if each of its infinite-dimensional complemented subspaces is isomorphic to X. Verify that c_0 and l_p $(1 \leq p \leq +\infty)$ are primary.

7.18. Let X and Y be Banach spaces. Take an operator T, a member of $B(X,\ Y)$, such that im T is nonmeager. Prove that T is normally solvable.

7.19. Let X_0 be a closed subspace of a normed space X. Assume further that X_0 and X/X_0 are Banach spaces. Show that X itself is a Banach space.

Chapter 8
Operators in Banach Spaces

8.1. Holomorphic Functions and Contour Integrals

8.1.1. DEFINITION. Let X be a Banach space. A subset Λ of the ball $B_{X'}$ in the dual space X' is called *norming* (for X) if $\|x\| = \sup\{|l(x)| : \ l \in \Lambda\}$ for all $x \in X$. If each subset U of X satisfies $\sup\|U\| < +\infty$ on condition that $\sup\{|l(u)| : \ u \in U\} < +\infty$ for all $l \in \Lambda$, then Λ is a *fully norming set.*

8.1.2. EXAMPLES.

(1) The ball $B_{X'}$ is a fully norming set in virtue of 5.1.10 (8) and 7.2.7.

(2) If Λ_0 is a (fully) norming set and $\Lambda_0 \subset \Lambda_1 \subset B_{X'}$ then Λ_1 itself is a (fully) norming set.

(3) The set ext $B_{X'}$ of the extreme points of $B_{X'}$ is norming in virtue of the Kreĭn–Milman Theorem in subdifferential form and the obvious equality $B_{X'} = |\partial|(\|\cdot\|_X)$ which has already been used many times. However, ext $B_{X'}$ can fail to be fully norming (in particular, the possibility is realized in the space $C([0,\ 1],\ \mathbb{R})$). ◁▷

(4) Let X and Y be Banach spaces (over the same ground field $\mathbb{F}$) and let Λ_Y be a norming set for Y. Put

$$\Lambda_B := \{\delta_{(y',x)} : \ y' \in \Lambda_Y,\ x \in B_X\},$$

where $\delta_{(y',x)}(T) := y'(Tx)$ for $y' \in Y,\ x \in X$ and $T \in B(X,\ Y)$. It is clear that

$$|\delta_{(y',x)}(T)| = |y'(Tx)| \le \|y'\|\ \|Tx\| \le \|y'\|\ \|T\|\ \|x\|;$$

i.e., $\delta_{(y',x)} \in B(X,\ Y)'$. Furthermore, given $T \in B(X,\ Y)$, infer that

$$\begin{aligned}\|T\| = \sup\{\|Tx\| : \ \|x\| \le 1\} &= \sup\{|y'(Tx)| : \ y' \in \Lambda_Y,\ \|x\| \le 1\}\\ &= \sup\{|\delta_{(y',x)}(T)| : \ \delta_{(y',x)} \in \Lambda_B\}.\end{aligned}$$

Therefore, Λ_B is a norming set for $B(X,\ Y)$. If Λ_Y is a fully norming set then Λ_B is also a fully norming set. Indeed, if U is such that the numeric set $\{|y'(Tx)| :$

$T \in U\}$ is bounded in $\mathbb{R}$ for all $x \in B_X$ and $y' \in \Lambda_Y$, then by hypothesis the set $\{Tx : T \in U\}$ is bounded in Y for every x in X. By virtue of the Uniform Boundedness Principle it means that $\sup \|U\| < +\infty$.

8.1.3. *Dunford–Hille Theorem.* *Let X be a complex Banach space and let Λ be a fully norming set for X. Assume further that $f : \mathscr{D} \to X$ is an X-valued function with domain $\mathscr{D}$ an open (in $\mathbb{C}_{\mathbb{R}} \simeq \mathbb{R}^2$) subset of $\mathbb{C}$. The following statements are equivalent:*

(1) *for every z_0 in $\mathscr{D}$ there is a limit*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0};$$

(2) *for all $z_0 \in \mathscr{D}$ and $l \in \Lambda$ there is a limit*

$$\lim_{z \to z_0} \frac{l \circ f(z) - l \circ f(z_0)}{z - z_0};$$

i.e., the function $l \circ f : \mathscr{D} \to \mathbb{C}$ is holomorphic for $l \in \Lambda$.

$\triangleleft$ (1) $\Rightarrow$ (2): This is obvious.

(2) $\Rightarrow$ (1): For the sake of simplicity assume that $z_0 = 0$ and $f(z_0) = 0$. Consider the disk of radius 2ε centered at zero and included in $\mathscr{D}$; i.e., $2\varepsilon\mathbb{D} \subset \mathscr{D}$, where $\mathbb{D} := B_{\mathbb{C}} := \{z \in \mathbb{C} : |z| \le 1\}$ is the *unit disk*. As is customary in complex analysis, treat the disk $\varepsilon\mathbb{D}$ as an (oriented) compact manifold with boundary $\varepsilon\mathbb{T}$, where $\mathbb{T}$ is the (properly oriented) *unit circle* $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ in the *complex plane* $\mathbb{C}_{\mathbb{R}}$. Take $z_1, z_2 \in \varepsilon\mathbb{D}\setminus 0$ and the holomorphic function $l \circ f$ (the functional l lies in Λ). Specifying the Cauchy Integral Formula, observe that

$$\frac{l \circ f(z_k)}{z_k} = \frac{1}{2\pi i} \int\limits_{2\varepsilon\mathbb{T}} \frac{l \circ f(z)}{z(z - z_k)} dz \quad (k := 1,\ 2).$$

If now $z_1 \ne z_2$ then, using the condition $|z - z_k| \ge \varepsilon \quad (k := 1,\ 2)$ for $z \in 2\varepsilon\mathbb{T}$ and the continuity property of the function $l \circ f$ on $\mathscr{D}$, find

$$\left| l \left(\frac{1}{z_1 - z_2} \left(\frac{f(z_1)}{z_1} - \frac{f(z_2)}{z_2} \right) \right) \right|$$

$$= \left| \frac{1}{z_1 - z_2} \cdot \frac{1}{2\pi i} \int\limits_{2\varepsilon\mathbb{T}} l \circ f(z) \left(\frac{1}{z(z - z_1)} - \frac{1}{z(z - z_2)} \right) dz \right|$$

$$= \frac{1}{2\pi} \left| \int\limits_{2\varepsilon\mathbb{T}} l \circ f(z) \frac{1}{z(z - z_1)(z - z_2)} dz \right| \le M \sup_{z \in 2\varepsilon\mathbb{T}} |l \circ f(z)| < +\infty$$

for a suitable $M > 0$. Since Λ is a fully norming set, conclude that

$$\sup_{\substack{z_1 \neq z_2; z_1, z_2 \neq 0 \\ |z_1| \le \varepsilon, |z_2| \le \varepsilon}} \frac{1}{|z_1 - z_2|} \left\| \frac{f(z_1)}{z_1} - \frac{f(z_2)}{z_2} \right\| < +\infty.$$

This final inequality ensures that the sought limit exists. ▷

8.1.4. Definition. A mapping $f : \mathscr{D} \to X$ satisfying 8.1.3 (1) (or, which is the same, 8.1.3 (2) for some fully norming set Λ) is called *holomorphic*.

8.1.5. Remark. A meticulous terminology is used sometimes. Namely, if f satisfies 8.1.3 (1) then f is called a *strongly holomorphic function*. If f satisfies 8.1.3 (2) with $\Lambda := B_{X'}$ then f is called *weakly holomorphic*. Under the hypotheses of 8.1.3 (2) and 8.1.2 (4), i.e. for $f : \mathscr{D} \to B(X, Y)$, $\Lambda_Y := B_{Y'}$ and the corresponding $\Lambda := \Lambda_B$, the expression, "f is *weakly operator holomorphic*," is employed. With regard to this terminology, the Dunford–Hille Theorem is often referred to as the *Holomorphy Theorem* and verbalized as follows: "A weakly holomorphic function is strongly holomorphic."

8.1.6. Remark. It is convenient in the sequel to use the integrals of the simplest smooth X-valued forms $f(z)dz$ over the simplest oriented manifolds, the boundaries of elementary planar compacta (cf. 4.8.5) which are composed of finitely many disjoint simple loops. An obvious meaning is ascribed to the integrals: Namely, given a loop γ, choose an appropriate (smooth) parametrization $\Psi : \mathbb{T} \to \gamma$ (with orientation accounted for) and put

$$\int_\gamma f(z)dz := \int_{\mathbb{T}} f \circ \Psi d\Psi,$$

with the integral treated for instance as a suitable Bochner integral (cf. 5.5.9 (6)). The soundness of the definition is beyond a doubt, since the needed Bochner integral exists independently of the choice of the parametrization Ψ.

8.1.7. *Cauchy–Wiener Integral Theorem.* *Let $\mathscr{D}$ be an nonempty open subset of the complex plane and let $f : \mathscr{D} \to X$ be a holomorphic X-valued function, with X a Banach space. Assume further that F is a rough draft for the pair $(\varnothing, \mathscr{D})$. Then*

$$\int_{\partial F} f(z)dz = 0.$$

Moreover,

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial F} \frac{f(z)}{z - z_0} dz$$

for $z_0 \in \operatorname{int} F$.

◁ By virtue of 8.1.3 the Bochner integrals exist. The sought equalities follow readily from the validity of their scalar versions basing on the Cauchy Integral Formula, the message of 8.1.2 (1) and the fact that the Bochner integral commutes with every bounded functional as mentioned in 5.5.9 (6). ▷

8.1.8. REMARK. The Cauchy–Wiener Integral Theorem enables us to infer analogs of the theorems of classical complex analysis for X-valued holomorphic functions on following the familiar patterns.

8.1.9. *Taylor Series Expansion Theorem.* *Let $f : \mathscr{D} \to X$ be a holomorphic X-valued function, with X a Banach space, and take $z_0 \in \mathscr{D}$. In every open disk $U := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ such that $\operatorname{cl} U$ lies in $\mathscr{D}$, the Taylor series expansion holds (in a compactly convergent power series, cf. 7.2.10):*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$

where the coefficients c_n, members of X, are calculated by the formulas:

$$c_n = \frac{1}{2\pi i} \int_{\partial U} \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{1}{n!} \frac{d^n f}{dz^n}(z_0).$$

◁ The proof results from a standard argument: Expand the kernel $u \mapsto (u - z)^{-1}$ of the formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial U'} \frac{f(u)}{u - z} du \quad (z \in \operatorname{cl} U)$$

in the powers of $z - z_0$; i.e.,

$$\frac{1}{u - z} = \frac{1}{(u - z_0)\left(1 - \frac{z - z_0}{u - z_0}\right)} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(u - z_0)^{n+1}}.$$

The last series converges uniformly in $u \in \partial U'$. (Here $U' = U + q\mathbb{D}$ for some $q > 0$ such that $\operatorname{cl} U' \subset \mathscr{D}$.) Taking it into account that $\sup \|f(\partial U')\| < +\infty$ and integrating, arrive at the sought presentation of $f(z)$ for $z \in \operatorname{cl} U$. Applying this to U' and using 8.1.7, observe that the power series under study converges in norm at every point of U'. This yields uniform convergence on every compact subset of U', and so on U. ▷

8.1.10. *Liouville Theorem.* *If an X-valued function $f : \mathbb{C} \to X$, with X a Banach space, is holomorphic and $\sup \|f(\mathbb{C})\| < +\infty$ then f is a constant mapping.*

◁ Considering the disk $\varepsilon\mathbb{D}$ with $\varepsilon > 0$ and taking note of 8.1.9, infer that

$$\|c_n\| \le \sup_{z\in\varepsilon\mathbb{T}} \|f(z)\| \cdot \varepsilon^{-n} \le \sup \|f(\mathbb{C})\| \cdot \varepsilon^{-n}$$

for all $n \in \mathbb{N}$ and $\varepsilon > 0$. Therefore, $c_n = 0$ for $n \in \mathbb{N}$. ▷

8.1.11. *Each bounded endomorphism of a nonzero complex Banach space has a nonempty spectrum.*

◁ Let T be such an endomorphism. If $\mathrm{Sp}(T) = \varnothing$ then the resolvent $R(T,\ \cdot\,)$ is holomorphic on the entire complex plane $\mathbb{C}$, for instance, by 5.6.21. Furthermore, by 5.6.15, $\|R(T,\ \lambda)\| \to 0$ as $|\lambda| \to +\infty$. By virtue of 8.1.10 conclude that $R(T,\ \cdot\,) = 0$. At the same time, using 5.6.15, observe that $R(T,\ \lambda)(\lambda - T) = 1$ for $|\lambda| > \|T\|$. A contradiction. ▷

8.1.12. *The Beurling–Gelfand formula holds:*

$$r(T) = \sup\{|\lambda| :\ \lambda \in \mathrm{Sp}(T)\}$$

for all $T \in B(X)$, with X a complex Banach space; i.e., the spectral radius of an operator T coincides with the radius of the spectrum of T.

◁ It is an easy matter that the spectral radius $r(T)$ is greater than the radius of the spectrum of T. So, there is nothing to prove if $r(T) = 0$. Assume now that $r(T) > 0$. Take $\lambda \in \mathbb{C}$ so that $|\lambda| > \sup\{|\mu| :\ \mu \in \mathrm{Sp}(T)\}$. Then the disk of radius $|\lambda|^{-1}$ lies entirely in the domain of the holomorphic function (cf. 5.6.15)

$$f(z) := \begin{cases} R(T,\ z^{-1}), & \text{for } z \neq 0 \text{ and } z^{-1} \in \mathrm{res}\,(T) \\ 0, & \text{for } z = 0. \end{cases}$$

Using 8.1.9 and 5.6.17, conclude that $|\lambda|^{-1} < r(T)^{-1}$. Consequently, $|\lambda| > r(T)$. ▷

8.1.13. *Let K be a nonempty compact subset of $\mathbb{C}$. Denote by $H(K)$ the set of all functions holomorphic in a neighborhood of K (i.e., $f \in H(K) \Leftrightarrow f :$ $\mathrm{dom}\, f \to \mathbb{C}$ is a holomorphic function with $\mathrm{dom}\, f \supset K$). Given $f_1,\ f_2 \in H(K)$, let the notation $f_1 \sim f_2$ mean that it is possible to find an open subset $\mathscr{D}$ of $\mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ satisfying $K \subset \mathscr{D}$ and $f_1|_{\mathscr{D}} = f_2|_{\mathscr{D}}$. Then $\sim$ is an equivalence in $H(K)$.* ◁▷

8.1.14. Definition. Under the hypotheses of 8.1.13, put $\mathscr{H}(K) := H(K)/\!\sim$. The element $\overline{f}$ in $\mathscr{H}(K)$ containing a function f in $H(K)$ is the *germ* of f on K.

8.1.15. *Let $\overline{f},\ \overline{g} \in \mathscr{H}(K)$. Take $f_1,\ f_2 \in \overline{f}$ and $g_1,\ g_2 \in \overline{g}$. Put*

$$x \in \mathrm{dom}\, f_1 \cap \mathrm{dom}\, g_1 \Rightarrow \varphi_1(x) := f_1(x) + g_1(x),$$
$$x \in \mathrm{dom}\, f_2 \cap \mathrm{dom}\, g_2 \Rightarrow \varphi_2(x) := f_2(x) + g_2(x).$$

Then $\varphi_1,\ \varphi_2 \in H(K)$ and $\overline{\varphi}_1 = \overline{\varphi}_2$.

◁ Choose open sets $\mathscr{D}_1$ and $\mathscr{D}_2$ such that $K \subset \mathscr{D}_1 \subset \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and $K \subset \mathscr{D}_2 \subset \operatorname{dom} g_1 \cap \operatorname{dom} g_2$, with $f_1|_{\mathscr{D}_1} = f_2|_{\mathscr{D}_1}$ and $g_1|_{\mathscr{D}_2} = g_2|_{\mathscr{D}_2}$. Observe now that φ_1 and φ_2 agree on $\mathscr{D}_1 \cap \mathscr{D}_2$. ▷

8.1.16. DEFINITION. The coset, introduced in 8.1.15, is the *sum of the germs* $\overline{f}_1$ and $\overline{f}_2$. It is denoted by $\overline{f}_1 + \overline{f}_2$. The *product of germs* and *multiplication of a germ by a complex number* are introduced by analogy.

8.1.17. *The set $\mathscr{H}(K)$ with operations defined in 8.1.16 is an algebra.* ◁▷

8.1.18. DEFINITION. The algebra $\mathscr{H}(K)$ is the *algebra of germs of holomorphic functions* on a compact set K.

8.1.19. *Let K be a compact subset of $\mathbb{C}$, and let $R : \mathbb{C} \setminus K \to X$ be an X-valued holomorphic function, with X a Banach space. Further, take $\overline{f} \in \mathscr{H}(K)$ and $f_1,\ f_2 \in \overline{f}$. If F_1 is a rough draft for the pair $(K,\ \operatorname{dom} f_1)$ and F_2 is a rough draft for the pair $(K,\ \operatorname{dom} f_2)$ then*

$$\int_{\partial F_1} f_1(z)R(z)dz = \int_{\partial F_2} f_2(z)R(z)dz.$$

◁ Let $K \subset \mathscr{D} \subset \operatorname{int} F_1 \cap \operatorname{int} F_2$, with $\mathscr{D}$ open and $f_1|_{\mathscr{D}} = f_2|_{\mathscr{D}}$. Choose a rough draft F for the pair $(K,\ D)$. Since $f_1 R$ is holomorphic on $\operatorname{dom} f_1 \setminus K$ and $f_2 R$ is holomorphic on $\operatorname{dom} f_2 \setminus K$, infer the equalities

$$\int_{\partial F} f_1(z)R(z)dz = \int_{\partial F_1} f_1(z)R(z)dz,$$

$$\int_{\partial F} f_2(z)R(z)dz = \int_{\partial F_2} f_2(z)R(z)dz$$

(from the nontrivial fact of the validity of their scalar analogs). Since f_1 and f_2 agree on $\mathscr{D}$, the proof is complete. ▷

8.1.20. DEFINITION. Under the hypotheses of 8.1.19, given an element h in $\mathscr{H}(K)$, define the *contour integral* of h with kernel R as the element

$$\oint h(z)R(z)dz := \int_{\partial F} f(z)R(z)dz,$$

where $h = \overline{f}$ and F is a rough draft for the pair $(K,\ \operatorname{dom} f)$.

8.1.21. REMARK. The notation $h(z)$ in 8.1.20 is far from being ad hoc. It is well justified by the fact that $w := f_1(z) = f_2(z)$ for every point z in K and every two members f_1 and f_2 of a germ h. In this connection the element w is said to be the *value of h at z*, which is expressed in writing as $h(z) = w$. It is also worth noting that the function R in 8.1.2 may be assumed to be given only in $U \setminus K$, where int $U \supset K$.

8.2. The Holomorphic Functional Calculus

8.2.1. DEFINITION. Let X be a (nonzero) complex Banach space and let T be a bounded *endomorphism* of X; i.e., $T \in B(X)$. For $h \in \mathscr{H}(\mathrm{Sp}(T))$, the contour integral with kernel the resolvent $R(T,\ \cdot\ )$ of T is denoted by

$$\mathscr{R}_T h := \frac{1}{2\pi i} \oint h(z) R(T,\ z)\,dz$$

and called the *Riesz–Dunford integral* (of the germ h). If f is a function holomorphic in a neighborhood about $\mathrm{Sp}(T)$ then put $f(T) := \mathscr{R}_T f := \mathscr{R}_T \overline{f}$. We also use more suggestive designations like

$$f(T) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - T}\,dz.$$

8.2.2. REMARK. In algebra, in particular, various representations of mathematical objects are under research. It is convenient to use the primary notions of representation theory for the most "algebraic" objects, namely, algebras. Recall the simplest of them.

Let A_1 and A_2 be two algebras (over the same field). A *morphism* from A_1 to A_2 or a *representation* of A_1 in A_2 (rarely, over A_2) is a *multiplicative linear operator* $\mathfrak{R}$, i.e. a member $\mathfrak{R}$ of $\mathscr{L}(A_1,\ A_2)$ such that $\mathfrak{R}(ab) = \mathfrak{R}(a)\mathfrak{R}(b)$ for all $a,\ b \in A_1$. The expression, "T represents A_1 in A_2," is also in common parlance. A representation $\mathfrak{R}$ is called *faithful* if $\ker \mathfrak{R} = 0$. The presence of a faithful representation $\mathfrak{R} : A_1 \to A_2$ makes it possible to treat A_1 as a subalgebra of A_2.

If A_2 is a (sub)algebra of the endomorphism algebra $\mathscr{L}(X)$ of some vector space X (over the same field), then a morphism of A_1 in A_2 is referred to as a (*linear*) *representation* of A_1 on X or as an *operator representation* of A_1. The space X is then called the *representation space* for the algebra A_1.

Given a representation $\mathfrak{R}$, suppose that the representation space X for A has a subspace X_1 invariant under all operators $\mathfrak{R}(a)$, $a \in A$. Then the representation $\mathfrak{R}_1 : A \to \mathscr{L}(X_1)$ arises naturally, acting by the rule $\mathfrak{R}_1(a)x_1 = \mathfrak{R}(a)x_1$ for $x_1 \in X_1$ and $a \in A$ and called a *subrepresentation* of $\mathfrak{R}$ (induced in X_1). If $X = X_1 \oplus X_2$ and the decomposition reduces each operator $\mathfrak{R}(a)$ for $a \in A$, then it is said that the representation $\mathfrak{R}$ *reduces* to the direct sum of subrepresentations $\mathfrak{R}_1$ and $\mathfrak{R}_2$ (induced in X_1 and X_2). We mention in passing the significance of studying arbitrary *irreducible representations*, each of which has only trivial subrepresentations by definition.

8.2.3. Gelfand–Dunford Theorem. *Let T be a bounded endomorphism of a Banach space X. The Riesz–Dunford integral $\mathscr{R}_T$ represents the algebra of germs of holomorphic functions on the spectrum of T on the space X. Moreover, if $f(z) = \sum_{n=0}^{\infty} c_n z^n$ (in a neighborhood about $\operatorname{Sp}(T)$) then $f(T) = \sum_{n=0}^{\infty} c_n T^n$ (summation is understood in the operator norm of $B(X)$).*

◁ There is no doubt that $\mathscr{R}_T$ is a linear operator. Check the multiplicativity property of $\mathscr{R}_T$. To this end, take $\overline{f}_1, \overline{f}_2 \in \mathscr{H}(\operatorname{Sp}(T))$ and choose rough drafts F_1 and F_2 such that $\operatorname{Sp}(T) \subset \operatorname{int} F_1 \subset F_1 \subset \operatorname{int} F_2 \subset F_2 \subset \mathscr{D}$, with the functions f_1 in $\overline{f}_1$ and f_2 in $\overline{f}_2$ holomorphic on $\mathscr{D}$.

Using the obvious properties of the Bochner integral, the Cauchy Integral Formula and the Hilbert identity, successively infer the chain of equalities

$$\mathscr{R}_T\overline{f}_1 \circ \mathscr{R}_T\overline{f}_2 = f_1(T)f_2(T) = \frac{1}{2\pi i}\frac{1}{2\pi i}\int\limits_{\partial F_1} \frac{f_1(z_1)}{z_1 - T}dz_1 \circ \int\limits_{\partial F_2} \frac{f_2(z_2)}{z_2 - T}dz_2$$

$$= \frac{1}{2\pi i}\frac{1}{2\pi i}\int\limits_{\partial F_2}\left(\int\limits_{\partial F_1} f_1(z_1)R(T,\ z_1)dz_1\right) f_2(z_2)R(T,\ z_2)dz_2$$

$$= \frac{1}{2\pi i}\frac{1}{2\pi i}\int\limits_{\partial F_1}\int\limits_{\partial F_2} f_1(z_1)f_2(z_2)R(T,\ z_1)R(T,\ z_2)dz_2dz_1$$

$$= \frac{1}{2\pi i}\frac{1}{2\pi i}\int\limits_{\partial F_1}\int\limits_{\partial F_2} f_1(z_1)f_2(z_2)\frac{R(T,\ z_1) - R(T,\ z_2)}{z_2 - z_1}dz_2dz_1$$

$$= \frac{1}{2\pi i}\int\limits_{\partial F_1} f_1(z_1)\left(\frac{1}{2\pi i}\int\limits_{\partial F_2}\frac{f_2(z_2)}{z_2 - z_1}dz_2\right)R(T,\ z_1)dz_1$$

$$-\frac{1}{2\pi i}\int\limits_{\partial F_2} f_2(z_2)\left(\frac{1}{2\pi i}\int\limits_{\partial F_1}\frac{f_1(z_1)}{z_2 - z_1}dz_1\right)R(T,\ z_2)dz_2$$

$$= \frac{1}{2\pi i}\int\limits_{\gamma} f_1(z_1)f_2(z_1)R(T,\ z_1)dz_1 - 0 = f_1f_2(T) = \mathscr{R}_T(\overline{f}_1\overline{f}_2).$$

Choose a circle $\gamma := \varepsilon\mathbb{T}$ that lies in $\operatorname{res}(T)$ as well as in the (open) disk of convergence of the series $f(z) = \sum_{n=0}^{\infty} c_n z^n$. From 5.6.16 and 5.5.9 (6) derive

$$f(T) = \frac{1}{2\pi i} \int_{\gamma} f(z) \sum_{n=0}^{\infty} z^{-n-1} T^n dz$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\gamma} f(z) z^{-n-1} T^n dz = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z^{n+1}} dz \right) T^n = \sum_{n=0}^{\infty} c_n T^n$$

in virtue of 8.1.9. ▷

8.2.4. REMARK. Theorem 8.2.3 is often referred to as the *Principal Theorem of the holomorphic functional calculus.*

8.2.5. *Spectral Mapping Theorem.* *For every function f holomorphic in a neighborhood about the spectrum of an operator T in $B(X)$, the equality holds:*

$$f(\mathrm{Sp}(T)) = \mathrm{Sp}(f(T)).$$

◁ Assume first that $\lambda \in \mathrm{Sp}(f(T))$ and $f^{-1}(\lambda) \cap \mathrm{Sp}(T) = \varnothing$. Given $z \in (\mathbb{C} \setminus f^{-1}(\lambda)) \cap \mathrm{dom}\, f$, put $g(z) := (\lambda - f(z))^{-1}$. Then g is a holomorphic function on a neighborhood of $\mathrm{Sp}(T)$, satisfying $\overline{g}(\overline{\lambda} - \overline{f}) = (\overline{\lambda} - \overline{f})\overline{g} = \overline{\mathbf{1}_{\mathbb{C}}}$. Using 8.2.3, observe that $\lambda \in \mathrm{res}\,(f(T))$, a contradiction. Consequently, $f^{-1}(\lambda) \cap \mathrm{Sp}(T) \neq \varnothing$; i.e., $\mathrm{Sp}(f(T)) \subset f(\mathrm{Sp}(T))$.

Now take $\lambda \in \mathrm{Sp}(T)$. Put

$$\lambda \neq z \Rightarrow g(z) := \frac{f(\lambda) - f(z)}{\lambda - z}; \quad g(\lambda) := f'(\lambda).$$

Clearly, g is a holomorphic function (the singularity is "removed"). From 8.2.3 obtain

$$g(T)(\lambda - T) = (\lambda - T)g(T) = f(\lambda) - f(T).$$

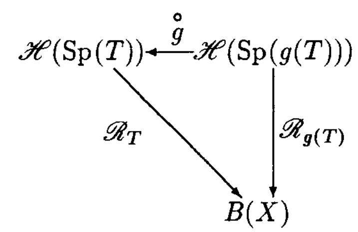
Consequently, if $f(\lambda) \in \mathrm{res}\,(f(T))$ then the operator $R(f(T),\ f(\lambda))g(T)$ is inverse to $\lambda - T$. In other words, $\lambda \in \mathrm{res}\,(T)$, which is a contradiction. Thus,

$$f(\lambda) \in \mathbb{C} \setminus \mathrm{res}\,(f(T)) = \mathrm{Sp}(f(T));$$

i.e., $f(\mathrm{Sp}(T)) \subset \mathrm{Sp}(f(T))$. ▷

8.2.6. *Let K be a nonempty compact subset of $\mathbb{C}$ and let $g : \mathrm{dom}\, g \to \mathbb{C}$ be a holomorphic function with $\mathrm{dom}\, g \supset K$. Given $f \in H(g(K))$, put $\mathring{g}(\overline{f}) := \overline{f \circ g}$. Then $\mathring{g}$ is a representation of the algebra $\mathscr{H}(g(K))$ in the algebra $\mathscr{H}(K)$.* ◁▷

8.2.7. Dunford Theorem. *For every function* $g : \operatorname{dom} g \to \mathbb{C}$ *holomorphic in the neighborhood* $\operatorname{dom} g$ *about the spectrum* $\operatorname{Sp}(T)$ *of an endomorphism* T *of a Banach space* X, *the following diagram of representations commutes:*



$\lhd$ Let $\overline{f} \in \mathscr{H}(g(\operatorname{Sp}(T)))$ with $f : \mathscr{D} \to \mathbb{C}$ such that $f \in \overline{f}$ and $\mathscr{D} \supset g(\operatorname{Sp}(T)) = \operatorname{Sp}(g(T))$. Let F_1 be a rough draft for the pair $(\operatorname{Sp}(g(T)),\ \mathscr{D})$ and let F_2 be a rough draft for the pair $(\operatorname{Sp}(T),\ g^{-1}(\operatorname{int} F_1))$. It is clear that now $g(\partial F_2) \subset \operatorname{int} F_1$ and, moreover, the function $z_2 \mapsto (z_1 - g(z_2))^{-1}$ is defined and holomorphic on $\operatorname{int} F_2$ for $z_1 \in \partial F_1$. Therefore, by 8.2.3

$$R(g(T),\ z_1) = \frac{1}{2\pi i} \int\limits_{\partial F_2} \frac{R(T,\ z_2)}{z_1 - g(z_2)} dz_2 \quad (z_1 \in \partial F_1).$$

From this equality, successively derive

$$\mathscr{R}_{g(T)} f = \frac{1}{2\pi i} \int\limits_{\partial F_1} \frac{f(z_1)}{z_1 - g(T)} dz_1 = \frac{1}{2\pi i} \frac{1}{2\pi i} \int\limits_{\partial F_1} f(z_1) \left(\int\limits_{\partial F_2} \frac{R(T,\ z_2)}{z_1 - g(z_2)} dz_2 \right) dz_1$$
$$= \frac{1}{2\pi i} \frac{1}{2\pi i} \int\limits_{\partial F_2} \left(\int\limits_{\partial F_1} \frac{f(z_1)}{z_1 - g(z_2)} dz_1 \right) R(T,\ z_2) dz_2.$$

Since $g(z_2) \in \operatorname{int} F_1$ for $z_2 \in \partial F_2$ by construction; therefore, the Cauchy Integral Formula yields the equality

$$f(g(z_2)) = \frac{1}{2\pi i} \int\limits_{\partial F_1} \frac{f(z_1)}{z_1 - g(z_2)} dz_1.$$

Consequently,

$$\mathscr{R}_{g(T)} f = \frac{1}{2\pi i} \int\limits_{\partial F_2} f(g(z_2)) R(T,\ z_2) dz_2 = \mathscr{R}_T \overset{\circ}{g}(\overline{f}).$$

8.2.8. Remark. The Dunford Theorem is often referred to as the *Composite Function Theorem* and written down symbolically as $f \circ g(T) = f(g(T))$.

8.2.9. DEFINITION. A subset σ of $\operatorname{Sp}(T)$ is a *clopen* or *isolated part* or rarely an *exclave* of $\operatorname{Sp}(T)$, if σ and its complement $\sigma' := \operatorname{Sp}(T) \setminus \sigma$ are closed.

8.2.10. *Let σ be a clopen part of $\operatorname{Sp}(T)$ and let $\varkappa_\sigma$ be some function that equals 1 in an open neighborhood of σ and 0 in an open neighborhood of σ'. Further, assign*

$$P_\sigma := \varkappa_\sigma(T) := \frac{1}{2\pi i} \oint \frac{\varkappa_\sigma(z)}{z - T} dz.$$

Then P_σ is a projection in X and the (closed) subspace $X_\sigma := \operatorname{im} P_\sigma$ is invariant under T.

◁ Since $\varkappa_\sigma^2 = \varkappa_\sigma$, it follows from 8.2.3 that $\varkappa_\sigma(T)^2 = \varkappa_\sigma(T)$. Furthermore, $T = \mathscr{R}_T I_{\mathbb{C}}$, where $I_{\mathbb{C}} : z \mapsto z$. Hence $TP_\sigma = P_\sigma T$ (because $\overline{I}_{\mathbb{C}} \overline{\varkappa}_\sigma = \overline{\varkappa}_\sigma \overline{I}_{\mathbb{C}}$). Consequently, in virtue of 2.2.9 (4), X_σ is invariant under T. ▷

8.2.11. DEFINITION. The projection P_σ is the *Riesz projection* or the *Riesz idempotent* corresponding to σ.

8.2.12. *Spectral Decomposition Theorem.* *Let σ be a clopen part of the spectrum of an operator T in $B(X)$. Then X splits into the direct sum decomposition of the invariant subspaces $X = X_\sigma \oplus X_{\sigma'}$ which reduces T to matrix form*

$$T \sim \begin{pmatrix} T_\sigma & 0 \\ 0 & T_{\sigma'} \end{pmatrix},$$

with the part T_σ of T in X_σ and the part $T_{\sigma'}$ of T in $X_{\sigma'}$ satisfying

$$\operatorname{Sp}(T_\sigma) = \sigma, \quad \operatorname{Sp}(T_{\sigma'}) = \sigma'.$$

◁ Since $\overline{\varkappa}_\sigma + \overline{\varkappa}_{\sigma'} = \overline{\varkappa_{\operatorname{Sp}(T)}} = \overline{1_{\mathbb{C}}}$, in view of 8.2.3 and 8.2.10 it suffices to establish the claim about the spectrum of T_σ.

From 8.2.5 and 8.2.3 obtain

$$\begin{aligned} \sigma \cup 0 = \varkappa_\sigma I_{\mathbb{C}}(\operatorname{Sp}(T)) &= \operatorname{Sp}(\varkappa_\sigma I_{\mathbb{C}}(T)) = \operatorname{Sp}(\mathscr{R}_T(\varkappa_\sigma I_{\mathbb{C}})) \\ &= \operatorname{Sp}(\mathscr{R}_T \varkappa_\sigma \circ \mathscr{R}_T I_{\mathbb{C}}) = \operatorname{Sp}(P_\sigma T). \end{aligned}$$

Moreover, in matrix form

$$P_\sigma T \sim \begin{pmatrix} T_\sigma & 0 \\ 0 & 0 \end{pmatrix}.$$

Let λ be a nonzero complex number. Then

$$\lambda - P_\sigma T \sim \begin{pmatrix} \lambda - T_\sigma & 0 \\ 0 & \lambda \end{pmatrix};$$

i.e., the operator $\lambda - P_\sigma T$ is not invertible if and only if the same is true of the operator $\lambda - T_\sigma$. Thus,

$$\mathrm{Sp}(T_\sigma) \setminus 0 \subset \mathrm{Sp}(P_\sigma T) \setminus 0 = (\sigma \cup 0) \setminus 0 \subset \sigma.$$

Suppose that $0 \in \mathrm{Sp}(T_\sigma)$ and $0 \notin \sigma$. Choose disjoint open sets $\mathscr{D}_\sigma$ and $\mathscr{D}_{\sigma'}$ so that $\sigma \subset \mathscr{D}_\sigma$, $0 \notin \mathscr{D}_\sigma$ and $\sigma' \subset \mathscr{D}_{\sigma'}$, and put

$$z \in \mathscr{D}_\sigma \Rightarrow h(z) := \frac{1}{z};$$
$$z \in \mathscr{D}_{\sigma'} \Rightarrow h(z) := 0.$$

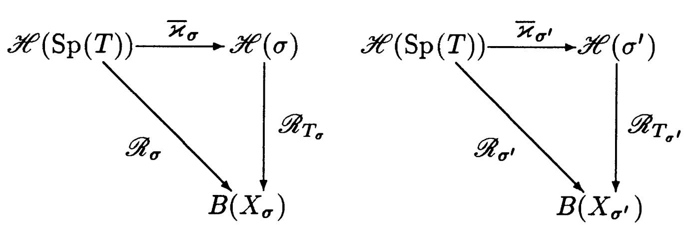
By 8.2.3, $h(T)T = Th(T) = P_\sigma$. Moreover, since $\bar{h}\overline{\varkappa}_\sigma = \overline{\varkappa}_\sigma \bar{h}$, the decomposition $X = X_\sigma \oplus X_{\sigma'}$ reduces $h(T)$ and $h(T)_\sigma T_\sigma = T_\sigma h(T)_\sigma = 1$ for the part $h(T)_\sigma$ of $h(T)$ in X_σ. So, T_σ is invertible; i.e., $0 \notin \mathrm{Sp}(T_\sigma)$. We thus arrive at a contradiction which implies that $0 \in \sigma$. In other words, $\mathrm{Sp}(T_\sigma) \subset \sigma$.

Observe now that $\mathrm{res}\,(T) = \mathrm{res}\,(T_\sigma) \cap \mathrm{res}\,(T_{\sigma'})$. Consequently, by the above

$$\mathrm{Sp}(T) = \mathbb{C} \setminus \mathrm{res}\,(T) = \mathbb{C} \setminus (\mathrm{res}\,(T_\sigma) \cap \mathrm{res}\,(T_{\sigma'}))$$
$$= (\mathbb{C} \setminus \mathrm{res}\,(T_\sigma)) \cup (\mathbb{C} \setminus \mathrm{res}\,(T_{\sigma'})) = \mathrm{Sp}(T_\sigma) \cup \mathrm{Sp}(T_{\sigma'}) \subset \sigma \cup \sigma' = \mathrm{Sp}(T).$$

Considering that $\sigma \cap \sigma' = \varnothing$, complete the proof. ▷

8.2.13. *Riesz–Dunford Integral Decomposition Theorem.* *Let σ be a clopen part of $\mathrm{Sp}(T)$ for an endomorphism T of a Banach space X. The direct sum decomposition $X = X_\sigma \oplus X_{\sigma'}$ reduces the representation $\mathscr{R}_T$ of the algebra $\mathscr{H}(\mathrm{Sp}(T))$ in X to the direct sum of the representations $\mathscr{R}_\sigma$ and $\mathscr{R}_{\sigma'}$. Moreover, the following diagrams of representations commute:*



Here $\overline{\varkappa}_\sigma(\bar{f}) := \overline{\varkappa_\sigma f}$ and $\overline{\varkappa}_{\sigma'}(\bar{f}) := \overline{\varkappa_{\sigma'} f}$ for $f \in H(\mathrm{Sp}(T))$ are the representations induced by restricting f onto σ and σ'. ◁▷

8.3. The Approximation Property

8.3.1. *Let X and Y be Banach spaces. For $K \in \mathscr{L}(X, Y)$ the following statements are equivalent:*

(1) *the operator K is compact: $K \in \mathscr{K}(X, Y)$;*

(2) *there are a neighborhood of zero U in X and a compact subset V of Y such that $K(U) \subset V$;*

(3) *the image under K of every bounded set in X is relatively compact in Y;*

(4) *the image under K of every bounded set in X is totally bounded in Y;*

(5) *for each sequence $(x_n)_{n\in\mathbb{N}}$ of points of the unit ball B_X, the sequence $(Kx_n)_{n\in\mathbb{N}}$ has a Cauchy subsequence.* ⊲⊳

8.3.2. ***Theorem.*** *Let X and Y be Banach spaces over a basic field $\mathbb{F}$. Then*

(1) *$\mathscr{K}(X, Y)$ is a closed subspace of $B(X, Y)$;*

(2) *for all Banach spaces W and Z, it holds that*

$$B(Y, Z) \circ \mathscr{K}(X, Y) \circ B(W, X) \subset \mathscr{K}(W, Z);$$

i.e., if $S \in B(W, X)$, $T \in B(Y, Z)$ and $K \in \mathscr{K}(X, Y)$ then $TKS \in \mathscr{K}(W, Z)$;

(3) *$I_{\mathbb{F}} \in \mathscr{K}(\mathbb{F}) := \mathscr{K}(\mathbb{F}, \mathbb{F})$.*

⊲ That $\mathscr{K}(X, Y)$ is a subspace of $B(X, Y)$ follows from 8.3.1. If $K_n \in \mathscr{K}(X, Y)$ and $K_n \to K$; then, given $\varepsilon > 0$, for n sufficiently large observe that $\|Kx - K_nx\| \leq \|K - K_n\| \, \|x\| \leq \varepsilon$ whenever $x \in B_X$. Therefore, $K_n(B_X)$ serves as an *ε-net* ($= B_\varepsilon$-net) for $K(B_X)$. It remains to refer to 4.6.4 and thus complete the proof of the closure property of $\mathscr{K}(X, Y)$. The other claims are evident. ⊳

8.3.3. REMARK. Theorem 8.3.2 is often verbalized as follows: "The class of all compact operators is an *operator ideal.*" Behind this lies a conspicuous analogy with the fact that $\mathscr{K}(X) := \mathscr{K}(X, X)$ presents a closed bilateral (two-sided) ideal in the (bounded) endomorphism algebra $B(X)$; i.e., $\mathscr{K}(X) \circ B(X) \subset \mathscr{K}(X)$ and $B(X) \circ \mathscr{K}(X) \subset \mathscr{K}(X)$.

8.3.4. ***Calkin Theorem.*** *The ideals 0, $\mathscr{K}(l_2)$, and $B(l_2)$ exhaust the list of closed bilateral ideals in the endomorphism algebra $B(l_2)$ of the Hilbert space l_2.*

8.3.5. REMARK. In view of 8.3.4 it is clear that a distinguishable role in operator theory should be performed by the algebra $B(X)/\mathscr{K}(X)$ called the *Calkin algebra* (on X). The performance is partly delivered in 8.5.

8.3.6. DEFINITION. An operator T, a member of $\mathscr{L}(X, Y)$, is called a *finite-rank* operator provided that $T \in B(X, Y)$ and im T is a finite-dimensional subspace of Y. In notation: $T \in F(X, Y)$. A hasty term "finite-dimensional operator" would abuse consistency since T as a subspace of $X \times Y$ is usually infinite-dimensional.

8.3.7. *The linear span of the set of (bounded) rank-one operators comprises all finite-rank operators:*

$$T \in F(X,\ Y)$$
$$\Leftrightarrow (\exists\, x'_1, \dots, x'_n \in X')\ (\exists\, y_1, \dots, y_n \in Y)\ T = \sum_{k=1}^{n} x'_k \otimes y_k. \ \triangleleft\triangleright$$

8.3.8. DEFINITION. Let Q be a (nonempty) compact set in X. Given $T \in B(X,\ Y)$, put
$$\|T\|_Q := \sup \|T(Q)\|.$$

The collection of all seminorms $B(X,\ Y)$ of type $\|\cdot\|_Q$ is the *Arens multinorm* in $B(X,\ Y)$, denoted by $\varkappa_{B(X,Y)}$. The corresponding topology is the *topology of uniform convergence on compact sets* or the *compact-open topology* (cf. 7.2.10).

8.3.9. Grothendieck Theorem. *Let X be a Banach space. The following conditions are equivalent:*

(1) *for every $\varepsilon > 0$ and every compact set Q in X there is a finite-rank endomorphism T of X, a member of $F(X) := F(X,\ X)$, such that $\|Tx - x\| \le \varepsilon$ for all $x \in Q$;*

(2) *for every Banach space W, the subspace $F(W,\ X)$ is dense in the space $B(W,\ X)$ with respect to the Arens multinorm $\varkappa_{B(W,X)}$;*

(3) *for every Banach space Y, the subspace $F(X,\ Y)$ is dense in the space $B(X,\ Y)$ with respect to the Arens multinorm $\varkappa_{B(X,Y)}$.*

$\triangleleft$ It is clear that (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1). Therefore, we are to show only that (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3).

(1) $\Rightarrow$ (2): If $T \in B(W,\ X)$ and Q is a nonempty compact set in W then, in view of the Weierstrass Theorem, $T(Q)$ is a nonempty compact set in X. So, for $\varepsilon > 0$, by hypothesis there is a member T_0 of $F(X)$ such that $\|T_0 - I_X\|_{T(Q)} = \|T_0T - T\|_Q \le \varepsilon$. Undoubtedly, $T_0T \in F(W,\ X)$.

(1) $\Rightarrow$ (3): Let $T \in B(X,\ Y)$. If $T = 0$ then there is nothing to be proven. Let $T \neq 0$, $\varepsilon > 0$ and Q be a nonempty compact set in X. By hypothesis there is a member T_0 of $F(X)$ such that $\|T_0 - I_X\|_Q \le \varepsilon\|T\|^{-1}$. Then $\|TT_0 - T\|_Q \le \|T\|\ \|T_0 - I_X\|_Q \le \varepsilon$. Furthermore, $TT_0 \in F(X,\ Y)$. $\triangleright$

8.3.10. DEFINITION. A Banach space satisfying one (and hence all) of the equivalent conditions 8.3.9 (1)–8.3.9 (3) is said to possess the *approximation property*.

8.3.11. Grothendieck Criterion. *A Banach space X possesses the approximation property if and only if, for every Banach space W, the equality holds: $\operatorname{cl} F(W,\ X) = \mathscr{K}(W,\ X)$, with the closure taken in operator norm.*

8.3.12. REMARK. For a long time there was an unwavering (and yet unprovable) belief that every Banach space possesses the approximation property. Therefore, P. Enflo's rather sophisticated example of a Banach space lacking the approximation property was acclaimed as sensational in the late seventies of the current century. Now similar counterexamples are in plenty:

8.3.13. SZANKOWSKI COUNTEREXAMPLE. *The space $B(l_2)$ lacks the approximation property.*

8.3.14. DAVIS–FIGIEL–SZANKOWSKI COUNTEREXAMPLES. *The spaces c_0 and l_p with $p \neq 2$ have closed subspaces lacking the approximation property.*

8.4. The Riesz–Schauder Theory

8.4.1. ***ε-Perpendicular Lemma.*** *Let X_0 be a closed subspace of a Banach space X and $X \neq X_0$. Given an $\varepsilon > 0$, there is an ε-perpendicular to X_0 in X; i.e., an element x_ε in X such that $\|x_\varepsilon\| = 1$ and $d(x_\varepsilon, X_0) := \inf d_{\|\cdot\|}(\{x_\varepsilon\} \times X_0) \geq 1 - \varepsilon$.*

◁ Take $1 > \varepsilon$ and $x \in X \setminus X_0$. It is clear that $d := d(x, X_0) > 0$. Find x' in the subspace X_0 satisfying $\|x - x'\| \leq d/(1-\varepsilon)$, which is possible because $d/(1-\varepsilon) > d$. Put $x_\varepsilon := (x - x')\|x - x'\|^{-1}$. Then $\|x_\varepsilon\| = 1$. Finally, for $x_0 \in X_0$, observe that

$$\|x_0 - x_\varepsilon\| = \left\| x_0 - \frac{x - x'}{\|x - x'\|} \right\|$$

$$= \frac{1}{\|x' - x\|} \, \|(\|x - x'\| x_0 + x') - x\| \geq \frac{d(x, X_0)}{\|x' - x\|} \geq 1 - \varepsilon. \ ▷$$

8.4.2. ***Riesz Criterion.*** *Let X be a Banach space. The identity operator in X is compact if and only if X is finite-dimensional.*

◁ Only the implication $\Rightarrow$ needs proving. If X fails to be finite-dimensional, then select a sequence of finite-dimensional subspaces $X_1 \subset X_2 \subset \dots$ in X such that $X_{n+1} \neq X_n$ for all $n \in \mathbb{N}$. In virtue of 8.4.1 there is a sequence (x_n) satisfying $x_{n+1} \in X_{n+1}$, $\|x_{n+1}\| = 1$ and $d(x_{n+1}, X_n) \geq {}^1/_2$, namely, some sequence of ${}^1/_2$-perpendiculars to X_n in X_{n+1}. It is clear that $d(x_m, x_k) \geq {}^1/_2$ for $m \neq k$. In other words, the sequence (x_n) lacks Cauchy subsequences. Consequently, by 8.3.1 the operator I_X is not compact. ▷

8.4.3. *Let $T \in \mathscr{K}(X, Y)$, with X and Y Banach spaces. The operator T is normally solvable if and only if T has finite rank.*

◁ Only the implication $\Rightarrow$ needs examining.

Let $Y_0 := \operatorname{im} T$ be a closed subspace in Y. By the Banach Homomorphism Theorem, the image $T(B_X)$ of the unit ball of X is a neighborhood of zero in Y_0. Furthermore, in virtue of the compactness property of T, the set $T(B_X)$ is relatively compact in Y_0. It remains to apply 8.4.2 to Y_0. ▷

8.4.4. *Let X be a Banach space and $K \in \mathscr{K}(X)$. Then the operator $1 - K$ is normally solvable.*

◁ Put $T := 1 - K$ and $X_1 := \ker T$. Undoubtedly, X_1 is finite-dimensional by 8.4.2. In accordance with 7.4.11 (1) a finite-dimensional subspace is complemented. Denote a topological complement of X_1 to X by X_2. Considering that X_2 is a Banach space and $T(X) = T(X_2)$, it suffices to verify that $\|Tx\| \geq t\|x\|$ for some $t > 0$ and all $x \in X_2$. In the opposite case, there is a sequence (x_n) such that $\|x_n\| = 1$, $x_n \in X_2$ and $Tx_n \to 0$. Using the compactness property of K, we may assume that (Kx_n) converges. Put $y := \lim Kx_n$. Then the sequence (x_n) converges to y, because $y = \lim(Tx_n + Kx_n) = \lim x_n$. Moreover, $Ty = \lim Tx_n = 0$; i.e., $y \in X_1$. It is beyond a doubt that $y \in X_2$. Thus, $y \in X_1 \cap X_2$; i.e., $y = 0$. We arrive at a contradiction: $\|y\| = \lim \|x_n\| = 1$. ▷

8.4.5. *For whatever strictly positive ε, there are only finitely many eigenvalues of a compact operator beyond the disk centered at zero and having radius ε.*

◁ Suppose by way of contradiction that there is a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of pairwise distinct eigenvalues of a compact operator K, with $|\lambda_n| \geq \varepsilon$ for all $n \in \mathbb{N}$ and $\varepsilon > 0$. Suppose further that x_n satisfying $0 \neq x_n \in \ker(\lambda_n - K)$ is an eigenvector with eigenvalue λ_n. Establish first that the set $\{x_n : n \in \mathbb{N}\}$ is linearly independent. To this end, assume the set $\{x_1, \dots, x_n\}$ linearly independent. In case $x_{n+1} = \sum_{k=1}^{n} \alpha_k x_k$, we would have $0 = (\lambda_{n+1} - K)x_{n+1} = \sum_{k=1}^{n} \alpha_k(\lambda_{n+1} - \lambda_k)x_k$. Consequently, $\alpha_k = 0$ for $k := 1, \dots, n$. Whence the false equality $x_{n+1} = 0$ would ensue.

Put $X_n := \operatorname{lin}(\{x_1, \dots, x_n\})$. By definition $X_1 \subset X_2 \subset \dots$; moreover, as was proven, $X_{n+1} \neq X_n$ for $n \in \mathbb{N}$. By virtue of 8.4.1 there is a sequence $(\overline{x}_n)$ such that $\overline{x}_{n+1} \in X_{n+1}$, $\|\overline{x}_{n+1}\| = 1$ and $d(\overline{x}_{n+1}, X_n) \geq {}^1/_2$. For $m > k$, straightforward calculation shows that $z := (\lambda_{m+1} - K)\overline{x}_{m+1} \in X_m$ and $z + K\overline{x}_k \in X_m + X_k \subset X_m$. Consequently,

$$\begin{aligned}\|K\overline{x}_{m+1} - K\overline{x}_k\| &= \| - \lambda_{m+1}\overline{x}_{m+1} + K\overline{x}_{m+1} + \lambda_{m+1}\overline{x}_{m+1} - K\overline{x}_k\| \\ &= \|\lambda_{m+1}\overline{x}_{m+1} - (z + Kx_k)\| \geq |\lambda_{m+1}| d(\overline{x}_{m+1}, X_m) \geq {}^\varepsilon/_2.\end{aligned}$$

In other words, the sequence $(K\overline{x}_n)$ has no Cauchy subsequences. ▷

8.4.6. Schauder Theorem. *Let X and Y be Banach spaces (over the same ground field $\mathbb{F}$). Then*

$$K \in \mathscr{K}(X, Y) \Leftrightarrow K' \in \mathscr{K}(Y', X').$$

◁ $\Rightarrow$: Observe first of all that the restriction mapping $x' \mapsto x'|_{B_X}$ implements an isometry of X' into $l_\infty(B_X)$. Therefore, to check that $K'(B_{Y'})$ is relatively compact we are to show the same for the set $V := \{K'y'|_{B_X} : y' \in B_{Y'}\}$. Since

$K'y'|_{B_X}(x) = y' \circ K|_{B_X}(x) = y'(Kx)$ for $x \in B_X$ and $y' \in B_{Y'}$, consider the compact set $Q := \operatorname{cl} K(B_X)$ and the mapping $\overset{\circ}{K} : C(Q,\ \mathbb{F}) \to l_\infty(B_X)$ defined by the rule $\overset{\circ}{K}g : x \mapsto g(Kx)$. Undoubtedly, the operator $\overset{\circ}{K}$ is bounded and, hence, continuous. Now put $S := \{y'|_Q :\ \ y' \in B_{Y'}\}$. It is clear that S is simultaneously an equicontinuous and bounded subset of $C(Q,\ \mathbb{F})$. Consequently, by the Ascoli–Arzelà Theorem, S is relatively compact. From the Weierstrass Theorem derive that $\overset{\circ}{K}(S)$ is a relatively compact set too. It remains to observe that $\overset{\circ}{K}y'|_Q = K'y'|_{B_X}$ for $y' \in B_{Y'}$; i.e., $\overset{\circ}{K}(S) = V$.

$\Leftarrow$: If $K' \in \mathscr{K}(Y',\ X')$ then, as is proven, $K'' \in \mathscr{K}(X'',\ Y'')$. By the Double Prime Lemma, $K''|_X = K$. Whence it follows that the operator K is compact. ▷

8.4.7. *Every nonzero point of the spectrum of a compact operator is isolated (i.e., such a point constitutes a clopen part of the spectrum).*

◁ Taking note of 8.4.4 and the Sequence Prime Principle, observe that each nonzero point of the spectrum of a compact operator K is either an eigenvalue of K or an eigenvalue of the dual of K. Using 8.4.5 and 8.4.6, conclude that, for each strictly positive ε, there are only finitely many points of $\operatorname{Sp}(K)$ beyond the disk centered at zero and having radius ε. ▷

8.4.8. *Riesz–Schauder Theorem.* *The spectrum of a compact operator K in an infinite-dimensional space contains zero. Each nonzero point of the spectrum of K is isolated and presents an eigenvalue of K with the corresponding eigenspace finite-dimensional.*

◁ Considering K, a compact endomorphism of a Banach space X, we must only demonstrate the implication

$$0 \neq \lambda \in \operatorname{Sp}(K) \Rightarrow \ker(\lambda - K) \neq 0.$$

First, settle the case $\mathbb{F} := \mathbb{C}$. Note that $\{\lambda\}$ is a clopen part of $\operatorname{Sp}(K)$. Putting $g(z) := {}^1/_z$ in some neighborhood about λ and $g(z) := 0$ for z in a suitable neighborhood about $\{\lambda\}'$, observe that $\varkappa_{\{\lambda\}} = \overline{g}\overline{I}_{\mathbb{C}}$. Thus, by 8.2.3 and 8.2.10, $P_{\{\lambda\}} = \overline{g}(K)K$. By virtue of 8.3.2 (2), $P_{\{\lambda\}} \in \mathscr{K}(X)$. From 8.4.3 it follows that $\operatorname{im} P_{\{\lambda\}}$ is a finite-dimensional space. It remains to invoke the Spectral Decomposition Theorem.

In the case of the reals, $\mathbb{F} := \mathbb{R}$, implement the process of *complexification.* Namely, furnish the space X^2 with multiplication by an element of $\mathbb{C}$ which is introduced by the rule $i(x,\ y) := (-y,\ x)$. The resulting complex vector space is denoted by $X \oplus iX$. Define the operator $\overline{K}(x,\ y) := (Kx,\ Ky)$ in the space $X \oplus iX$. Equipping $X \oplus iX$ with an appropriate norm (cf. 7.3.2), observe that the operator $\overline{K}$ is compact and $\lambda \in \operatorname{Sp}(\overline{K})$. Consequently, λ is an eigenvalue of $\overline{K}$ by what was proven. Whence it follows that λ is an eigenvalue of K. ▷

8.4.9. Theorem. *Let X be a complex Banach space. Given $T \in B(X)$, assume further that $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function vanishing only at zero and such that $f(T) \in \mathscr{K}(X)$. Then every nonzero point λ of the spectrum of T is isolated and the Riesz projection $P_{\{\lambda\}}$ is compact.*

◁ Suppose the contrary; i.e., find a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of distinct points of $\operatorname{Sp}(T)$ such that $\lambda_n \to \lambda \neq 0$ (in particular, X is infinite-dimensional). Then $f(\lambda_n) \to f(\lambda)$ and $f(\lambda) \neq 0$ by hypothesis. By the Spectral Mapping Theorem, $\operatorname{Sp}(f(T)) = f(\operatorname{Sp}(T))$. Thus, by 8.4.8, $f(\lambda_n) = f(\lambda)$ for all sufficiently large n. Whence it follows that $f(z) = f(\lambda)$ for all $z \in \mathbb{C}$ and so $f(T) = f(\lambda)$. By the Riesz Criterion in this case X is finite-dimensional. We come to a contradiction meaning that λ is an isolated point of $\operatorname{Sp}(T)$. Letting $g(z) := f(z)^{-1}$ in some neighborhood about λ disjoint from zero, infer that $\overline{gf} = \overline{\varkappa_{\{\lambda\}}}$. Consequently, by the Gelfand–Dunford Theorem, $P_{\{\lambda\}} = g(T)f(T)$; i.e., in virtue of 8.3.2 (2) the Riesz projection $P_{\{\lambda\}}$ is compact. ▷

8.4.10. Remark. Theorem 8.4.9 is sometimes referred to as the *Generalized Riesz–Schauder Theorem.*

8.5. Fredholm Operators

8.5.1. Definition. Let X and Y be Banach spaces (over the same ground field $\mathbb{F}$). An operator T, a member of $B(X, Y)$, is a *Fredholm operator* (in symbols, $T \in \mathscr{F}r(X, Y)$) if $\ker T := T^{-1}(0)$ and $\operatorname{coker} T := Y/\operatorname{im} T$ are finite-dimensional; i.e., if the following quantities, called the *nullity* and the *deficiency* of T, are finite:

$$\alpha(T) := \operatorname{nul} T := \dim \ker T; \quad \beta(T) := \operatorname{def} T := \dim \operatorname{coker} T.$$

The integer $\operatorname{ind} T := \alpha(T) - \beta(T)$, a member of $\mathbb{Z}$, is the *index* or, fully, the *Fredholm index* of T.

8.5.2. Remark. In the Russian literature, a Fredholm operator is usually called a Noether operator, whereas the term "Fredholm operator" is applied only to an index-zero Fredholm operator.

8.5.3. *Every Fredholm operator is normally solvable.*

◁ Immediate from the Kato Criterion. ▷

8.5.4. *For $T \in B(X, Y)$, the equivalence holds:*

$$T \in \mathscr{F}r(X, Y) \Leftrightarrow T' \in \mathscr{F}r(Y', X').$$

Moreover, $\operatorname{ind} T = -\operatorname{ind} T'$.

◁ By virtue of 2.3.5 (6), 8.5.3, 5.5.4 and the Sequence Prime Principle, the next pairs of sequences are exact simultaneously:

$$0 \to \ker T \to X \xrightarrow{T} Y \to \operatorname{coker} T \to 0;$$

$$0 \leftarrow (\ker T)' \leftarrow X' \xleftarrow{T'} Y' \leftarrow (\operatorname{coker} T)' \leftarrow 0;$$

$$0 \to \ker (T') \to Y' \xrightarrow{T'} X' \to \operatorname{coker} (T') \to 0;$$

$$0 \leftarrow (\ker (T'))' \leftarrow Y \xleftarrow{T} X \leftarrow (\operatorname{coker} (T'))' \leftarrow 0.$$

Moreover, $\alpha(T) = \beta(T')$ and $\beta(T) = \alpha(T')$ (cf. 7.6.14). ▷

8.5.5. *An operator T is an index-zero Fredholm operator if and only if so is the dual of T.*

◁ This is a particular case of 8.5.4. ▷

8.5.6. *Fredholm Alternative.* *For an index-zero Fredholm operator T either of the following mutually exclusive events takes place:*

(1) *The homogeneous equation $Tx = 0$ has a sole solution, zero. The homogeneous conjugate equation $T'y' = 0$ has a sole solution, zero. The equation $Tx = y$ is solvable and has a unique solution given an arbitrary right side. The conjugate equation $T'y' = x'$ is solvable and has a unique solution given an arbitrary right side.*

(2) *The homogeneous equation $Tx = 0$ has a nonzero solution. The homogeneous conjugate equation $T'y' = 0$ has a nonzero solution. The homogeneous equation $Tx = 0$ has finitely many linearly independent solutions $x_1, \dots, x_n$. The homogeneous conjugate equation $T'y' = 0$ has finitely many linearly independent solutions $y'_1, \dots, y'_n$.*

The equation $Tx = y$ is solvable if and only if $y'_1(y) = \dots = y'_n(y) = 0$. Moreover, the general solution x is the sum of a partial solution x_0 and the general solution of the homogeneous equation; i.e., it has the form

$$x = x_0 + \sum_{k=1}^{n} \lambda_k x_k \quad (\lambda_k \in \mathbb{F}).$$

The conjugate equation $T'y' = x'$ is solvable if and only if $x'(x_1) = \dots = x'(x_n) = 0$. Moreover, the general solution y' is the sum of a partial solution y'_0 and the general solution of the homogeneous equation; i.e., it has the form

$$y' = y'_0 + \sum_{k=1}^{n} \mu_k y'_k \quad (\mu_k \in \mathbb{F}).$$

◁ This is a reformulation of 8.5.5 with account taken of the Polar Lemma. ▷

8.5.7. EXAMPLES.

(1) If T is invertible then T is an index-zero Fredholm operator.

(2) Let $T \in \mathscr{L}(\mathbb{F}^n, \mathbb{F}^m)$. Let rank $T :=$ dim im T be the *rank* of T. Then $\alpha(T) = n-\text{rank}\ T$ and $\beta(T) = m-\text{rank}\ T$. Consequently, $T \in \mathscr{F}r(\mathbb{F}^n, \mathbb{F}^m)$ and ind $T = n - m$.

(3) Let $T \in B(X)$ and $X = X_1 \oplus X_2$. Assume that this direct sum decomposition of X reduces T to matrix form

$$T \sim \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}.$$

Undoubtedly, T is a Fredholm operator if and only if its parts are Fredholm operators. Moreover, $\alpha(T) = \alpha(T_1) + \alpha(T_2)$ and $\beta(T) = \beta(T_1) + \beta(T_2)$; i.e., ind $T =$ ind $T_1 +$ ind T_2. ◁▷

8.5.8. Fredholm Theorem. *Let $K \in \mathscr{K}(X)$. Then $1 - K$ is an index-zero Fredholm operator.*

◁ First, settle the case $\mathbb{F} := \mathbb{C}$. If $1 \notin \text{Sp}(K)$ then $1 - K$ is invertible and $\text{ind}(1 - K) = 0$. If $1 \in \text{Sp}(K)$ then in virtue of the Riesz–Schauder Theorem and the Spectral Decomposition Theorem there is a decomposition $X = X_1 \oplus X_2$ such that X_1 is finite-dimensional and $1 \notin \text{Sp}(K_2)$, with K_2 the part of K in X_2. Furthermore,

$$1 - K \sim \begin{pmatrix} 1 - K_1 & 0 \\ 0 & 1 - K_2 \end{pmatrix}.$$

By 8.5.7 (2), $\text{ind}(1 - K_1) = 0$ and, by 8.5.7 (3), $\text{ind}(1 - K) = \text{ind}(1 - K_1) + \text{ind}(1 - K_2) = 0$.

In the case of the reals, $\mathbb{F} := \mathbb{R}$, proceed by way of complexification as in the proof of 8.4.8. Namely, consider the operator $\overline{K}(x, y) := (Kx, Ky)$ in the space $X \oplus iX$. By above, $\text{ind}(1 - \overline{K}) = 0$. Considering the difference between $\mathbb{R}$ and $\mathbb{C}$, observe that $\alpha(1 - K) = \alpha(1 - \overline{K})$ and $\beta(1 - K) = \beta(1 - \overline{K})$. Finally, $\text{ind}(1 - K) = 0$. ▷

8.5.9. DEFINITION. Let $T \in B(X, Y)$. An operator L, a member of $B(Y, X)$, is a *left approximate inverse* of T if $LT - 1 \in \mathscr{K}(X)$. An operator R, a member of $B(Y, X)$, is a *right approximate inverse* of T if $TR - 1 \in \mathscr{K}(Y)$. An operator S, a member of $B(Y, X)$, is an *approximate inverse* of T if S is simultaneously a left and right approximate inverse of T. If an operator T has an approximate inverse S then T is called *approximately invertible*. The terms "regularizer" and "parametrix" are all current in this context with regard to S.

8.5.10. *Let L and R be a left approximate inverse and a right approximate inverse of T, respectively. Then $L - R \in \mathscr{K}(Y, X)$.*

◁ $LT = 1 + K_X\ (K_X \in \mathscr{K}(X)) \Rightarrow LTR = R + K_X R$;

$TR = 1 + K_Y\ (K_Y \in \mathscr{K}(Y)) \Rightarrow LTR = L + LK_Y$ ▷

8.5.11. *If L is a left approximate inverse of T and $K \in \mathscr{K}(Y, X)$ then $L+K$ is also a left approximate inverse of T.*

$\lhd$ $(L + K)T - 1 = (LT - 1) + KT \in \mathscr{K}(X)$ $\rhd$

8.5.12. *An operator is approximately invertible if and only if it has a right approximate inverse and a left approximate inverse.*

$\lhd$ Only the implication $\Leftarrow$ needs examining. Let L and R be a left approximate inverse and a right approximate inverse of T, respectively. By 8.5.10, $K := L - R \in \mathscr{K}(Y, X)$. Consequently, by 8.5.11, $R = L - K$ is a left approximate inverse of T. Thus, R is an approximate inverse of T. $\rhd$

8.5.13. REMARK. The above shows that, in the case $X = Y$, an operator S is an approximate inverse of T if and only if $\varphi(S)\varphi(T) = \varphi(T)\varphi(S) = 1$, where $\varphi : B(X) \to B(X)/\mathscr{K}(X)$ is the coset mapping to the Calkin algebra. In other words, a left approximate inverse is the inverse image of a left inverse in the Calkin algebra, etc.

8.5.14. *Noether Criterion.* *An operator is a Fredholm operator if and only if it is approximately invertible.*

$\lhd$ $\Rightarrow$: Let $T \in \mathscr{F}r(X, Y)$. Using the Complementation Principle, consider the decompositions $X = \ker T \oplus X_1$ and $Y = \operatorname{im} T \oplus Y_1$ and the respective finite-rank projections P which carries X onto $\ker T$ along X_1 and Q which carries Y onto Y_1 along $\operatorname{im} T$. It is clear that the restriction $T_1 := T|_{X_1}$ is an invertible operator $T_1 : X_1 \to \operatorname{im} T$. Put $S := T_1^{-1}(1 - Q)$. The operator S may be viewed as a member of $B(Y, X)$. Moreover, it is beyond a doubt that $ST + P = 1$ and $TS + Q = 1$.

$\Leftarrow$: Let S be an approximate inverse of T; i.e., $ST = 1 + K_X$ and $TS = 1 + K_Y$ for appropriate compact operators K_X and K_Y. Consequently, $\ker T \subset \ker(1 + K_X)$; i.e., $\ker T$ is finite-dimensional since so is $\ker(1 + K_X)$ in virtue of 8.5.8. Furthermore, $\operatorname{im} T \supset \operatorname{im}(1 + K_Y)$; and so the range of T is of finite codimension because $1 + K_Y$ is an index-zero Fredholm operator. $\rhd$

8.5.15. Corollary. *If $T \in \mathscr{F}r(X, Y)$ and S is an approximate inverse of T then $S \in \mathscr{F}r(Y, X)$.* $\lhd\rhd$

8.5.16. Corollary. *The product of Fredholm operators is itself a Fredholm operator.*

$\lhd$ The composition of approximate inverses (taken in due succession) is an approximate inverse to the composition of the originals. $\rhd$

8.5.17. *Consider an exact sequence*

$$0 \to X_1 \to X_2 \to \ldots \to X_{n-1} \to X_n \to 0$$

of finite-dimensional vector spaces. Then the Euler identity holds:

$$\sum_{k=1}^{n}(-1)^k \dim X_k = 0.$$

◁ For $n = 1$ the exactness of the sequence $0 \to X_1 \to 0$ means that $X_1 = 0$, and for $n = 2$ the exactness of $0 \to X_1 \to X_2 \to 0$ amounts to isomorphy between X_1 and X_2 (cf. 2.3.5 (4)). Therefore, the Euler identity is beyond a doubt for $n := 1,\ 2$.

Suppose now that for $m \le n - 1$, where $n > 2$, the desired identity is already established. The exact sequence

$$0 \to X_1 \to X_2 \to \ldots \to X_{n-2} \xrightarrow{T_{n-2}} X_{n-1} \xrightarrow{T_{n-1}} X_n \to 0$$

reduces to the exact sequence

$$0 \to X_1 \to X_2 \to \ldots \to X_{n-2} \xrightarrow{T_{n-2}} \ker T_{n-1} \to 0.$$

By hypothesis,

$$\sum_{k=1}^{n-2}(-1)^k \dim X_k + (-1)^{n-1} \dim\ker T_{n-1} = 0.$$

Furthermore, since T_{n-1} is an epimorphism,

$$\dim X_{n-1} = \dim\ker T_{n-1} + \dim X_n.$$

Finally,

$$0 = \sum_{k=1}^{n-2}(-1)^k \dim X_k + (-1)^{n-1}(\dim X_{n-1} - \dim X_n)$$

$$= \sum_{k=1}^{n}(-1)^k \dim X_k. \ \triangleright$$

8.5.18. ***Atkinson Theorem.*** *The index of the product of Fredholm operators equals the sum of the indices of the factors.*

◁ Let $T \in \mathscr{F}r(X,\ Y)$ and $S \in \mathscr{F}r(Y,\ Z)$. By virtue of 8.5.16, $ST \in \mathscr{F}r(X,\ Z)$. Using the Snowflake Lemma, obtain the exact sequence of finite-dimensional spaces

$$0 \to \ker T \to \ker ST \to \ker S \to \operatorname{coker} T \to \operatorname{coker} ST \to \operatorname{coker} S \to 0.$$

Applying 8.5.17, infer that

$$\alpha(T) - \alpha(ST) + \alpha(S) - \beta(T) + \beta(ST) - \beta(S) = 0;$$

whence $\operatorname{ind}(ST) = \operatorname{ind} S + \operatorname{ind} T$. ▷

8.5.19. *Corollary.* *Let T be a Fredholm operator and let S be an approximate inverse of T. Then* $\operatorname{ind} T = -\operatorname{ind} S$.

◁ $\operatorname{ind}(ST) = \operatorname{ind}(1 + K)$ for some compact operator K. By Theorem 8.5.8, $1 + K$ is an index-zero Fredholm operator. ▷

8.5.20. *Compact Index Stability Theorem.* *The property of being a Fredholm operator and the index of a Fredholm operator are preserved under compact perturbations: if $T \in \mathscr{F}r(X, Y)$ and $K \in \mathscr{K}(X, Y)$ then $T + K \in \mathscr{F}r(X, Y)$ and* $\operatorname{ind}(T + K) = \operatorname{ind} T$.

◁ Let S be an approximate inverse to T; i.e.,

$$ST = 1 + K_X; \quad TS = 1 + K_Y$$

with some $K_X \in \mathscr{K}(X)$ and $K_Y \in \mathscr{K}(Y)$ (that S exists is ensured by 8.5.14). It is clear that

$$S(T + K) = ST + SK = 1 + K_X + SK \in 1 + \mathscr{K}(X);$$
$$(T + K)S = TS + KS = 1 + K_Y + KS \in 1 + \mathscr{K}(Y);$$

i.e., S is an approximate inverse of $T+K$. By virtue of 8.5.14, $T+K \in \mathscr{F}r(X, Y)$. Finally, from 8.5.19 infer the equalities $\operatorname{ind}(T + K) = -\operatorname{ind} S$ and $\operatorname{ind} T = -\operatorname{ind} S$. ▷

8.5.21. *Bounded Index Stability Theorem.* *The property of being a Fredholm operator and the index of a Fredholm operator are preserved under sufficiently small bounded perturbations: the set $\mathscr{F}r(X, Y)$ is open in the space of bounded operators, and the index of a Fredholm operator* $\operatorname{ind} : \mathscr{F}r(X, Y) \to \mathbb{Z}$ *is a continuous function.*

◁ Let $T \in \mathscr{F}r(X, Y)$. By 8.5.14 there are operators $S \in B(Y, X)$, $K_X \in \mathscr{K}(X)$ and $K_Y \in \mathscr{K}(Y)$ such that

$$ST = 1 + K_X; \quad TS = 1 + K_Y.$$

If $S = 0$ then the spaces X and Y are finite-dimensional by the Riesz Criterion, i.e., nothing is left to proof: it suffices to refer to 8.5.7 (2). If $S \neq 0$ then for all $V \in B(X, Y)$ with $\|V\| < 1/\|S\|$, from the inequality of 5.6.1 it follows: $\|SV\| < 1$ and $\|VS\| < 1$. Consequently, in virtue of 5.6.10 the operators $1 + SV$ and $1 + VS$ are invertible in $B(X)$ and in $B(Y)$, respectively.

Observe that

$$(1 + SV)^{-1}S(T + V) = (1 + SV)^{-1}(1 + K_X + SV)$$
$$= 1 + (1 + SV)^{-1}K_X \in 1 + \mathscr{K}(X);$$

i.e., $(1 + SV)^{-1}S$ is a left approximate inverse of $T + V$. By analogy, show that $S(1 + VS)^{-1}$ is a right approximate inverse of $T + V$. Indeed,

$$(T + V)S(1 + VS)^{-1} = (1 + K_Y + VS)(1 + VS)^{-1}$$
$$= 1 + K_Y(1 + VS)^{-1} \in 1 + \mathscr{K}(Y).$$

By 8.5.12, $T + V$ is approximately invertible. In virtue of 8.5.14, $T + V \in \mathscr{F}r(X, Y)$. This proves the openness property of $\mathscr{F}r(X, Y)$. When left and right approximate inverses of a Fredholm operator W exist, each of them is an approximate inverse to W (cf. 8.5.12), Therefore, from 8.5.19 and 8.5.18 obtain

$$\operatorname{ind}(T + V) = -\operatorname{ind}((1 + SV)^{-1}S)$$
$$= -\operatorname{ind}(1 + SV)^{-1} - \operatorname{ind} S = -\operatorname{ind} S = \operatorname{ind} T$$

(because $(1 + SV)^{-1}$ is a Fredholm operator by 8.5.7 (1)). This means that the Fredholm index is continuous. ▷

8.5.22. *Nikol'skiĭ Criterion.* *An operator is an index-zero Fredholm operator if and only if it is the sum of an invertible operator and a compact operator.*

◁ $\Rightarrow$: Let $T \in \mathscr{F}r(X, Y)$ and $\operatorname{ind} T = 0$. Consider the direct sum decompositions $X = X_1 \oplus \ker T$ and $Y = \operatorname{im} T \oplus Y_1$. It is beyond a doubt that the operator T_1, the restriction of T to X_1, implements an isomorphism between X_1 and $\operatorname{im} T$. Furthermore, in virtue of 8.5.5, $\dim Y_1 = \beta(T) = \alpha(T)$; i.e., there is a natural isomorphism $\mathrm{Id} : \ker T \to Y_1$. Therefore, T admits the matrix presentation

$$T \sim \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_1 & 0 \\ 0 & \mathrm{Id} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\mathrm{Id} \end{pmatrix}.$$

$\Leftarrow$: If $T := S + K$ with $K \in \mathscr{K}(X, Y)$ and $S^{-1} \in B(Y, X)$ then, by 8.5.20 and 8.5.7 (1), $\operatorname{ind} T = \operatorname{ind}(S + K) = \operatorname{ind} S = 0$. ▷

8.5.23. REMARK. Let $\operatorname{Inv}(X, Y)$ stand as before for the set of all invertible operators from X to Y (this set is open by Theorem 5.6.12). Denote by $\mathscr{F}(X, Z)$ the set of all Fredholm operators acting from X to Y and having index zero. The Nikol'skiĭ Criterion may now be written down as

$$\mathscr{F}(X, Y) = \operatorname{Inv}(X, Y) + \mathscr{K}(X, Y).$$

As is seen from the proof of 8.5.22, it may also be asserted that

$$\mathscr{F}(X, Y) = \operatorname{Inv}(X, Y) + F(X, Y),$$

where, as usual, $F(X, Y)$ is the subspace of $B(X, Y)$ comprising all finite-rank operators. ◁▷

Exercises

8.1. Study the Riesz–Dunford integral in finite-dimensions.

8.2. Describe the kernel of the Riesz–Dunford integral.

8.3. Given $n \in \mathbb{N}$, let f_n be a function holomorphic in a neighborhood U about the spectrum of an operator T. Prove that the uniform convergence of (f_n) to zero on U follows from the convergence of $(f_n(T))$ to zero in operator norm.

8.4. Let σ be an isolated part of the spectrum of an operator T. Assume that the part $\sigma' := \operatorname{Sp}(T) \setminus \sigma$ is separated from σ by some open disk with center a and radius r so that $\sigma \subset \{z \in \mathbb{C} : |z - a| < r\}$. Considering the Riesz projection P_σ, prove that

$$P_\sigma = \lim_n (1 - z^{-n}(T - a)^n)^{-1};$$

$$x \in \operatorname{im}(P_\sigma) \Leftrightarrow \limsup_n \|(a - T)^n x\|^{1/n} < r.$$

8.5. Find conditions for a projection to be a compact operator.

8.6. Prove that every closed subspace lying in the range of a compact operator in a Banach space is finite-dimensional.

8.7. Prove that a linear operator carries each closed linear subspace onto a closed set if and only if the operator is normally solvable and its kernel is finite-dimensional or finite-codimensional (the latter means that the kernel has a finite-dimensional algebraic complement).

8.8. Let $1 \le p < r < +\infty$. Prove that every bounded operator from l_r to l_p or from c_0 to l_p is compact.

8.9. Let H be a separable Hilbert space. Given an operator T in $B(H)$ and a Hilbert basis (e_n) for H, define the *Hilbert–Schmidt norm* as

$$\|T\|_2 := \left(\sum_{n=1}^{\infty} \|Te_n\|^2 \right)^{1/2}.$$

(Examine soundness!) An operator with finite Hilbert–Schmidt norm is a *Hilbert–Schmidt operator*. Demonstrate that an operator T is a Hilbert–Schmidt operator if and only if T is compact and $\sum_{n=1}^{\infty} \lambda_n^2 < +\infty$, where (λ_n) ranges over the eigenvalues of some operator $(T^*T)^{1/2}$ (define the latter!).

8.10. Let T be an endomorphism. Then

$$\operatorname{im}(T^0) \supset \operatorname{im}(T^1) \supset \operatorname{im}(T^2) \supset \dots .$$

If there is a number n satisfying $\operatorname{im}(T^n) = \operatorname{im}(T^{n+1})$ then say that T has *finite descent*. The least number n with which stabilization begins is the *descent* of T, denoted by $d(T)$. By analogy, considering the kernels

$$\ker(T^0) \subset \ker(T^1) \subset \ker(T^2) \subset \dots ,$$

introduce the concept of *ascent* and the denotation $a(T)$. Demonstrate that, for an operator T with finite descent and finite ascent, the two quantities, $a(T)$ and $d(T)$, coincide.

8.11. An operator T is a *Riesz–Schauder* operator, if T is a Fredholm operator and has finite descent and finite ascent. Prove that an operator T is a Riesz–Schauder operator if and only if T is of the form $T = U + V$, where U is invertible and V is of finite rank (or compact) and commutes with U.

8.12. Let T be a bounded endomorphism of a Banach space X which has finite descent and finite ascent, $r := a(T) = d(T)$. Prove that the subspaces $\operatorname{im}(T^r)$ and $\ker(T^r)$ are closed, the decomposition $X = \ker(T^r) \oplus \operatorname{im}(T^r)$ reduces T, and the restriction of T to $\operatorname{im}(T^r)$ is invertible.

8.13. Let T be a normally solvable operator. If either of the next quantities is finite

$$\alpha(T) := \dim \ker T, \quad \beta(T) := \dim \operatorname{coker} T,$$

then T is called *semi-Fredholm*. Put

$$\Phi_+(X) := \{T \in B(X) : \operatorname{im} T \in \operatorname{Cl}(X),\ \alpha(T) < +\infty\};$$
$$\Phi_-(X) := \{T \in B(X) : \operatorname{im} T \in \operatorname{Cl}(X),\ \beta(T) < +\infty\}.$$

Prove that

$$T \in \Phi_+(X) \Leftrightarrow T' \in \Phi_-(X');$$
$$T \in \Phi_-(X) \Leftrightarrow T' \in \Phi_+(X').$$

8.14. Let T be a bounded endomorphism. Prove that T belongs to $\Phi_+(X)$ if and only if for every bounded but not totally bounded set U, the image $T(U)$ is not a totally bounded set in X.

8.15. A bounded endomorphism T in a Banach space is a *Riesz operator*, if for every nonzero complex λ the operator $(\lambda - T)$ is a Fredholm operator. Prove that T is a Riesz operator if and only if for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$, the following conditions are fulfilled:

(1) the operator $(\lambda - T)$ has finite descent and finite ascent;
(2) the kernel of $(\lambda - T)^k$ is finite-dimensional for every $k \in \mathbb{N}$;
(3) the range of $(\lambda - T)^k$ has finite deficiency for $k \in \mathbb{N}$,

and, moreover, all nonzero points of the spectrum of T are eigenvalues, with zero serving as the only admissible limit point (that is, for whatever strictly positive ε, there are only finitely many points of $\operatorname{Sp}(T)$ beyond the disk centered at zero with radius ε).

8.16. Establish the isometric isomorphisms: $(X/Y)' \simeq Y^\perp$ and $X'/Y^\perp \simeq Y'$ for Banach spaces X and Y such that Y is embedded into X.

8.17. Prove that for a normal operator T in a Hilbert space and a holomorphic function f, a member of $H(\operatorname{Sp}(T))$, the operator $f(T)$ is normal. (An operator is ***normal*** if it commutes with its adjoint, cf. 11.7.1.)

8.18. Show that a continuous endomorphism T of a Hilbert space is a Riesz operator if and only if T is the sum of a compact operator and a *quasinilpotent* operator. (Quasinilpotency of an operator means triviality of its spectral radius.)

8.19. Given two Fredholm operators S and T, members of $\mathscr{F}r(X, Y)$, with $\operatorname{ind} S = \operatorname{ind} T$, demonstrate that there is a Jordan arc joining S and T within $\mathscr{F}r(X, Y)$.

Chapter 9
An Excursion into General Topology

9.1. Pretopologies and Topologies

9.1.1. DEFINITION. Let X be a set. A mapping $\tau : X \to \mathscr{P}(\mathscr{P}(X))$ is a *pretopology* on X if

(1) $x \in X \Rightarrow \tau(x)$ is a filter on X;

(2) $x \in X \Rightarrow \tau(x) \subset \operatorname{fil}\{x\}$.

A member of $\tau(x)$ is a *(pre)neighborhood* about x or of x. The pair (X, τ), as well as the set X itself, is called a *pretopological space*.

9.1.2. DEFINITION. Let $\mathscr{T}(X)$ be the collection of all pretopologies on X. If $\tau_1, \tau_2 \in \mathscr{T}(X)$ then τ_1 is said to be *stronger* than τ_2 or *finer* than τ_2 (in symbols, $\tau_1 \geq \tau_2$) provided that $x \in X \Rightarrow \tau_1(x) \supset \tau_2(x)$. Of course, τ_2 is weaker or *coarser* than τ_1.

9.1.3. *The set $\mathscr{T}(X)$ with the relation "to be stronger" presents a complete lattice.*

◁ If $X = \varnothing$ then $\mathscr{T}(X) = \{\varnothing\}$ and there is nothing to be proven. If $X \neq \varnothing$ then refer to 1.3.13. ▷

9.1.4. DEFINITION. A subset G of X is an *open set* in X, if G is a (pre)neighborhood of its every point (in symbols, $G \in \operatorname{Op}(\tau) \Leftrightarrow (\forall x \in G)\ (G \in \tau(x))$). A subset F of X is a *closed set* in X if the complement of F to X is open; that is, $F \in \operatorname{Cl}(\tau) \Leftrightarrow X \setminus F \in \operatorname{Op}(\tau)$.

9.1.5. *The union of a family of open sets and the intersection of a finite family of open sets are open. The intersection of a family of closed sets and the union of a finite family of closed sets are closed.* ◁▷

9.1.6. *Let (X, τ) be a pretopological space. Given $x \in X$, put*

$$U \in t(\tau)(x) \Leftrightarrow (\exists V \in \operatorname{Op}(\tau))\quad x \in V \ \&\ U \supset V.$$

The mapping $t(\tau) : x \mapsto t(\tau)(x)$ is a pretopology on X. ◁▷

9.1.7. DEFINITION. A pretopology τ on X is a *topology* if $\tau = t(\tau)$. The pair (X, τ), as well as the underlying set X itself, is then called a *topological space.* The set of all topologies on X is denoted by the symbol $\mathrm{T}(X)$.

9.1.8. EXAMPLES.

(1) A metric topology.

(2) The topology of a multinormed space.

(3) Let $\tau_\circ := \inf \mathscr{T}(X)$. It is clear that $\tau_\circ(x) = \{X\}$ for $x \in X$. Consequently, $\mathrm{Op}(\tau_\circ) = \{\varnothing, X\}$ and so $\tau_\circ = t(\tau_\circ)$; i.e., $\tau_\circ$ is a topology. This topology is called *trivial*, or *antidiscrete*, or even *indiscrete.*

(4) Let $\tau^\circ := \sup \mathscr{T}(X)$. It is clear that $\tau^\circ(x) = \mathrm{fil}\,\{x\}$ for $x \in X$. Consequently, $\mathrm{Op}(\tau^\circ) = 2^X$ and so $\tau^\circ = t(\tau^\circ)$; i.e., τ° is a topology. This topology is called *discrete.*

(5) *Let* Op *be a collection of subsets in X which is stable under the taking of the union of each of its subfamilies and the intersection of each of its finite subfamilies. Then there is a unique topology τ on X such that* $\mathrm{Op}(\tau) = \mathrm{Op}$.

◁ Put $\tau(x) := \mathrm{fil}\,\{V \in \mathrm{Op} : x \in V\}$ for $x \in X$ (in case $X = \varnothing$ there is nothing to prove). Observe that $\tau(x) \neq \varnothing$ since the intersection of the empty family equals X (cf. $\inf \varnothing = +\infty$). From the construction derive that $t(\tau) = \tau$ and $\mathrm{Op} \subset \mathrm{Op}(\tau)$. If $G \in \mathrm{Op}(\tau)$ then $G = \cup\{V : V \in \mathrm{Op},\ V \subset G\}$ and so $G \in \mathrm{Op}$ by hypothesis. The claim of uniqueness raises no doubts. ▷

9.1.9. *Let the mapping $t : \mathscr{T}(X) \to \mathscr{T}(X)$ act by the rule $t : \tau \mapsto t(\tau)$. Then*

(1) $\mathrm{im}\, t = \mathrm{T}(X)$; *i.e.*, $\tau \in \mathscr{T}(X) \Rightarrow t(\tau) \in \mathrm{T}(X)$;

(2) $\tau_1 \le \tau_2 \Rightarrow t(\tau_1) \le t(\tau_2)$ $(\tau_1, \tau_2 \in \mathscr{T}(X))$;

(3) $t \circ t = t$;

(4) $\tau \in \mathscr{T}(X) \Rightarrow t(\tau) \le \tau$;

(5) $\mathrm{Op}(\tau) = \mathrm{Op}(t(\tau))$ $(\tau \in \mathscr{T}(X))$.

◁ The inclusion $\mathrm{Op}(\tau) \supset \mathrm{Op}(t(\tau))$ holds because it is easier to be open in τ. The reverse inclusion $\mathrm{Op}(\tau) \subset \mathrm{Op}(t(\tau))$ follows from the definition of $t(\tau)$. The equality $\mathrm{Op}(\tau) = \mathrm{Op}(t(\tau))$ makes everything evident. ▷

9.1.10. *A pretopology τ on X is a topology if and only if*

$$(\forall U \in \tau(x))(\exists V \in \tau(x)\ \&\ V \subset U)(\forall y)\ (y \in V \Rightarrow V \in \tau(y))$$

for $x \in X$.

◁ Straightforward from 9.1.9 (5). ▷

9.1.11. *Let $\tau_1, \tau_2 \in \mathrm{T}(X)$. The following statements are equivalent:*

(1) $\tau_1 \ge \tau_2$;

(2) $\mathrm{Op}(\tau_1) \supset \mathrm{Op}(\tau_2)$;

(3) $\mathrm{Cl}(\tau_1) \supset \mathrm{Cl}(\tau_2)$. ◁▷

9.1.12. REMARK. As follows from 9.1.8 (5) and 9.1.11, the topology of a space is uniquely determined from the collection of its open sets. Therefore, the set $\operatorname{Op}(X)$ itself is legitimately called the *topology* of the space X. In particular, the collection of open sets of a pretopological space $(X,\ \tau)$ makes X into the topological space $(X,\ t(\tau))$ with the same open sets in stock. Therefore, given a pretopology τ, the topology $t(\tau)$ is usually called the *topology associated with* τ.

9.1.13. ***Theorem.*** *The set* $\mathrm{T}(X)$ *of all topologies on* X *with the relation "to be stronger" presents a complete lattice. Moreover, for every subset* $\mathscr{E}$ *of* $\mathrm{T}(X)$ *the equality holds*:

$$\sup_{\mathrm{T}(X)} \mathscr{E} = \sup_{\mathscr{T}(X)} \mathscr{E}.$$

◁ Evidently, $t(\sup_{\mathscr{T}(X)} \mathscr{E}) \geq \sup_{\mathscr{T}(X)} t(\mathscr{E}) \geq \sup_{\mathscr{T}(X)} \mathscr{E} \geq t(\sup_{\mathscr{T}(X)} \mathscr{E})$. Thus, $\tau := \sup_{\mathscr{T}(X)} \mathscr{E}$ belongs to $\mathrm{T}(X)$. It is clear that $\tau \geq \mathscr{E}$. Furthermore, if $\tau_0 \geq \mathscr{E}$ and $\tau_0 \in \mathrm{T}(X)$ then $\tau_0 \geq \tau$ and so $\tau = \sup_{\mathrm{T}(X)} \mathscr{E}$. It remains to refer to 1.2.14. ▷

9.1.14. REMARK. The explicit formula for the greatest lower bound of $\mathscr{E}$ is more involved:

$$\inf_{\mathrm{T}(X)} \mathscr{E} = t(\inf_{\mathscr{T}(X)} \mathscr{E}).$$

However, the matter becomes simpler when the topologies are given by means of their collections of open sets in accordance with 9.1.12. Namely,

$$U \in \operatorname{Op}(\inf_{\mathrm{T}(X)} \mathscr{E}) \Leftrightarrow (\forall \tau \in \mathscr{E})\ U \in \operatorname{Op}(\tau).$$

In other words,

$$\operatorname{Op}(\inf_{\mathrm{T}(X)} \mathscr{E}) = \bigcap_{\tau \in \mathscr{E}} \operatorname{Op}(\tau).$$

In this connection it is in common parlance to speak of the *intersection* of the set $\mathscr{E}$ of topologies (rather than of the greatest lower bound of $\mathscr{E}$). ◁▷

9.2. Continuity

9.2.1. REMARK. The presence of a topology on a set obviously makes it possible to deal with such things as the interior and closure of a subset, convergence of filters and nets, etc. We have already made use of this circumstance while introducing multinormed spaces. Observe for the sake of completeness that in every topological space the following analogs of 4.1.19 and 4.2.1 are valid:

9.2.2. ***Birkhoff Theorem.*** *For a nonempty subset* U *and a point* x *of a topological space the following statements are equivalent*:

(1) *x is an adherent point of U*;

(2) *there is some filter containing U and converging to x*;

(3) *there is a net of elements of U which converges to x.* ◁▷

9.2.3. *For a mapping f between topological spaces the following conditions are equivalent:*

(1) *the inverse image under f of an open set is open;*

(2) *the inverse image under f of a closed set is closed;*

(3) *the image under f of the neighborhood filter of an arbitrary point x is coarser than the neighborhood filter of $f(x)$;*

(4) *for all x, the mapping f transforms each filter convergent to x into a filter convergent to $f(x)$;*

(5) *for all x, the mapping f sends a net that converges to x to a net that converges to $f(x)$.* ◁▷

9.2.4. DEFINITION. A mapping acting between topological spaces X and Y and satisfying one (and hence all) of the equivalent conditions 9.2.3 (1)–9.2.3 (5) is called *continuous*. A continuous one-to-one mapping f from X onto Y whose inverse f^{-1} acts continuously from Y to X is a *homeomorphism* or a *topological mapping* or a *topological isomorphism* between X and Y.

9.2.5. REMARK. If $f : (X,\ \tau_X) \to (Y,\ \tau_Y)$ meets 9.2.3 (5) at some point x in X then it is customary to say that f is *continuous at* x (cf. 4.2.2). Observe that the difference is immaterial between the definitions of the continuity property at a point of X and the general continuity property (on X). Indeed, if we let $\tau_x(x) := \tau_X(x)$ and $\tau_x(\bar{x}) := \text{fil}\ \{\bar{x}\}$ for $\bar{x} \in X$, $\bar{x} \neq x$, then the continuity property of f at x (with respect to the topology τ_X in X) amounts to that of $f : (X,\ \tau_x) \to (Y,\ \tau_Y)$ (at every point of the space X with topology τ_x).

9.2.6. *Let $\tau_1, \tau_2 \in \mathrm{T}(X)$. Then $\tau_1 \geq \tau_2$ if and only if $I_X : (X,\ \tau_1) \to (X,\ \tau_2)$ is continuous.* ◁▷

9.2.7. *Let $f : (X,\ \tau) \to (Y,\ \omega)$ be a continuous mapping and let $\tau_1 \in \mathrm{T}(X)$ and $\omega_1 \in \mathrm{T}(Y)$ be such that $\tau_1 \geq \tau$ and $\omega \geq \omega_1$. Then $f : (X,\ \tau_1) \to (Y,\ \omega_1)$ is continuous.*

◁ By hypothesis the following diagram commutes:

$$\begin{array}{ccc} (X,\tau) & \xrightarrow{f} & (Y,\omega) \\ I_X \uparrow & & \downarrow I_Y \\ (X,\tau_1) & \xrightarrow{f} & (Y,\omega_1) \end{array}$$

It suffices to observe that every composition of continuous mappings is continuous. ▷

9.2.8. ***Inverse Image Topology Theorem.*** *Let $f : X \to (Y,\ \omega)$. Put*

$$T_0 := \{\tau \in \mathrm{T}(X) :\ f : (X,\ \tau) \to (Y,\ \omega) \text{ is continuous}\}.$$

Then the topology $f^{-1}(\omega) := \inf T_0$ belongs to T_0.

◁ From 9.2.3 (1) it follows that

$$\tau \in T_0 \Leftrightarrow (x \in X \Rightarrow f^{-1}(\omega(f(x))) \subset \tau(x)).$$

Let $\overline{\tau}(x) := f^{-1}(\omega(f(x)))$. Undoubtedly, $t(\overline{\tau}) = \overline{\tau}$. Furthermore, $f(\overline{\tau}(x)) = f(f^{-1}(\omega(f(x)))) \supset \omega(f(x))$; i.e., $\overline{\tau} \in T_0$ by 9.2.3 (3). Thus, $f^{-1}(\omega) = \overline{\tau}$. ▷

9.2.9. DEFINITION. The topology $f^{-1}(\omega)$ is the *inverse image* of ω under a mapping f or simply the *inverse image topology* under f.

9.2.10. REMARK. Theorem 9.2.8 is often verbalized as follows: "The inverse image topology under a mapping is the weakest topology on the set of departure in which the mapping is continuous." Moreover, it is easy for instance from 9.1.14 that the open sets of the inverse image topology are precisely the inverse images of open sets. In particular, $(x_\xi \to x$ in $f^{-1}(\omega)) \Leftrightarrow (f(x_\xi) \to f(x)$ in $\omega)$; likewise, $(\mathscr{F} \to x$ in $f^{-1}(\omega)) \Leftrightarrow (f(\mathscr{F}) \to f(x)$ in $\omega)$ for a filter $\mathscr{F}$. ◁▷

9.2.11. *Image Topology Theorem.* *Let $f : (X,\ \tau) \to Y$. Put*

$$\Omega_0 := \{\omega \in \mathrm{T}(Y) :\ f : (X,\ \tau) \to (Y,\ \omega) \text{ is continuous}\}.$$

Then the topology $f(\tau) := \sup \Omega_0$ belongs to Ω_0.

◁ Appealing to 9.1.13, observe that

$$f(\tau)(y) = (\sup\nolimits_{\mathrm{T}(Y)} \Omega_0)(y) = (\sup\nolimits_{\mathscr{F}(Y)} \Omega_0)(y) = \sup\{\omega(y) :\ \omega \in \Omega_0\}$$

for $y \in Y$. By virtue of 9.2.3 (3),

$$\omega \in \Omega_0 \Leftrightarrow (x \in X \Rightarrow f(\tau(x)) \supset \omega(f(x))).$$

Comparing the formulas, infer that $f(\tau) \in \Omega_0$. ▷

9.2.12. DEFINITION. The topology $f(\tau)$ is the *image of* τ under a mapping f or simply the *image topology* under f.

9.2.13. REMARK. Theorem 9.2.11 is often verbalized as follows: "The image of a topology under a mapping is the strongest topology on the set of arrival in which the mapping is continuous."

9.2.14. *Theorem.* *Let $(f_\xi : X \to (Y_\xi,\ \omega_\xi))_{\xi \in \Xi}$ be a family of mappings. Further, put $\tau := \sup_{\xi \in \Xi} f_\xi^{-1}(\omega_\xi)$. Then τ is the weakest (= least) topology on X making all the mappings f_ξ $(\xi \in \Xi)$ continuous.*

◁ Using 9.2.8, note that

$$(f_\xi : (X,\ \overline{\tau}) \to (Y_\xi,\ \omega_\xi) \text{ is continuous}) \Leftrightarrow \overline{\tau} \geq f_\xi^{-1}(\omega_\xi). \ \triangleright$$

9.2.15. *Theorem.* *Let $(f_\xi : (X_\xi,\ \tau_\xi) \to Y)_{\xi \in \Xi}$ be a family of mappings. Further, assign $\omega := \inf_{\xi \in \Xi} f_\xi(\tau_\xi)$. Then ω is the strongest (= greatest) topology on Y making all the mappings f_ξ $(\xi \in \Xi)$ continuous.*

◁ Appealing to 9.2.11, conclude that

$$(f_\xi : (X_\xi,\ \tau_\xi) \to (Y,\ \overline{\omega}) \text{ is continuous}) \Leftrightarrow \overline{\omega} \le f_\xi(\tau_\xi). \ \triangleright$$

9.2.16. REMARK. The messages of 9.2.14 and 9.2.15 are often referred to as the *theorems on topologizing by a family of mappings.*

9.2.17. EXAMPLES.

(1) Let $(X,\ \tau)$ be a topological space and let X_0 be a subset of X. Denote the identical embedding of X_0 into X by $\iota : X_0 \to X$. Put $\tau_0 := \iota^{-1}(\tau)$. The topology τ_0 is the *induced topology* (by τ in X_0), or the *relative* or *subspace topology*; and the space $(X_0,\ \tau_0)$ is a *subspace* of $(X,\ \tau)$.

(2) Let $(X_\xi,\ \tau_\xi)_{\xi\in\Xi}$ be a family of topological spaces and let $\mathfrak{X} := \prod_{\xi\in\Xi} X_\xi$ be the product of $(X_\xi)_{\xi\in\Xi}$. Put $\tau := \sup_{\xi\in\Xi} \mathrm{Pr}_\xi^{-1}(\tau_\xi)$, where $\mathrm{Pr}_\xi : \mathfrak{X} \to X_\xi$ is the coordinate projection (onto X_ξ); i.e., $\mathrm{Pr}_\xi\, x = x_\xi$ $(\xi \in \Xi)$. The topology τ is the *product topology* or the *product* of the topologies $(\tau_\xi)_{\xi\in\Xi}$, or the *Tychonoff topology* of $\mathfrak{X}$. The space $(\mathfrak{X},\ \tau)$ is the *Tychonoff product* of the topological spaces under study. In particular, if $X_\xi := [0,\ 1]$ for all $\xi \in \Xi$ then $\mathfrak{X} := [0,\ 1]^\Xi$ (with the Tychonoff topology) is a *Tychonoff cube*. When $\Xi := \mathbb{N}$, the term "Hilbert cube" is applied.

9.3. Types of Topological Spaces

9.3.1. *For a topological space the following conditions are equivalent:*

(1) *every singleton of the space is closed;*

(2) *the intersection of all neighborhoods of each point in the space consists solely of the point;*

(3) *each one of any two points in the space has a neighborhood disjoint from the other.*

◁ To prove, it suffices to observe that

$$y \in \mathrm{cl}\,\{x\} \Leftrightarrow (\forall\, V \in \tau(y)) \quad x \in V \Leftrightarrow x \in \cap\{V :\ V \in \tau(y)\},$$

where x and y are points of a space with topology τ. ▷

9.3.2. DEFINITION. A topological space satisfying one (and hence all) of the equivalent conditions 9.3.1 (1)–9.3.1 (3), is called a *separated space* or a T_1-*space.* The topology of a T_1-space is called a *separated topology* or (rarely) a T_1-*topology.*

9.3.3. REMARK. By way of expressiveness, one often says: "A T_1-space is a space with closed points."

9.3.4. *For a topological space the following conditions are equivalent:*

(1) *each filter has at most one limit;*

(2) *the intersection of all closed neighborhoods of a point in the space consists of the sole point;*

(3) *each one of any two points of the space has a neighborhood disjoint from some neighborhood of the other point.*

$\lhd$ (1) $\Rightarrow$ (2): If $y \in \bigcap_{U \in \tau(x)} \mathrm{cl}\, U$ then $U \cap V \neq \varnothing$ for all $V \in \tau(y)$, provided that $U \in \tau(x)$. Therefore, the join $\mathscr{F} := \tau(x) \vee \tau(y)$ is available. Clearly, $\mathscr{F} \to x$ and $\mathscr{F} \to y$. By hypothesis, $x = y$.

(2) $\Rightarrow$ (3): Let $x,\ y \in X$, $x \neq y$ (if such points are absent then either $X = \varnothing$ or X is a singleton and nothing is left unproven). There is an neighborhood U in $\tau(x)$ such that $U = \mathrm{cl}\, U$ and $y \notin U$. Consequently, the complement V of U to X is open. Furthermore, $U \cap V = \varnothing$.

(3) $\Rightarrow$ (1): Let $\mathscr{F}$ be a filter on X. If $\mathscr{F} \to x$ and $\mathscr{F} \to y$ then $\mathscr{F} \supset \tau(x)$ and $\mathscr{F} \supset \tau(y)$. Thus, $U \cap V \neq \varnothing$ for $U \in \tau(x)$ and $V \in \tau(y)$, which means that $x = y$. $\rhd$

9.3.5. DEFINITION. A topological space satisfying one (and hence all) of the equivalent conditions 9.3.4 (1)–9.3.4 (3) is a *Hausdorff space* or a T_2-*space*. A natural meaning is ascribed to the term "Hausdorff topology."

9.3.6. REMARK. By way of expressiveness, one often says: "A T_2-space is a space with unique limits."

9.3.7. DEFINITION. Let U and V be subsets of a topological space. It is said that V is a *neighborhood* of U or about U, provided $\mathrm{int}\, V \supset U$. If U is nonempty then all neighborhoods of U constitute some filter that is the *neighborhood filter* of U.

9.3.8. *For a topological space the following conditions are equivalent:*

(1) *the intersection of all closed neighborhoods of an arbitrary closed set consists only of the members of the set;*

(2) *the neighborhood filter of each point has a base of closed sets;*

(3) *if F is a closed set and x is a point not in F then there are disjoint neighborhoods of F and x, respectively.*

$\lhd$ (1) $\Rightarrow$ (2): If $x \in X$ and $U \in \tau(x)$ then $V := X \setminus \mathrm{int}\, U$ is closed and $x \notin V$. By hypothesis there is a set F in $\mathrm{Cl}\,(\tau)$ such that $x \notin F$ and $\mathrm{int}\, F \supset V$. Put $G := X \setminus F$. Clearly, $G \in \tau(x)$. Moreover, $G \subset X \setminus \mathrm{int}\, F = \mathrm{cl}\,(X \setminus \mathrm{int}\, F) \subset X \setminus V \subset \mathrm{int}\, U \subset U$. Consequently, $\mathrm{cl}\, G \subset U$.

(2) $\Rightarrow$ (3): If $x \in X$ and $F \in \mathrm{Cl}\,(\tau)$ with $x \notin F$ then $X \setminus F \in \tau(x)$. Thus, there is a closed neighborhood U in $\tau(x)$ lying in $X \setminus F$. Thus, $X \setminus U$ is a neighborhood of F disjoint from U.

(3) $\Rightarrow$ (1): If $F \in \mathrm{Cl}\,(\tau)$ and $\mathrm{int}\, G \supset F \Rightarrow y \in \mathrm{cl}\, G$, then $U \cap G \neq \varnothing$ for every U in $\tau(y)$ and every neighborhood G of F. This means that $y \in F$. $\rhd$

9.3.9. DEFINITION. A T_3-*space* is a topological space satisfying one (and hence all) of the equivalent conditions 9.3.8 (1)–9.3.8 (3). A separated T_3-space is called *regular*.

9.3.10. *Urysohn Little Lemma.* *For a topological space the following conditions are equivalent:*

(1) *the neighborhood filter of each nonempty closed set has a base of closed sets;*

(2) *if two closed sets are disjoint then they have disjoint neighborhoods.*

◁ (1) ⇒ (2): Let F_1 and F_2 be closed sets in some space X with $F_1 \cap F_2 = \varnothing$. Put $G := X \setminus F_1$. Obviously, G is open and $G \supset F_2$. If $F_2 = \varnothing$, then there is nothing to be proven. It may be assumed consequently that $F_2 \neq \varnothing$. Then there is a closed set V_2 such that $G \supset V_2 \supset \operatorname{int} V_2 \supset F_2$. Put $V_1 := X \setminus V_2$. It is clear that V_1 is open, and $V_1 \cap V_2 = \varnothing$. Moreover, $V_1 \supset X \setminus G = X \setminus (X \setminus F_1) = F_1$.

(2) ⇒ (1): Let $F = \operatorname{cl} F$, $G = \operatorname{int} G$ and $G \supset F$. Put $F_1 := X \setminus G$. Then $F_1 = \operatorname{cl} F_1$ and so there are open sets U and U_1 satisfying $U \cap U_1 = \varnothing$, with $F \subset U$ and $F_1 \subset U_1$. Finally, $\operatorname{cl} U \subset X \setminus U_1 \subset X \setminus F_1 = G$. ▷

9.3.11. DEFINITION. A T_4-*space* is a topological space meeting one (and hence both) of the equivalent conditions 9.3.10 (1) and 9.3.10 (2). A separated T_4-space is called *normal*.

9.3.12. *Continuous Function Recovery Lemma.* *Let a subset T be dense in $\overline{\mathbb{R}}$ and let $t \mapsto U_t$ $(t \in T)$ be a family of subsets of a topological space X. There is a unique continuous function $f : X \to \overline{\mathbb{R}}$ such that*

$$\{f < t\} \subset U_t \subset \{f \le t\} \quad (t \in T)$$

if and only if

$$(t,\ s \in T \ \&\ t < s) \Rightarrow \operatorname{cl} U_t \subset \operatorname{int} U_s.$$

◁ ⇒: Take $t < s$. Since $\{f \le t\}$ is closed and $\{f < s\}$ is open, the inclusions hold:

$$\operatorname{cl} U_t \subset \{f \le t\} \subset \{f < s\} \subset \operatorname{int} U_s.$$

⇐: Since $U_t \subset \operatorname{cl} U_t \subset \operatorname{int} U_s \subset U_s$ for $t < s$, the family $t \mapsto U_t$ $(t \in T)$ increases by inclusion. Therefore, f exists by 3.8.2 and is unique by 3.8.4. Consider the families $t \mapsto V_t := \operatorname{cl} U_t$ and $t \mapsto W_t := \operatorname{int} U_t$. These families increase by inclusion. Consequently, on applying 3.8.2 once again, find functions $g,\ h : X \to \overline{\mathbb{R}}$ satisfying

$$\{g < t\} \subset V_t \subset \{g \le t\}, \quad \{h < t\} \subset W_t \subset \{h \le t\}$$

for all $t \in T$. If $t,\ s \in T$ and $t < s$, then in view of 3.8.3

$$W_t = \mathrm{int}\ U_t \subset U_t \subset U_s \Rightarrow f \le h;$$
$$V_t = \mathrm{cl}\ U_t \subset \mathrm{int}\ U_s = W_s \Rightarrow h \le g;$$
$$U_t \subset U_s \subset \mathrm{cl}\ U_s = V_s \Rightarrow g \le f.$$

So, $f = g = h$. Taking account of 3.8.4 and 9.1.5 and given $t \in \overline{\mathbb{R}}$, find

$$\{f < t\} = \{h < t\} = \cup\{W_s :\ s < t,\ s \in T\} \in \mathrm{Op}(\tau_X);$$
$$\{f \le t\} = \{g \le t\} = \cap\{V_s :\ t < s,\ s \in T\} \in \mathrm{Cl}(\tau_X).$$

These inclusions readily provide continuity for f. ▷

9.3.13. Urysohn Great Lemma. *Let X be a T_4-space. Assume further that F is a closed set in X and G is a neighborhood of F. Then there is a continuous function $f : X \to [0,\ 1]$ such that $f(x) = 0$ for $x \in F$ and $f(x) = 1$ for $x \notin G$.*

◁ Put $U_t := \varnothing$ for $t < 0$ and $U_t := X$ for $t > 1$. Consider the set $\overline{T}$ of the *dyadic-rational points* of the interval $[0,\ 1]$; i.e., $\overline{T} := \cup_{n \in \mathbb{N}} T_n$ with $T_n := \{k2^{-n+1} : k := 0,\ 1, \dots, 2^{n-1}\}$. It suffices to define U_t for all t in $\overline{T}$ so that the family $t \mapsto U_t$ $(t \in T := \overline{T} \cup (\overline{\mathbb{R}} \setminus [0,\ 1]))$ satisfy the criterion of 9.3.12. This is done by way of induction.

If $t \in T_1$, i.e., $t \in \{0,\ 1\}$; then put $U_0 := F$ and $U_1 := G$. Assume now that, for $t \in T_n$ and $n \ge 1$, some set U_t has already been constructed, satisfying $\mathrm{cl}\ U_t \subset \mathrm{int}\ U_s$ whenever $t,\ s \in T_n$ and $t < s$. Take $t \in T_{n+1}$ and find the two points t_l and t_r in T_n nearest to t:

$$t_l := \sup\{s \in T_n :\ s \le t\};$$
$$t_r := \inf\{s \in T_n :\ t \le s\}.$$

If $t = t_l$ or $t = t_r$, then U_t exists by the induction hypothesis. If $t \ne t_l$ and $t \ne t_r$, then $t_l < t < t_r$ and again by the induction hypothesis $\mathrm{cl}\ U_{t_l} \subset \mathrm{int}\ U_{t_r}$. By virtue of 9.3.11 there is a closed set U_t such that

$$\mathrm{cl}\ U_{t_l} \subset \mathrm{int}\ U_t \subset U_t = \mathrm{cl}\ U_t \subset \mathrm{int}\ U_{t_r}.$$

It remains to show that the resulting family satisfies the criterion of 9.3.12.

To this end, take $t,\ s \in T_{n+1}$ with $t < s$. If $t_r = s_l$, then for $s > s_l$ by construction

$$\mathrm{cl}\ U_t \subset \mathrm{cl}\ U_{t_r} = \mathrm{cl}\ U_{s_l} \subset \mathrm{int}\ U_s.$$

For $t < t_r = s_l$ similarly deduce the following:

$$\text{cl } U_t \subset \text{int } U_{t_r} = \inf U_{s_l} \subset \text{int } U_s.$$

If $t_r < s_l$ then, on using the induction hypothesis, infer that

$$\text{cl } U_t \subset \text{cl } U_{t_r} \subset \text{int } U_{s_l} \subset \text{int } U_s,$$

what was required. ▷

9.3.14. *Urysohn Theorem.* *A topological space X is a T_4-space if and only if to every pair of disjoint closed sets F_1 and F_2 in X there corresponds a continuous function $f : X \to [0,\ 1]$ such that $f(x) = 0$ for $x \in F_1$ and $f(x) = 1$ for $x \in F_2$.*

◁ $\Rightarrow$: It suffices to apply 9.3.13 with $F := F_1$ and $G := X \setminus F_2$.

$\Leftarrow$: If $F_1 \cap F_2 = \varnothing$ and F_1 and F_2 are closed sets, then for a corresponding function f the sets $G_1 := \{f < {}^1/_2\}$ and $G_2 := \{f > {}^1/_2\}$ are open and disjoint. Moreover, $G_1 \supset F_1$ and $G_2 \supset F_2$. ▷

9.3.15. DEFINITION. A topological space X is a $\mathrm{T}_{3^1/_2}$-*space*, if to a closed set F in X and a point x not in F there corresponds a continuous function f : $X \to [0,\ 1]$ such that $f(x) = 1$ and $y \in F \Rightarrow f(y) = 0$. A separated $\mathrm{T}_{3^1/_2}$-space is a *Tychonoff space* or a *completely regular space*.

9.3.16. *Every normal space is a Tychonoff space.*

◁ Straightforward from 9.3.1 and 9.3.14. ▷

9.4. Compactness

9.4.1. *Let $\mathscr{B}$ be a filterbase on a topological space and let*

$$\text{cl } \mathscr{B} := \cap\{\text{cl } B : \ B \in \mathscr{B}\}$$

be the set of adherent points of $\mathscr{B}$ (also called the adherence of $\mathscr{B}$). Then

(1) $\text{cl } \mathscr{B} = \text{cl fil } \mathscr{B}$;

(2) $\mathscr{B} \to x \Rightarrow x \in \text{cl } \mathscr{B}$;

(3) *($\mathscr{B}$ is an ultrafilter and $x \in \text{cl } \mathscr{B}$) $\Rightarrow \mathscr{B} \to x$.*

◁ Only (3) needs demonstrating, since (1) and (2) are evident. Given $U \in \tau(x)$ and $B \in \mathscr{B}$, observe that $U \cap B \neq \varnothing$. In other words, the join $\mathscr{F} := \tau(x) \vee \mathscr{B}$ is available. It is clear that $\mathscr{F} \to x$. Furthermore, $\mathscr{F} = \mathscr{B}$, because $\mathscr{B}$ is an ultrafilter. ▷

9.4.2. DEFINITION. A subset C of a topological space X is a *compact set* in X if each open cover of C has a finite subcover (cf. 4.4.1).

9.4.3. Theorem. *Let X be a topological space and let C be a subset of X. The following statements are equivalent:*

(1) *C is compact;*

(2) *if a filterbase $\mathscr{B}$ lacks adherent points in C then there is a member B of $\mathscr{B}$ such that $B \cap C = \varnothing$;*

(3) *each filterbase containing C has an adherent point in C;*

(4) *each ultrafilter containing C has a limit in C.*

◁ (1) ⇒ (2): Since $\mathrm{cl}\,\mathscr{B} \cap C = \varnothing$; therefore, $C \subset X \setminus \mathrm{cl}\,\mathscr{B}$. Thus, $C \subset X \setminus \cap\{\mathrm{cl}\,B : B \in \mathscr{B}\} = \cup\{X \setminus \mathrm{cl}\,B : B \in \mathscr{B}\}$. Consequently, there is a finite subset $\mathscr{B}_0$ of $\mathscr{B}$ such that $C \subset \cup\{X \setminus \mathrm{cl}\,B_0 : B_0 \in \mathscr{B}_0\} = X \setminus \cap\{\mathrm{cl}\,B_0 : B_0 \in \mathscr{B}_0\}$. Let $B \in \mathscr{B}$ satisfy $B \subset \cap\{B_0 : B_0 \in \mathscr{B}_0\} \subset \cap\{\mathrm{cl}\,B_0 : B_0 \in \mathscr{B}_0\}$. Then $C \cap B \subset C \cap (\cap\{\mathrm{cl}\,B_0 : B_0 \in \mathscr{B}_0\}) = \varnothing$.

(2) ⇒ (3): In case $C = \varnothing$, there is nothing to be proven. If $C \neq \varnothing$, then for $B \in \mathscr{B}$ by hypothesis $B \cap C \neq \varnothing$ because $C \in \mathscr{B}$. Thus, $\mathrm{cl}\,\mathscr{B} \cap C \neq \varnothing$.

(3) ⇒ (4): It suffices to appeal to 9.4.1.

(4) ⇒ (1): Assume that $C \neq \varnothing$ (otherwise, nothing is left to proof).

Suppose that C is not compact. Then there is a set $\mathscr{E}$ of open sets such that $C \subset \cup\{G : G \in \mathscr{E}\}$ and at the same time, for every finite subset $\mathscr{E}_0$ of $\mathscr{E}$, the inclusion $C \subset \cup\{G : G \in \mathscr{E}_0\}$ fails. Put

$$\mathscr{B} := \left\{ \bigcap_{G \in \mathscr{E}_0} X \setminus G : \ \mathscr{E}_0 \text{ is a finite subset of } \mathscr{E} \right\}.$$

It is clear that $\mathscr{B}$ is a filterbase. Furthermore,

$$\begin{aligned} \mathrm{cl}\,\mathscr{B} &= \cap\{\mathrm{cl}\,B : B \in \mathscr{B}\} = \cap\{X \setminus G : G \in \mathscr{E}\} \\ &= X \setminus \cup\{G : G \in \mathscr{E}\} \subset X \setminus C. \end{aligned}$$

Now choose an ultrafilter $\mathscr{F}$ that is coarser than $\mathscr{B}$, which is guaranteed by 1.3.10. By supposition each member of $\mathscr{B}$ meets C. We may thus assume that $C \in \mathscr{F}$ (adjusting the choice of $\mathscr{F}$, if need be). Then $\mathscr{F} \to x$ for some x in C and so, by 9.4.1 (2), $\mathrm{cl}\,\mathscr{F} \cap C \neq \varnothing$. At the same time $\mathrm{cl}\,\mathscr{F} \subset \mathrm{cl}\,\mathscr{B}$. We arrive at a contradiction. ▷

9.4.4. Remark. The equivalence (1) ⇔ (4) in Theorem 9.4.3 is called the *Bourbaki Criterion* and verbalized for $X = C$ as follows: "A space is compact if and only if every ultrafilter on it converges" (cf. 4.4.7). An *ultranet* is a net whose tail filter is an ultrafilter. The Bourbaki Criterion can be expressed as follows: "Compactness amounts to convergence of ultranets." Many convenient tests for compactness are formulated in the language of nets. For instance: "A space X is compact if and only if each net in X has a convergent subnet."

9.4.5. *Weierstrass Theorem.* *The image of a compact set under a continuous mapping is compact (cf. 4.4.5).* ⊲⊳

9.4.6. *Let X_0 be a subspace of a topological space X and let C be a subset of X_0. Then C is compact in X_0 if and only if C is compact in X.*

⊲ $\Rightarrow$: Immediate from 9.4.5 and 9.2.17 (1).

$\Leftarrow$: Let $\mathscr{B}$ be a filterbase on X_0. Further, let $V := \mathrm{cl}_{X_0} \mathscr{B}$ stand for the adherence of $\mathscr{B}$ relative to X_0. Suppose that $V \cap C = \varnothing$. Since $\mathscr{B}$ is also a filterbase on X, it makes sense to speak of the adherence $W := \mathrm{cl}_X \mathscr{B}$ of $\mathscr{B}$ relative to X. It is clear that $V = W \cap X_0$ and, consequently, $W \cap C = \varnothing$. Since C is compact in X, by 9.4.3 there is some B in $\mathscr{B}$ such that $B \cap C = \varnothing$. Using 9.4.3 once again, infer that C is compact in X_0. ⊳

9.4.7. Remark. The claim of 9.4.6 is often expressed as follows: "Compactness is an absolute concept." It means that for C to be or not to be compact depends on the topology induced in C rather than on the ambient space inducing the topology. For that reason, it is customary to confine study to *compact spaces*, i.e. to sets "compact in themselves." A topology τ on a set C, making C into a compact space, is usually called a *compact topology* on C. Also, such C is referred to as "compact with respect to τ."

9.4.8. *Tychonoff Theorem.* *The Tychonoff product of compact spaces is compact.*

⊲ Let $\mathfrak{X} := \prod_{\xi \in \Xi} X_\xi$ be the product of such spaces. If at least one of the spaces X_ξ is nonempty then $\mathfrak{X} = \varnothing$ and nothing is left to proof. Let $\mathfrak{X} \neq \varnothing$ and let $\mathscr{F}$ be an ultrafilter on $\mathfrak{X}$. By 1.3.12, given $\xi \in \Xi$ and considering the coordinate projection $\mathrm{Pr}_\xi : \mathfrak{X} \to X_\xi$, observe that $\mathrm{Pr}_\xi(\mathscr{F})$ is an ultrafilter on X_ξ. Consequently, in virtue of 9.4.3 there is some x_ξ in X_ξ such that $\mathrm{Pr}_\xi(\mathscr{F}) \to x_\xi$. Let $x : \xi \mapsto x_\xi$. It is clear that $\mathscr{F} \to x$ (cf. 9.2.10). Appealing to 9.4.3 once more, infer that $\mathfrak{X}$ is compact. ⊳

9.4.9. *Every closed subset of a compact space is compact.*

⊲ Let X be compact and $C \in \mathrm{Cl}(X)$. Assume further that $\mathscr{F}$ is an ultrafilter on X and $C \in \mathscr{F}$. By Theorem 9.4.3, $\mathscr{F}$ has a limit x in X: that is, $\mathscr{F} \to x$. By the Birkhoff Theorem, $x \in \mathrm{cl}\, C = C$. Using 9.4.3 again, conclude that C is compact. ⊳

9.4.10. *Every compact subset of a Hausdorff space is closed.*

⊲ Let C be compact in a Hausdorff space X. If $C = \varnothing$ then there is nothing to prove. Let $C \neq \varnothing$ and $x \in \mathrm{cl}\, C$. By virtue of 9.2.2 there is a filter $\mathscr{F}_0$ on X such that $C \in \mathscr{F}_0$ and $\mathscr{F}_0 \to x$. Let $\mathscr{F}$ be an ultrafilter finer than $\mathscr{F}_0$. Then $\mathscr{F} \to x$ and $C \in \mathscr{F}$. By 9.4.3, $\mathscr{F}$ has a limit in C. By 9.3.4 every limit in X is unique. Consequently, $x \in C$. ⊳

9.4.11. *Let $f : (X,\ \tau) \to (Y,\ \omega)$ be a continuous one-to-one mapping with $f(X) = Y$. If τ is a compact topology and ω is a Hausdorff topology, then f is a homomorphism.*

◁ It suffices to establish that f^{-1} is continuous. To this end, we are to demonstrate that $F \in \mathrm{Cl}(\tau) \Rightarrow f(F) \in \mathrm{Cl}(\omega)$. Take $F \in \mathrm{Cl}(\tau)$. Then F is compact by 9.4.9. Successively applying 9.4.5 and 9.4.10, infer that $f(F)$ is closed. ▷

9.4.12. *Let τ_1 and τ_2 be two topologies on a set X. If $(X,\ \tau_1)$ is a compact space and $(X,\ \tau_2)$ is a Hausdorff space with $\tau_1 \geq \tau_2$, then $\tau_1 = \tau_2$.* ◁▷

9.4.13. REMARK. The message of 9.4.12 is customarily verbalized as follows: "A compact topology is minimal among Hausdorff topologies."

9.4.14. ***Theorem.*** *Every Hausdorff compact space is normal.*

◁ Let X be the space under study and let $\mathscr{B}$ be some filterbase on X. Assume further that U is a neighborhood of $\mathrm{cl}\,\mathscr{B}$. It is clear that $X \setminus \mathrm{int}\,U$ is compact (cf. 9.4.9) and $\mathrm{cl}\,\mathscr{B} \cap (X \setminus \mathrm{int}\,U) = \varnothing$. By Theorem 9.4.3 there is a member B of $\mathscr{B}$ such that $B \cap (X \setminus \mathrm{int}\,U) = \varnothing$; i.e., $B \subset U$. Putting, if need be, $\mathscr{B} := \{\mathrm{cl}\,B : B \in \mathscr{B}\}$, we may assert that $\mathrm{cl}\,B \subset U$.

To begin with, take $x \in X$ and put $\mathscr{B} := \tau(x)$. By virtue of 9.3.4, $\mathrm{cl}\,\mathscr{B} = \{x\}$ and, consequently, the filter $\tau(x)$ has a base of closed sets. Thus, X is regular.

Now take nonempty closed subset F of X. Take as $\mathscr{B}$ the neighborhood filter of F. By 9.3.8, $\mathrm{cl}\,\mathscr{B} = F$, and, as is already established, $\mathscr{B}$ has a base of closed sets. In accordance with 9.3.9, X is a normal space. ▷

9.4.15. ***Corollary.*** *Each Hausdorff compact space is (to within a homeomorphism) a closed subset of a Tychonoff cube.*

◁ The compactness property of a closed subset of a Tychonoff cube follows from 9.4.8 and 9.4.9. Moreover, every cube is a Hausdorff space and so such is each of its subspaces.

Now take some Hausdorff compact space X. Let Q be the collection of all continuous function from X to $[0,\ 1]$. Define the mapping $\Psi : X \to [0,\ 1]^Q$ as $\Psi(x)(f) := f(x)$ where $x \in X$ and $f \in Q$. From 9.4.14 and 9.3.14 infer that Ψ carries X onto $\Psi(X)$ in a one-to-one fashion. Furthermore, Ψ is continuous. Application to 9.4.11 completes the proof. ▷

9.4.16. REMARK. Corollary 9.4.15 presents a part of a more general assertion. Namely, a Tychonoff space is (to within a homeomorphism) a subspace of a Tychonoff cube. ◁▷

9.4.17. REMARK. Sometimes a Hausdorff compact space is also called a *compactum* (cf. 4.5 and 4.6).

9.4.18. ***Diedonnè Lemma.*** *Let F be a closed set and let $G_1, \dots, G_n$ be open sets in a normal topological space, with $F \subset G_1 \cup \dots \cup G_n$. There are closed sets $F_1, \dots, F_n$ such that $F = F_1 \cup \dots \cup F_n$ and $F_k \subset G_k$ $(k := 1, \dots, n)$.*

◁ It suffices to settle the case $n := 2$. For $k := 1, 2$ the set $U_k := F \backslash G_k$ is closed. Moreover, $U_1 \cap U_2 = \varnothing$. By 9.3.10 there are open V_1 and V_2 such that $U_1 \subset V_1$, $U_2 \subset V_2$ and $V_1 \cap V_2 = \varnothing$. Put $F_k := F \setminus V_k$. It is clear that F_k is closed and $F_k \subset F \backslash U_k = F \backslash (F \backslash G_k) \subset G_k$ for $k := 1, 2$. Finally, $F_1 \cup F_2 = F \backslash (V_1 \cup V_2) = F$. ▷

9.4.19. REMARK. From 9.3.14 we deduce that under the hypotheses of 9.4.18 there are continuous functions $h_1, \dots, h_n : X \to [0,\ 1]$ such that $h_k|_{G'_k} = 0$ and $\sum_{k=1}^n h_k(x) = 1$ for every point x in some neighborhood about F. (As usual, $G'_k := X \setminus G_k$.)

9.4.20. DEFINITION. A topology is called *locally compact* if each point possesses a compact neighborhood. A *locally compact space* is a set furnished with a Hausdorff locally compact topology.

9.4.21. *A topological space is locally compact if and only if it is homeomorphic with a punctured compactum (= a compactum with a deleted point), i.e. the complement of a singleton to a compactum.*

◁ $\Leftarrow$: In virtue of the Weierstrass Theorem it suffices to observe that each point of a punctured compactum possesses a closed neighborhood (since every compactum is regular). It remains to make use of 9.4.9 and 9.4.6.

$\Rightarrow$: Put the initial space X in $X^{\cdot} := X \cup \{\infty\}$, adjoining to X a point ∞ taken elsewhere. Take the complements to $X^{\cdot}$ of compact subsets of X as a base for the neighborhood filter about ∞. A neighborhood of a point x of X in $X^{\cdot}$ is declared to be a superset of a neighborhood of x in X. If $\mathfrak{A}$ is an ultrafilter in $X^{\cdot}$ and K is a compactum in X then $\mathfrak{A}$ converges to a point in K provided that $K \in \mathfrak{A}$. If $\mathfrak{A}$ contains the complement of each compactum K in X to X, then $\mathfrak{A}$ converges to ∞. ▷

9.4.22. REMARK. If a locally compact space X is not compact in its own right then $X^{\cdot}$ of 9.4.20 is the *one-point* or *Alexandroff compactification* of X.

9.5. Uniform and Multimetric Spaces

9.5.1. DEFINITION. Let X be a nonempty set and let $\mathscr{U}_X$ be a filter on X^2. The filter $\mathscr{U}_X$ is a *uniformity* on X if

(1) $\mathscr{U}_X \subset \operatorname{fil}\{I_X\}$;
(2) $U \in \mathscr{U}_X \Rightarrow U^{-1} \in \mathscr{U}_X$;
(3) $(\forall U \in \mathscr{U}_X)(\exists V \in \mathscr{U}_X)\ V \circ V \subset U$.

The uniformity of the empty set is by definition $\{\varnothing\}$. The pair $(X,\ \mathscr{U}_X)$, as well as the underlying set X, is called a *uniform space*.

9.5.2. *Given a uniform space $(X,\ \mathscr{U}_X)$, put*

$$x \in X \Rightarrow \tau(x) := \{U(x) :\ U \in \mathscr{U}_X\}.$$

The mapping $\tau : x \mapsto \tau(x)$ is a topology on X.

◁ Clearly, τ is a pretopology. If $W \in \tau(x)$ then $W = U(x)$ for some U in $\mathscr{U}_X$. Choose a member V of $\mathscr{U}_X$ so that $V \circ V \subset U$. If $y \in V(x)$ then $V(y) \subset V(V(x)) = V \circ V(x) \subset U(x) \subset W$. In other words, the set W is a neighborhood about y for every y in $V(x)$. Therefore, $V(x)$ lies in int W. Consequently, int W is a neighborhood about x. It remains to refer to 9.1.6. ▷

9.5.3. DEFINITION. The topology τ appearing in 9.5.2 is the *topology of the uniform space* $(X, \mathscr{U}_X)$ under consideration or the *uniform topology* on X. It is also denoted by $\tau(\mathscr{U}_X)$, τ_X, etc.

9.5.4. DEFINITION. A topological space (X, τ) is called *uniformizable* provided that there is a uniformity $\mathscr{U}$ on X such that τ coincides with the uniform topology $\tau(\mathscr{U})$.

9.5.5. EXAMPLES.

(1) A metric space (with its metric topology) is uniformizable (with its metric uniformity).

(2) A multinormed space (with its topology) is uniformizable (with its uniformity).

(3) Let $f : X \to (Y, \mathscr{U}_Y)$ and $f^{-1}(\mathscr{U}_Y) := f^{\times -1}(\mathscr{U}_Y)$, where as usual $f^{\times}(x_1, x_2) := (f(x_1), f(x_2))$ for $(x_1, x_2) \in X^2$. Evidently, $f^{-1}(\mathscr{U}_Y)$ is a uniformity on X. Moreover,

$$\tau(f^{-1}(\mathscr{U}_Y)) = f^{-1}(\tau(\mathscr{U}_Y)).$$

The uniformity $f^{-1}(\mathscr{U}_Y)$ is the *inverse image* of $\mathscr{U}_Y$ under f. Therefore, the inverse image of a uniform topology is uniformizable.

(4) Let $(X_\xi, \mathscr{U}_\xi)_{\xi\in\Xi}$ be a family of uniform spaces. Assume further that $\mathfrak{X} := \prod_{\xi\in\Xi} X_\xi$ is the product of the family. Put $\mathscr{U}_{\mathfrak{X}} := \sup_{\xi\in\Xi} \Pr_\xi^{-1}(\mathscr{U}_\xi)$. The uniformity $\mathscr{U}_{\mathfrak{X}}$ is the *Tychonoff uniformity.* It is beyond a doubt that the uniform topology $\tau(\mathscr{U}_{\mathfrak{X}})$ is the Tychonoff topology of the product of $(X_\xi, \tau(\mathscr{U}_\xi))_{\xi\in\Xi}$. ◁▷

(5) *Each Hausdorff compact space is uniformizable in a unique fashion.*

◁ By virtue of 9.4.15 such a space X may be treated as a subspace of a Tychonoff cube. From 9.5.5 (3) and 9.5.5 (4) it follows that X is uniformizable. Since each entourage of a uniform space includes a closed entourage; therefore, the compactness property of the diagonal I_X of X^2 implies that every neighborhood of I_X belongs to $\mathscr{U}_X$. On the other hand, each entourage is always a neighborhood of the diagonal. ▷

(6) Assume that X and Y are nonempty sets, $\mathscr{U}_Y$ is a uniformity on Y and $\mathscr{B}$ is an upward-filtered subset of $\mathscr{P}(X)$. Given $B \in \mathscr{B}$ and $\theta \in \mathscr{U}_Y$, put

$$U_{B,\theta} := \{(f, g) \in Y^X \times Y^X : g \circ I_B \circ f^{-1} \subset \theta\}.$$

Then $\mathscr{U} :=$ fil $\{U_{B,\theta} : B \in \mathscr{B}, \theta \in \mathscr{U}_Y\}$ is a uniformity on Y^X. This uniformity has a cumbersome (but exact) title, the "uniformity of uniform convergence on the members of $\mathscr{B}$." Such is, for instance, the uniformity of the Arens multinorm (cf. 8.3.8). If $\mathscr{B}$ is the collection of all finite subsets of X, then $\mathscr{U}$ coincides with the Tychonoff uniformity on Y^X. This uniformity is called *weak*, and the corresponding uniform topology is called the topology of *pointwise convergence* or, rarely, that of *simple convergence*. If $\mathscr{B}$ is a singleton $\{X\}$, then the uniformity $\mathscr{U}$ is called *strong* and the corresponding topology $\tau(\mathscr{U})$ in Y^X is the topology of *uniform convergence* on X.

9.5.6. REMARK. It is clear that, in a uniform (or uniformizable) space, the concepts make sense such as uniform continuity, total boundedness, completeness, etc. It is beyond a doubt that in such a space the analogs of 4.2.4–4.2.9, 4.5.8, 4.5.9, and 4.6.1–4.6.7 are preserved. It is a rewarding practice to ponder over a possibility of completing a uniform space, to validate a uniform version of the Hausdorff Criterion, to inspect the proof of the Ascoli–Arcelà Theorem in an abstract uniform setting, etc.

9.5.7. DEFINITION. Let X be a set and put $\mathbb{R}^{\cdot}_{+} := \{x \in \mathbb{R}^{\cdot} : \ x \geq 0\}$. A mapping $d : X^2 \to \mathbb{R}^{\cdot}_{+}$ is called a *semimetric* or a *pseudometric* on X, provided that

(1) $d(x,\ x) = 0\ (x \in X)$;

(2) $d(x,\ y) = d(y,\ x)\ (x,\ y \in X)$;

(3) $d(x,\ y) \leq d(x,\ z) + d(z,\ y)\ (x,\ y,\ z \in X)$.

A pair $(X,\ d)$ is a *semimetric space.*

9.5.8. *Given a semimetric space $(X,\ d)$, let $\mathscr{U}_d := \operatorname{fil}\{\{d \leq \varepsilon\} :\ \varepsilon > 0\}$. Then $\mathscr{U}_d$ is a uniformity.* ◁▷

9.5.9. DEFINITION. Let $\mathfrak{M}$ be a (nonempty) set of semimetrics on X. Then the pair $(X,\ \mathfrak{M})$ is a *multimetric space* with *multimetric* $\mathfrak{M}$. The *multimetric uniformity* on X is defined as $\mathscr{U}_{\mathfrak{M}} := \sup\{\mathscr{U}_d :\ d \in \mathfrak{M}\}$.

9.5.10. DEFINITION. A uniform space is called *multimetrizable*, if its uniformity coincides with some multimetric uniformity. A *multimetrizable topological space* is defined by analogy.

9.5.11. *Assume that X, Y, and Z are sets, T is a dense subset of $\overline{\mathbb{R}}$, and $(U_t)_{t\in T}$ and $(V_t)_{t\in T}$ are increasing families of subsets of $X \times Z$ and $Z \times Y$, respectively. Then there are unique functions $f : X \times Z \to \overline{\mathbb{R}}$, $g : Z \times Y \to \overline{\mathbb{R}}$ and $h : X \times Y \to \overline{\mathbb{R}}$ such that*

$$\{f < t\} \subset U_t \subset \{f \leq t\}, \quad \{g < t\} \subset V_t \subset \{g \leq t\},$$
$$\{h < t\} \subset V_t \circ U_t \subset \{h \leq t\} \quad (t \in T).$$

Moreover, the presentation holds:

$$h(x,\ y) = \inf\{f(x,\ z) \vee g(z,\ y) :\ z \in Z\}.$$

◁ The sought functions exist by 3.8.2. The claim of uniqueness is straightforward from 3.8.4. The presentation of h via f and g raises no doubts. ▷

9.5.12. DEFINITION. Let $f : X \times Z \to \overline{\mathbb{R}}$ and $g : Z \times Y \to \overline{\mathbb{R}}$. The function h, given by 9.5.11, is called the $\vee$-*convolution* (read: vel-convolution) of f and g and is denoted by

$$f \square_{\vee} g(x,\ y) := \inf\{f(x,\ z) \vee g(z,\ y) :\ z \in Z\}.$$

By analogy, the $+$-*convolution* of f and g is defined by the rule

$$f \square_{+} g(x,\ y) := \inf\{f(x,\ z) + g(z,\ y) :\ z \in Z\}.$$

9.5.13. DEFINITION. A mapping $f : X^2 \to \mathbb{R}^{\cdot}_{+}$ is a K-*ultrametric* with $K \in \mathbb{R}$, $K \geq 1$, if

(1) $f(x,\ x) = 0\ (x \in X)$;
(2) $f(x,\ y) = f(y,\ x)\ (x,\ y \in X)$;
(3) $^1/_K f(x,\ u) \leq f(x,\ y) \vee f(y,\ z) \vee f(z,\ u)\ (x,\ y,\ z,\ u \in X)$.

9.5.14. REMARK. Condition 9.5.13 (3) is often referred to as the (strong) *ultrametric inequality.* In virtue of 9.5.12 this inequality may be rewritten as $K^{-1}f \leq f \square_{\vee} f \square_{\vee} f$.

9.5.15. 2-*Ultrametric Lemma.* *To every 2-ultrametric $f : X^2 \to \mathbb{R}^{\cdot}_{+}$ there corresponds a semimetric d such that $^1/_2 f \leq d \leq f$.*

◁ Let $f_1 := f$ and $f_{n+1} := f_n \square_{+} f\ (n \in \mathbb{N})$. Then

$$f_{n+1}(x,\ y) \leq f_n(x,\ y) + f(y,\ y) = f_n(x,\ y) \quad (x,\ y \in X).$$

Thus, (f_n) is a decreasing sequence. Put

$$d(x,\ y) := \lim f_n(x,\ y) = \inf_{n \in \mathbb{N}} f_n(x,\ y).$$

Since

$$d(x,\ y) \leq f_{2n}(x,\ y) = f_n \square_{+} f_n(x,\ y) \leq f_n(x,\ z) + f_n(z,\ y),$$

for $n \in \mathbb{N}$, it follows that $d(x,\ y) \leq d(x,\ z) + d(z,\ y)$. The validity of 9.5.7 (1) and 9.5.7 (2) is immediate.

We are left with proving that $^1/_2 f \leq d$. To this end, show that $f_n \geq {}^1/_2 f$ for $n \in \mathbb{N}$. Proceed by way of induction.

The desired inequalities are obvious when $n := 1, 2$. Assume now that $f \geq f_1 \geq \dots \geq f_n \geq {}^1/_2 f$ and at the same time $f_{n+1}(x,\ y) < {}^1/_2 f(x,\ y)$ for some $(x,\ y)$ in X^2 and $n \geq 2$.

For suitable $z_1, \dots, z_n$ in X by construction

$$t := f(x,\ z_1) + f(z_1,\ z_2) + \dots + f(z_{n-1},\ z_n) + f(z_n,\ y) < {}^1/_2 f(x,\ y).$$

If $f(x,\ z_1) \geq {}^t/_2$ then ${}^t/_2 \geq f(z_1,\ z_2) + \dots + f(z_n,\ y) \geq {}^1/_2 f(z_1,\ y)$. It follows that $t \geq f(x,\ z_1)$ and $t \geq f(z_1,\ y)$. On account of 9.5.13 (3), ${}^1/_2 f(x,\ y) \leq f(x,\ z_1) \vee f(z_1,\ y) \leq t$. Whence we come to ${}^1/_2 f(x,\ y) > t \geq {}^1/_2 f(x,\ y)$, which is false.

Thus, $f(x,\ z_1) < {}^t/_2$. Find $m \in \mathbb{N},\ m < n$, satisfying

$$f(x,\ z_1) + \dots + f(z_{m-1},\ z_m) < {}^t/_2;$$

$$f(x,\ z_1) + \dots + f(z_m,\ z_{m+1}) \geq {}^t/_2.$$

This is possible, since assuming $m = n$ would entail the false inequality $f(z_n,\ y) \geq {}^t/_2$. (We would have ${}^t/_2 \geq f(x,\ z_1) + \dots + f(z_{n-1},\ z_n) \geq {}^1/_2 f(x,\ z_n)$ and so ${}^1/_2 f(x,\ y) > t \geq f(x,\ z^n) \vee f(z_n,\ y) \geq {}^1/_2 f(x,\ y)$.)

We obtain the inequality

$$f(z_{m+1}, z_{m+2}) + \dots + f(z_{n-1},\ z_n) + f(z_n,\ y) < {}^t/_2.$$

Using the induction hypothesis, conclude that

$$f(x,\ z_m) \leq 2(f(x,\ z_1) + \dots + f(z_{m-1},\ z_m)) \leq t;$$
$$f(z_m,\ z_{m+1}) \leq t;$$
$$f(z_{m+1},\ y) \leq 2(f(z_{m+1},\ z_{m+2}) + \dots + f(z_n,\ y)) \leq t.$$

Consequently, by the definition of 2-ultrametric

$${}^1/_2 f(x,\ y) \leq f(x,\ z_m) \vee f(z_m,\ z_{m+1}) \vee f(z_{m+1},\ y) \leq t < {}^1/_2 f(x,\ y).$$

We arrive at a contradiction, completing the proof. ▷

9.5.16. ***Theorem.*** *Every uniform space is multimetrizable.*

◁ Let $(X,\ \mathcal{U}_X)$ be a uniform space. Take $V \in \mathcal{U}_X$. Put $V_1 := V \cap V^{-1}$. If now $V_n \in \mathcal{U}_X$ then find a symmetric entourage $\overline{V} = \overline{V}^{-1}$, a member of $\mathcal{U}_X$, satisfying $\overline{V} \circ \overline{V} \circ \overline{V} \subset V_n$. Define $V_{n+1} := \overline{V}$. Since by construction

$$V_n \supset V_{n+1} \circ V_{n+1} \circ V_{n+1} \supset V_{n+1} \circ I_X \circ I_X \supset V_{n+1};$$

therefore, $(V_n)_{n\in\mathbb{N}}$ is a decreasing family.

Given $t \in \mathbb{R}$, define a set U_t by the rule

$$U_t := \begin{cases} \varnothing, & t < 0 \\ I_X, & t = 0 \\ V_{\inf\{n\in\mathbb{N}:\, t \geq 2^{-n}\}}, & 0 < t < 1 \\ V_1, & t = 1 \\ X^2, & t > 1. \end{cases}$$

By definition, the family $t \mapsto U_t$ $(t \in \mathbb{R})$ increases. Consider a unique function $f : X^2 \to \overline{\mathbb{R}}$ satisfying the next conditions (cf. 3.8.2 and 3.8.4)

$$\{f < t\} \subset U_t \subset \{f \leq t\} \quad (t \in \mathbb{R}).$$

If $W_t := U_{2t}$ for $t \in \mathbb{R}$ then

$$U_s \circ U_s \circ U_s \subset W_t$$

for $s < t$. Consequently, in virtue of 3.8.3 and 9.2.1 the mapping f is a 2-ultrametric.

Using 9.5.15, find a semimetric d_V such that ${}^1/_2 f \leq d_V \leq f$. Clearly, $\mathscr{U}_{d_V} = \operatorname{fil}\{V_n : n \in \mathbb{N}\}$. Also, it is also beyond a doubt that $\mathscr{U}_{\mathfrak{M}} = \mathscr{U}_X$ for the multimetric $\mathfrak{M} := \{d_V : V \in \mathscr{U}_X\}$. ▷

9.5.17. Corollary. *A topological space is uniformizable if and only if it is a* $\mathrm{T}_{3^1/_2}$*-space.* ◁▷

9.5.18. Corollary. *A Tychonoff space is the same as a separated multimetric space.* ◁▷

9.6. Covers, and Partitions of Unity

9.6.1. DEFINITION. Let $\mathscr{E}$ and $\mathscr{F}$ be *covers* of a subset of U in X; i.e., $\mathscr{E}, \mathscr{F} \subset \mathscr{P}(X)$ and $U \subset (\cup\mathscr{E}) \cap (\cup\mathscr{F})$. It is said that $\mathscr{F}$ *coarsens* $\mathscr{E}$ or $\mathscr{E}$ *refines* $\mathscr{F}$, if each member of $\mathscr{E}$ is included in some member of $\mathscr{F}$; i.e., $(\forall E \in \mathscr{E})$ $(\exists F \in \mathscr{F})$ $E \subset F$. It is also said that $\mathscr{E}$ is a *refinement* of $\mathscr{F}$. Observe that if $\mathscr{F}$ is a *subcover* of $\mathscr{E}$ (i.e., $\mathscr{F} \subset \mathscr{E}$) then $\mathscr{E}$ refines $\mathscr{F}$.

9.6.2. DEFINITION. A cover $\mathscr{E}$ of a set X is called *locally finite* (with respect to a topology τ on X), if each point in X possesses a neighborhood (in the sense of τ) meeting only finite many members of $\mathscr{E}$. In the case of the discrete topology on X, such a cover is called *point finite.* If X is regarded as furnished with a prescribed topology τ then, speaking of a locally finite cover of X, we imply the topology τ.

9.6.3. *Lefschetz Lemma.* *Let $\mathscr{E}$ be a point finite open cover of a normal space X. Then there is an open cover $\{G_E : E \in \mathscr{E}\}$ such that $\mathrm{cl}\, G_E \subset E$ for all $E \in \mathscr{E}$.*

◁ Let the set S comprise the mappings $s : \mathscr{E} \to \mathrm{Op}(X)$ such that $\cup s(\mathscr{E}) = X$ and for $E \in \mathscr{E}$ either $s(E) = E$ or $\mathrm{cl}\, s(E) \subset E$. Given functions s_1 and s_2, put $s_1 \le s_2 := (\forall E \in \mathscr{E})\ (s_1(E) \ne E \Rightarrow s_2(E) = s_1(E))$. It is evident that $(S, \le)$ is an ordered set and $I_{\mathscr{E}} \in S$. Show that S is inductive.

Given a chain S_0 in S, for all $E \in \mathscr{E}$ put $s_0(E) := \cap\{s(E) : s \in S_0\}$. If $s_0(E) = E$ then $s(E) = E$ for all $s \in S_0$. If the case $s_0(E) \ne E$ observe that $s_0(E) = \cap\{s(E) : s(E) \ne E,\ s \in S_0\}$.

Since the order of S_0 is linear, infer that $s_0(E) = s(E)$ for $s \in S_0$ with $s(E) \ne E$. Hence, $s_0(\mathscr{E}) \subset \mathrm{Op}(X)$ and $s_0 \ge S_0$. It remains to verify that s_0 is a cover of X (and so $s_0 \in S$). By the hypothesis of point finiteness, given $x \in X$, there are some $E_1, \dots, E_n$ in $\mathscr{E}$ such that $x \in E_1 \cap \dots \cap E_n$ and $x \notin E$ for the other members E of $\mathscr{E}$. If $s(E_k) = E_k$ for some k, then there is nothing to prove, for $x \in \cup s_0(\mathscr{E})$. In the case when $s_0(E_k) \ne E_k$ for every k, there are $s_1, \dots, s_n \in S_0$ meeting the conditions $s_k(E_k) \ne E_k$ $(k := 1, 2 \dots, n)$. Since S_0 is a chain, it may be assumed that $s_n \ge \{s_1, \dots, s_{n-1}\}$. Moreover, $x \in s_n(\overline{E}) \subset \overline{E}$ for an appropriate $\overline{E}$ in $\mathscr{E}$. It is clear that $\overline{E} \in \{E_1, \dots, E_n\}$ (because $x \notin E$ for the other members E of $\mathscr{E}$). Since $s_0(\overline{E}) = s_n(\overline{E})$, it follows that $x \in s_0(\overline{E})$.

By the Kuratowski–Zorn Lemma there is a maximal element $\overline{s}$ in S. Take $E \in \mathscr{E}$. If $F := X \setminus \cup \overline{s}(\mathscr{E} \setminus \{E\})$, then F is closed and $\overline{s}(E)$ is a neighborhood of F. For a suitable G in $\mathrm{Op}(X)$ by 9.3.10 $F \subset G \subset \mathrm{cl}\, G \subset \overline{s}(E)$. Put $s(E) := G$ and $s(\overline{E}) := \overline{s}(\overline{E})$ for $\overline{E} \ne E$ $(\overline{E} \in \mathscr{E})$. It is clear that $s \in S$. If $\overline{s}(E) = E$, then $s \ge \overline{s}$ and so $s = \overline{s}$. Moreover, $\overline{s}(E) \subset \mathrm{cl}\, G \subset \overline{s}(E) = E$; i.e., $\mathrm{cl}\, \overline{s}(E) \subset E$. If $\overline{s}(E) \ne E$, then $\mathrm{cl}\, \overline{s}(E) \subset E$ by definition. Thus, $\overline{s}$ is a sought cover. ▷

9.6.4. DEFINITION. Let f be a *numeric* or *scalar-valued function* on a topological space X, i.e. $f : X \to \mathbb{F}$. The set $\mathrm{supp}(f) := \mathrm{cl}\,\{x \in X : f(x) \ne 0\}$ is the *support* of f. If $\mathrm{supp}(f)$ is a compact set then f is a *compactly-supported function* or a *function of compact support.* The designation $\mathrm{spt}(f) := \mathrm{supp}(f)$ is used sometimes.

9.6.5. *Let $(f_e)_{e \in \mathscr{E}}$ be a family of numeric functions on X and let $\overline{\mathscr{E}} := \{\mathrm{supp}(f_e) : e \in \mathscr{E}\}$ be the family of their supports. If $\overline{\mathscr{E}}$ is a point finite cover of X then the family $(f_e)_{e \in \mathscr{E}}$ is summable pointwise. If in addition $\overline{\mathscr{E}}$ is locally finite and every member of $(f_e)_{e \in \mathscr{E}}$ is continuous, then the sum $\sum_{e \in \mathscr{E}} f_e$ is also continuous.*

◁ It suffices to observe that in a suitable neighborhood about a point in X only finitely many members of the family $(f_e)_{e \in \mathscr{E}}$ are distinct from zero. ▷

9.6.6. DEFINITION. It is said that a family of functions $(f : X \to [0,\ 1])_{f \in F}$ is a *partition of unity* on a subset U of X, if the supports of the members of the family composes a point finite cover of X, and $\sum_{f\in F} f(x) = 1$ for all $x \in U$. The empty family of functions in this context is treated as summable to unity at each point. The term "continuous partition of unity" and the like are understood naturally.

9.6.7. DEFINITION. Let $\mathscr{E}$ be a cover of a subset U of a topological space and let F be a continuous partition of unity on U. If the family of supports $\{\operatorname{supp}(f) : f \in F\}$ refines $\mathscr{E}$ then F is a *partition of unity subordinate to* $\mathscr{E}$. A possibility of finding such an F for $\mathscr{E}$ is also verbalized as follows: "$\mathscr{E}$ admits a partition of unity."

9.6.8. *Each locally finite open cover of a normal space admits a partition of unity.*

◁ By the Lefschetz Lemma, such a cover $\{U_\xi : \xi \in \Xi\}$ has an open refinement $\{V_\xi : \xi \in \Xi\}$ satisfying the condition $\operatorname{cl} V_\xi \subset U_\xi$ for all $\xi \in \Xi$. By the Urysohn Theorem, there is a continuous function $g_\xi : X \to [0,\ 1]$ such that $g_\xi(x) = 1$ for $x \in V_\xi$ and $g_\xi(x) = 0$ for $x \in X \setminus U_\xi$. Consequently, $\operatorname{supp}(g_\xi) \subset U_\xi$. In virtue of 9.6.5 the family $(g_\xi)_{\xi\in\Xi}$ is summable pointwise to a continuous function g. Moreover, $g(x) > 0$ for all $x \in X$ by construction. Put $f_\xi := g_\xi/g$ $(\xi \in \Xi)$. The family $(f_\xi)_{\xi\in\Xi}$ is what we need. ▷

9.6.9. DEFINITION. A topological space X is called *paracompact*, if each cover of X has a locally finite open refinement.

9.6.10. REMARK. The theory of paracompactness contains deep and surprising facts.

9.6.11. ***Theorem.*** *Every metric space is paracompact.*

9.6.12. ***Theorem.*** *A Hausdorff topological space is paracompact if and only if its every open cover admits a partition of unity.*

9.6.13. REMARK. The metric space $\mathbb{R}^N$ possesses a number of additional structures providing a stock of well-behaved, *smooth* (= infinitely differentiable) functions (cf. 4.8.1).

9.6.14. DEFINITION. A *mollifier* or a *mollifying kernel* on $\mathbb{R}^N$ is a real-valued smooth function a having unit (Lebesque) integral and such that $a(x) > 0$ for $|x| < 1$ and $a(x) = 0$ for $|x| \geq 1$. In this event, $\operatorname{supp}(a) = \{x \in \mathbb{R}^N : |x| \leq 1\}$ is the (unit Euclidean) *ball* $\mathbb{B} := B_{\mathbb{R}^N}$.

9.6.15. DEFINITION. A *delta-like sequence* is a family of real-valued (smooth) functions $(b_\varepsilon)_{\varepsilon>0}$ such that, first, $\lim_{\varepsilon\to 0}(\sup|\operatorname{supp}(b_\varepsilon)|) = 0$ and, second, the equality holds $\int_{\mathbb{R}^N} b_\varepsilon(x)\,dx = 1$ for all $\varepsilon > 0$. The terms "δ-sequence" and "δ-like sequence" are also in current usage. Such a sequence is often assumed countable without further specification.

9.6.16. EXAMPLE. The function $a(x) := t\exp(-(|x|^2 - 1)^{-1})$ is taken as a most popular mollifier when extended by zero beyond the *open ball* $\operatorname{int}\mathbb{B}$, with the constant t determined from the condition $\int_{\mathbb{R}^N} a(x)\,dx = 1$. Each mollifier generates the delta-like sequence $a_\varepsilon(x) := \varepsilon^{-N} a(x/\varepsilon)$ $(x \in \mathbb{R}^N)$.

9.6.17. DEFINITION. Let $f \in L_{1,\mathrm{loc}}(\mathbb{R}^N)$; i.e., let f be a *locally integrable function*, that is, a function whose restriction to each compact subset of $\mathbb{R}^N$ is integrable. For a compactly-supported integrable function g the *convolution* $f * g$ is defined as

$$f * g(x) := \int_{\mathbb{R}^N} f(x-y)g(y)\,dy \quad (x \in \mathbb{R}^N).$$

9.6.18. REMARK. The role of a mollifying kernel and the corresponding delta-like sequence $(a_\varepsilon)_{\varepsilon>0}$ becomes clear from inspecting the aftermath of applying the *smoothing process* $f \mapsto (f * a_\varepsilon)_{\varepsilon>0}$ to a function f belonging to $L_{1,\mathrm{loc}}(\mathbb{R}^N)$ (cf. 10.10.7 (5)).

9.6.19. *The following statements are valid:*

(1) *to every compact set K in the space $\mathbb{R}^N$ and every neighborhood U of K there corresponds a truncator (= a bump function) $\psi := \psi_{K,U}$, i.e. a smooth mapping $\psi : \mathbb{R}^N \to [0,\ 1]$ such that $K \subset \operatorname{int}\{\psi = 1\}$ and $\operatorname{supp}(\psi) \subset U$;*

(2) *assume that $U_1, \dots, U_n \in \operatorname{Op}(\mathbb{R}^N)$ and $U_1 \cup \dots \cup U_n$ is a neighborhood of a compact set K; there are smooth functions $\psi_1, \dots, \psi_n : \mathbb{R}^N \to [0,\ 1]$ such that $\operatorname{supp}(\psi_k) \subset U_k$ and $\sum_{k=1}^n \psi_k(x) = 1$ for x in some neighborhood of K.*

◁ (1) Put $\varepsilon := d(K,\ \mathbb{R}^N \setminus U) := \inf\{|x-y| :\ x \in K,\ y \notin U\}$. It is clear that $\varepsilon > 0$. Given $\beta > 0$, denote the characteristic function of $K + \varepsilon\mathbb{B}$ by χ_β. Take a delta-like sequence $(b_\gamma)_{\gamma>0}$ of positive functions and put $\psi := \chi_\beta * b_\gamma$. When $\overline{\gamma} \le \beta$ and $\beta + \overline{\gamma} \le \varepsilon$ with $\overline{\gamma} := \sup|\operatorname{supp}(b_\gamma)|$, observe that ψ is a sought function.

(2) By the Diedonnè Lemma there are closed sets F_k, with $F_k \subset U_k$, composing a cover of K. Put $K_k := F_k \cap K$ and choose some truncators $\psi_k := \psi_{K_k,U_k}$. The functions $\psi_k / \sum_{k=1}^n \psi_k$ $(k := 1, \dots, n)$, defined on $\{\sum_{k=1}^n \psi_k > 0\}$, meet the claim after extension by zero onto $\{\sum_{k=1}^n \psi_k = 0\}$ and multiplication by a truncator corresponding to an appropriate neighborhood of K. ▷

9.6.20. *Countable Partition Theorem.* *Let $\mathscr{E}$ be a family of open sets in $\mathbb{R}^N$ and $\Omega := \cup\mathscr{E}$. There is a countable partition of unity which is composed of smooth compactly-supported functions on $\mathbb{R}^N$ and subordinate to the cover $\mathscr{E}$ of Ω.*

◁ Refine from $\mathscr{E}$ a countable locally finite cover A of Ω with compact sets so that the family $(\overline{\alpha} := \operatorname{int}\alpha)_{\alpha\in\mathrm{A}}$ be also an open cover of Ω. Choose an open cover $(V_\alpha)_{\alpha\in\mathrm{A}}$ of Ω from the condition $\operatorname{cl} V_\alpha \subset \overline{\alpha}$ for $\alpha \in \mathrm{A}$. In virtue of 9.6.19 (1) there are truncators $\overline{\psi}_\alpha := \psi_{\operatorname{cl} V_\alpha, \overline{\alpha}}$. Putting $\psi_\alpha(x) := \overline{\psi}_\alpha(x) / \sum_{\alpha\in\mathrm{A}} \overline{\psi}_\alpha(x)$ for $x \in \Omega$ and $\psi_\alpha(x) := 0$ for $x \in \mathbb{R}^N \setminus \Omega$, arrive to a sought partition. ▷

9.6.21. REMARK. It is worth observing that the so-constructed partition of unity $(\psi_\alpha)_{\alpha\in \mathrm{A}}$ possesses the property that to each compact subset K of Ω there correspond a finite subset A_0 of A and a neighborhood U of K such that $\sum_{\alpha\in \mathrm{A}_0} \psi_\alpha(x) = 1$ for all $x \in U$ (cf. 9.3.17 and 9.6.19 (2)).

Exercises

9.1. Give examples of pretopological and topological spaces and constructions leading to them.

9.2. Is it possible to introduce a topology by indicating convergent filters or sequences?

9.3. Establish relations between topologies and preorders on a finite set.

9.4. Describe topological spaces in which the union of every family of closed sets is closed. What are the continuous mappings between such spaces?

9.5. Let $(f_\xi : X \to (Y_\xi,\ \tau_\xi))_{\xi\in\Xi}$ be a family of mappings. A topology σ on X is called admissible (in the case under study), if for every topological space $(Z,\ \omega)$ and every mapping $g : Z \to X$ the following statement holds: $g : (Z,\ \omega) \to (X,\ \sigma)$ is continuous if and only if so is each mapping $f_\xi \circ g$ $(\xi \in \Xi)$. Demonstrate that the weakest topology on X making every f_ξ $(\xi \in \Xi)$ continuous is the strongest admissible topology (in the case under study).

9.6. Let $(f_\xi : (X_\xi,\ \sigma_\xi) \to Y)_{\xi\in\Xi}$ be a family of mappings. A topology τ on Y is called admissible (in the case under study), if for every topological space $(Z,\ \omega)$ and every mapping $g : Y \to Z$ the following statement is true: $g : (Y,\ \tau) \to (Z,\ \omega)$ is continuous if and only if each mapping $g \circ f_\xi$ $(\xi \in \Xi)$ is continuous. Demonstrate that the strongest topology on Y making every f_ξ $(\xi \in \Xi)$ continuous is the weakest admissible topology (in the case under study).

9.7. Prove that in the Tychonoff product of topological spaces, the closure of the product of subsets of the factors is the product of closures:

$$\mathrm{cl}\left(\prod_{\xi\in\Xi} A_\xi\right) = \prod_{\xi\in\Xi} \mathrm{cl}\ A_\xi.$$

9.8. Show that a Tychonoff product is a Hausdorff space if and only if so is every factor.

9.9. Establish compactness criteria for subsets of classical Banach spaces.

9.10. A Hausdorff space X is called *H-closed*, if X is closed in every ambient Hausdorff space. Prove that a regular H-closed space is compact.

9.11. Study possibilities of compactifying a topological space.

9.12. Prove that the Tychonoff product of uncountably many real axes fails to be a normal space.

9.13. Show that each continuous function on the product of compact spaces depends on at most countably many coordinates in an evident sense (specify it!).

9.14. Let A be a compact subset and let B be a closed subset of a uniform space, with $A \cap B = \varnothing$. Prove that $V(A) \cap V(B) = \varnothing$ for some entourage V.

9.15. Prove that a completion (in an appropriate sense) of the product of uniform spaces is uniformly homeomorphic (specify!) to the product of completions of the factors.

9.16. A subset A of a separated uniform space is called ***precompact*** if a completion of A is compact. Prove that a set is precompact if and only if it is totally bounded.

9.17. Which topological spaces are metrizable?

9.18. Given a uniformizable space, describe the strongest uniformity among those inducing the initial topology.

9.19. Verify that the product of a paracompact space and a compact space is paracompact. Is paracompactness preserved under general products?

Chapter 10
Duality and Its Applications

10.1. Vector Topologies

10.1.1. DEFINITION. Let $(X,\ \mathbb{F},\ +,\ \cdot)$ be a vector space over a basic field $\mathbb{F}$. A topology τ on X is a *topology compatible with vector structure* or, briefly, a *vector topology*, if the following mappings are continuous:

$$+ : (X \times X,\ \tau \times \tau) \to (X,\ \tau),$$
$$\cdot : (\mathbb{F} \times X,\ \tau_{\mathbb{F}} \times \tau) \to (X,\ \tau).$$

The space $(X,\ \tau)$ is then referred to as a *topological vector space.*

10.1.2. *Let τ_X be a vector topology. The mappings*

$$x \mapsto x + x_0, \quad x \mapsto \alpha x \quad (x_0 \in X,\ \alpha \in \mathbb{F} \setminus 0)$$

are topological isomorphisms in $(X,\ \tau_X)$. ◁▷

10.1.3. REMARK. It is beyond a doubt that a vector topology τ on a space X possesses the next "linearity" property:

$$\tau(\alpha x + \beta y) = \alpha\tau(x) + \beta\tau(y) \quad (\alpha,\ \beta \in \mathbb{F} \setminus 0;\ x,\ y \in X),$$

where in accordance with the general agreements (cf. 1.3.5 (1))

$$U_{\alpha x + \beta y} \in \alpha\tau(x) + \beta\tau(y)$$
$$\Leftrightarrow (\exists\, U_x \in \tau(x)\ \&\ U_y \in \tau(y))\ \alpha U_x + \beta U_y \subset U_{\alpha x + \beta y}.$$

In this regard a vector topology is often called a *linear topology* and a topological vector space, a *linear topological space.* This terminology should be used on the understanding that a topology may possess the "linearity" property while failing to be linear. For instance, such is the discrete topology of a nonzero vector space.

10.1.4. Theorem. *Let X be a vector space and let $\mathscr{N}$ be a filter on X. There is a vector topology τ on X such that $\mathscr{N} = \tau(0)$ if and only if the following conditions are fulfilled:*

(1) $\mathscr{N} + \mathscr{N} = \mathscr{N}$;

(2) *$\mathscr{N}$ consists of absorbing sets;*

(3) *$\mathscr{N}$ has a base of balanced sets.*

Moreover, $\tau(x) = x + \mathscr{N}$ for all $x \in X$.

$\triangleleft$ $\Rightarrow$: Let τ be a vector topology and $\mathscr{N} = \tau(0)$. From 10.1.2 infer that $\tau(x) = x + \mathscr{N}$ for $x \in X$. It is also clear that (1) reformulates the continuity property of addition at zero (of the space X^2). Condition (2) may be rewritten as $\tau_{\mathbb{F}}(0)x \supset \mathscr{N}$ for every x in X, which is the continuity property of the mapping $\alpha \mapsto \alpha x$ at zero (of the space $\mathbb{R}$) for every fixed x in X. Condition (3) with account taken of (2) may in turn be rendered in the form $\tau_{\mathbb{F}}(0)\mathscr{N} = \mathscr{N}$, which is the continuity property of scalar multiplication at zero (of the space $\mathbb{F} \times X$).

$\Leftarrow$: Let $\mathscr{N}$ be a filter satisfying (1)–(3). It is evident that $\mathscr{N} \subset \operatorname{fil}\{0\}$. Put $\tau(x) := x + \mathscr{N}$. Then τ is a pretopology. From the definition of τ and (1) it follows that τ is a topology, with every translation continuous and addition continuous at zero in X^2. Thus, addition is continuous at every point of X^2. The validity of (2) and (3) means that the mapping $(\lambda,\ x) \mapsto \lambda x$ is jointly continuous at zero and continuous at zero in the first argument with the second argument fixed. By virtue of the identity

$$\lambda x - \lambda_0 x_0 = \lambda_0(x - x_0) + (\lambda - \lambda_0)x_0 + (\lambda - \lambda_0)(x - x_0),$$

we are left with examining the continuity property of scalar multiplication at zero in the second argument with the first argument fixed. In other words, it is necessary to show that $\lambda\mathscr{N} \supset \mathscr{N}$ for $\lambda \in \mathbb{F}$. With this in mind, find $n \in \mathbb{N}$ such that $|\lambda| \leq n$. Let V in $\mathscr{N}$ and W in $\mathscr{N}$ be such that W is balanced and $W_1 + \ldots + W_n \subset V$, where $W_k := W$. Then $\lambda W = n\left({}^{\lambda}/{}_{n}W\right) \subset nW \subset W_1 + \ldots + W_n \subset V$. $\triangleright$

10.1.5. Theorem. *The set $\mathrm{VT}(X)$ of all vector topologies on X presents a complete lattice. Moreover,*

$$\sup\nolimits_{\mathrm{VT}(X)} \mathscr{E} = \sup\nolimits_{\mathrm{T}(X)} \mathscr{E}$$

for every subset $\mathscr{E}$ of $\mathrm{VT}(X)$.

$\triangleleft$ Let $\overline{\tau} := \sup_{\mathrm{T}(X)} \mathscr{E}$. Since for $\tau \in \mathscr{E}$ each translation by a vector is a topological isomorphism in $(X,\ \tau)$; therefore, this mapping is a topological isomorphism in $(X,\ \overline{\tau})$. Using 9.1.13, observe that the filter $\overline{\tau}(0)$ meets conditions 10.1.4 (1)–10.1.4 (3), since these conditions are fulfilled for every filter $\tau(0)$ with $\tau \in \mathscr{E}$. It remains to refer to 1.2.14. $\triangleright$

10.1.6. Theorem. *The inverse image of a vector topology under a linear operator is a vector topology.*

◁ Take $T \in \mathscr{L}(X, Y)$ and $\omega \in \mathrm{VT}(Y)$. Put $\tau := T^{-1}(\omega)$. If $x_\gamma \to x$ and $y_\gamma \to y$ in (X, τ) then by 9.2.8 $Tx_\gamma \to Tx$ and $Ty_\gamma \to Ty$. So $T(x_\gamma + y_\gamma) \to T(x + y)$. This means in virtue of 9.2.10 that $x_\gamma + y_\gamma \to x + y$ in (X, τ). Thus, $\tau(x) = x + \tau(0)$ for all $x \in X$ and, moreover, $\tau(0) + \tau(0) = \tau(0)$. Successively applying 3.4.10 and 3.1.8 to the linear correspondence T^{-1}, observe that the filter $\tau(0) = T^{-1}(\omega(0))$ consists of absorbing sets and has a base of balanced sets. The reason is as follows: by 10.1.4 the filter $\omega(0)$ possesses these two properties. Once again using 10.1.4, conclude that $\tau \in \mathrm{VT}(X)$. ▷

10.1.7. *The product of vector topologies is a vector topology.*

◁ Immediate from 10.1.5 and 10.1.6. ▷

10.1.8. Definition. Let A and B be subsets of a vector space. It is said that A is *B-stable* if $A + B \subset A$.

10.1.9. *To every vector topology τ on X there corresponds a unique uniformity $\mathscr{U}_\tau$ having a base of I_X-stable sets and such that $\tau = \tau(\mathscr{U}_\tau)$.*

◁ Given $U \in \tau(0)$, put $V_U := \{(x, y) \in X^2 : y - x \in U\}$. Observe the obvious properties:

$$I_X \subset V_U; \quad V_U + I_X = V_U; \quad (V_U)^{-1} = V_{-U};$$
$$V_{U_1 \cap U_2} \subset V_{U_1} \cap V_{U_2}; \quad V_{U_1} \circ V_{U_2} \subset V_{U_1 + U_2}$$

for all U, U_1, $U_2 \in \tau(0)$. Using 10.1.4, infer that $\mathscr{U}_\tau := \mathrm{fil}\,\{V_U : U \in \tau(0)\}$ is a uniformity and $\tau = \tau(\mathscr{U}_\tau)$. It is also beyond a doubt that $\mathscr{U}_\tau$ has a base of I_X-stable sets.

If now $\mathscr{U}$ is another uniformity such that $\tau(\mathscr{U}) = \tau$, and W is some I_X-stable entourage in $\mathscr{U}$; then $W = V_{W(0)}$. Whence the sought uniqueness follows. ▷

10.1.10. Definition. Let (X, τ) be a topological vector space. The uniformity $\mathscr{U}_\tau$, constructed in 10.1.9, is the *uniformity* of X.

10.1.11. Remark. Considering a topological vector space, we assume it to be furnished with the corresponding uniformity without further specification.

10.2. Locally Convex Topologies

10.2.1. Definition. A vector topology is *locally convex* if the neighborhood filter of each point has a base of convex sets.

10.2.2. Theorem. *Let X be a vector space and let $\mathscr{N}$ be a filter on X. There is a locally convex topology τ on X such that $\mathscr{N} = \tau(0)$ if and only if*

(1) $^1/_2\mathscr{N} = \mathscr{N}$;

(2) *$\mathscr{N}$ has a base of absorbing absolutely convex sets.*

◁ ⇒: By virtue of 10.1.2 the mapping $x \mapsto 2x$ is a topological isomorphism. This means that $^1/_2\mathscr{N} = \mathscr{N}$. Now take $U \in \mathscr{N}$. By hypothesis there is a convex set V in $\mathscr{N}$ such that $V \subset U$. Applying 10.1.4, find a balanced set W satisfying $W \subset V$. Using the Motzkin formula and 3.1.14, show that the convex hull $\operatorname{co}(W)$ is absolutely convex. Moreover, $W \subset \operatorname{co}(W) \subset V \subset U$.

⇐: An absolutely convex set is balanced. Consequently, $\mathscr{N}$ satisfies 10.1.4 (2) and 10.1.4 (3). If $V \in \mathscr{N}$ and W is a convex set, $W \in \mathscr{N}$ and $W \subset V$; then $^1/_2 W \in \mathscr{N}$. Furthermore, $^1/_2 W + {}^1/_2 W \subset W \subset V$ because of the convexity property of W. This means that $\mathscr{N} + \mathscr{N} = \mathscr{N}$. It remains to refer to 10.1.4. ▷

10.2.3. *Corollary.* *The set* $\operatorname{LCT}(X)$ *of all locally convex topologies on* X *is a complete lattice. Moreover,*

$$\sup\nolimits_{\operatorname{LCT}(X)} \mathscr{E} = \sup\nolimits_{\operatorname{T}(X)} \mathscr{E}$$

for every subset $\mathscr{E}$ *of* $\operatorname{LCT}(X)$. ◁▷

10.2.4. *Corollary.* *The inverse image of a locally convex topology under a linear operator is a locally convex topology.* ◁▷

10.2.5. *Corollary.* *The product of locally convex topologies is a locally convex topology.* ◁▷

10.2.6. *The topology of a multinormed space is locally convex.* ◁▷

10.2.7. DEFINITION. Let τ be a locally convex topology on X. The set of all everywhere-defined continuous seminorms on X is called the *mirror* (rarely, the *spectrum*) of τ and is denoted by $\mathfrak{M}_\tau$. The multinormed space $(X,\ \mathfrak{M}_\tau)$ is called *associated* with $(X,\ \tau)$.

10.2.8. *Theorem.* *Each locally convex topology coincides with the topology of the associated multinormed space.*

◁ Let τ be a locally convex topology on X and let $\omega := \tau(\mathfrak{M}_\tau)$ be the topology of the associated space $(X,\ \mathfrak{M}_\tau)$. Take $V \in \tau(0)$. By 10.2.2 there is an absolutely convex neighborhood B of zero, $B \in \tau(0)$, such that $B \subset V$. In virtue of 3.8.7

$$\{p_B < 1\} \subset B \subset \{p_B \le 1\}.$$

It is obvious that p_B is continuous (cf. 7.5.1); i.e., $p_B \in \mathfrak{M}_\tau$, and so $\{p_B < 1\} \in \omega(0)$. Consequently, $V \in \omega(0)$. Using 5.2.10, infer that $\omega(x) = x + \omega(0) \supset x + \tau(0) = \tau(x)$; i.e., $\omega \ge \tau$. Furthermore, $\tau \ge \omega$ by definition. ▷

10.2.9. DEFINITION. A vector space, endowed with a separated locally convex topology, is a *locally convex space.*

10.2.10. REMARK. Theorem 10.2.8 in slightly trimmed form is often verbalized as follows: "The concept of locally convex space and the concept of separated multinormed space have the same scope." For that reason, the terminology connected with the associated multinormed space is lavishly applied to studying a locally convex space (cf. 5.2.13).

10.2.11. DEFINITION. Let τ be a locally convex topology on X. The symbol $(X,\ \tau)'$ (or, in short, X') denotes the subspace of $X^{\#}$ that comprises all continuous linear functionals. The space $(X,\ \tau)'$ is the *dual* (or τ-*dual*) of $(X,\ \tau)$.

10.2.12. $(X,\ \tau)' = \cup\{|\partial|(p) :\ p \in \mathfrak{M}_\tau\}$. ◁▷

10.2.13. ***Prime Theorem.*** *The prime mapping* $\tau \mapsto (X,\ \tau)'$ *from* LCT(X) *to* Lat$(X^{\#})$ *preserves suprema; i.e.,*

$$(X,\ \sup \mathscr{E})' = \sup\{(X,\ \tau)' :\ \tau \in \mathscr{E}\}$$

for every subset $\mathscr{E}$ *of* LCT(X).

◁ If $\mathscr{E} = \varnothing$ then $\sup \mathscr{E}$ is the trivial topology $\tau_\circ$ of X and, consequently, $(X,\ \tau_\circ)' = 0 = \inf \mathrm{Lat}\,(X^{\#}) = \sup_{\mathrm{Lat}\,(X^{\#})} \varnothing$. By virtue of 9.2.7 the prime mapping increases. Given a nonempty $\mathscr{E}$, from 2.1.5 infer that

$$(X,\ \sup \mathscr{E})' \geq \sup\{(X,\ \tau)' :\ \tau \in \mathscr{E}\}.$$

If $f \in (X,\ \sup \mathscr{E})'$, then in view of 10.2.12 and 9.1.13 there are topologies $\tau_1, \dots, \tau_n \in \mathscr{E}$ such that $f \in (X,\ \tau_1 \vee \ldots \vee \tau_n)'$. Using 10.2.12 and 5.3.7, find $p_1 \in \mathfrak{M}_{\tau_1}, \dots, p_n \in \mathfrak{M}_{\tau_n}$ satisfying $f \in |\partial|(p_1 \vee \ldots \vee p_n)$. Recalling 3.5.7 and 3.7.9, observe that $|\partial|(p_1 + \ldots + p_n) = |\partial|(p_1) + \ldots + |\partial|(p_n)$. Finally,

$$f \in (X,\ \tau_1)' + \ldots + (X,\ \tau_n)' = (X,\ \tau_1)' \vee \ldots \vee (X,\ \tau_n)'. \ ▷$$

10.3. Duality Between Vector Spaces

10.3.1. DEFINITION. Let X and Y be vector spaces over the same ground field $\mathbb{F}$. Assume further that there is fixed a bilinear form (or, as it is called sometimes, a *bracketing*) $\langle \cdot \,|\, \cdot \rangle$ acting from $X \times Y$ to $\mathbb{F}$, i.e. a mapping linear in each of its arguments. Given $x \in X$ and $y \in Y$, put

$$\langle x \,| : y \mapsto \langle x \,|\, y \rangle, \quad \langle \cdot \,| : X \to \mathbb{F}^Y, \quad \langle X \,| \subset Y^{\#};$$
$$|\, y \rangle : x \mapsto \langle x \,|\, y \rangle, \quad |\, \cdot \rangle : Y \to \mathbb{F}^X, \quad |\, Y \rangle \subset X^{\#}.$$

The mappings $\langle \cdot \,|$ and $|\, \cdot \rangle$ are the *bra-mapping* and the *ket-mapping* of the initial bilinear form. By analogy, a member of $\langle X \,|$ is a *bra-functional* on X and a member of $|\, Y \rangle$ is a *ket-functional* on Y.

10.3.2. *The bra-mapping and the ket-mapping are linear operators.* ◁▷

10.3.3. DEFINITION. A bracketing of some vector spaces X and Y is a *pairing*, if its bra-mapping and ket-mapping are monomorphisms. In this case we say that X and Y are (set) in duality, or present a *duality pair*, or that Y is the *pair-dual* of X, etc. This is written down as $X \leftrightarrow Y$. Each of the bra-mapping and the ket-mapping is then referred to as *dualization*. For suggestiveness, the corresponding pairing of some spaces in duality is also called their *duality bracket*.

10.3.4. EXAMPLES.

(1) Let $X \leftrightarrow Y$ with duality bracket $\langle \cdot \mid \cdot \rangle$. Given $(y,\ x) \in Y \times X$, put $\langle y \mid x \rangle := \langle x \mid y \rangle$. It is immediate that the new bracketing is a pairing of Y and X. Moreover, the pairs of dualizations with respect to the old (direct) and new (reverse) duality brackets are the same. It thus stands to reason to draw no distinction between the two duality brackets unless in case of an emergency (cf. 10.3.3). For instance, Y is the pair-dual of X in the direct duality bracket if and only if X is the pair-dual of Y in the reverse duality bracket. Therefore, the hair-splitting is neglected and a unified term "pair-dual" is applied to each of the spaces in duality, with duality treated as a whole abstract phenomenon. Observe immediately that the mapping $\langle x \mid y \rangle_{\mathbb{R}} := \operatorname{Re} \langle x \mid y \rangle$ sets in duality the real carriers $X_{\mathbb{R}}$ and $Y_{\mathbb{R}}$. By way of taking liberties, the previous notation is sometimes reserved for the arising duality $X_{\mathbb{R}} \leftrightarrow Y_{\mathbb{R}}$; i.e., it is assumed that $\langle x \mid y \rangle := \langle x \mid y \rangle_{\mathbb{R}}$, on considering x and y as members of the real carriers.

(2) Let H be a Hilbert space. The inner product on H sets H and H_* in duality. The prime mapping is then coincident with the ket-mapping.

(3) Let $(X,\ \tau)$ be a locally convex space and let X' be the dual of X. The natural *evaluation mapping* $(x,\ x') \mapsto x'(x)$ sets X and X' in duality.

(4) Let X be a vector space and let $X^{\#} := \mathscr{L}(X,\ \mathbb{F})$ be the (algebraic) dual of X. It is clear that the evaluation mapping $(x,\ x^{\#}) \mapsto x^{\#}(x)$ sets the spaces in duality.

10.3.5. DEFINITION. Let $X \leftrightarrow Y$. The inverse image in X of the Tychonoff topology on $\mathbb{F}^Y$ under the bra-mapping, further denoted by $\sigma(X,\ Y)$, is the *bra-topology* or the *weak topology on* X induced by Y. The bra-topology $\sigma(Y,\ X)$ of $Y \leftrightarrow X$ is the *ket-topology* of $X \leftrightarrow Y$ or the *weak topology* on Y induced by X.

10.3.6. *The bra-topology is the weakest topology making every ket-functional continuous. The ket-topology is the weakest topology making every bra-functional continuous.*

$\lhd$ $x_\gamma \to x$ (in $\sigma(X,\ Y)$) $\Leftrightarrow$ $\langle x_\gamma \mid \to \langle x \mid$ (in $\mathbb{F}^Y$) $\Leftrightarrow$ $(\forall y \in Y)\ \langle x_\gamma \mid (y) \to \langle x \mid (y)$ $\Leftrightarrow$ $(\forall y \in Y)\ \langle x_\gamma \mid y \rangle \to \langle x \mid y \rangle$ $\Leftrightarrow$ $(\forall y \in Y)\ \mid y \rangle (x_\gamma) \to \mid y \rangle (x)$ $\Leftrightarrow$ $(\forall y \in Y)$ $x_\gamma \to x$ (in $\mid y \rangle^{-1}(\tau_{\mathbb{F}})$) $\rhd$

10.3.7. REMARK. The notation $\sigma(X,\ Y)$ agrees perfectly with that of the weak multinorm in 5.1.10 (4). Namely, $\sigma(X,\ Y)$ is the topology of the multinorm

$\{|\langle\cdot\,|\,y\rangle| :\ y \in Y\}$. Likewise, $\sigma(Y,\ X)$ is the topology of the multinorm $\{|\langle x\,|\,\cdot\rangle| :\ x \in X\}$. ◁▷

10.3.8. *The spaces $(X,\ \sigma(X,\ Y))$ and $(Y,\ \sigma(Y,\ X))$ are locally convex.*

◁ Immediate from 10.2.4 and 10.2.5. ▷

10.3.9. ***Dualization Theorem.*** *Each dualization is an isomorphism between the pair-dual and the weak dual of the pertinent member of a duality pair.*

◁ Consider a duality pair $X \leftrightarrow Y$. We are to prove exactness for the sequences

$$0 \to X \xrightarrow{\langle\cdot|} (Y,\ \sigma(Y,\ X))' \to 0; \quad 0 \to Y \xrightarrow{|\cdot\rangle} (X,\ \sigma(X,\ Y))' \to 0.$$

Since the ket-mapping of $X \leftrightarrow Y$ is the bra-mapping of $Y \leftrightarrow X$, it suffices to show that the first sequence is exact. The bra-mapping is a monomorphism by definition. Furthermore, from 10.2.13 and 10.3.6 it follows that

$$(Y,\ \sigma(Y,\ X))' = (Y,\ \sup\{\langle x\,|^{-1}(\tau_{\mathbb{F}}) :\ x \in X\})'$$
$$= \sup\{(Y,\ \langle x\,|^{-1}(\tau_{\mathbb{F}}))' :\ x \in X\} = \operatorname{lin}(\{(Y,\ f^{-1}(\tau_{\mathbb{F}}))' :\ f \in \langle X\,|\}) = \langle X\,|,$$

since in view of 5.3.7 and 2.3.12 $(Y,\ f^{-1}(\tau_{\mathbb{F}}))' = \{\lambda f :\ \lambda \in \mathbb{F}\}$ $(f \in Y^{\#})$. ▷

10.3.10. REMARK. Theorem 10.3.9 is often referred to as the *theorem on the general form of a weakly continuous functional.* Here a useful convention reveals itself: apply the base form "weak" when using objects and properties that are related to weak topologies. Observe immediately that, in virtue of 10.3.9, Example 10.3.4 (3) actually lists all possible duality brackets. That is why in what follows we act in accordance with 5.1.11, continuing the habitual use of the designation $(x,\ y) := \langle x\,|\,y\rangle$, since it leads to no misunderstanding. For the same reason, given a vector space X, we draw no distinction between the pair-dual of a space X and the weak dual of X. In other words, considering a duality pair $X \leftrightarrow Y$, we sometimes identify X with $(Y,\ \sigma(Y,\ X))'$ and Y with $(X,\ \sigma(X,\ Y))'$, which justifies writing $X' = Y$ and $Y' = X$.

10.3.11. REMARK. A somewhat obsolete convention relates to $X \leftrightarrow X'$ with X a normed space. The ket-topology $\sigma(X',\ X)$ is customarily called the *weak* topology* (read: weak-star topology) in X', which reflects the concurrent notation X^* for X'. The term "weak*" proliferates in a routine fashion elsewhere.

10.4. Topologies Compatible with Duality

10.4.1. DEFINITION. Take a duality pair $X \leftrightarrow Y$ and let τ be a locally convex topology on X. It is said that τ is *compatible with duality* (between X and Y by pairing $X \leftrightarrow Y$), provided that $(X,\ \tau)' = |\,Y\rangle$. A locally convex topology ω on Y is compatible with duality (by pairing $X \leftrightarrow Y$), if ω is compatible with duality (by pairing $Y \leftrightarrow X$); i.e., if the equality holds: $(Y,\ \omega)' = \langle X\,|$. A unified concise term "compatible topology" is also current in each of the above cases.

10.4.2. *Weak topologies are compatible.*

◁ Follows from 10.3.9. ▷

10.4.3. *Let $\tau(X, Y)$ stand for the least upper bound of the set of all locally convex topologies on X compatible with duality (between X and Y). Then the topology $\tau(X, Y)$ is also compatible.*

◁ Denote the set of all compatible topologies on X by $\mathscr{E}$. Theorem 10.2.13 readily yields the equalities

$$(X, \tau(X, Y))' = \sup\{(X, \tau)' : \tau \in \mathscr{E}\} = \sup\{|Y\rangle : \tau \in \mathscr{E}\} = |Y\rangle,$$

because $\mathscr{E}$ is nonempty by 10.4.2. ▷

10.4.4. Definition. The topology $\tau(X, Y)$, constructed in 10.4.3 (i.e., the finest locally convex topology on X compatible with duality by pairing $X \leftrightarrow Y$), is the *Mackey topology* (on X induced by $X \leftrightarrow Y$).

10.4.5. Mackey–Arens Theorem. *A locally convex topology τ on X is compatible with duality between X and Y if and only if*

$$\sigma(X, Y) \le \tau \le \tau(X, Y).$$

◁ By 10.2.13 the prime mapping $\tau \mapsto (X, \tau)'$ preserves suprema and, in particular, increases. Therefore, given τ in the interval of topologies, from 10.4.2 and 10.4.3 obtain

$$|Y\rangle = (X, \sigma(X, Y)) \subset (X, \tau)' \subset (X, \tau(X, Y))' = |Y\rangle.$$

The remaining claim is obvious. ▷

10.4.6. Mackey Theorem. *All compatible topologies have the same bounded sets in stock.*

◁ A stronger topology has fewer bounded sets. So, to prove the theorem it suffices to show that if some set U is weakly bounded in X (= is bounded in the bra-topology) then U is bounded in the Mackey topology.

Take a seminorm p from the mirror of the Mackey topology and demonstrate that $p(U)$ is bounded in $\mathbb{R}$. Put $X_0 := X/\ker p$ and $p_0 := p_{X/\ker p}$. By 5.2.14, p_0 is clearly a norm. Let $\varphi : X \to X_0$ be the coset mapping. It is beyond a doubt that $\varphi(U)$ is weakly bounded in (X_0, p_0). From 7.2.7 it follows that $\varphi(U)$ is bounded in the norm p_0. Since $p_0 \circ \varphi = p$; therefore, U is bounded in (X, p). ▷

10.4.7. Corollary. *Let X be a normed space. Then the Mackey topology $\tau(X, X')$ coincides with the initial norm topology on X.*

◁ It suffices to refer to the Kolmogorov Normability Criterion implying that the space X with the topology $\tau(X, X')$ finer than the original topology is normable. Appealing to 5.3.4 completes the proof. ▷

10.4.8. *Strict Separation Theorem.* *Let (X, τ) be a locally convex space. Assume further that K and V are nonempty convex subsets of X with K compact, V closed and $K \cap V = \varnothing$. Then there is a functional f, a member of $(X, \tau)'$, such that*

$$\sup \operatorname{Re} f(K) < \inf \operatorname{Re} f(V).$$

◁ A locally convex space is obviously a regular space. Since K is compact, it thus follows that, for an appropriate convex neighborhood of zero, say W, the set $U := K + W$ does not meet V (it suffices to consider the filterbases comprising all subsets of the form $K + \overline{W}$ and $V + \overline{W}$, with $\overline{W}$ a closed neighborhood of zero). By 3.1.10, U is convex. Furthermore, $K \subset \operatorname{int} U = \operatorname{core} U$. By the Eidelheit Separation Theorem, there is a functional l, a member of $(X_{\mathbb{R}})^{\#}$, such that the hyperplane $\{l = 1\}$ in $X_{\mathbb{R}}$ separates V from U and does not meet the core of U. Obviously, l is bounded above on W and so $l \in (X_{\mathbb{R}}, \tau)'$ by 7.5.1. If $f := \mathbb{R}\mathrm{e}^{-1} l$ then, of view of 3.7.5, $f \in (X, \tau)'$. It is clear that f is a sought functional. ▷

10.4.9. *Mazur Theorem.* *All compatible topologies have the same closed convex sets in stock.*

◁ A stronger topology has more closed sets. So, to prove the theorem it suffices in view of 10.4.5 to show that if U is a convex set closed in the Mackey topology then U is weakly closed. The last claim is beyond a doubt since by Theorem 10.4.8 U is the intersection of weakly closed sets of the form $\{\operatorname{Re} f \leq t\}$, with f a (weakly) continuous linear functional and $t \in \mathbb{R}$. ▷

10.5. Polars

10.5.1. DEFINITION. Let X and Y be sets and let $F \subset X \times Y$ be a correspondence. Given a subset U of X and a subset V of Y, put

$$\pi(U) := \pi_F(U) := \{y \in Y : F^{-1}(y) \supset U\};$$
$$\pi^{-1}(V) := \pi_F^{-1}(V) := \{x \in X : F(u) \supset V\}.$$

The set $\pi(U)$ is the (***direct***) ***polar*** of U (under F), and the set $\pi^{-1}(V)$ is the (***reverse***) ***polar*** of V (under F).

10.5.2. *The following statements are valid:*

(1) $\pi(u) := \pi(\{u\}) = F(u)$ *and* $\pi(U) = \bigcap_{u \in U} \pi(u)$;

(2) $\pi(\bigcup_{\xi \in \Xi} U_\xi) = \bigcap_{\xi \in \Xi} \pi(U_\xi)$;

(3) $\pi_F^{-1}(V) = \pi_{F^{-1}}(V)$;

(4) $U_1 \subset U_2 \Rightarrow \pi(U_1) \supset \pi(U_2)$;

(5) $U \times V \subset F \Rightarrow (V \subset \pi(U)$ & $U \subset \pi^{-1}(V)$;

(6) $U \subset \pi^{-1}(\pi(U))$. ◁▷

10.5.3. *Akilov Criterion.* *A subset U of X is the polar of some subset of Y if and only if given $x \in X \setminus U$ there is an element y in Y such that*

$$U \subset \pi^{-1}(y), \quad x \notin \pi^{-1}(y).$$

◁ ⇒: If $U = \pi^{-1}(V)$ then $U = \cap_{v \in V} \pi^{-1}(v)$ by 10.5.2 (1).

⇐: The inclusion $U \subset \pi^{-1}(y)$ means that $y \in \pi(U)$. Thus, by hypothesis $U = \cap_{y \in \pi(U)} \pi^{-1}(y) = \pi^{-1}(\pi(U))$. ▷

10.5.4. *Corollary.* *The set $\pi^{-1}(\pi(U))$ is the (inclusion) least polar greater than U.* ◁▷

10.5.5. Definition. The set $\pi_F^{-1}(\pi_F(U))$ is the *bipolar* of a subset U (under the correspondence F).

10.5.6. Examples.

(1) Let (X, σ) be an ordered set and let U be a subset of X. Then $\pi_\sigma(U)$ is the collection of all upper bounds of U (cf. 1.2.7).

(2) Let $(H, (\cdot, \cdot)_H)$ be a Hilbert space and $F := \{(x, y) \in H^2 : (x, y)_H = 0\}$. Then $\pi(U) = \pi^{-1}(U) = U^\perp$ for every subset U of H. The bipolar of U in this case coincides with the *closed linear span of U*, that is, the closure of the linear span of U.

(3) Let X be a normed space and let X' be the dual of X. Consider $F := \{(x, x') : x'(x) = 0\}$. Then $\pi(X_0) = X_0^\perp$ and $\pi^{-1}(\mathscr{X}_0) = {}^\perp\mathscr{X}_0$ for a subspace X_0 of X and a subspace $\mathscr{X}_0$ of X' (cf. 7.6.8). Moreover, $\pi^{-1}(\pi(X_0)) = \mathrm{cl}\, X_0$ by 7.5.14.

10.5.7. Definition. Let $X \leftrightarrow Y$. Put

$$\mathrm{pol} := \{(x, y) \in X \times Y : \ \mathrm{Re}\,\langle x \mid y\rangle \le 1\};$$
$$\mathrm{abs\ pol} := \{(x, y) \in X \times Y : \ |\langle x \mid y\rangle| \le 1\}.$$

To refer to direct or inverse polars under pol, we use the unified term "polar" (with respect to $X \leftrightarrow Y$) and the unified designations $\pi(U)$ and $\pi(V)$. In the case of the correspondence abs pol, we speak of *absolute polars* (with respect to $X \leftrightarrow Y$) and write U° and V° (for $U \subset X$ and $V \subset Y$).

10.5.8. *Bipolar Theorem.* *The bipolar $\pi^2(U) := \pi(\pi(U))$ is the (inclusion) least weakly closed conical segment greater than U.*

◁ Straightforward from 10.4.8 and the Akilov Criterion. ▷

10.5.9. *Absolute Bipolar Theorem.* *The absolute bipolar $U^{\circ\circ} := (U^\circ)^\circ$ is the (inclusion) least weakly closed absolutely convex set greater than U.*

◁ It suffices, first, to observe that the polar of a balanced set U coincides with the absolute polar of U and, second, to apply 10.5.8. ▷

10.6. Weakly Compact Convex Sets

10.6.1. *Let X be a locally convex real vector space and let $p : X \to \mathbb{R}$ be a continuous sublinear functional on X. Then the (topological) subdifferential $\partial(p)$ is compact in the topology $\sigma(X', X)$.*

◁ Put $Q := \prod_{x \in X}[-p(-x),\ p(x)]$ and endow Q with the Tychonoff topology. Evidently $\partial(p) \subset Q$, with the Tychonoff topology on Q and $\sigma(X', X)$ inducing the same topology on $\partial(p)$. It is beyond a doubt that the set $\partial(p)$ is closed in Q by the continuity of p. Taking note of the Tychonoff Theorem and 9.4.9, conclude that $\partial(p)$ is a $\sigma(X', X)$-compact set. ▷

10.6.2. *The balanced subdifferential of each continuous seminorm is weakly compact.* ◁▷

10.6.3. Theorem. *Let X be a real vector space. A subset U of $X^{\#}$ is the subdifferential of a (unique total) sublinear functional $s_U : X \to \mathbb{R}$ if and only if U is nonempty, convex and $\sigma(X^{\#}, X)$-compact.*

◁ $\Rightarrow$: Let $U = \partial(s_U)$ for some s_U. The uniqueness of s_U is ensured by 3.6.6. In view of 10.2.12 it is easy that the mirror of the Mackey topology $\tau(X, X^{\#})$ is the strongest multinorm on X (cf. 5.1.10 (2)). Whence we infer that the functional s_U is continuous with respect to $\tau(X, X^{\#})$. In virtue of 10.6.1 the set U is compact in $\sigma(X^{\#}, X)$. The convexity and nonemptiness of U are obvious.

$\Leftarrow$: Put $s_U(x) := \sup\{l(x) : l \in U\}$. Undoubtedly, s_U is a sublinear functional and $\operatorname{dom} s_U = X$. By definition, $U \subset \partial(s_U)$. If $l \in \partial(s_U)$ and $l \notin U$, then by the Strict Separation Theorem and the Dualization Theorem $s_U(x) < l(x)$ for some x in X. This is a contradiction. ▷

10.6.4. DEFINITION. The sublinear functional s_U, constructed in 10.6.3, is the *supporting function* of U. The term "support function" is also in current usage.

10.6.5. *Kreĭn–Milman Theorem.* *Each compact convex set in a locally convex space is the closed convex hull (= the closure of the convex hull) of the set of its extreme points.*

◁ Let U be such a subset of a space X. It may be assumed that the space X is real and $U \neq \varnothing$. By virtue of 9.4.12, U is compact with respect to the topology $\sigma(X, X')$. Since $\sigma(X, X')$ is induced in X by the topology $\sigma(X'^{\#}, X')$ on $X'^{\#}$; therefore, $U = \partial(s_U)$. Here (cf. 10.6.3) $s_U : X' \to \mathbb{R}$ acts by the rule $s_U(x') := \sup x'(U)$. By the Kreĭn–Milman Theorem in subdifferential form, the set $\operatorname{ext} U$ of the extreme points of U is not empty. The closure of the convex hull of $\operatorname{ext} U$ is a subdifferential by Theorem 10.6.3. Moreover, this set has s_U as its supporting function, thus coinciding with U (cf. 3.6.6). ▷

10.6.6. *Let $X \leftrightarrow Y$ and let S be a conical segment in X. Assume further that p_S is the Minkowski functional of S. The polar $\pi(S)$ is the inverse image* *of the (algebraic) subdifferential $\partial(p_S)$ under the ket-mapping; i.e.,*

$$\pi(S) = |\,\partial(p_S)\,\rangle_{\mathbb{R}}^{-1}.$$

If S is absolutely convex, then the absolute polar S° is the inverse image of the (algebraic) balanced subdifferential $|\partial|(p_S)$ under the ket-mapping; i.e.,

$$S^\circ = |\,|\partial|(p_S)\,\rangle^{-1}.$$

$\triangleleft$ If $y \in Y_{\mathbb{R}}$ and $y \in |\,\partial(p_S)\,\rangle_{\mathbb{R}}^{-1}$ then $|\,y\rangle_{\mathbb{R}}$ belongs to $\partial(p_S)$. Hence, $\mathrm{Re}\,\langle x\,|\,y\rangle = \langle x\,|\,y\rangle_{\mathbb{R}} = |\,y\rangle_{\mathbb{R}}(x) \le p_S(x) \le 1$ for $x \in S$, because $S \subset \{p_S \le 1\}$ by the Gauge Theorem. Consequently, $y \in \pi(S)$.

If, in turn, $y \in \pi(S)$ then $|\,y\rangle_{\mathbb{R}}$ belongs to $\partial(p_S)$. Indeed, $1 > p_S(\alpha^{-1}x)$ for all x in $X_{\mathbb{R}}$ and $\alpha > p_S(x)$; i.e., $\alpha^{-1}x \in \{p_S < 1\} \subset S$. Whence $\langle \alpha^{-1}x\,|\,y\rangle_{\mathbb{R}} = \mathrm{Re}\,\langle \alpha^{-1}x\,|\,y\rangle = \alpha^{-1}\mathrm{Re}\,\langle x\,|\,y\rangle \le 1$. Finally, observe that $|\,y\rangle_{\mathbb{R}}(x) \le \alpha$. Since α is arbitrary, this inequality means that $|\,y\rangle_{\mathbb{R}}(x) \le p_S(x)$. In other words, $y \in |\,\partial(p_s)\rangle_{\mathbb{R}}^{-1}$, which implies that $\pi(S) = |\,\partial(p_S)\rangle_{\mathbb{R}}^{-1}$. The remaining claim follows from the properties of the complexifier (cf. 3.7.3 and 3.7.9). $\triangleright$

10.6.7. *Alaoglu–Bourbaki Theorem.* *The polar of a neighborhood of zero of each compatible topology is a weakly compact convex set.*

$\triangleleft$ Let U be a neighborhood of zero in a space X and let $\pi(U)$ be the polar of U (with respect to $X \leftrightarrow X'$). Since $U \supset \{p \le 1\}$ for some continuous seminorm p, by 10.5.2 (4), $\pi(U) \subset \pi(\{p \le 1\}) = \pi(B_p) = B_p^\circ$. Using 10.6.6 and recalling that p is the Minkowski functional of B_p, obtain the inclusion $\pi(U) \subset |\partial|(p)$. By virtue of 10.6.2 the topological balanced subdifferential $|\partial|(p)$ is $\sigma(X',\ X)$-compact. By definition $\pi(U)$ is weakly closed. To infer the $\sigma(X',\ X)$-compactness of $\pi(U)$, it remains to appeal to 9.4.9. The convexity property of $\pi(U)$ is beyond a doubt. $\triangleright$

10.7. Reflexive Spaces

10.7.1. *Kakutani Criterion.* *A normed space is reflexive if and only if its unit ball is weakly compact.*

$\triangleleft$ $\Rightarrow$: Let X be reflexive, i.e. $''(X) = X''$. In other words, the image of X under the double prime mapping coincides with X''. Since the ball $B_{X''}$ is the polar of the ball $B_{X'}$ with respect to $X'' \leftrightarrow X'$; therefore, $B_{X''}$ is a $\sigma(X'',\ X')$-compact set by the Alaoglu–Bourbaki Theorem. It remains to observe that $B_{X''}$ is (the image under the double prime mapping of) B_X, and $\sigma(X,\ X')$ is (the inverse image under the double prime mapping of) $\sigma(X'',\ X')$.

$\Leftarrow$: Consider the duality pair $X'' \leftrightarrow X'$. By definition, the ball $B_{X''}$ presents the bipolar of B_X (more precisely, the bipolar of $(B_X)''$). Using the Absolute Bipolar Theorem and observing that the weak topology $\sigma(X,\ X')$ is induced in X

by the topology $\sigma(X'',\ X')$, conclude that $B_{X''} = B_X$ (because of the obvious convexity and closure properties of B_X, the latter following from compactness since X is separated). Thus, X is reflexive. ▷

10.7.2. *Corollary.* *A space X is reflexive if and only if every bounded closed convex set in X is weakly compact.* ◁▷

10.7.3. *Corollary.* *Every closed subspace of a reflexive space is reflexive.*

◁ By the Mazur Theorem, such a subspace and, hence, the unit ball of it are weakly closed. It thus suffices to apply the Kakutani Criterion twice. ▷

10.7.4. *Pettis Theorem.* *A Banach space and its dual are (or are not) reflexive simultaneously.*

◁ If X is reflexive then $\sigma(X',\ X)$ coincides with $\sigma(X',\ X'')$. Therefore, by the Alaoglu–Bourbaki Theorem, $B_{X'}$ is $\sigma(X',\ X'')$-compact. Consequently, X' is reflexive. If, in turn, X' is reflexive, then so is X'' by what was proven. However, X, as a Banach space, is a closed subspace of X''. Thus, X is reflexive by 10.7.3. ▷

10.7.5. *James Theorem.* *A Banach space is reflexive if and only if each continuous (real) linear functional attains its supremum on the unit ball of the space.*

10.8. The Space $C(Q,\ \mathbb{R})$

10.8.1. REMARK. Throughout Section 10.8 let Q stand for a nonempty Hausdorff compact space, denoting by $C(Q,\ \mathbb{R})$ the set of continuous real-valued functions on Q. Unless specified otherwise, $C(Q,\ \mathbb{R})$ is furnished with the natural pointwise algebraic operations and order and equipped with the sup-*norm* $\|\cdot\| := \|\cdot\|_\infty$ related to the Chebyshëv metric (cf. 4.6.8). Keeping this in mind, we treat the statements like "$C(Q,\ \mathbb{R})$ is a vector lattice," "$C(Q,\ \mathbb{R})$ is a Banach algebra," etc. Other structures, if ever introduced in $C(Q,\ \mathbb{R})$, are specified deliberately.

10.8.2. DEFINITION. A subset L of $C(Q,\ \mathbb{R})$ is a *sublattice* (in $C(Q,\ \mathbb{R})$) if $f_1 \vee f_2 \in L$ and $f_1 \wedge f_2 \in L$ for $f_1,\ f_2 \in L$, where, as usual,

$$f_1 \vee f_2(q) := f_1(q) \vee f_2(q),$$
$$f_1 \wedge f_2(q) := f_1(q) \wedge f_2(q) \quad (q \in Q).$$

10.8.3. REMARK. Observe that to be a sublattice in $C(Q,\ \mathbb{R})$ means more than to be a lattice with respect to the order induced from $C(Q,\ \mathbb{R})$.

10.8.4. EXAMPLES.

(1) $\varnothing$, $C(Q,\ \mathbb{R})$, and the closure of a sublattice.

(2) The intersection of each family of sublattices is also a sublattice.

(3) Let L be a sublattice and let Q_0 be a subset of Q. Put

$$L_{Q_0} := \{f \in C(Q,\ \mathbb{R}) :\ (\exists g \in L)\ g(q) = f(q)\ (q \in Q_0)\}.$$

Then L_{Q_0} is a sublattice. Moreover, $L \subset L_{Q_0}$.

(4) Let Q_0 be a compact subset of Q. Given a sublattice L in $C(Q,\ \mathbb{R})$, put

$$L\big|_{Q_0} := \{f\big|_{Q_0} :\ f \in L\}.$$

Therefore,

$$L_{Q_0} = \{f \in C(Q,\ \mathbb{R}) :\ f\big|_{Q_0} \in L\big|_{Q_0}\}.$$

It is clear that $L\big|_{Q_0}$ is a sublattice of $C(Q_0,\ \mathbb{R})$. Furthermore, if L is a *vector sublattice* of $C(Q,\ \mathbb{R})$ (i.e., a vector subspace and simultaneously a sublattice of $C(Q,\ \mathbb{R})$); then $L\big|_{Q_0}$ is a vector sublattice of $C(Q_0,\ \mathbb{R})$ (certainly, if $Q_0 \neq \varnothing$).

(5) Let $Q := \{1,\ 2\}$. Then $C(Q,\ \mathbb{R}) \simeq \mathbb{R}^2$. Each nonzero vector sublattice of $\mathbb{R}^2$ is given as

$$\{(x_1,\ x_2) \in \mathbb{R}^2 :\ \alpha_1 x_1 = \alpha_2 x_2\}$$

for some $\alpha_1,\ \alpha_2 \in \mathbb{R}_+$.

(6) Let L be a vector sublattice of $C(Q,\ \mathbb{R})$. For $q \in Q$, the alternative is offered: either $L_{\{q\}} = C(Q,\ \mathbb{R})$ or $L_{\{q\}} = \{f \in C(Q,\ \mathbb{R}) :\ f(q) = 0\}$. If q_1 and q_2 are distinct points of Q and $L\big|_{\{q_1,q_2\}} \neq 0$, then by 10.8.4 (5) there are some α_1, $\alpha_2 \in \mathbb{R}_+$ such that

$$L_{\{q_1,q_2\}} = \{f \in C(Q,\ \mathbb{R}) :\ \alpha_1 f(q_1) = \alpha_2 f(q_2)\}.$$

Moreover, if L contains a *constant function* other than zero (i.e. a nonzero multiple of the constantly-one function **1**) then as α_1 and α_2 in the above presentation of $L_{\{q_1,q_2\}}$ the unity, 1, may be taken. ⊲⊳

10.8.5. *Let L be a sublattice of $C(Q,\ \mathbb{R})$. A function f in $C(Q,\ \mathbb{R})$ belongs to the closure of L if and only if for all $\varepsilon > 0$ and $(x,\ y) \in Q^2$ there is a function $\overline{f} := f_{x,y,\varepsilon}$ in L satisfying the conditions*

$$\overline{f}(x) - f(x) < \varepsilon,\quad \overline{f}(y) - f(y) > -\varepsilon.$$

⊲ $\Rightarrow$: This is obvious.

$\Leftarrow$: Basing it on 3.2.10 and 3.2.11, assume that $f = 0$. Take $\varepsilon > 0$. Fix $x \in Q$ and consider the function $g_y := f_{x,y,\varepsilon} \in L$. Let $V_y := \{g \in Q :\ g_y(q) > -\varepsilon\}$. Then

V_y is an open set and $y \in V_y$. By a standard compactness argument, there are $y_1, \dots, y_n \in Q$ such that $Q = V_{y_1} \cup \dots \cup V_{y_n}$. Put $f_x := g_{y_1} \vee \dots \vee g_{y_n}$. It is clear that $f_x \in L$. Furthermore, $f_x(x) < \varepsilon$ and $f_x(y) > -\varepsilon$ for all $y \in Q$. Now let $U_x := \{q \in Q : f_x(q) < \varepsilon\}$. Then U_x is open and $x \in U_x$. Using the compactness of Q once again, find $x_1, \dots, x_m \in Q$ such that $Q = U_{x_1} \cup \dots \cup U_{x_m}$. Finally, put $l := f_{x_1} \wedge \dots \wedge f_{x_m}$. It is beyond a doubt that $l \in L$ and $\|l\| < \varepsilon$. ▷

10.8.6. REMARK. The message of 10.8.5 is often referred to as the *Generalized Dini Theorem* (cf. 7.2.10).

10.8.7. *Kakutani Lemma.* *Every sublattice* L *of* $C(Q, \mathbb{R})$ *is expressible as*

$$\operatorname{cl} L = \bigcap_{(q_1,q_2)\in Q^2} \operatorname{cl}\left(L_{\{q_1,q_2\}}\right).$$

◁ The inclusion of $\operatorname{cl} L$ into $\operatorname{cl}\left(L_{\{q_1,q_2\}}\right)$ for all $(q_1, q_2) \in Q^2$ raises no doubts. If $f \in \operatorname{cl}\left(L_{\{q_1,q_2\}}\right)$ for all such q_1 and q_2, then by 10.8.5, $f \in \operatorname{cl} L$. ▷

10.8.8. *Corollary.* *Every vector sublattice* L *of* $C(Q, \mathbb{R})$ *is expressible as*

$$\operatorname{cl} L = \bigcap_{(q_1,q_2)\in Q^2} L_{\{q_1,q_2\}}.$$

◁ Observe that every set of the form $L_{\{q_1,q_2\}}$ is closed. ▷

10.8.9. DEFINITION. A subset U of $\mathbb{F}^Q$ *separates the points* of Q, if for all $q_1, q_2 \in Q$ such that $q_1 \neq q_2$ there is a function $u \in U$ assuming different values at q_1 and q_2, i.e. $u(q_1) \neq u(q_2)$.

10.8.10. *Stone Theorem.* *If a vector sublattice of* $C(Q, \mathbb{R})$ *contains constant functions and separates the points of* Q, *then it is dense in* $C(Q, \mathbb{R})$.

◁ Given such a sublattice L, observe that

$$L_{\{q_1,q_2\}} = C(Q, \mathbb{R})_{\{q_1,q_2\}}$$

for every pair (q_1, q_2) in Q^2 (cf. 10.8.4 (6)). It remains to appeal to 10.8.8. ▷

10.8.11. *Let* $\mu \in C(Q, \mathbb{R})'$. *Put*

$$\mathscr{N}(\mu) := \{f \in C(Q, \mathbb{R}) : [0, |f|] \subset \ker \mu\}.$$

Then there is a unique closed subset $\operatorname{supp}(\mu)$ *of* Q *such that*

$$f \in \mathscr{N}(\mu) \Leftrightarrow f|_{\operatorname{supp}(\mu)} = 0.$$

◁ By the Interval Addition Lemma

$$[0,\ |f|] + [0,\ |g|] = [0,\ |f| + |g|].$$

Thus, $f,\ g \in \mathscr{N}(\mu) \Rightarrow |f| + |g| \in \mathscr{N}(\mu)$. Since $\mathscr{N}(\mu)$ is an *order ideal*; i.e., $(f \in \mathscr{N}(\mu)\ \&\ 0 \le |g| \le |f| \Rightarrow g \in \mathscr{N}(\mu))$, conclude that $\mathscr{N}(\mu)$ is a linear set. Moreover, $\mathscr{N}(\mu)$ is closed. Indeed, assuming $f_n \ge 0$, $f_n \to f$ and $f_n \in \mathscr{N}(\mu)$, for $g \in [0,\ f]$ find $g \wedge f_n \to g$ and $g \wedge f_n \in [0,\ f_n]$. Whence it follows that $\mu(g) = 0$; i.e., $f \in \mathscr{N}(\mu)$.

Since $\mathscr{N}(\mu)$ is an order ideal, from 10.8.8 deduce that

$$\mathscr{N}(\mu) = \bigcap_{q \in Q} \mathscr{N}(\mu)_{\{q\}}.$$

Define the set $\operatorname{supp}(\mu)$ as

$$q \in \operatorname{supp}(\mu) \Leftrightarrow \mathscr{N}(\mu)_{\{q\}} \ne C(Q,\ \mathbb{R}) \Leftrightarrow (f \in \mathscr{N}(\mu) \Rightarrow f(q) = 0).$$

It is beyond a doubt that $\operatorname{supp}(\mu)$ is closed. Moreover, the equalities hold:

$$\mathscr{N}(\mu) = \bigcap_{q \in \operatorname{supp}(\mu)} \mathscr{N}(\mu)_{\{q\}} = \{f \in C(Q,\ \mathbb{R}) :\ f|_{\operatorname{supp}(\mu)} = 0\}.$$

The claim of uniqueness follows from the normality of Q (cf. 9.4.14) and the Urysohn Theorem. ▷

10.8.12. Definition. The set $\operatorname{supp}(\mu)$ under discussion in 10.8.11 is the *support* of μ (cf. 10.9.4 (5)).

10.8.13. Remark. If μ is positive then

$$\mathscr{N}(\mu) = \{f \in C(Q,\ \mathbb{R}) :\ \mu(|f|) = 0\}.$$

Consequently, when $\mu(fg) = 0$ for all $g \in C(Q,\ \mathbb{R})$, observe that $f|_{\operatorname{supp}(\mu)} = 0$. By analogy $\operatorname{supp}(\mu) = \varnothing \Leftrightarrow \mathscr{N}(\mu) = C(Q,\ \mathbb{R}) \Leftrightarrow \mu = 0$. Therefore, it is quite convenient to work with the support of a positive functional.

Let F be a closed subset of Q. It is said that F *supports* or *carries* μ or that $X \setminus F$ *lacks* μ or is *void of* μ if $\mu(|f|) = 0$ for every continuous function f with $\operatorname{supp}(f) \subset Q \setminus F$. The support $\operatorname{supp}(\mu)$ of μ carries μ; moreover, $\operatorname{supp}(\mu)$ is included in every closed subset of Q supporting μ. In other words, the support of μ is the complement of the greatest open set void of μ (cf. 10.10.5 (6)).

It stands to reason to observe that in virtue of 3.2.14 and 3.2.15 to every bounded functional μ there correspond some positive (and hence bounded) functionals μ_+, μ_-, and $|\mu|$ defined as

$$\mu_+(f) = \sup \mu[0,\ f]; \quad \mu_-(f) = -\inf \mu[0,\ f]; \quad |\mu| = \mu_+ + \mu_-,$$

given $f \in C(Q,\ \mathbb{R})_+$.

Moreover, $C(Q,\ \mathbb{R})'$ is a Kantorovich space (cf. 3.2.16). ◁▷

10.8.14. *The supports of* μ *and* $|\mu|$ *coincide.*

$\triangleleft$ By definition $\mathscr{N}(\mu) = \mathscr{N}(|\mu|)$. $\triangleright$

10.8.15. *Considering* $a \in C(Q,\ \mathbb{R})$ *with* $0 \le a \le \mathbf{1}$, *define* $a\mu : f \mapsto \mu(af)$ *for* $f \in C(Q,\ \mathbb{R})$ *and* $\mu \in C(Q,\ \mathbb{R})'$. *Then* $|a\mu| = a|\mu|$.

$\triangleleft$ Given $f \in C(Q,\ \mathbb{R})_+$, infer that

$$(a\mu)_+(f) = \sup\{\mu(ag) :\ 0 \le g \le f\} \le \sup \mu[0,\ af]$$
$$= \mu_+(af) = a\mu_+(f).$$

Furthermore,

$$\mu_+ = (a\mu + (\mathbf{1} - a)\mu)_+ \le (a\mu)_+ + ((\mathbf{1} - a)\mu)_+ \le a\mu_+ + (\mathbf{1} - a)\mu_+ = \mu_+.$$

Consequently, $(a\mu)_+ = a\mu_+$, whence the claim follows. $\triangleright$

10.8.16. ***De Branges Lemma.*** *Let* A *be a subalgebra of* $C(Q,\ \mathbb{R})$ *containing constant functions. Take* $\mu \in \operatorname{ext}(A^\perp \cap B_{C(Q,\mathbb{R})'})$. *Then the restriction of each member of* A *to the support of* μ *is a constant function.*

$\triangleleft$ If $\mu = 0$ then $\operatorname{supp}(\mu) = \varnothing$, and there is nothing to be proven. If $\mu \ne 0$ then, certainly, $\|\mu\| = \mathbf{1}$. Take $a \in A$. Since the subalgebra A contains constant functions, it suffices to settle the case in which $0 \le a \le \mathbf{1}$ and

$$q \in \operatorname{supp}(\mu) \Rightarrow 0 < a(q) < \mathbf{1}.$$

Put $\mu_1 := a\mu$ and $\mu_2 := (\mathbf{1} - a)\mu$. It is clear that $\mu_1 + \mu_2 = \mu$ and the functionals μ_1 and μ_2 are both nonzero. Moreover,

$$\|\mu\| \le \|\mu_1\| + \|\mu_2\|$$

$$= \sup_{\|f\|\le \mathbf{1}} \mu(af) + \sup_{\|g\|\le \mathbf{1}} \mu((\mathbf{1} - a)g) = \sup_{\|f\|\le \mathbf{1}, \|g\|\le \mathbf{1}} \mu(af + (\mathbf{1} - a)g) \le \|\mu\|,$$

because it is obvious that

$$aB_{C(Q,\mathbb{R})} + (\mathbf{1} - a)B_{C(Q,\mathbb{R})} \subset B_{C(Q,\mathbb{R})}.$$

Thus, $\|\mu\| = \|\mu_1\| + \|\mu_2\|$. Since

$$\mu = \|\mu_1\| \frac{\mu_1}{\|\mu_1\|} + \|\mu_2\| \frac{\mu_2}{\|\mu_2\|},$$

and $\mu_1,\ \mu_2 \in A^\perp$, conclude that $\mu_1 = \|\mu_1\|\mu$. By 10.8.15, $a|\mu| = |a\mu| = |\mu_1| = \|\mu_1\|\,|\mu|$. Consequently, $|\mu|((a - \|\mu_1\|\mathbf{1})g) = 0$ for all $g \in C(Q,\ \mathbb{R})$. Using 10.8.13 and 10.8.14, infer that the function a is constant on the support of μ. $\triangleright$

10.8.17. *Stone–Weierstrass Theorem.* *Let A be a subalgebra of the algebra $C(Q, \mathbb{R})$. Suppose that A contains constant functions and separates the points of Q. Then A is dense in $C(Q, \mathbb{R})$.*

◁ Proceeding by way of contradiction, assume the contrary. By the Absolute Bipolar Theorem, the subspace $A^{\perp}$ (coincident with A°) of $C(Q, \mathbb{R})'$ is nonzero. Using the Alaoglu–Bourbaki Theorem, observe that $A^{\perp} \cap B_{C(Q,\mathbb{R})'}$ is a nonempty absolutely convex weakly compact set. Thus, by the Kreĭn–Milman Theorem, the set has an extreme point, say, μ.

Undoubtedly, μ is a nonzero functional. By the de Branges Lemma the support of μ fails to contain two distinct points, since A separates the points of Q. The support of μ is not a singleton, since μ annihilates constant functions. Thus, $\operatorname{supp}(\mu)$ is empty. But then μ is zero (cf. 10.8.13). We arrive at a contradiction completing the proof. ▷

10.8.18. *Corollary.* *The closure of a subalgebra of $C(Q, \mathbb{R})$ is a vector sublattice of $C(Q, \mathbb{R})$.*

◁ Using the Stone–Weierstrass Theorem, find a polynomial p_n satisfying

$$|p_n(t) - |t|\,| \leq {}^1/_{2n}$$

for all $t \in [-1, 1]$. Then $|p_n(0)| \leq {}^1/_{2n}$. Therefore, the polynomial

$$\overline{p}_n(t) := p_n(t) - p_n(0)$$

maintains the inequality $|\overline{p}_n(t) - |t|\,| \leq {}^1/_n$ when $-1 \leq t \leq 1$. By construction, $\overline{p}_n$ lacks the constant term. Now, if a function a lies in a subalgebra A of $C(Q, \mathbb{R})$ and $\|a\| \leq 1$, then

$$|\overline{p}_n(a(q)) - |a(q)|\,| \leq {}^1/_n \quad (q \in Q).$$

Moreover, the function $q \mapsto \overline{p}_n(a(q))$ is clearly a member of A. ▷

10.8.19. Remark. Corollary 10.8.18 (together with 10.8.8) completely describes all closed subalgebras of $C(Q, \mathbb{R})$. In turn, as the proof prompts, the claim of 10.8.18 is immediate on providing some sequence of polynomials which converges uniformly to the function $t \mapsto |t|$ on the interval $[-1, 1]$. It takes no pains to demonstrate 10.8.17, with 10.8.18 available.

10.8.20. *Tietze–Urysohn Theorem.* *Let Q_0 be a compact subset of a compact set Q and $f_0 \in C(Q_0, \mathbb{R})$. Then there is a function f in $C(Q, \mathbb{R})$ such that $f|_{Q_0} = f_0$.*

◁ Suppose that $Q_0 \neq \varnothing$ (otherwise, there is nothing to prove). Consider the identical embedding $\iota : Q_0 \to Q$ and the bounded linear operator $\mathring{\iota} : C(Q, \mathbb{R}) \to C(Q_0, \mathbb{R})$ acting by the rule $\mathring{\iota} f := f \circ \iota$. We have to show that $\mathring{\iota}$ is an epimorphism.

It is beyond a doubt that im $\overset{\circ}{\iota}$ is a subalgebra of $C(Q_0, \mathbb{R})$ separating the points of Q_0 and containing constant functions. In virtue of 10.8.17 it is thus sufficient (and, clearly, necessary) to examine the closure of im $\overset{\circ}{\iota}$.

Consider the monoquotient $\bar{\iota}$ of the operator $\overset{\circ}{\iota}$ and the coset mapping $\varphi : C(Q, \mathbb{R}) \to \operatorname{coim} \overset{\circ}{\iota}$. Given $f \in C(Q, \mathbb{R})$, put

$$g := (f \wedge \sup |f(Q_0)|\mathbf{1}) \vee (-\sup |f(Q_0)|\mathbf{1}).$$

By definition $f\big|_{Q_0} = g\big|_{Q_0}$; i.e., $\overline{f} := \varphi(f) = \varphi(g)$. Consequently, $\|g\| \geq \|\overline{f}\|$. Furthermore,

$$\|\overline{f}\| = \inf\{\|h\|_{C(Q,\mathbb{R})} : \overset{\circ}{\iota}(h - f) = 0\} = \inf\{\|h\|_{C(Q,\mathbb{R})} : h\big|_{Q_0} = f\big|_{Q_0}\}$$
$$\geq \inf\{\|h\big|_{Q_0}\|_{C(Q,\mathbb{R})} : h\big|_{Q_0} = f\big|_{Q_0}\} = \sup |f(Q_0)| = \|g\| \geq \|\overline{f}\|.$$

Therefore,

$$\|\bar{\iota}\overline{f}\| = \|\overset{\circ}{\iota}g\| = \|\overset{\circ}{\iota}g\|_{C(Q_0,\mathbb{R})}$$
$$= \|g \circ \iota\|_{C(Q_0,\mathbb{R})} = \sup |g(Q_0)| = \|g\| = \|\overline{f}\|;$$

i.e., $\bar{\iota}$ is an isometry. Successively applying 5.5.4 and 4.5.15, infer first that coim $\overset{\circ}{\iota}$ is a Banach space and second that im $\bar{\iota}$ is closed in $C(Q_0, \mathbb{R})$. It suffices to observe that im $\overset{\circ}{\iota} = \operatorname{im} \bar{\iota}$. ▷

10.9. Radon Measures

10.9.1. DEFINITION. Let Ω be a locally compact topological space. Put $K(\Omega) := K(\Omega, \mathbb{F}) := \{f \in C(\Omega, \mathbb{F}) : \operatorname{supp}(f) \text{ is compact}\}$. If Q is compact in Ω then let $K(Q) := K_\Omega(Q) := \{f \in K(\Omega) : \operatorname{supp}(f) \subset Q\}$. The space $K(Q)$ is furnished with the norm $\|\cdot\|_\infty$. Given $E \in \operatorname{Op}(\Omega)$, put $K(E) := \cup\{K(Q) : Q \Subset E\}$. (The notation $Q \Subset E$ for a subset E of Ω means that Q is compact and Q lies in the interior of E relative to Ω.)

10.9.2. *The following statements are valid:*

(1) *if $Q \Subset \Omega$ and $f \in C(Q, \mathbb{F})$ then*

$$f\big|_{\partial Q} = 0 \Leftrightarrow (\exists g \in K(Q))\ g\big|_Q = f;$$

moreover, $K(Q)$ is a Banach space;

(2) *let Q, Q_1, and Q_2 be compact sets and $Q \Subset Q_1 \times Q_2$; the linear span in the space $C(Q, \mathbb{F})$ of the restrictions to Q of the functions like*

$u_1 \cdot u_2(q_1, q_2) := u_1 \otimes u_2(q_1, q_2) := u_1(q_1)u_2(q_2)$ *with* $u_s \in K(Q_s)$ *is dense in* $C(Q, \mathbb{F})$;

(3) *if* Ω *is compact then* $K(\Omega) = C(\Omega, \mathbb{F})$; *if* Ω *fails to be compact and is embedded naturally into* $C(\Omega^{\cdot}, \mathbb{F})$, *with* $\Omega^{\cdot} := \Omega \cup \{\infty\}$ *the Alexandroff compactification of* Ω, *then the space* $K(\Omega)$ *is dense in the hyperplane* $\{f \in C(\Omega^{\cdot}, \mathbb{F}) : f(\infty) = 0\}$;

(4) *the mapping* $E \in \operatorname{Op}(\Omega) \mapsto K(E) \in \operatorname{Lat}(K(\Omega))$ *preserves suprema;*

(5) *for* $E', E'' \in \operatorname{Op}(\Omega)$ *the following sequence is exact:*

$$0 \to K(E' \cap E'') \xrightarrow{\iota_{(E',E'')}} K(E') \times K(E'') \xrightarrow{\sigma_{(E',E'')}} K(E' \cup E'') \to 0,$$

with $\iota_{(E',E'')}f := (f, -f)$, *and* $\sigma_{(E',E'')}(f, g) := f + g$.

◁ (1): The boundary ∂Q of Q is at the same time the boundary of the exterior of Q, the set $\operatorname{int}(\Omega \setminus Q)$.

(2): The set under study is a subalgebra. Apply 9.3.13 and 10.8.17 (cf. 11.8.2).

(3): It may be assumed that $\mathbb{F} = \mathbb{R}$. The claim will then follow from 10.8.8 since $K(\Omega)$ is an order ideal separating the points of $\Omega^{\cdot}$ (cf. 10.8.11).

(4): Clearly, $K(\sup \varnothing) = K(\varnothing) = 0$. If $\mathscr{E} \subset \operatorname{Op}(\Omega)$ and $\mathscr{E}$ is filtered upward then, for $f \in K(\cup\mathscr{E})$, observe that $\operatorname{supp}(f) \subset E$ for some $E \in \mathscr{E}$ by the compactness of $\operatorname{supp}(f)$. Whence $K(\cup\mathscr{E}) = \cup\{K(E) : E \in \mathscr{E}\}$. To conclude, take $E_1, \dots, E_n \in \operatorname{Op}(\Omega)$ and $f \in K(E_1 \cup \dots \cup E_n)$. In accordance with 9.4.18 there are some $\psi_k \in K(E_k)$ such that $\sum_{k=1}^n \psi_k = 1$. Moreover, $f = \sum_{k=1}^n \psi_k f$ and $\operatorname{supp}(f\psi_k) \subset E_k$ $(k := 1, \dots, n)$.

(5): This is straightforward from (4). ▷

10.9.3. Definition. A functional μ, a member of $K(\Omega, \mathbb{F})^{\#}$, is a *measure* (or, amply, a *Radon* $\mathbb{F}$*-measure*) on Ω, in symbols, $\mu \in \mathscr{M}(\Omega) := \mathscr{M}(\Omega, \mathbb{F})$; if $\mu|_{K(Q)} \in K(Q)'$ whenever $Q \Subset \Omega$. The following notation is current:

$$\int_{\Omega} f\,d\mu := \int f\,d\mu := \int f(x)\,d\mu(x) := \mu(f) \quad (f \in K(\Omega)).$$

The scalar $\mu(f)$ is the *integral of* f *with respect to* μ. In this connection μ is also called an *integral.*

10.9.4. Examples.

(1) For $q \in \Omega$ the *Dirac measure* $f \mapsto f(q)$ $(f \in K(\Omega))$ presents a Radon measure. It is usually denoted by the symbol δ_q and called the *delta-function at* q.

Suppose also that Ω is furnished with group structure so that the taking of an inverse $q \in \Omega \mapsto q^{-1} \in \Omega$ and the group operation $(s, t) \in \Omega \times \Omega \mapsto st \in \Omega$ are continuous; i.e., Ω is a *locally compact group.* By δ we denote δ_e where e is the *unity* of Ω (recall the concurrent terms: "identity," "unit," and "neutral element," all meaning the same: $es = se = s$ for all $s \in \Omega$). The nomenclature pertinent to addition is routinely involved in abelian (commutative) groups.

Given $a \in \Omega$, acknowledge the operation of (*left* or *right*) *translation* or *shift* by a in $K(\Omega)$ (in fact, every function is shifted in $\Omega \times \mathbb{F}$):

$$({}_a\tau f)(q) := {}_af(q) := f(a^{-1}q), \quad (\tau_a f)(q) := f_a(q) := f(qa^{-1})$$

with $f \in K(\Omega)$ and $q \in \Omega$. Clearly, ${}_a\tau$, $\tau_a \in \mathscr{L}(K(\Omega))$.

A propitious circumstance of paramount importance is the presence of a nontrivial measure on Ω, a member of $\mathscr{M}(\Omega, \mathbb{R})$, which is invariant under left (or right) translations. All (left)invariant Radon measures are proportional. Each nonzero (left)invariant positive Radon measure is a (*left*) *Haar measure* (rarely *Haar integral*). In the case of right translations, the term "(right) Haar measure" is in common parlance. In the abelian case, we speak only of a *Haar measure* and even of *Haar measure*, neglecting the necessity of scaling. The familiar *Lebesque measure* on $\mathbb{R}^N$ is Haar measure on the abelian group $\mathbb{R}^N$. That is why the conventional notation of Lebesque integration is retained in the case of an abstract Haar measure. In particular, the left-invariance condition is written down as

$$\int\limits_{\Omega} f(a^{-1}x)\,dx = \int\limits_{\Omega} f(x)\,dx \quad (f \in K(\Omega),\ a \in \Omega).$$

(2) Let $M(\Omega) := (K(\Omega),\ \|\cdot\|_\infty)'$. A member of $M(\Omega)$ is a *bounded Radon measure*. It is clear that a bounded measure belongs to the space $C(\Omega^{\cdot},\ \mathbb{F})'$ (cf. 10.9.2 (2)).

(3) Given $\mu \in \mathscr{M}(\Omega)$, put $\mu^*(f) = \mu(f^*)^*$, where $f^*(q) := f(q)^*$ for $q \in \Omega$ and $f \in K(\Omega)$. The measure μ^* is the *conjugate* of μ. Distinction between μ^* and μ is perceptible only if $\mathbb{F} = \mathbb{C}$. In case $\mu = \mu^*$, we speak of a *real* $\mathbb{C}$*-measure*. It is clear that $\mu = \mu_1 + i\mu_2$, where μ_1 and μ_2 are uniquely-determined real $\mathbb{C}$-measures. In turn, each real $\mathbb{C}$-measure is generates by two $\mathbb{R}$*-measures* (also called *real measures*), members of $\mathscr{M}(\Omega, \mathbb{R})$, because $K(\Omega, \mathbb{C})$ coincides with the complexification $K(\Omega, \mathbb{R}) \oplus iK(\Omega, \mathbb{R})$ of $K(\Omega, \mathbb{R})$. The real $\mathbb{R}$-measures obviously constitute a Kantorovich space. Moreover, the integral with respect to such a measure serves as (pre)integral. So, an opportunity is offered to deal with the corresponding Lebesque extension and spaces of scalar-valued or vector-valued functions (cf. 5.5.9 (4) and 5.5.9 (5)). We seize the opportunity without much ado and specification.

To every Radon measure μ we assign the positive measure $|\mu|$ that is defined for $f \in K(\Omega, \mathbb{R})$, $f \geq 0$, by the rule

$$|\mu|(f) := \sup\{|\mu(g)| :\ g \in K(\Omega,\ \mathbb{F}),\ |g| \leq f\}.$$

Observe that, in current usage, the word "measure" often means a positive measure, whereas a "general" measure is referred to as a *signed measure* or a *charge.*

If μ and ν are measures and $|\mu| \wedge |\nu| = 0$ then μ and ν are called *disjoint* or *independent* (of one another). A measure ν is *absolutely continuous* with respect to μ on condition that ν is independent of every measure independent of μ. Such a measure ν may be given as $\nu = f\mu$, where $f \in L_{1,\mathrm{loc}}(\mu)$ and the measure $f\mu$ having *density* f with respect to μ acts by the rule $(f\mu)(g) := \mu(fg)$ $(g \in K(\Omega))$ (this is the *Radon–Nikodým Theorem*).

(4) Given $\Omega' \in \mathrm{Op}(\Omega)$ and $\mu \in \mathscr{M}(\Omega)$, consider the *restriction* $\mu_{\Omega'} := \mu\big|_{K(\Omega')}$ of μ to $K(\Omega')$. The *restriction operator* $\mu \mapsto \mu_{\Omega'}$ from $\mathscr{M}(\Omega)$ to $\mathscr{M}(\Omega')$, viewed as depending on a subset of Ω, meets the *agreement condition*: if $\Omega'' \subset \Omega' \subset \Omega$ and $\mu \in \mathscr{M}(\Omega)$ then $\mu_{\Omega''} = (\mu_{\Omega'})_{\Omega''}$. This situation is verbalized as follows: "The mapping $\mathscr{M} : E \in \mathrm{Op}(\Omega) \mapsto \mathscr{M}(E)$ and the restriction operator (referred to jointly as the *functor* $\mathscr{M}$) defines a *presheaf* (of vector spaces over Ω)." It stands to reason to convince oneself that the (values of the) restriction operator need not be an epimorphism.

(5) Let $E \in \mathrm{Op}(\Omega)$ and $\mu \in \mathscr{M}(\Omega)$. It is said that E *lacks* μ or is *void of* μ or that $\Omega \setminus E$ *supports* or *carries* μ if $\mu_E = 0$. By 10.9.2 (4) there is a least closed set $\mathrm{supp}(\mu)$ supporting μ, the *support* of μ. It may be shown that $\mathrm{supp}(\mu) = \mathrm{supp}(|\mu|)$. The above definition agrees with 10.8.12. The Dirac measure δ_q is a unique Radon measure supported at $\{q\}$ to within scaling.

(6) Let Ω_k be a locally compact space and $\mu_k \in \mathscr{M}(\Omega_k)$ $(k := 1, 2)$. There is a unique measure μ on the product $\Omega_1 \times \Omega_2$ such that

$$\int\limits_{\Omega_1 \times \Omega_2} u_1(x)u_2(y)\,d\mu(x,y) = \int\limits_{\Omega_1} u_1(x)\,d\mu_1(x) \int\limits_{\Omega_2} u_2(y)\,d\mu_2(y)$$

with $u_k \in K(\Omega_k)$. The next designations are popular: $\mu_1 \times \mu_2 := \mu_1 \otimes \mu_2 := \mu$. Using 10.9.2 (4), infer that for $f \in K(\Omega_1 \times \Omega_2)$ the value $\mu_1 \times \mu_2(f)$ may be calculated by repeated integration (this is the *Fubini Theorem*).

(7) Let G be a locally compact group, and μ, $\nu \in M(G)$. For $f \in K(G)$ the function $\dot{f}(s,\ t) := f(st)$ is continuous and $|(\mu \times \nu)(\dot{f})| \le \|\mu\|\,\|\nu\|\,\|f\|_\infty$. This defines the Radon measure $\mu * \nu(f) := (\mu \times \nu)(\dot{f})$ $(f \in K(G))$, the *convolution* of μ and ν. Using vector integrals, obtain the presentations:

$$\mu * \nu = \int\limits_{G \times G} \delta_s * \delta_t\, d\mu(s) d\nu(t) = \int\limits_{G} \delta_s * \nu\, d\mu(s) = \int\limits_{G} \mu * \delta_t\, d\nu(t).$$

The space $M(G)$ of bounded measures with convolution as multiplication, exemplifies a Banach *convolution algebra.*

This algebra is commutative if and only if G is an abelian group. In that event the space $L_1(G)$, constructed with respect to Haar measure m, also possesses a natural structure of a convolution algebra (namely, that of a subalgebra of $M(G)$). The algebra $(L_1(G), *)$ is the *group algebra* of G. Thus, for $f,\ g \in L_1(G)$, the definitions of convolution for functions and measures agree with one another (cf. 9.6.17): $(f * g)dm = fdm * gdm$. By analogy, the *convolution* of $\mu \in M(G)$ and $f \in L_1(G)$ is defined as $(\mu * f)dm := \mu * (fdm)$; i.e., as the density of the convolution of μ and fdm with respect to Haar measure dm. In particular,

$$f * g = \int_G (\delta_x * g)f(x)\,dm(x) = \int_G \tau_x(g)f(x)\,dm(x).$$

Wendel Theorem. *Let $T \in B(L_1(G))$. Then the following statements are equivalent*:

(i) *there is a measure $\mu \in M(G)$ such that $Tf = \mu * f$ for $f \in L_1(G)$;*
(ii) *T commutes with translations: $T\tau_a = \tau_a T$ for $a \in G$, where τ_a is a unique bounded extension to $L_1(G)$ of translation by the element a in $K(G)$;*
(iii) *$T(f * g) = (Tf) * g$ for $f,\ g \in L_1(G)$;*
(iv) *$T(f * \nu) = (Tf) * \nu$ for $\nu \in M(G)$ and $f \in L_1(G)$.*

10.9.5. Definition. The spaces $K(\Omega)$ and $\mathscr{M}(\Omega)$ are set in duality (induced by the duality bracket $K(\Omega) \leftrightarrow K(\Omega)^{\#}$). In this case, the space $\mathscr{M}(\Omega)$ is furnished with the topology $\sigma(\mathscr{M}(\Omega),\ K(\Omega))$, usually called *vague*. The space $K(\Omega)$ is furnished with the Mackey topology $\tau_{K(\Omega)} := \tau(K(\Omega),\ \mathscr{M}(\Omega))$ (thereby, in particular, $(K(\Omega),\ \tau_{K(\Omega)})' = \mathscr{M}(\Omega)$). The space of bounded measures $M(\Omega)$ is usually considered with the dual norm:

$$\|\mu\| := \sup\{|\mu(f)| :\ \|f\|_\infty \le 1,\ f \in K(\Omega)\} \quad (\mu \in M(\Omega)).$$

10.9.6. *The topology $\tau_{K(\Omega)}$ is strongest among all locally convex topologies making the identical embedding of $K(Q)$ to $K(\Omega)$ continuous for every Q with $Q \Subset \Omega$ (i.e., $\tau_{K(\Omega)}$ is the so-called inductive limit topology (cf. 9.2.15)).*

◁ If τ is the inductive limit topology and $\mu \in (K(\Omega),\ \tau)'$ then by definition $\mu \in \mathscr{M}(\Omega)$, because $\mu \circ \iota_{K(Q)}$ is continuous for $Q \Subset \Omega$. In turn, for $\mu \in \mathscr{M}(\Omega)$ the set $V_Q := \{f \in K(Q) :\ |\mu(f)| \le 1\}$ is a neighborhood of zero in $K(Q)$. From the definition of τ, infer that $\cup\{V_Q\ :\ Q \Subset \Omega\} = \{f \in K(\Omega)\ :\ |\mu(f)| \le 1\}$ is a neighborhood of zero in τ. Thus, $\mu \in (K(\Omega),\ \tau)'$ and τ is compatible with duality. Therefore, $\tau \le \tau_{K(\Omega)}$.

On the other hand, if p is a seminorm in the mirror of the Mackey topology then p is the supporting function of a subdifferential lying in $\mathscr{M}(\Omega)$. Consequently, the restriction $q := p \circ \iota_{K(Q)}$ of p to $K(Q)$ is lower semicontinuous anyway. By the Gelfand Theorem (in view of the barreledness of $K(Q)$) the seminorm q is continuous. Consequently, the identical embedding $\iota_{K(Q)} : K(Q) \to (K(\Omega),\ \tau_{K(\Omega)})$ is continuous and $\tau \geq \tau_{K(\Omega)}$ by the definition of inductive limit topology. ▷

10.9.7. *A subset A of $K(\mathbb{R}^N)$ is bounded (in the inductive limit topology), if* $\sup \|A\|_\infty < +\infty$ *and, moreover, the supports of the members of A lie in a common compact set.*

◁ Suppose to the contrary that, for some Q such that $Q \Subset \mathbb{R}^N$, the inclusion $A \subset K(Q)$ fails. In other words, for $n \in \mathbb{N}$ there are some $q_n \in \mathbb{R}^N$ and $a_n \in A$ satisfying $a_n(q_n) \neq 0$ and $|q_n| > n$. Put $B := \{n|a_n(q_n)|^{-1}\delta_{q_n} : n \in \mathbb{N}\}$ and observe that this set of Radon measures is vaguely bounded and so the seminorm $p(f) := \sup\{|\mu|(|f|) : \mu \in B\}$ is continuous. Moreover, $p(a_n) \geq n|a_n(q_n)|^{-1}\delta_{q_n}(|a_n|) = n$, which contradicts the boundedness property of A. ▷

10.9.8. REMARK. Let $(f_n) \subset K(\mathbb{R}^N)$. The notation $f_n \underset{K}{\twoheadrightarrow} 0$ symbolizes the proposition $(\exists\, Q \Subset \mathbb{R}^N)(\forall n)\ \operatorname{supp}(f_n) \subset Q\ \&\ \|f_n\|_\infty \to 0$. From 10.9.7 it is immediate that $\mu \in K(\mathbb{R}^N)^\#$ *is a Radon measure if $\mu(f_n) \to 0$ whenever $f_n \underset{K}{\twoheadrightarrow} 0$.* Observe also that the same holds for every locally compact Ω *countable at infinity* or *σ-compact*, i.e. for Ω presenting the union of countably many compact sets.

10.9.9. REMARK. There is a sequence (p_n) of real-valued positive polynomials on $\mathbb{R}$ such that the measures $p_n dx$ converge vaguely to δ as $n \to +\infty$. Considering products of the measures, arrive at some polynomials P_n on the $\mathbb{R}^N$, with $(P_n dx)$ converging vaguely to δ (here, as usual, $dx := dx_1 \times \ldots \times dx_N$ is the Lebesque measure on $\mathbb{R}^N$). Now, take $f \in K(\mathbb{R}^N)$ such that f is of *class* $C^{(m)}$ in some neighborhood of a compact set Q (i.e., f has continuous derivatives up to order m on Q). Arranging the convolutions $(f * P_n)$, obtain a sequence of polynomials which approximates not only f but also the derivatives of f up to order m uniformly on Q. The possibility of such a reqularization is referred to as the *Generalized Weierstrass Theorem* in $\mathbb{R}^N$ (cf. 10.10.2 (4)).

10.9.10. ***Measure Localization Principle.*** *Let $\mathscr{E}$ be an open cover of Ω and let $(\mu_E)_{E\in\mathscr{E}}$ be a family of Radon measures, $\mu_E \in \mathscr{M}(E)$. Assume further that, for every pair $(E',\ E'')$ of the members of $\mathscr{E}$, the restrictions of $\mu_{E'}$ and $\mu_{E''}$ to $E' \cap E''$ coincide. Then there is a unique measure μ on Ω whose restriction to E equals μ_E for all $E \in \mathscr{E}$.*

◁ Using 10.9.2 (5), arrange the sequence

$$\sum_{\substack{\{E',E''\}\\ E',E''\in\mathscr{E},\ E'\neq E''}} K(E' \cap E'') \xrightarrow{\ \iota\ } \sum_{E\in\mathscr{E}} K(E) \xrightarrow{\ \sigma\ } K(\Omega) \to 0,$$

where ι is generated by summation of the coordinate identical embeddings $\iota_{(E',E'')}$ and σ is the standard addition. Every direct sum is always assumed to be furnished with the inductive limit topology (cf. 10.9.6).

Examine the exactness of the sequence. Since $K(\Omega) = \cup_{Q \Subset \Omega} K(Q)$ and in view of 10.9.2 (4), it suffices to settle the case of a finite cover by checking exactness at the second term. Thus, proceeding by induction, suppose that for every n-element cover $\{E_1, \dots, E_n\}$ $(n \geq 2)$ the following sequence is exact:

$$K_n \xrightarrow{\iota_n} \prod_{k=1}^{n} K(E_k) \xrightarrow{\sigma_n} K(E_1 \cup \ldots \cup E_n) \to 0,$$

where ι_n is the "restriction" of ι to K_n, the symbol σ_n stands for addition and

$$K_n := \prod_{\substack{k<l \\ k,l \in \{1,\dots,n\}}} K(E_k \cap E_l).$$

By hypothesis, $\operatorname{im} \iota_n = \ker \sigma_n$. If $\sigma_{n+1}(\widetilde{f}, f_{n+1}) = 0$ where $\widetilde{f} := (f_1, \dots, f_n)$, then $\sigma_n \widetilde{f} = -f_{n+1}$ and $f_{n+1} \in K((E_1 \cup \ldots \cup E_n) \cap E_{n+1})$. Since σ_n is an epimorphism by 10.9.2 (5), there are some $\theta_k \in K(E_k \cap E_{n+1})$ such that $\sigma_n \theta = -f_{n+1}$ for $\theta := (\theta_1, \dots, \theta_n)$. Whence $(\widetilde{f} - \theta) \in \ker \sigma_n$, and by hypothesis there is a member $\varkappa$ of K_n such that $\iota_n \varkappa = \widetilde{f} - \theta$. Clearly,

$$K_{n+1} = K_n \times \prod_{k=1}^{n} K(E_k \cap E_{n+1})$$

(to within isomorphism), $\overline{\varkappa} := (\varkappa, \ \theta_1, \dots, \theta_n) \in K_{n+1}$, and $\iota_{n+1} \overline{\varkappa} = (\widetilde{f}, \ f_{n+1})$.

Passing to the diagram prime (cf. 7.6.13), arrive at the exact sequence

$$0 \to \mathscr{M}(\Omega) \xrightarrow{\sigma'} \prod_{E \in \mathscr{E}} \mathscr{M}(E) \xrightarrow{\iota'} \prod_{\substack{\{E',E''\} \\ E',E'' \in \mathscr{E}, E' \neq E''}} \mathscr{M}(E' \cap E'').$$

The proof is complete. ▷

10.9.11. REMARK. In topology, a presheaf recoverable from its *patches* or *local data* in the above manner is a *sheaf*. In this regard, the claim of 10.9.10 is verbalized as follows: *the presheaf* $\Omega \mapsto \mathscr{M}(\Omega)$ *of Radon measures is a sheaf* or, putting it more categorically, *the functor* $\mathscr{M}$ *is a sheaf* (of vector spaces over Ω, cf. 10.9.4 (4)).

10.10. The Spaces $\mathscr{D}(\Omega)$ and $\mathscr{D}'(\Omega)$

10.10.1. DEFINITION. A compactly-supported smooth function $f : \mathbb{R}^N \to \mathbb{F}$ is a *test function*; in symbols, $f \in \mathscr{D}(\mathbb{R}^N) := \mathscr{D}(\mathbb{R}^N, \mathbb{F})$. Given $Q \Subset \mathbb{R}^N$ and $\Omega \in \operatorname{Op}(\mathbb{R}^N)$, designate $\mathscr{D}(Q) := \{f \in \mathscr{D}(\mathbb{R}^N) : \operatorname{supp}(f) \subset Q\}$ and $\mathscr{D}(\Omega) := \cup\{\mathscr{D}(Q) : Q \Subset \Omega\}$.

10.10.2. *The following statements are valid:*

(1) $\mathscr{D}(Q) = 0 \Leftrightarrow \operatorname{int} Q = \varnothing$;

(2) *given* $Q \Subset \mathbb{R}^N$, *put*

$$\|f\|_{n,Q} := \sum_{|\alpha| \le n} \|\partial^\alpha f\|_{C(Q)} := \sum_{\substack{\alpha \in (\mathbb{Z}_+)^N \\ \alpha_1 + \dots + \alpha_N \le n}} \sup |(\partial^{\alpha_1} \dots \partial^{\alpha_n} f)(Q)|$$

for a function f smooth in a neighborhood of Q (as usual, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$); the multinorm $\mathfrak{M}_Q := \{\|\cdot\|_{n,Q} : n \in \mathbb{N}\}$ makes $\mathscr{D}(Q)$ into a Fréchet space;

(3) *the space of smooth functions $C_\infty(\Omega) := \mathscr{E}(\Omega)$ on Ω in $\operatorname{Op}(\mathbb{R}^N)$ with the multinorm $\mathfrak{M}_\Omega := \{\|\cdot\|_{n,Q} : n \in \mathbb{N},\ Q \Subset \Omega\}$ is a Fréchet space; moreover, $\mathscr{D}(\Omega)$ is dense in $C_\infty(\Omega)$;*

(4) *if $Q_1 \Subset \mathbb{R}^N$, $Q_2 \Subset \mathbb{R}^M$ and $Q \Subset Q_1 \times Q_2$; then the linear span in $\mathscr{D}(Q)$ of the restrictions to Q of the functions like $f_1 f_2(q_1, q_2) := f_1 \otimes f_2(q_1, q_2) := f_1(q_1) f_2(q_2)$, with $q_k \in Q_k$ and $f_k \in \mathscr{D}(Q_k)$, is dense in $\mathscr{D}(Q)$;*

(5) *the mapping $E \in \operatorname{Op}(\Omega) \mapsto \mathscr{D}(E) \in \operatorname{Lat}(\mathscr{D}(\Omega))$ preserves suprema:*

$$\mathscr{D}(E' \cap E'') = \mathscr{D}(E') \cap \mathscr{D}(E''),\ \mathscr{D}(E' \cup E'') = \mathscr{D}(E') + \mathscr{D}(E'');$$
$$\mathscr{D}(\cup \mathscr{E}) = \mathscr{L}(\cup\{\mathscr{D}(E) : E \in \mathscr{E}\}) \quad (\mathscr{E} \subset \operatorname{Op}(\Omega)).$$

Moreover, the next sequence is exact (cf. 10.9.2 (5)*):*

$$0 \to \mathscr{D}(E' \cap E'') \xrightarrow{\iota(E', E'')} \mathscr{D}(E') \times \mathscr{D}(E'') \xrightarrow{\sigma(E', E'')} \mathscr{D}(E' \cup E'') \to 0.$$

◁ (1) and (2) are obvious.

(3): Choose a sequence $(Q_m)_{m \in \mathbb{N}}$ such that $Q_m \Subset \Omega$, $Q_m \Subset Q_{m+1}$ and $\cup_{m \in \mathbb{N}} Q_m = \Omega$. The multinorm $\{\|\cdot\|_{n,Q_m} : n \in \mathbb{N},\ m \in \mathbb{N}\}$ is countable and equivalent to $\mathfrak{M}_\Omega$. A reference to 5.4.2 yields the metrizability of $C_\infty(\Omega)$. The completeness of $C_\infty(\Omega)$ raises no doubts.

To show that $\mathscr{D}(\Omega)$ is dense in $C_\infty(\Omega)$, consider the *truncator set* $\operatorname{Tr}(\Omega) := \{\psi \in \mathscr{D}(\Omega) : 0 \le \psi \le \mathbf{1}\}$. Make $\operatorname{Tr}(\Omega)$ a *direction* on letting $\psi_1 \le \psi_2 \Leftrightarrow \operatorname{supp}(\psi_1) \subset \operatorname{int}\{\psi_2 = 1\}$. It is clear that, for $f \in C_\infty(\Omega)$, the net $(\psi f)_{\psi \in \operatorname{Tr}(\Omega)}$ approximates f as is needed.

(4): Take $q' \in \mathbb{R}^N$ and $q'' \in \mathbb{R}^M$. Let $a(q',\ q'') := a'(q')a''(q'')$, where a' and a'' are mollifiers on $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively. Given $f \in \mathscr{D}(Q)$, $n \in \mathbb{N}$, and $\varepsilon > 0$, choose χ from the condition $\|f - f * a_\chi\|_{n,Q} \le {}^\varepsilon/_2$. Using the equicontinuity property of the family $\mathscr{F} := \{\partial^\alpha f(q)\tau_q(a_\chi) :\ |\alpha| \le n,\ q \in Q_1 \times Q_2\}$, find finite sets Δ' and Δ'', with $\Delta' \subset Q_1$ and $\Delta'' \subset Q_2$, so that the integral of each function in $\mathscr{F}$ be approximated to within ${}^1/_2(N+1)^{-n}\varepsilon$ by a Riemann sum of it using the points of $\Delta' \times \Delta''$. This yields a function $\overline{f}$, a member of the linear span of $\mathscr{D}(Q_1) \times \mathscr{D}(Q_2)$, which was sought; i.e., $\|f - \overline{f}\|_{n,Q} \le \varepsilon$.

(5): Check this is as in 10.9.2 (4), on replacing 9.4.18 with 9.6.19 (2). ▷

10.10.3. Remark. The Generalized Weierstrass Theorem may be applied to the demonstration of 10.10.2 (4), when combined with due truncation providing that the constructed approximation has compact support.

10.10.4. Definition. A functional u, a member of $\mathscr{D}(\Omega,\ \mathbb{F})^\#$, is a *distribution* or a *generalized function* whenever $u|_{\mathscr{D}(Q)} \in \mathscr{D}'(Q) := \mathscr{D}(Q)'$ for all $Q \Subset \Omega$. This is expressed in writing as $u \in \mathscr{D}'(\Omega) := \mathscr{D}'(\Omega,\ \mathbb{F})$. Sometimes a reference is appended to the nature of the ground field $\mathbb{F}$.

The usual designations are as follows: $\langle u,\ f\rangle := \langle f \mid u\rangle := u(f)$. Often we use the most suggestive and ubiquitous symbol

$$\int f(x)u(x)\,dx := u(f) \quad (f \in \mathscr{D}(\Omega)).$$

10.10.5. Examples.

(1) Let $g \in L_{1,\mathrm{loc}}(\mathbb{R}^N)$ be a locally integrable function. The mapping

$$u_g(f) := \int f(x)g(x)\,dx \quad (f \in \mathscr{D}(\Omega))$$

determines a distribution. A distribution of this type is *regular*. To denote a regular distribution u_g, a more convenient symbol g is also employed. In this connection, we write $\mathscr{D}(\Omega) \subset \mathscr{D}'(\Omega)$ and $u_g = |\,g\rangle$.

(2) Every Radon measure is a distribution. Each *positive distribution* u (i.e., such that $f \ge 0 \Rightarrow u(f) \ge 0$) is determined by a positive measure.

(3) A distribution u is said to *has order at most* m, if to every Q such that $Q \Subset \mathbb{R}^N$ there corresponds a number t_Q satisfying

$$|u(f)| \le t_Q\|f\|_{m,Q} \quad (f \in \mathscr{D}(Q)).$$

The notions of the *order of a distribution* and of a *distribution of finite order* are understood in a matter-of-fact fashion. Evidently, it is false that every distribution has finite order.

(4) Let α be a multi-index, $\alpha \in (\mathbb{Z}_+)^N$; and let u be a distribution, $u \in \mathscr{D}'(\Omega)$. Given $f \in \mathscr{D}(\Omega)$, put $(\partial^\alpha u)(f) := (-1)^{|\alpha|}u(\partial^\alpha f)$. The distribution $\partial^\alpha u$ is the *derivative* of u (of order α). We also speak of *generalized differentiation*, of *derivatives in the distribution sense*, etc. and use the conventional symbols of differential calculus.

A derivative (of nonzero order) of a Dirac measure is not a measure. At the same time $\delta \in \mathscr{D}'(\mathbb{R})$ is the derivative of the *Heaviside function* $\delta^{(-1)} := H$, where $H : \mathbb{R} \to \mathbb{R}$ is the characteristic function of $\mathbb{R}_+$. If a derivative of a (regular) distribution u is some regular distribution u_g, then g is a *weak derivative* of u or a *generalized derivative* of u in the Sobolev sense. For a test function, such a derivative coincides with its ordinary counterpart.

(5) Given $u \in \mathscr{D}'(\Omega)$, put $u^*(f) := u(f^*)^*$. The distribution u^* is the *conjugate* of u. The presence of the involution $*$ routinely justifies speaking of *real distributions* and *complex distributions* (cf. 10.9.3 (3)).

(6) Let $E \in \mathrm{Op}(\Omega)$ and $u \in \mathscr{D}'(\Omega)$. For $f \in \mathscr{D}(E)$, the scalar $u(f)$ is easily determined. This gives rise to the distribution u_E, a member of $\mathscr{D}'(E)$, called the *restriction* of u to E. The functor $\mathscr{D}$ is clearly a presheaf.

Given $u \in \mathscr{D}'(\Omega)$ and $E \in \mathrm{Op}(\Omega)$, say that E *lacks* or is *void of* u, if $u_E = 0$. By 10.10.4 (5), if the members of a family of open subsets of Ω are void of u then so is the union of the family. The complement (to $\mathbb{R}^N$) of the greatest open set void of u is the *support* of u, denoted by $\mathrm{supp}(u)$. Observe that $\mathrm{supp}(\partial^\alpha u) \subset \mathrm{supp}(u)$. Moreover, a distribution with compact support (= *compactly-supported distribution*) has finite order.

(7) Let $u \in \mathscr{D}'(\Omega)$ and $f \in C_\infty(\Omega)$. If $g \in \mathscr{D}(\Omega)$ then $fg \in \mathscr{D}(\Omega)$. Put $(fu)(g) := u(fg)$. The resulting distribution fu is the *product* of f and u. Consider the truncator direction $\mathrm{Tr}(\Omega)$. If there is a limit $\lim_{\psi \in \mathrm{Tr}(\Omega)} u(f\psi)$ then say that u *applies* to f. It is clear that a compactly-supported distribution u applies to every function in $C_\infty(\Omega)$. Moreover, $u \in \mathscr{E}'(\Omega) := C_\infty(\Omega)'$. In turn, each element u of $\mathscr{E}'(\Omega)$ (cf. 10.10.2 (3)) uniquely determines a distribution with compact support, which is implicit in the notation $u \in \mathscr{D}'(\Omega)$.

If $f \in C_\infty(\Omega)$ and $\partial^\alpha f\big|_{\mathrm{supp}(u)} = 0$ for all α, $|\alpha| \le m$, where u is a compactly-supported distribution of order at most m, then it is easy that $u(f) = 0$. In consequence, only linear combinations of a Dirac measure and its derivatives are supported at a singleton. ◁▷

(8) Let $\Omega_1, \Omega_2 \in \mathrm{Op}(\mathbb{R}^N)$ and $u_k \in \mathscr{D}'(\Omega_k)$. There is a unique distribution u on $\Omega_1 \times \Omega_2$ such that $u(f_1 f_2) = u_1(f_1)u_2(f_2)$ for all $f_k \in \mathscr{D}(\Omega_k)$. The distribution is denoted by $u_1 \times u_2$ or $u_1 \otimes u_2$. Using 10.10.2 (4), infer that for $f \in \mathscr{D}(\Omega_1 \times \Omega_2)$ the value $u(f)$ of u at f appears from successive application of u_1 and u_2. Strictly speaking,

$$u(f) = u_2(y \in \Omega_2 \mapsto u_1(f(\,\cdot\,,\ y))) = u_1(x \in \Omega_1 \mapsto u_2(f(x,\ \cdot\,))).$$

More suggestive designations prompt the *Fubini Theorem:*

$$\iint\limits_{\Omega_1\times\Omega_2} f(x,\ y)(u_1\times u_2)(x,\ y)\,dxdy$$

$$=\int\limits_{\Omega_2}\left(\int\limits_{\Omega_1} f(x,\ y)u_1(x)\,dx\right)u_2(y)\,dy=\int\limits_{\Omega_1}\left(\int\limits_{\Omega_2} f(x,\ y)u_2(y)\,dy\right)u_1(x)\,dx.$$

It is worth noting that

$$\operatorname{supp}(u_1\times u_2)=\operatorname{supp}(u_1)\times\operatorname{supp}(u_2).$$

(9) Let $u,\ v\in\mathscr{D}'(\mathbb{R}^N)$. Given $f\in\mathscr{D}(\mathbb{R}^N)$, put $\overset{+}{f}:=f\circ +$. It is clear that $\overset{+}{f}\in C_\infty(\mathbb{R}^N\times\mathbb{R}^N)$. Say that the distributions u and v *convolute* or *admit convolution* or are *convolutive* provide that the product $u\times v$ applies to the function $\overset{+}{f}\subset C_\infty(\mathbb{R}^N\times\mathbb{R}^N)$ for each f in $\mathscr{D}(\mathbb{R}^N)$. It is easy (cf. 10.10.10) that the resulting linear functional $f\mapsto(u\times v)(\overset{+}{f})$ $(f\in\mathscr{D}(\mathbb{R}^N))$ is a distribution called the *convolution* of u and v and denoted by $u*v$. It is beyond a doubt that the convolutions of functions (cf. 9.6.17) and measures on $\mathbb{R}^N$ (cf. 10.9.4 (7)) are particular cases of the convolution of distributions. In some classes, every two distributions convolute. For instance, the space $\mathscr{E}'(\mathbb{R}^N)$ of compactly-supported distributions with convolution as multiplication presents an (associative and commutative) algebra with unity the delta-function δ. Further, $\partial^\alpha u=\partial^\alpha\delta*u$ and $\partial^\alpha(u*v)=\partial^\alpha u*v=u*\partial^\alpha v$. Moreover, the following remarkable equality, the *Lions Theorem of Supports*, holds:

$$\operatorname{co}(\operatorname{supp}(u*v))=\operatorname{co}(\operatorname{supp}(u))+\operatorname{co}(\operatorname{supp}(v)).$$

It is worth emphasizing that the pairwise convolutivity of distributions fails in general to guarantee the associativity of convolution (for instance, $(\mathbf{1}*\delta')*\delta^{(-1)}=0$ whereas $\mathbf{1}*(\delta'*\delta^{(-1)})=\mathbf{1}$, with $\mathbf{1}:=\mathbf{1}_{\mathbb{R}}$).

Each distribution u convolutes with a test function f, yielding some regular distribution $(u*f)(x)=u(\tau_x(f\,\check{}\,))$, where $f\,\check{}:=\widetilde{f}$ is the *reflection* of f; i.e., $f\,\check{}(x):=f(-x)$ $(x\in\mathbb{R}^N)$. The operator $u*:f\mapsto u*f$ from $\mathscr{D}(\mathbb{R}^N)$ to $C_\infty(\mathbb{R}^N)$ is continuous and commutes with translations: $(u*)\tau_x=\tau_x u*$ for $x\in\mathbb{R}^N$. It is easily seen that the above properties are characteristic of $u*$; i.e., if an operator T, a member of $\mathscr{L}(\mathscr{D}(\mathbb{R}^N),\ C_\infty(\mathbb{R}^N))$, is continuous and commutes with translations, then there is a unique distribution u such that $T=u*$; namely, $u(f):=(T'\delta)(\widetilde{f})$ for $f\in\mathscr{D}(\mathbb{R}^N)$ (cf. the Wendel Theorem).

10.10.6. DEFINITION. The spaces $\mathscr{D}(\Omega)$ and $\mathscr{D}'(\Omega)$ are set in duality (induced by the duality bracket $\mathscr{D}(\Omega) \leftrightarrow \mathscr{D}(\Omega)^{\#}$). Moreover, $\mathscr{D}'(\Omega)$ is furnished with the *topology of the distribution space* $\sigma(\mathscr{D}'(\Omega), \mathscr{D}(\Omega))$, and $\mathscr{D}(\Omega)$ is furnished with the *topology of the test function space*, the Mackey topology $\tau_{\mathscr{D}} := \tau_{\mathscr{D}(\Omega)} := \tau(\mathscr{D}(\Omega), \mathscr{D}'(\Omega))$.

10.10.7. *Let* $\Omega \in \mathrm{Op}(\mathbb{R}^N)$. *Then*

(1) $\tau_{\mathscr{D}}$ *is the strongest of the locally convex topologies making the identical embedding of* $\mathscr{D}(Q)$ *into* $\mathscr{D}(\Omega)$ *continuous for all* $Q \Subset \Omega$ *(i.e.,* $\tau_{\mathscr{D}}$ *is the inductive limit topology);*

(2) *a subset* A *of* $\mathscr{D}(\Omega)$ *is bounded if and only if* A *lies in* $\mathscr{D}(Q)$ *for some* Q *such that* $Q \Subset \Omega$ *and is bounded in* $\mathscr{D}(Q)$*;*

(3) *a sequence* (f_n) *converges to* f *in* $(\mathscr{D}(\Omega), \tau_{\mathscr{D}})$ *if and only if there is a compact set* Q *such that* $Q \Subset \Omega$, $\mathrm{supp}\,(f_n) \subset Q$, $\mathrm{supp}\,(f) \subset Q$ *and* $(\partial^\alpha f_n)$ *converges to* f *uniformly on* Q *for every multi-index* α *(in symbols,* $f_n \twoheadrightarrow f$*);*

(4) *an operator* T, *a member of* $\mathscr{L}(\mathscr{D}(\Omega), Y)$ *with* Y *a locally convex space, is continuous if and only if* $Tf_n \to 0$ *provided that* $f_n \twoheadrightarrow 0$*;*

(5) *a delta-like sequence* (b_n) *serves as a (convolution) approximate unity in* $\mathscr{D}(\mathbb{R}^N)$ *as well as in* $\mathscr{D}'(\mathbb{R}^N)$*; i.e.,* $b_n * f \twoheadrightarrow f$ *(in* $\mathscr{D}(\mathbb{R}^N)$*) and* $b_n * u \to u$ *(in* $\mathscr{D}'(\mathbb{R}^N)$*) for* $f \in \mathscr{D}(\mathbb{R}^N)$ *and* $u \in \mathscr{D}'(\mathbb{R}^N)$.

$\triangleleft$ (1): This is established similarly as 10.9.6; and (2), by analogy with 10.9.7 using the presentation of Ω as the union $\Omega = \cup_{n\in\mathbb{N}} Q_n$, where $Q_n \Subset Q_{n+1}$ for $n \in \mathbb{N}$.

(3): Note that a convergent countable sequence is bounded, and apply 10.10.7 (2) (cf. 10.9.8).

(4): In virtue of 10.10.7 (1) the continuity of T amounts to that of the restriction $T|_{\mathscr{D}(Q)}$ for all $Q \Subset \Omega$. By 10.10.2 (2) the space $\mathscr{D}(Q)$ is metrizable. It remains to refer to 10.10.7 (3).

(5): It is clear that all of the supports $\mathrm{supp}\,(b_n * f)$ lie in some compact neighborhood about $\mathrm{supp}\,(f)$. Furthermore, for $g \in C(\mathbb{R}^N)$, it is evident that $b_n * g \to g$ uniformly on compact subsets of $\mathbb{R}^N$. Applying this to $\partial^\alpha f$ and considering (3), infer that $b_n * f \twoheadrightarrow f$.

On account of 10.10.6 (8), observe the following:

$$
\begin{aligned}
u(\tilde{f}) &= (u * f)(0) = \lim_n (u * (b_n * f))(0) \\
&= \lim_n ((u * b_n) * f)(0) = \lim_n (b_n * u)(\tilde{f})
\end{aligned}
$$

for $f \in \mathscr{D}(\mathbb{R}^N)$. $\triangleright$

10.10.8. REMARK. In view of 10.10.7 (3), for $\Omega \in \mathrm{Op}(\mathbb{R}^N)$ and $m \in \mathbb{Z}_+$ it is often convenient to consider the space $\mathscr{D}^{(m)}(\Omega) := C_0^{(m)}(\Omega)$ comprising all compactly-supported functions f whose derivatives $\partial^\alpha f$ are continuous for all $|\alpha| \leq m$. The space $\mathscr{D}^{(m)}(Q) := \{f \in \mathscr{D}^{(m)}(\Omega) : \operatorname{supp}(f) \subset Q\}$ for $Q \Subset \Omega$ is furnished with the norm $\|\cdot\|_{m,Q}$ making it into a Banach space. In that event, $\mathscr{D}^{(m)}(\Omega)$ is endowed with the inductive limit topology. Thus, $\mathscr{D}^{(0)}(\Omega) = K(\Omega)$ and $\mathscr{D}(\Omega) = \cap_{m\in\mathbb{N}}\mathscr{D}^{(m)}(\Omega)$. For a sequence (f_n) to converge in $\mathscr{D}^{(m)}(\Omega)$ means to converge uniformly with all derivatives up to order m on some Q such that $Q \Subset \Omega$ and $\operatorname{supp}(f_n) \subset Q$ for all sufficiently large n. Note that $\mathscr{D}^{(m)}(\Omega)'$ comprises all distributions of order at most m. Correspondingly,

$$\mathscr{D}'_F(\Omega) := \bigcup_{m\in\mathbb{N}} \mathscr{D}^{(m)}(\Omega)'$$

is the *space of finite-order distributions.*

10.10.9. *Let* $\Omega \in \mathrm{Op}(\mathbb{R}^N)$. *Then*

(1) *the space* $\mathscr{D}(\Omega)$ *is barreled; i.e., each* ***barrel****, a closed absorbing absolutely convex subset, is a neighborhood of zero;*

(2) *every bounded closed set in* $\mathscr{D}(\Omega)$ *is compact; i.e.,* $\mathscr{D}(\Omega)$ *is a* ***Montel space****;*

(3) *every absolutely convex subset of* $\mathscr{D}(\Omega)$*, absorbing each bounded set, is a neighborhood of zero; i.e.,* $\mathscr{D}(\Omega)$ *is a* ***bornological space****;*

(4) *the test functions are dense in the distribution space.*

◁ (1): A barrel V in $\mathscr{D}(\Omega)$ such that $V_Q := V \cap \mathscr{D}(Q)$ is a barrel in $\mathscr{D}(Q)$ for all $Q \Subset \Omega$. Thus, V_Q is a neighborhood of zero in $\mathscr{D}(Q)$ (cf. 7.1.8).

(2): Such a set lies in $\mathscr{D}(Q)$ for some $Q \Subset \Omega$ by 10.10.7 (2). In virtue of 10.10.2 (2), $\mathscr{D}(Q)$ is metrizable. Applying 4.6.10 and 4.6.11 proves the claim.

(3): This follows from the barreledness of $\mathscr{D}(Q)$ for $Q \Subset \Omega$.

(4): Let $g \in |\mathscr{D}(\Omega))^\circ$, with the polar taken with respect to $\mathscr{D}(\Omega) \leftrightarrow \mathscr{D}'(\Omega)$. It is clear that $u_f(g) = 0$ for $f \in \mathscr{D}(\Omega)$; i.e., $\int g(x)f(x)\,dx = 0$. Thus, $g = 0$. It remains to refer to 10.5.9. ▷

10.10.10. ***Schwartz Theorem.*** *Let* $(u_k)_{k\in\mathbb{N}}$ *be a sequence of distributions. Assume that for every* f *in* $\mathscr{D}(\Omega)$ *there is a sum*

$$u(f) := \sum_{k=1}^{\infty} u_k(f).$$

Then u *is a distribution and*

$$\partial^\alpha u = \sum_{k=1}^{\infty} \partial^\alpha u_k$$

for every multi-index α.

◁ The continuity of u is guaranteed by 10.10.9 (1). Furthermore, for $f \in \mathscr{D}(\Omega)$ by definition (cf. 10.10.5 (4))

$$\partial^\alpha u(f) = u\left((-1)^{|\alpha|}\partial^\alpha f\right) = \sum_{k=1}^{\infty} u_k\left((-1)^{|\alpha|}\partial^\alpha f\right) = \sum_{k=1}^{\infty} \partial^\alpha u_k(f). \ ▷$$

10.10.11. *Theorem.* *The functor $\mathscr{D}'$ is a sheaf.*

◁ It is immediate (cf. 10.9.10 and 10.9.11). ▷

10.10.12. REMARK. The possibility of recovering a distribution from local data, the ***Distribution Localization Principle*** stated in 10.10.11, admits clarification in view of the paracompactness of $\mathbb{R}^N$. Namely, consider an open cover $\mathscr{E}$ of Ω and a distribution $u \in \mathscr{D}'(\Omega)$ with local data $(u_E)_{E\in\mathscr{E}}$. Take a countable (locally finite) partition of unity $(\psi_k)_{k\in\mathbb{N}}$ subordinate to $\mathscr{E}$. Evidently, $u = \sum_{k=1}^{\infty} \psi_k u_k$, where $u_k := u_{E_k}$ and $\operatorname{supp}(\psi_k) \subset E_k$ $(k \in \mathbb{N})$.

10.10.13. *Theorem.* *Each distribution u on Ω of order at most m may be expressed as sum of derivatives of Radon measures:*

$$u = \sum_{|\alpha|\le m} \partial^\alpha \mu_\alpha,$$

where $\mu_\alpha \in \mathscr{M}(\Omega)$.

◁ To begin with, assume that u has compact support. Let Q with $Q \Subset \Omega$ be a compact neighborhood of $\operatorname{supp}(u)$. By hypothesis (cf. 10.10.5 (7) and 10.10.8)

$$|u(f)| \le t \sum_{|\alpha|\le m} \|\partial^\alpha f\|_\infty \quad (f \in \mathscr{D}(Q))$$

for some $t \ge 0$.

Using 3.5.7 and 3.5.3, from 10.9.4 (2) obtain

$$u = t \sum_{|\alpha|\le m} \nu_\alpha \circ \partial^\alpha = t \sum_{|\alpha|\le m} (-1)^{|\alpha|}\partial^\alpha \nu_\alpha$$

for a suitable family $(\nu_\alpha)_{|\alpha|\le m}$, where $\nu_\alpha \in |\partial|(\|\cdot\|_\infty)$.

Passing to the general case and invoking the Countable Partition Theorem, find a partition of unity $(\psi_k)_{k\in\mathbb{N}}$ with $\psi_k \in \mathscr{D}(\Omega)$ such that some neighborhoods Q_k of $\operatorname{supp}(\psi_k)$ compose a locally finite cover of Ω (cf. 10.10.12). For each of the distributions $(\psi_k u)_{k\in\mathbb{N}}$ it is already proven that

$$\psi_k u = \sum_{|\alpha|\le m} \partial^\alpha \mu_{k,\alpha},$$

where every $\mu_{k,\alpha}$ is a Radon measure on Ω and $\operatorname{supp}(\mu_{k,\alpha}) \subset Q_k$.

From the Schwartz Theorem it is immediate that the sum

$$\mu_\alpha(f) := \sum_{k=1}^{\infty} \mu_{k,\alpha}(f)$$

exists for all $f \in K(\Omega)$. Moreover, the resulting distribution μ_α is a Radon measure. Once again appealing to 10.10.10, infer that

$$u = \sum_{k=1}^{\infty} \psi_k u = \sum_{k=1}^{\infty} \sum_{|\alpha| \le m} \partial^\alpha \mu_{k,\alpha} = \sum_{|\alpha| \le m} \partial^\alpha \left(\sum_{k=1}^{\infty} \mu_{k,\alpha} \right) = \sum_{|\alpha| \le m} \partial^\alpha \mu_\alpha,$$

which was required. ▷

10.10.14. REMARK. The claim of 10.10.13 is often referred to as the *theorem on the general form of a distribution.* Further abstraction and clarification are available. For instance, it may be verified that a compactly-supported Radon measure serves as a derivative (in the distribution sense) of suitable order of some continuous function. This enables us to view each distribution as a result of generalized differentiation of a conventional function.

10.11. The Fourier Transform of a Distribution

10.11.1. *Let χ be a nonzero functional over the space $L_1(\mathbb{R}^N) := L_1(\mathbb{R}^N, \mathbb{C})$. The following statements are equivalent:*

(1) *χ is a character of the group algebra $(L_1(\mathbb{R}^N), *)$; i.e., $\chi \ne 0$, $\chi \in L_1(\mathbb{R}^N)'$ and*

$$\chi(f * g) = \chi(f)\chi(g) \quad (f,\ g \in L_1(\mathbb{R}^N))$$

(in symbols, $\chi \in X(L_1(\mathbb{R}^N))$, cf. 11.6.4);

(2) *there is a unique vector t in $\mathbb{R}^N$ such that*

$$\chi(f) = \widehat{f}(t) := (f * e_t)(0) := \int_{\mathbb{R}^N} f(x) e^{i(x,t)}\, dx$$

for all $f \in L_1(\mathbb{R}^N)$.

◁ (1) ⇒ (2): Suppose that $\chi(f)\chi(g) \ne 0$. If $x \in \mathbb{R}^N$ then

$$\chi(\delta_x * f * g) = \chi(\delta_x * f)\chi(g) = \chi(\delta_x * g)\chi(f).$$

Put $\psi(x) := \chi(f)^{-1}\chi(\delta_x * f)$. This soundly defines some continuous mapping $\psi : \mathbb{R}^N \to \mathbb{C}$. Moreover,

$$\begin{aligned}&\psi(x+y)\\ &= \chi(f*g)^{-1}\chi(\delta_{x+y}*(f*g)) = \chi(f)^{-1}\chi(g)^{-1}\chi(\delta_x * f * \delta_y * g)\\ &= \chi(f)^{-1}\chi(\delta_x * f)\chi(g)^{-1}\chi(\delta_y * g) = \psi(x)\psi(y)\end{aligned}$$

for $x,\ y \in \mathbb{R}^N$; i.e., ψ is a (unitary) group character: $\psi \in \mathrm{X}(\mathbb{R}^N)$. Calculus shows that $\psi = e_t$ for some (obviously, unique) $t \in \mathbb{R}^N$. Further, by the properties of the Bochner integral

$$\chi(f)\chi(g) = \chi(f*g) = \chi\left(\int_{\mathbb{R}^N} (\delta_x * g) f(x)\,dx\right)$$

$$= \int_{\mathbb{R}^N} \chi(\delta_x * g) f(x)\,dx = \int_{\mathbb{R}^N} f(x)\chi(g)\psi(x)\,dx = \chi(g)\int_{\mathbb{R}^N} f(x)\psi(x)\,dx.$$

Thus,

$$\chi(f) = \int_{\mathbb{R}^N} f(x)\psi(x)\,dx \quad (f \in L_1(\mathbb{R}^N)).$$

(2) $\Rightarrow$ (1): Given $t \in \mathbb{R}^N$ and treating f, g and $f*g$ as distributions, infer that

$$\widehat{f*g}(t) = u_{f*g}(e_t)$$

$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)g(y)e_t(x+y)\,dx dy = \int_{\mathbb{R}^N} f(x)e_t(x)\,dx \int_{\mathbb{R}^N} g(y)e_t(y)\,dy$$

$$= u_f(e_t)u_g(e_t) = \widehat{f}(t)\widehat{g}(t). \ \triangleright$$

10.11.2. REMARK. The essential steps of the above argument remain valid for every locally compact abelian group G, and so there is a one-to-one correspondence between the character space $\mathrm{X}(L_1(G))$ of the group algebra and the set $\mathrm{X}(G)$ comprising all (unitary) group characters of G. Recall that such a character is a continuous mapping $\psi : G \to \mathbb{C}$ satisfying

$$|\psi(x)| = 1, \quad \psi(x+y) = \psi(x)\psi(y) \quad (x,\ y \in G).$$

Endowed with pointwise multiplication, the set $\widehat{G} := \mathrm{X}(G)$ becomes a commutative group. In virtue of the Alaoglu–Bourbaki Theorem, $\mathrm{X}(L_1(G))$ is locally compact in the weak topology $\sigma((L_1(G))', L_1(G))$. So, $\widehat{G}$ may be treated as a locally compact abelian group called the *character group* of G or the *dual group* of G. Each element q in G defines the character $\widehat{\widehat{q}} : \widehat{q} \in \widehat{G} \mapsto \widehat{q}(q) \in \mathbb{C}$ of the dual group $\widehat{G}$. The resulting embedding of G into $\widehat{\widehat{G}}$ is surprisingly an isomorphism of the locally compact abelian groups G and $\widehat{\widehat{G}}$ (the *Pontryagin–van Kampen Duality Theorem*).

10.11.3. Definition. For a function f in $L_1(\mathbb{R}^N)$, the mapping $\widehat{f} : \mathbb{R}^N \to \mathbb{C}$, defined by the rule

$$\widehat{f}(t) := f^{\wedge}(t) := (f * e_t)(0),$$

is the *Fourier transform* of f.

10.11.4. Remark. By way of taking convenient liberties, we customarily use the term "Fourier transform" expansively. First, it is retained not only for the operator $\mathscr{F} : L_1(\mathbb{R}^N) \to \mathbb{C}^{\mathbb{R}^N}$ acting by the rule $\mathscr{F}f := \widehat{f}$ but also for its modifications (cf. 10.11.13). Second, the Fourier transform $\mathscr{F}$ is identified with each operator $\mathscr{F}_\theta f := \widehat{f} \circ \theta$, where θ is an *automorphism* (= isomorphism with itself) of $\mathbb{R}^N$. Among the most popular are the functions: $\theta(x) := {}_{\sim}(x) := -x$, $\theta(x) := {}_{2\pi}(x) := 2\pi x$ and $\theta(x) := {}_{-2\pi}(x) := -2\pi x$ $(x \in \mathbb{R}^N)$. In other words, the Fourier transform is usually defined by one the following formulas:

$$\mathscr{F}_{\sim} f(t) = \int\limits_{\mathbb{R}^N} f(x) e^{-i(x,t)}\, dx,$$

$$\mathscr{F}_{2\pi} f(t) = \int\limits_{\mathbb{R}^N} f(x) e^{2\pi i(x,t)}\, dx,$$

$$\mathscr{F}_{-2\pi} f(t) = \int\limits_{\mathbb{R}^N} f(x) e^{-2\pi i(x,t)}\, dx.$$

Since the character groups of isomorphic groups are also isomorphic, there are grounds to using the same notation $\widehat{f}$ for generally distinct functions $\mathscr{F}f$, $\mathscr{F}_{\sim}f$, and $\mathscr{F}_{\pm 2\pi}f$. The choice of the symbol $\widehat{\ }$ for $\mathscr{F}_{2\pi}$ $(\mathscr{F}_{-2\pi})$ dictates the denotation $^{\vee}$ for $\mathscr{F}_{-2\pi}$ $(\mathscr{F}_{2\pi})$ (cf. 10.11.12).

10.11.5. Examples.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be the characteristic function of the interval $[-1,\ 1]$.

Clearly, $\widehat{f}(t) = 2t^{-1}\sin t$. Observe that if $k\pi \geq t_0 > 0$ then

$$\int\limits_{[t_0,+\infty)} |\widehat{f}(t)|\,dt \geq \int\limits_{[k\pi,+\infty)} |\widehat{f}(t)|\,dt = \sum_{n=k}^{\infty} \int\limits_{[n\pi,(n+1)\pi]} |\widehat{f}(t)|\,dt$$

$$\geq \sum_{n=k}^{\infty} \int\limits_{[n\pi,(n+1)\pi]} \frac{2|\sin t|}{(n+1)\pi}\,dt = 4\sum_{n=k}^{\infty} \frac{1}{(n+1)\pi} = +\infty.$$

Thus, $\widehat{f} \notin L_1(\mathbb{R})$.

(2) *For $f \in L_1(\mathbb{R}^N)$ the function $\widehat{f}$ is continuous, with the inequality $\|\widehat{f}\|_\infty \leq \|f\|_1$ holding.*

◁ The continuity of $\widehat{f}$ follows from the Lebesgue Dominated Convergence Theorem; and the boundedness of $\widehat{f}$, from the obvious estimate

$$|\widehat{f}(t)| \leq \int\limits_{\mathbb{R}^N} |f(x)|\,dx = \|f\|_1 \quad (t \in \mathbb{R}^N). \ \triangleright$$

(3) *If $f \in L_1(\mathbb{R}^N)$, then $|\widehat{f}(t)| \to 0$ as $|t| \to +\infty$ (= the Riemann–Lebesque Lemma).*

◁ The claim is obvious for compactly-supported step functions. It suffices to refer to 5.5.9 (6) and the containment $\mathscr{F} \in B(L_1(\mathbb{R}^N),\ l_\infty(\mathbb{R}^N))$. ▷

(4) *Let $f \in L_1(\mathbb{R}^N)$, $\varepsilon > 0$, and $f_\varepsilon(x) := f(\varepsilon x)$ $(x \in \mathbb{R}^N)$. Then $\widehat{f_\varepsilon}(t) = \varepsilon^{-N}\widehat{f}({}^t/_\varepsilon)$ $(t \in \mathbb{R}^N)$.*

◁ $\widehat{f_\varepsilon}(t) = \int\limits_{\mathbb{R}^N} f(\varepsilon x)e_t(x)\,dx = \varepsilon^{-N} \int\limits_{\mathbb{R}^N} f(\varepsilon x)e_{t/\varepsilon}(\varepsilon x)\,d\varepsilon x = \varepsilon^{-N}\widehat{f}({}^t/_\varepsilon)$ ▷

(5) $\mathscr{F}(f^*) = (\mathscr{F}_\sim f)^*$, $\widehat{\tau_x f} = e_x\widehat{f}$, and $\widehat{e_x f} = \tau_x\widehat{f}$ for all $f \in L_1(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$.

◁ We will only check the first equality. Since $a^*b = (ab^*)^*$ for $a,\ b \in \mathbb{C}$; on using the properties of conjugation and integration, given $t \in \mathbb{R}^N$, infer that

$$\mathscr{F}(f^*)(t) = \int\limits_{\mathbb{R}^N} f(x)e^{i(x,t)}\,dx = \left(\int\limits_{\mathbb{R}^N} f(x)\big(e^{i(x,t)}\big)^*\,dx\right)^*$$

$$= \left(\int\limits_{\mathbb{R}^N} f(x)e^{-i(x,t)}\,dx\right)^* = (\mathscr{F}_\sim f)^*(t). \ \triangleright$$

(6) *If $f,\ g \in L_1(\mathbb{R}^N)$ then*

$$\widehat{f * g} = \widehat{f}\,\widehat{g};\quad \int_{\mathbb{R}^N} \widehat{f} g = \int_{\mathbb{R}^N} f\widehat{g}.$$

◁ The first equality is straightforward from 10.11.1. The second, the *multiplication formula*, follows on applying the Fubini Theorem:

$$\int_{\mathbb{R}^N} \widehat{f} g = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)e_t(x)\,dx g(t)\,dt$$

$$= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} g(t)e_t(x)\,dt \right) f(x)\,dx = \int_{\mathbb{R}^N} f\widehat{g}. \ \triangleright$$

(7) *If $\widehat{f}$, f and g belong to $L_1(\mathbb{R}^N)$ then $(\widehat{f}g)^{\widehat{}} = f^{\sim} * \widehat{g}$.*

◁ Given $x \in \mathbb{R}^N$, observe that

$$(\widehat{f}g)^{\widehat{}}(x) = \int_{\mathbb{R}^N} g(t)\widehat{f}(t)e_t(x)\,dt = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g(t)f(y)e_t(y)e_t(x)\,dydt$$

$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(y)g(t)e_t(x+y)\,dtdy$$

$$= \int_{\mathbb{R}^N} f(y)\widehat{g}(x+y)\,dy = \int_{\mathbb{R}^N} f(y-x)\widehat{g}(y)\,dy = f^{\sim} * g(x). \ \triangleright$$

(8) *If $f \in \mathscr{D}(\mathbb{R}^N)$ and $\alpha \in (\mathbb{Z}_+)^N$ then*

$$\mathscr{F}(\partial^\alpha f) = i^{|\alpha|} t^\alpha \mathscr{F} f, \quad \partial^\alpha(\mathscr{F} f) = i^{|\alpha|}\mathscr{F}(x^\alpha f);$$
$$\mathscr{F}_{2\pi}(\partial^\alpha f) = (2\pi i)^{|\alpha|} t^\alpha \mathscr{F}_{2\pi} f, \quad \partial^\alpha(\mathscr{F}_{2\pi} f) = (2\pi i)^{|\alpha|}\mathscr{F}_{2\pi}(x^\alpha f)$$

(these equalities take the rather common liberty of designating $x^\alpha := t^\alpha := (\cdot)^\alpha :$ $y \in \mathbb{R}^N \mapsto y_1^{\alpha_1} \cdot \ldots \cdot y_N^{\alpha_N}$).

◁ It suffices (cf. 10.11.4) to examine only the first row. Since $\partial^\alpha e_t = i^{|\alpha|} t^\alpha e_t$; therefore,

$$\mathscr{F}(\partial^\alpha f)(t) = \big(e_t * \partial^\alpha f\big)(0)$$
$$= \big(\partial^\alpha e_t * f\big)(0) = i^{|\alpha|} t^\alpha (e_t * f)(0) = i^{|\alpha|} t^\alpha \widehat{f}(t).$$

By analogy, on differentiating under the integral sign, infer that

$$\frac{\partial}{\partial t_1}(\mathscr{F}f)(t) = \frac{\partial}{\partial t_1}\int_{\mathbb{R}^N} f(x)e^{i(x,t)}\,dx = \int_{\mathbb{R}^N} f(x)ix_1e^{i(x,t)}\,dx = \mathscr{F}(ix_1f)(t). \ \triangleright$$

(9) *If* $f_N(x) := \exp\left(-{}^1/_2|x|^2\right)$ *for* $x \in \mathbb{R}^N$, *then* $\widehat{f_N} = (2\pi)^{N/2} f_N$.

$\triangleleft$ It is clear that

$$\widehat{f_N}(t) = \prod_{k=1}^N \int_{\mathbb{R}} e^{it_kx_k}e^{-{}^1/_2|x|_k^2}\,dx_k \quad (t \in \mathbb{R}^N).$$

Consequently, the matter reduces to the case $N = 1$. Now, given $y \in \mathbb{R}$, observe that

$$\widehat{f_1}(y) = \int_{\mathbb{R}} e^{-{}^1/_2x^2}e^{ixy}\,dx = \int_{\mathbb{R}} e^{-{}^1/_2(x-iy)^2-{}^1/_2y^2}\,dx$$
$$= f_1(y)\int_{\mathbb{R}} e^{-{}^1/_2(x-iy)^2}\,dx.$$

To calculate the integral A that we are interested in, consider (concurrently oriented) straight lines λ_1 and λ_2 parallel to the real axis $\mathbb{R}$ in the complex plane $\mathbb{C}_{\mathbb{R}} \simeq \mathbb{R}^2$. Applying the Cauchy Integral Theorem to the holomorphic function $f(z) := \exp\left(-{}^{z^2}/_2\right)$ $(z \in \mathbb{C})$ and a rectangular with vertices on λ_1 and λ_2 and properly passing to the limit, conclude that $\int_{\lambda_1} f(z)\,dz = \int_{\lambda_2} f(z)\,dz$. Whence it follows that

$$A = \int_{\mathbb{R}} e^{-{}^1/_2(x-iy)^2}\,dx = \int_{\mathbb{R}} e^{-{}^1/_2x^2}\,dx = \sqrt{2\pi}. \ \triangleright$$

10.11.6. DEFINITION. The *Schwartz space* is the set of *tempered* (or *rapidly decreasing*) functions, cf. 10.11.17 (2),

$$\mathscr{S}(\mathbb{R}^N) := \left\{f \in C_\infty(\mathbb{R}^N) : \ (\forall \alpha, \beta \in (\mathbb{Z}_+)^N)\ |x| \to +\infty \Rightarrow x^\alpha \partial^\beta f(x) \to 0\right\}$$

with the multinorm $\{p_{\alpha,\beta} : \ \alpha,\ \beta \in (\mathbb{Z}_+)^N\}$, where $p_{\alpha,\beta}(f) := \|x^\alpha\partial^\beta f\|_\infty$.

This space is treated as a subspace of the space of all functions from $\mathbb{R}^N$ to $\mathbb{C}$.

10.11.7. *The following statements are valid:*

(1) $\mathscr{S}(\mathbb{R}^N)$ *is a Fréchet space;*

(2) *the operators of multiplication by a polynomial and differentiation are continuous endomorphisms of* $\mathscr{S}(\mathbb{R}^N)$;

(3) *the topology* $\mathscr{S}(\mathbb{R}^N)$ *may equivalently be given by the multinorm* $\{p_n : n \in \mathbb{N}\}$, *where*

$$p_n(f) := \sum_{|\alpha| \le n} \|(1 + |\cdot|^2)^n \partial^\alpha f\|_\infty \quad (f \in \mathscr{S}(\mathbb{R}^N))$$

(as usual, $|x|$ *stands for the Euclidean norm of a vector* x *in* $\mathbb{R}^N$*);*

(4) $\mathscr{D}(\mathbb{R}^N)$ *is dense in* $\mathscr{S}(\mathbb{R}^N)$*; furthermore, the identical embedding of* $\mathscr{D}(\mathbb{R}^N)$ *into* $\mathscr{S}(\mathbb{R}^N)$ *is continuous and* $\mathscr{S}(\mathbb{R}^N)' \subset \mathscr{D}'(\mathbb{R}^N)$;

(5) $\mathscr{S}(\mathbb{R}^N) \subset L_1(\mathbb{R}^N)$.

◁ We will check (4), because the other claims are easier.

Take $f \in \mathscr{S}(\mathbb{R}^N)$ and let ψ be a truncator in $\mathscr{D}(\mathbb{R}^N)$ such that $\mathbb{B} \subset \{\psi = 1\}$. Given $x \in \mathbb{R}^N$ and $\xi > 0$, put $\psi_\xi(x) := \psi(\xi x)$, and $f_\xi = \psi_\xi f$. Evidently, $f_\xi \in \mathscr{D}(\mathbb{R}^N)$. Let $\varepsilon > 0$ and α, $\beta \in (\mathbb{Z}_+)^N$. It is an easy matter to show that $\sup\{\|\partial^\gamma(\psi_\xi - 1)\|_\infty : \gamma \le \beta, \gamma \in (\mathbb{Z}_+)^N\} < +\infty$ for $0 < \xi \le 1$. Considering that $x^\alpha \partial^\beta f(x) \to 0$ as $|x| \to +\infty$, find $r > 1$ satisfying $|x^\alpha \partial^\beta((\psi_\xi(x) - 1) f(x))| < \varepsilon$ whenever $|x| > r$. Moreover, $f_\xi(x) - f(x) = (\psi(\xi x) - 1) f(x) = 0$ for $|x| \le \xi^{-1}$. Therefore,

$$\begin{aligned} p_{\alpha,\beta}(f_\xi - f) &= \sup_{|x| > \xi^{-1}} |x^\alpha \partial^\beta((\psi_\xi(x) - 1) f(x))| \\ &\le \sup_{|x| > r} |x^\alpha \partial^\beta((\psi_\xi(x) - 1) f(x))| < \varepsilon \end{aligned}$$

for $\xi \le r^{-1}$. Thus, $p_{\alpha,\beta}(f_\xi - f) \to 0$ as $\xi \to 0$; i.e., $f_\xi \to f$ in $\mathscr{S}(\mathbb{R}^N)$. The required continuity of the identical embedding raises no doubt. ▷

10.11.8. *The Fourier transform is a continuous endomorphism of* $\mathscr{S}(\mathbb{R}^N)$.

◁ Given $f \in \mathscr{D}(\mathbb{R}^N)$, from 10.11.5 (8), 10.11.5 (2) and the Hölder inequality obtain

$$\|t^\alpha \widehat{f}\|_\infty = \|(\partial^\alpha f)\widehat{\ }\|_\infty \le \|\partial^\alpha f\|_1 \le K \|\partial^\alpha f\|_\infty.$$

Thus,

$$\|t^\alpha \partial^\beta \widehat{f}\|_\infty = \|t^\alpha (x^\beta f)\widehat{\ }\|_\infty \le K' \|\partial^\alpha (x^\beta f)\|_\infty.$$

Whence it is easily seen that $\widehat{f} \in \mathscr{S}(\mathbb{R}^N)$ and the restriction of $\mathscr{F}$ to $\mathscr{D}(Q)$ with $Q \Subset \mathbb{R}^N$ is continuous. It remains to refer to 10.10.7 (4) and 10.11.7 (4). ▷

10.11.9. Theorem. *The repeated Fourier transform in the space $\mathscr{S}(\mathbb{R}^N)$ is proportional to the reflection.*

◁ Let $f \in \mathscr{S}(\mathbb{R}^N)$ and $g(x) := f_N(x) = \exp\left(-{}^1/_2|x|^2\right)$. From 10.11.8 and 10.11.7 derive that $\widehat{f}$, f, $g \in L_1(\mathbb{R}^N)$ and so, by 10.11.5 (7), $(\widehat{f}g)^{\widehat{}} = f^{\sim} * \widehat{g}$. Put $g_\varepsilon(x) := g(\varepsilon x)$ for $x \in \mathbb{R}^N$ and $\varepsilon > 0$. Then for the same x, in view of 10.11.5 (4) find

$$\int_{\mathbb{R}^N} g(\varepsilon t)\widehat{f}(t)e_t(x)\,dt = \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N} f(y-x)\widehat{g}\left(\frac{y}{\varepsilon}\right)dy = \int_{\mathbb{R}^N} f(\varepsilon y - x)\widehat{g}(y)\,dy.$$

Using 10.11.5 (9) and the Lebesgue Dominated Convergence Theorem as $\varepsilon \to 0$, infer that

$$g(0)\int_{\mathbb{R}^N} \widehat{f}(t)e_t(x)\,dt = f(-x)\int_{\mathbb{R}^N} \widehat{g}(y)\,dy$$

$$= (2\pi)^{N/2} f(x)\int_{\mathbb{R}^N} e^{-{}^1/_2|x|^2}\,dx = (2\pi)^N f(-x).$$

Finally, $\mathscr{F}^2 f = (2\pi)^N f^{\sim}$. ▷

10.11.10. Corollary. *$\mathscr{F}_{2\pi}^2$ is the reflection and $(\mathscr{F}_{2\pi})^{-1} = \mathscr{F}_{-2\pi}$.*

◁ Given $f \in \mathscr{S}(\mathbb{R}^N)$ and $t \in \mathbb{R}^N$, deduce that

$$f(-t) = (2\pi)^N \int_{\mathbb{R}^N} e^{i(x,t)}\widehat{f}(x)\,dx = \int_{\mathbb{R}^N} e^{2\pi i(x,t)}\widehat{f}(2\pi x)\,dx$$

$$= (\mathscr{F}_{2\pi}(\mathscr{F}_{2\pi} f))(t).$$

Since $\mathscr{F}_{2\pi} f^{\sim} = \mathscr{F}_{-2\pi} f$, the proof is complete. ▷

10.11.11. Corollary. *$\mathscr{S}(\mathbb{R}^N)$ is a convolution algebra (= an algebra with convolution as multiplication).*

◁ For f, $g \in \mathscr{S}(\mathbb{R}^N)$ the product fg is an element of $\mathscr{S}(\mathbb{R}^N)$ and so $\widehat{f}\widehat{g} \in \mathscr{S}(\mathbb{R}^N)$. From 10.11.5 (6) infer that $\mathscr{F}_{2\pi}(f * g) \in \mathscr{S}(\mathbb{R}^N)$ and, consequently, by 10.11.10, $f * g = \mathscr{F}_{-2\pi}(\mathscr{F}_{2\pi}(f * g)) \in \mathscr{S}(\mathbb{R}^N)$. ▷

10.11.12. Inversion Theorem. *The Fourier transform $\mathfrak{F} := \mathscr{F}_{2\pi}$ is a topological automorphism of the Schwartz space $\mathscr{S}(\mathbb{R}^N)$, with convolution carried into pointwise multiplication. The inverse transform $\mathfrak{F}^{-1}$ equals $\mathscr{F}_{-2\pi}$, with pointwise multiplication carried into convolution. Moreover, the Parseval identity holds:*

$$\int_{\mathbb{R}^N} f g^* = \int_{\mathbb{R}^N} \widehat{f}\widehat{g}^* \quad (f,\ g \in \mathscr{S}(\mathbb{R}^N)).$$

◁ In view of 10.11.10 and 10.11.5 (5), only the sought identity needs examining. Moreover, given f and g, from 10.11.5 (7) and 10.11.7 (4), obtain $(\widehat{f}g)(0) = (\widetilde{f} * \widehat{g})(0)$. Using 10.11.5 (5), conclude that

$$\int\limits_{\mathbb{R}^N} fg^* = (\mathfrak{F}(\mathfrak{F}^{-1}f)g^*)\widehat{\ }(0) = ((\mathfrak{F}^{-1}f)^{\sim} * \mathfrak{F}g^*)(0)$$

$$= \int\limits_{\mathbb{R}^N} \mathfrak{F}f(\mathfrak{F}g^*)^{\sim} dx = \int\limits_{\mathbb{R}^N} \mathfrak{F}_f\mathfrak{F}_{\sim}(g^*)\, dx = \int\limits_{\mathbb{R}^N} \mathfrak{F}f(\mathfrak{F}g)^*. \ ▷$$

10.11.13. Remark. In view of 10.11.9, the *theorem on the repeated Fourier transform*, the following mutually inverse operators

$$\overline{\mathfrak{F}}f(t) = \frac{1}{(2\pi)^{N/2}} \int\limits_{\mathbb{R}^N} f(x)e^{i(x,t)}\, dx;$$

$$\overline{\mathfrak{F}}^{-1}f(x) = \frac{1}{(2\pi)^{N/2}} \int\limits_{\mathbb{R}^N} f(t)e^{-i(x,t)}\, dt$$

are considered alongside $\mathfrak{F}$. In this case, an analog of 10.11.12 is valid on condition that convolution is redefined as $f \overline{*} g := (2\pi)^{-N/2} f * g \quad (f,\ g \in L_1(\mathbb{R}^N))$. The merits of $\overline{\mathfrak{F}}$ and $\overline{\mathfrak{F}}^{-1}$ are connected with some simplification of 10.11.5 (8). In the case of $\mathfrak{F}$, a similar goal is achieved by introducing the differential operator $D^\alpha := (2\pi i)^{-|\alpha|}\partial^\alpha$ with $\alpha \in (\mathbb{Z}_+)^N$.

10.11.14. *Plancherel Theorem.* *The Fourier transform in the Schwartz space $\mathscr{S}(\mathbb{R}^N)$ is uniquely extendible to an isometric automorphism of $L_2(\mathbb{R}^N)$.*

◁ Immediate from 10.11.12 and 4.5.10 since $\mathscr{S}(\mathbb{R}^N)$ is dense in $L_2(\mathbb{R}^N)$. ▷

10.11.15. Remark. The extension, guaranteed by 10.11.14, retains the previous name and notation. Rarely (in search of emphasizing distinctions and subtleties) one speaks of the *Fourier–Plancherel transform* or the *L_2-Fourier transform*. It that event it stands to reason to specify the understanding of the integral formulas for $\mathfrak{F}f$ and $\mathfrak{F}^{-1}f$ with $f \in L_2(\mathbb{R}^N)$ which are treated as the results of appropriate passage to the limit in $L_2(\mathbb{R}^N)$.

10.11.16. Definition. Let $u \in \mathscr{S}'(\mathbb{R}^N) := \mathscr{S}(\mathbb{R}^N)'$. Such a distribution u is referred to as a *tempered distribution* (variants: a *distribution of slow growth*, a *slowly increasing distribution*, etc.). The space $\mathscr{S}'(\mathbb{R}^N)$ of tempered distributions is furnished with the weak topology $\sigma(\mathscr{S}'(\mathbb{R}^N),\ \mathscr{S}(\mathbb{R}^N))$ and is sometimes called the *Schwartz space* (as well as $\mathscr{S}(\mathbb{R}^N)$).

10.11.17. Examples.

(1) *$L_p(\mathbb{R}^N) \subset \mathscr{S}'(\mathbb{R}^N)$ for $1 \le p \le +\infty$.*

◁ Let $f \in L_p(\mathbb{R}^N)$, $\psi \in \mathscr{S}(\mathbb{R}^N)$, $p < +\infty$ and $^1/_{p'} + {}^1/_p = 1$. Using Hölder inequality, for suitable positive K, K', and K'' successively infer that

$$\|\psi\|_{p'} \le \left(\int\limits_{\mathbb{B}} |\psi|^p\right)^{1/p} + \left(\int\limits_{\mathbb{R}^N \setminus \mathbb{B}} |(1+|x|^2)^N (1+|x|^2)^{-N}\psi(x)|^p \, dx\right)^{1/p}$$

$$\le K'\|\psi\|_\infty + \|(1+|\cdot|^2)^N \psi\|_\infty \left(\int\limits_{\mathbb{R}^N\setminus\mathbb{B}} \frac{dx}{(1+|x|^2)^{Np}}\right)^{1/p} \le K'' p_1(\psi).$$

Once again using the Hölder inequality, observe that

$$|u_f(\psi)| = |\langle \psi \mid f\rangle| = \left|\int\limits_{\mathbb{R}^N} f\psi\right| \le \|f\|_p \, \|\psi\|_{p'} \le K p_1(\psi).$$

The case $p = +\infty$ raises no doubts. ▷

(2) *$\mathscr{S}(\mathbb{R}^N)$ is dense in $\mathscr{S}'(\mathbb{R}^N)$.*

◁ Follows from 10.11.7 (4), 10.11.17 (1), 10.11.7 (5), and 10.10.9 (4). ▷

(3) Let $\mu \in \mathscr{M}(\mathbb{R}^N)$ be a *tempered Radon measure*; i.e.,

$$\int\limits_{\mathbb{R}^N} \frac{d|\mu|(x)}{(1+|x|^2)^n} < +\infty$$

for some $n \in \mathbb{N}$. Evidently, μ is a tempered distribution.

(4) If $u \in \mathscr{S}'(\mathbb{R}^N)$, $f \in \mathscr{S}(\mathbb{R}^N)$ and $\alpha \in (\mathbb{Z}_+)^N$ then $fu \in \mathscr{S}'(\mathbb{R}^N)$ and $\partial^\alpha u \in \mathscr{S}'(\mathbb{R}^N)$ in virtue of 10.11.7 (2). By a similar argument, putting $D^\alpha u(f) := (-1)^{|\alpha|} u D^\alpha f$ for $f \in \mathscr{S}(\mathbb{R}^N)$, infer that $D^\alpha u \in \mathscr{S}'(\mathbb{R}^N)$ and $D^\alpha u = (2\pi i)^{-|\alpha|}\partial^\alpha u$.

(5) *Every compactly-supported distribution is tempered.*

◁ In accordance with 10.10.5 (7) such u in $\mathscr{D}'(\mathbb{R}^N)$ may be identified with a member of $\mathscr{E}'(\mathbb{R}^N)$. Since the topology of $\mathscr{S}(\mathbb{R}^N)$ is stronger than that induced by the identical embedding in $C_\infty(\mathbb{R}^N)$, conclude that $u \in \mathscr{S}'(\mathbb{R}^N)$. ▷

(6) Let $u \in \mathscr{S}'(\mathbb{R}^N)$. If $f \in \mathscr{S}(\mathbb{R}^N)$ then u convolutes with f and $u * f \in \mathscr{S}(\mathbb{R}^N)$. In may be shown that u also convolutes with every distribution v, a member of $\mathscr{E}'(\mathbb{R}^N)$, and $u * v \in \mathscr{S}'(\mathbb{R}^N)$.

(7) Take $u \in \mathscr{D}'(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$. Let $\tau_x u := (\tau_{-x})' u = u \circ \tau_{-x}$ be the corresponding *translation* of u. A distribution u is *periodic* (with period x) if $\tau_x u = u$. Every periodic distribution is tempered. Periodicity is preserved under differentiation and convolution.

(8) If $u_n \in \mathscr{S}'(\mathbb{R}^N)$ ($u \in \mathbb{N}$) and for every $f \in \mathscr{S}(\mathbb{R}^N)$ there is a sum $u(f) := \sum_{n=1}^{\infty} u_n(f)$, then $u \in \mathscr{S}'(\mathbb{R}^N)$ and $\partial^\alpha u = \sum_{n=1}^{\infty} \partial^\alpha u_n$ (cf. 10.10.10).

10.11.18. Theorem. *Each tempered distribution is the sum of derivatives of tempered measures.*

◁ Let $u \in \mathscr{S}'(\mathbb{R}^N)$. On account of 10.11.7 (3) and 5.3.7, for some $n \in \mathbb{N}$ and $K > 0$, observe that

$$|u(f)| \le K \sum_{|\alpha| \le n} \left\| (1 + |\cdot|^2)^n \partial^\alpha f \right\|_\infty \quad (f \in \mathscr{S}(\mathbb{R}^N)).$$

From 3.5.3 and 3.5.7, for some $\mu_\alpha \in M(\mathbb{R}^N)$, obtain

$$u(f) = \int_{|\alpha| \le n} \mu_\alpha \left((1 + |\cdot|^2)^n \partial^\alpha f \right) \quad (f \in \mathscr{S}(\mathbb{R}^N)).$$

Let $\nu_\alpha := (-1)^{|\alpha|}(1 + |\cdot|^2)^n \mu_\alpha$. Then ν_α is tempered and $u = \sum_{|\alpha| \le n} \partial^\alpha \nu_\alpha$. ▷

10.11.19. DEFINITION. For $u \in \mathscr{S}'(\mathbb{R}^N)$, the distribution $\mathfrak{F} u$ acting as

$$\langle f \mid \mathfrak{F} u \rangle = \langle \mathfrak{F} f \mid u \rangle \quad (f \in \mathscr{S}(\mathbb{R}^N))$$

is the *Fourier transform* or, amply, the *Fourier–Schwartz transform* of u.

10.11.20. Theorem. *The Fourier–Schwartz transform $\mathfrak{F}$ is a unique extension of the Fourier transform in $\mathscr{S}(\mathbb{R}^N)$ to a topological automorphism of $\mathscr{S}'(\mathbb{R}^N)$. The inverse $\mathfrak{F}^{-1}$ of $\mathfrak{F}$ is a unique continuous extension of the inverse Fourier transform in $\mathscr{S}(\mathbb{R}^N)$.*

◁ The Fourier–Schwartz transform in $\mathscr{S}'(\mathbb{R}^N)$ is the dual of the Fourier transform in $\mathscr{S}(\mathbb{R}^N)$. It remains only to appeal to 10.11.7 (5), 10.11.12, 10.11.17 (2), and 4.5.10. ▷

Exercises

10.1. Give examples of linear topological spaces and locally convex spaces as well as constructions leading to them.

10.2. Prove that a Hausdorff topological vector space is finite-dimensional if and only if it is locally compact.

10.3. Characterize weakly continuous sublinear functionals.

10.4. Prove that the weak topology of a locally convex space is normable or metrizable if and only if the space is finite-dimensional.

10.5. Describe weak convergence in classical Banach spaces.

10.6. Prove that a normed space is finite-dimensional if and only if its ***unit sphere***, comprising all norm-one vectors, is weakly closed.

10.7. Assume that an operator T carries each weakly convergent net into a norm convergent net. Prove that T has finite rank.

10.8. Let X and Y be Banach spaces and let T be a linear operator from X to Y. Prove that T is bounded if and only if T is weakly continuous (i.e., continuous as a mapping from $(X, \sigma(X, X'))$ to $(Y, \sigma(Y, Y')))$.

10.9. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms making X into a Banach space. Assume further that $(X, \|\cdot\|_1)' \cap (X, \|\cdot\|_2)'$ separates the points of X. Prove that these norms are equivalent.

10.10. Let S act from Y' to X'. When does S serve as the dual of some operator from X to Y?

10.11. What is the Mackey topology $\tau(X, X^{\#})$?

10.12. Let $(X_\xi)_{\xi\in\Xi}$ be a family of locally convex spaces, and let $X := \prod_{\xi\in\Xi} X_\xi$ be the product of the family. Validate the presentations:

$$\sigma(X, X') = \prod_{\xi\in\Xi} \sigma(X_\xi, X'_\xi); \quad \tau(X, X') = \prod_{\xi\in\Xi} \tau(X_\xi, X'_\xi).$$

10.13. Let X and Y be Banach spaces and let T be a element of $B(X, Y)$ satisfying $\operatorname{im} T = Y$. Demonstrate that Y is reflexive provided so is X.

10.14. Show that the spaces $(X')''$ and $(X'')'$ coincide.

10.15. Prove that the space c_0 has no infinite-dimensional reflexive subspaces.

10.16. Let p be a continuous sublinear functional on Y, and let $T \in \mathscr{L}(X, Y)$ be a continuous linear operator. Establish the following inclusion of the sets of extreme points: $\operatorname{ext} T'(\partial p) \subset T'(\operatorname{ext} \partial p)$.

10.17. Let p be a continuous seminorm on X and let $\mathscr{X}$ be a subspace of X. Prove that $f \in \operatorname{ext}(\mathscr{X}^\circ \cap \partial p)$ if and only if the next equality holds:

$$X = \operatorname{cl} \mathscr{X} + \{p - f \le 1\} - \{p - f \le 1\}.$$

10.18. Prove that the absolutely convex hull of a totally bounded subset of a locally convex space is also totally bounded.

10.19. Establish that barreledness is preserved under passage to the inductive limit. What happens with other linear topological properties?

Chapter 11
Banach Algebras

11.1. The Canonical Operator Representation

11.1.1. DEFINITION. An element e of an algebra A is called a *unity element* if $e \neq 0$ and $ea = ae = a$ for all $a \in A$. Such an element is obviously unique and is also referred to as the *unity* or the *identity* or the *unit* of A. An algebra A is *unital* provided that A has unity.

11.1.2. REMARK. Without further specification, we only consider unital algebras over a basic field $\mathbb{F}$. Moreover, for simplicity, it is assumed that $\mathbb{F} := \mathbb{C}$, unless stated otherwise. In studying a representation of unital algebras, we naturally presume that it preserves unity. In other words, given some algebras A_1 and A_2, by a representation of A_1 in A_2 we henceforth mean a *morphism*, a multiplicative linear operator, from A_1 to A_2 which sends the unity element of A_1 to the unity element of A_2.

For an algebra A without unity, the process of *unitization* or *adjunction of unity* is in order. Namely, the vector space $\mathscr{A}_e := A \times \mathbb{C}$ is transformed into an algebra by putting $(a, \lambda)(b, \mu) := (ab + \mu a + \lambda b, \lambda\mu)$, where $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$. In the normed case, it is also taken for granted that $\|(a, \lambda)\|_{\mathscr{A}_e} := \|a\|_A + |\lambda|$.

11.1.3. DEFINITION. An element a_r in A is a *right inverse* of a if $aa_r = e$. An element a_l of A is a *left inverse* of a if $a_l a = e$.

11.1.4. *If an element has left and right inverses then the latter coincide.*

$\lhd$ $a_r = (a_l a)a_r = a_l(aa_r) = a_l e = a_l$ $\rhd$

11.1.5. DEFINITION. An element a of an algebra A is called *invertible*, in writing $a \in \operatorname{Inv}(A)$, if a has a left and right inverse. Denote $a^{-1} := a_r = a_l$. The element a^{-1} is the *inverse* of a. A subalgebra (with unity) B of an algebra A is called *pure* or *full* or *inverse-closed* in A if $\operatorname{Inv}(B) = \operatorname{Inv}(A) \cap B$.

11.1.6. ***Theorem.*** *Let A be a Banach algebra. Given $a \in A$, put $L_a : x \mapsto ax$ $(x \in A)$. Then the mapping*

$$L_A := L : a \mapsto L_a \quad (a \in A)$$

is a faithful operator representation. Moreover, $L(A)$ is a pure closed subalgebra of $B(A)$ and $L : A \to L(A)$ is a topological isomorphism.

◁ Clearly,

$$L(ab) : x \mapsto L_{ab}(x) = abx = a(bx) = L_a(L_b x) = (La)(Lb)x$$

for $x,\ a,\ b \in A$; i.e., L is a representation (because the linearity of L is obvious). If $La = 0$ then $0 = La(e) = ae = a$, so that L is a faithful representation. To prove the closure property of the range $L(A)$, consider the algebra A_r coinciding with A as a vector space and equipped with the *reversed multiplication* $ab := ba$ $(a,\ b \in A)$.

Let $R := L_{A_r}$, i.e. $R_a := Ra : x \mapsto xa$ for $a \in A$. Check that $L(A)$ is in fact the *centralizer* of the range $R(A)$, i.e. the closed subalgebra

$$Z(\operatorname{im} R) := \{T \in B(A) :\ TR_a = R_a T\ (a \in A)\}.$$

Indeed, if $T \in L(A)$, i.e. $T = L_a$ for some $a \in A$, then $L_a R_b(x) = axb = R_b(L_a(x)) = R_b L_a(x)$ for all $b \in A$. Hence, $T \in Z(R(A))$. If, in turn, $T \in Z(R(A))$ then, putting $a := Te$, find

$$L_a x = ax = (Te)x = R_x(Te) = (R_x T)e = (TR_x)e = T(R_x e) = Tx$$

for all $x \in A$. Consequently, $T = L_a \in L(A)$. Thus, $L(A)$ is a Banach subalgebra of $B(A)$.

For $T = L_a$ there is a T^{-1} in $B(A)$. Put $b := T^{-1}e$ and observe that $ab = L_a b = Tb = TT^{-1}e = e$. Moreover, $ab = e \Rightarrow aba = a \Rightarrow T(ba) = L_a ba = aba = a = L_a e = Te$. Whence $ba = e$, because T is a monomorphism. Thus, $L(A)$ is a subalgebra of A.

By the definition of a Banach algebra, the norm is submultiplicative, providing

$$\|L\| = \sup\{\|La\| :\ \|a\| \le 1\} = \sup\{\|ab\| :\ \|a\| \le 1,\ \|b\| \le 1\} \le 1.$$

Using the Banach Isomorphism Theorem, conclude that L is a topological isomorphism (i.e., L^{-1} is a continuous operator from $L(A)$ onto A). ▷

11.1.7. Definition. The representation L_A, constructed in 11.1.6, is the *canonical operator representation* of A.

11.1.8. Remark. The presence of the canonical operator representation allows us to confine the subsequent exposition to studying Banach algebras with norm-one unity.

For such an algebra A the canonical operator representation L_A implements an isometric embedding of A into the endomorphism algebra $B(A)$ or, in short, an *isometric representation* of A in $B(A)$. In this case, L_A implements an *isometric* *isomorphism* between the algebras A and $L(A)$. The same natural terminology is used for studying representations of arbitrary Banach algebras. Observe immediately that the canonical operator representation of A, in particular, justifies the use of the symbol λ in place of λe for $\lambda \in \mathbb{C}$, where e is the unity of A (cf. 5.6.5). In other words, henceforth the isometric representation $\lambda \mapsto \lambda e$ is considered as the identification of $\mathbb{C}$ with the subalgebra $\mathbb{C}e$ of A.

11.2. The Spectrum of an Element of an Algebra

11.2.1. DEFINITION. Let A be a Banach algebra and $a \in A$. A scalar λ in $\mathbb{C}$ is a *resolvent value* of a, in writing $\lambda \in \operatorname{res}(a))$, if $(\lambda - a) \in \operatorname{Inv}(A)$. The *resolvent* $R(a, \lambda)$ of a at λ is $R(a, \lambda) := \frac{1}{\lambda - a} := (\lambda - a)^{-1}$. The set $\operatorname{Sp}(a) := \mathbb{C} \setminus \operatorname{res}(a)$ is the *spectrum* of a, with a point of $\operatorname{Sp}(a)$ a *spectral value* of a. When it is necessary, more detailed designations like $\operatorname{Sp}_A(a)$ are in order.

11.2.2. *For $a \in A$ the equalities hold*:

$$\operatorname{Sp}_A(a) = \operatorname{Sp}_{L(A)}(L_a) = \operatorname{Sp}(L_a);$$
$$LR(a, \lambda) = R(L_a, \lambda) \quad (\lambda \in \operatorname{res}(a) = \operatorname{res}(L_a)). \ \triangleleft\triangleright$$

11.2.3. ***Gelfand–Mazur Theorem.*** *The field of complex numbers is up to isometric isomorphism the sole Banach division algebra (or skew field); i.e., each complex Banach algebra with norm-one unity and invertible nonzero elements has an isometric representation on $\mathbb{C}$.*

$\triangleleft$ Consider $\Psi : \lambda \mapsto \lambda e$, with e the unity of A and $\lambda \in \mathbb{C}$. It is clear that Ψ represents $\mathbb{C}$ in A. Take $a \in A$. By virtue of 11.2.2 and 8.1.11, $\operatorname{Sp}(a) \neq \varnothing$. Consequently, there is $\lambda \in \mathbb{C}$ such that the element $(\lambda - a)$ is not invertible; i.e., $a = \lambda e$ by hypothesis. Hence, Ψ is an epimorphism. Moreover, $\|\Psi(\lambda)\| = \|\lambda e\| = |\lambda| \, \|e\| = |\lambda|$ so that Ψ is an isometry. $\triangleright$

11.2.4. ***Shilov Theorem.*** *Let A be a Banach algebra and let B be a closed (unital) subalgebra of A. Then*

$$\operatorname{Sp}_B(b) \supset \operatorname{Sp}_A(b), \quad \partial \operatorname{Sp}_A(b) \supset \partial \operatorname{Sp}_B(b)$$

for all $b \in B$.

$\triangleleft$ If $\bar{b} := \lambda - b \in \operatorname{Inv}(B)$ then surely $\bar{b} \in \operatorname{Inv}(A)$. Whence $\operatorname{res}_B(b) \subset \operatorname{res}_A(b)$; i.e.,

$$\operatorname{Sp}_B(b) = \mathbb{C} \setminus \operatorname{res}_B(b) \supset \mathbb{C} \setminus \operatorname{res}_A(b) = \operatorname{Sp}_A(b).$$

If $\lambda \in \partial \operatorname{Sp}_B(b)$ then $\bar{b} \in \partial \operatorname{Inv}(B)$. Therefore, there is a sequence (b_n), $b_n \in \operatorname{Inv}(B)$, convergent to $\bar{b}$. Putting $t := \sup_{n \in \mathbb{N}} \|b_n^{-1}\|$, deduce that

$$\|b_n^{-1} - b_m^{-1}\| = \|b_n^{-1}(1 - b_n b_m^{-1})\| = \|b_n^{-1}(b_m - b_n) b_m^{-1}\| \le t^2 \|b_n - b_m\|.$$

In other words, if $t < +\infty$ then there is a limit $a := \lim b_n^{-1}$ in B. Multiplication is jointly continuous, and so $a\bar{b} = \bar{b}a = 1$; i.e., $\bar{b} \in \mathrm{Inv}\,(B)$. Since $\mathrm{Inv}\,(B)$ is open by the Banach Inversion Stability Theorem and 11.1.6, arrive at a contradiction to the containment $\bar{b} \in \partial\,\mathrm{Inv}\,(B)$.

Therefore, it may be assumed (on dropping to a subsequence, if need be) that $\left\|b_n^{-1}\right\| \to +\infty$. Put $a_n := \left\|b_n^{-1}\right\|^{-1} b_n^{-1}$. Then

$$\left\|\bar{b}a_n\right\| = \left\|(\bar{b} - b_n)a_n + b_n a_n\right\|$$

$$\leq \left\|\bar{b} - b_n\right\| \left\|a_n\right\| + \left\|b_n^{-1}\right\|^{-1} \left\|b_n b_n^{-1}\right\| \to 0.$$

Whence it follows that $\bar{b}$ is not invertible. Indeed, in the opposite case for $a := \bar{b}^{-1}$ it would hold that

$$1 = \|a_n\| = \left\|a\bar{b}a_n\right\| \leq \|a\|\,\left\|\bar{b}a_n\right\| \to 0.$$

Finally, conclude that $\lambda - b$ does not belong to $\mathrm{Inv}\,(A)$; i.e., $\lambda \in \mathrm{Sp}_A(b)$. Since λ is a boundary point of a larger set $\mathrm{Sp}_B(b)$; undoubtedly, $\lambda \in \partial\mathrm{Sp}_A(b)$. ▷

11.2.5. *Corollary.* *If* $\mathrm{Sp}_B(b)$ *lacks interior points then* $\mathrm{Sp}_B(b) = \mathrm{Sp}_A(b)$.

◁ $\mathrm{Sp}_B(b) = \partial\mathrm{Sp}_B(b) \subset \partial\mathrm{Sp}_B(b) \subset \partial\mathrm{Sp}_A(b) \subset \mathrm{Sp}_A(b) \subset \mathrm{Sp}_B(b)$ ▷

11.2.6. REMARK. The Shilov Theorem is often referred to as the *Unremovable Spectral Boundary Theorem* and verbalized as follows: "A boundary spectral value is unremovable."

11.3. The Holomorphic Functional Calculus in Algebras

11.3.1. DEFINITION. Let a be an element of a Banach algebra A, and let $h \in \mathscr{H}(\mathrm{Sp}(a))$ be a germ of a holomorphic function on the spectrum of a. Put

$$\mathscr{R}_a h := \frac{1}{2\pi i} \oint \frac{h(z)}{z - a}\, dz.$$

The element $\mathscr{R}_a h$ of A is the *Riesz–Dunford integral* of h. If, in particular, $f \in H(\mathrm{Sp}(a))$ is a function holomorphic in a neighborhood about the spectrum of a, then $f(a) := \mathscr{R}_a \bar{f}$.

11.3.2. *Gelfand–Dunford Theorem.* *The Riesz–Dunford integral* $\mathscr{R}_a$ *represents the algebra of germs of holomorphic functions on the spectrum of an element* a *of a Banach algebra* A *in* A. *Moreover, if* $f(z) := \sum_{n=0}^{\infty} c_n z^n$ *(in a neighborhood of* $\mathrm{Sp}(a)$*) then* $f(a) := \sum_{n=0}^{\infty} c_n a^n$.

From 11.2.3, 8.2.1, and 11.2.2 obtain

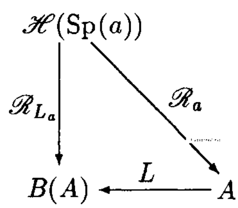
$$(L\mathscr{R}_a h)(b) = L_{\mathscr{R}_a h} b = (\mathscr{R}_a h) b$$

$$= \frac{1}{2\pi i} \oint h(z) R(a,\ z)\, dz b = \frac{1}{2\pi i} \oint h(z) R(a,\ z)\, b dz$$

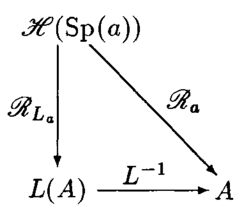
$$= \frac{1}{2\pi i} \oint h(z) R(L_a,\ z)\, b dz = \frac{1}{2\pi i} \oint h(z) R(L_a,\ z)\, dz b$$

$$= \mathscr{R}_{L_a} h(b)$$

for all $b \in A$. In particular, im L includes the range of $\mathscr{R}_{L_a}(\mathscr{H}(\mathrm{Sp}(a)))$. Therefore, the already-proven commutativity of the diagram



implies that the following diagram also commutes:



It remains to appeal to 11.1.6 and the Gelfand–Dunford Theorem in an operator setting. ▷

11.3.3. REMARK. All that we have established enables us to use in the sequel the rules of the holomorphic functional calculus which were exposed in 8.2 for the endomorphism algebra $B(X)$, with X a Banach space.

11.4. Ideals of Commutative Algebras

11.4.1. DEFINITION. Let A be a commutative algebra. A subspace J of A is an *ideal* of A, in writing $J \lhd A$, provided that $AJ \subset J$.

11.4.2. *The set $J(A)$ of all ideals of A, ordered by inclusion, is a complete lattice. Moreover,*

$$\sup_{J(A)} \mathscr{E} = \sup_{\mathrm{Lat}\,(A)} \mathscr{E}, \quad \inf_{J(A)} \mathscr{E} = \inf_{\mathrm{Lat}\,(A)} \mathscr{E},$$

for every subset $\mathscr{E}$ of $J(A)$; i.e., $J(A)$ is embedded into the complete lattice Lat (A) *of all subspaces of A with preservation of suprema and infima of arbitrary subsets.*

$\lhd$ Clearly, 0 is the least ideal, whereas A is the greatest ideal. Furthermore, the intersection of ideals and the sum of finitely many ideals are ideals. It remains to refer to 2.1.5 and 2.1.6. $\rhd$

11.4.3. *Let $J_0 \lhd A$. Assume further that $\varphi : A \to A/J_0$ is the coset mapping of A onto the quotient algebra $\overline{A} := A/J_0$. Then*

$$J \lhd A \Rightarrow \varphi(J) \lhd \overline{A}; \quad \overline{J} \lhd \overline{A} \Rightarrow \varphi^{-1}(\overline{J}) \lhd A.$$

$\lhd$ Since by definition $\overline{a}\overline{b} := \varphi(\varphi^{-1}(\overline{a})\varphi^{-1}(\overline{b}))$ for $\overline{a}$, $\overline{b} \in \overline{A}$, the operator φ is multiplicative: $\varphi(ab) = \varphi(a)\varphi(b)$ for a, $b \in A$. Whence successively derive

$$\varphi(J) \subset \overline{A}\varphi(J) = \varphi(A)\varphi(J) \subset \varphi(AJ) \subset \varphi(J);$$
$$\varphi^{-1}(\overline{J}) \subset A\varphi^{-1}(\overline{J}) \subset \varphi^{-1}(\varphi(A)\overline{J}) = \varphi^{-1}(\overline{A\overline{J}}) \subset \varphi^{-1}(\overline{J}). \ \rhd$$

11.4.4. *Let $J \lhd A$ and $J \neq 0$. The following conditions are equivalent:*

(1) $A \neq J$;
(2) $1 \notin J$;
(3) *no element of J has a left inverse.* $\lhd\rhd$

11.4.5. DEFINITION. An ideal J of A is called *proper* if J is other than A. A maximal element of the set of proper ideals (ordered by inclusion) is a *maximal ideal.*

11.4.6. *A commutative algebra is a division algebra if and only if it has no proper ideals other than zero.* $\lhd\rhd$

11.4.7. *Let J be a proper ideal of A. Then (J is maximal) $\Leftrightarrow$ (A/J is a field).*

$\lhd$ $\Rightarrow$: Let $\overline{J} \lhd A/J$. Then, by 11.4.3, $\varphi^{-1}(\overline{J}) \lhd A$. Since, beyond a doubt, $J \subset \varphi^{-1}(\overline{J})$; therefore, either $J = \varphi^{-1}(\overline{J})$ and $0 = \varphi(J) = \varphi(\varphi^{-1}(\overline{J})) = \overline{J}$, or $A = \varphi^{-1}(\overline{J})$ and $\overline{J} = \varphi(\varphi^{-1}(\overline{J})) = \varphi(A) = A/J$ in virtue of 1.1.6. Consequently, A/J has no proper ideals other than zero. It remains to refer to 11.4.6.

$\Leftarrow$: Let $J_0 \lhd A$ and $J_0 \subset J$. Then, by 11.4.3, $\varphi(J_0) \lhd A/J$. In virtue of 11.4.6, either $\varphi(J_0) = 0$ or $\varphi(J_0) = A/J$. In the first case, $J_0 \subset \varphi^{-1} \circ \varphi(J_0) \subset \varphi^{-1}(0) = J$ and $J = J_0$. In the second case, $\varphi(J_0) = \varphi(A)$; i.e., $A = J_0 + J \subset J_0 + J_0 = J_0 \subset A$. Thus, J is a maximal ideal. $\rhd$

11.4.8. *Krull Theorem.* *Each proper ideal is included in some maximal ideal.*

◁ Let J_0 be a proper ideal of an algebra A. Assume further that $\mathscr{E}$ is the set comprising all proper ideals J of A such that $J_0 \subset J$. In virtue of 11.4.2 each chain $\mathscr{E}_0$ in $\mathscr{E}$ has a least upper bound: $\sup \mathscr{E} = \cup\{J :\ J \in \mathscr{E}_0\}$. By 11.4.4 the ideal $\sup \mathscr{E}_0$ is proper. Thus, $\mathscr{E}$ is inductive and the claim follows from the Kuratowski–Zorn Lemma. ▷

11.5. Ideals of the Algebra $C(Q,\ \mathbb{C})$

11.5.1. *Minimal Ideal Theorem.* *Let J be an ideal of the algebra $C(Q,\ \mathbb{C})$ of complex continuous functions on a compactum Q. Assume further that*

$$Q_0 := \cap\{f^{-1}(0) :\ f \in J\};$$
$$J_0 := \{f \in C(Q,\ \mathbb{C}) :\ \operatorname{int} f^{-1}(0) \supset Q_0\}.$$

Then $J_0 \lhd C(Q,\ \mathbb{C})$ and $J_0 \subset J$.

◁ Let $Q_1 := \operatorname{cl}(Q \setminus f^{-1}(0))$ for a function f, a member of J_0. By hypothesis, $Q_1 \cap Q_0 = \varnothing$. To prove the containment $f \in J$ it is necessary (and, certainly, sufficient) to find $u \in J$ satisfying $u(q) = 1$ for all $q \in Q_1$. Indeed, in that event $uf = f$.

With this in mind, observe first that for $q \in Q_1$ there is a function f_q in J such that $f_q(q) \neq 0$. Putting $g_q := f_q^* f_q$, where as usual $f_q^* : x \mapsto f_q(x)^*$ is the conjugate of f_q, observe that $g_q \geq 0$ and, moreover, $g_q(q) > 0$. It is also clear that $g_q \in J$ for $q \in Q_1$. The family $(U_q)_{q \in Q_1}$, with $U_q := \{x \in Q_1 :\ g_q(x) > 0\}$, is an open cover of Q_1. Using a standard compactness argument, choose a finite subset $\{q_1, \dots, q_n\}$ of Q_1 such that $Q_1 \subset U_{q_1} \cup \dots \cup U_{q_n}$. Put $g := g_{q_1} + \dots + g_{q_n}$. Undoubtedly, $g \in J$ and $g(q) > 0$ for $q \in Q_1$. Let $h_0(q) := g(q)^{-1}$ for $q \in Q_1$. By the Tietze–Urysohn Theorem, there is a function h in $C(Q,\ \mathbb{R})$ satisfying $h|_{Q_1} = h_0$. Finally, let $u := hg$. This u is a sought function.

We have thus demonstrated that $J_0 \subset J$. Moreover, J_0 is an ideal of $C(Q,\ \mathbb{C})$ for obvious reasons. ▷

11.5.2. *For every closed ideal J of the algebra $C(Q,\ \mathbb{C})$ there is a unique compact subset Q_0 of Q such that*

$$J = J(Q_0) := \{f \in C(Q,\ \mathbb{C}) :\ q \in Q_0 \Rightarrow f(q) = 0\}.$$

◁ Uniqueness follows from the Urysohn Theorem. Define Q_0 as in 11.5.1. Then, surely, $J \subset J(Q_0)$. Take $f \in J(Q_0)$ and, given $n \in \mathbb{N}$, put

$$U_n := \left\{|f| \leq {}^1/_{2n}\right\}, \quad V_n := \left\{|f| \geq {}^1/_n\right\}.$$

Once again using the Urysohn Theorem, find $h_n \in C(Q, \mathbb{R})$ satisfying $0 \leq h_n \leq 1$ with $h_n|_{U_n} = 0$ and $h_n|_{V_n} = 1$. Consider $f_n := fh_n$. Since

$$\text{int } f_n^{-1}(0) \supset \text{int } U_n \supset Q_0,$$

therefore, from 11.5.1 derive $f_n \in J$. It suffices to observe that $f_n \to f$ by construction. ▷

11.5.3. ***Maximal Ideal Theorem.*** *A maximal ideal of $C(Q, \mathbb{C})$ has the form*

$$J(q) := J(\{q\}) = \{f \in C(Q, \mathbb{C}) : f(q) = 0\},$$

with q a point of Q.

◁ Follows from 11.5.2, because the closure of an ideal is also an ideal. ▷

11.6. The Gelfand Transform

11.6.1. *Let A be a commutative Banach algebra, and let J be a closed ideal of A other than A. Then the quotient algebra A/J, endowed with the quotient norm, is a Banach algebra. If $\varphi : A \to A/J$ is the coset mapping then $\varphi(1)$ is the unity of A/J, the operator φ is multiplicative and $\|\varphi\| = 1$.*

◁ Given $a, b \in A$, from 5.1.10 (5) derive

$$\begin{aligned} \|\varphi(a)\varphi(b)\|_{A/J} &= \inf\{\|a'b'\|_A : \varphi(a') = \varphi(a),\ \varphi(b') = \varphi(b)\} \\ &\leq \inf\{\|a'\|_A\|b'\|_A : \varphi(a') = \varphi(a),\ \varphi(b') = \varphi(b)\} \\ &= \|\varphi(a)\|_{A/J}\|\varphi(b)\|_{A/J}. \end{aligned}$$

In other words, the norm of A/J is submultiplicative. Consequently, $\|\varphi(1)\| \geq 1$. Furthermore,

$$\|\varphi(1)\|_{A/J} = \inf\{\|a\|_A : \varphi(a) = \varphi(1)\} \leq \|1\|_A = 1;$$

i.e., $\|\varphi(1)\| = 1$. Whence, in particular, the equality $\|\varphi\| = 1$ follows. The remaining claims are evident. ▷

11.6.2. REMARK. The message of 11.6.1 remains valid for a noncommutative Banach algebra A under the additional assumption that J is a *bilateral ideal* of A; i.e., J is a subspace of A satisfying the condition $AJA \subset J$.

11.6.3. *Let $\chi : A \to \mathbb{C}$ be a nonzero multiplicative linear functional on A. Then χ is continuous and $\|\chi\| = \chi(1) = 1$ (in particular, χ is a representation of A in $\mathbb{C}$).*

◁ Since $\chi \neq 0$; therefore, $0 \neq \chi(a) = \chi(a1) = \chi(a)\chi(1)$ for some a in A. Consequently, $\chi(1) = 1$. If now a in A and λ in $\mathbb{C}$ are such that $|\lambda| > \|a\|$, then $\lambda - a \in \mathrm{Inv}(A)$ (cf. 5.6.15). So, $1 = \chi(1)\chi(\lambda - a)\chi((\lambda - a)^{-1})$. Whence $\chi(\lambda - a) \neq 0$; i.e., $\chi(a) \neq \lambda$. Thus, $|\chi(a)| \leq \|a\|$ and $\|\chi\| \leq 1$. Since $\|\chi\| = \|\chi\| \, \|1\| \geq |\chi(1)| = 1$, conclude that $\|\chi\| = 1$. ▷

11.6.4. DEFINITION. A nonzero multiplicative linear functional on an algebra A is a *character* of A. The set of all characters of A is denoted by $\mathrm{X}(A)$, furnished with the topology of pointwise convergence (induced in $\mathrm{X}(A)$ by the weak topology $\sigma(A', A)$) and called the *character space* of A.

11.6.5. *The character space is a compactum.*

◁ It is beyond a doubt that $\mathrm{X}(A)$ is a Hausdorff space. By virtue of 11.6.3, $\mathrm{X}(A)$ is a $\sigma(A', A)$-closed subset of the ball $B_{A'}$. The latter is $\sigma(A', A)$-compact by the Alaoglu–Bourbaki Theorem. It remains to refer to 9.4.9. ▷

11.6.6. ***Ideal and Character Theorem.*** *Each maximal ideal of a commutative Banach algebra A is the kernel of a character of A. Moreover, the mapping $\chi \mapsto \ker \chi$ from the character space $\mathrm{X}(A)$ onto the set $\mathrm{M}(A)$ of all maximal ideals of A is one-to-one.*

◁ Let $\chi \in \mathrm{X}(A)$ be a character of A. Clearly, $\ker \chi \lhd A$. From 2.3.11 it follows that the monoquotient $\overline{\chi} : A/\ker \chi \to \mathbb{C}$ of χ is a monomorphism. In view of 11.6.1, $\overline{\chi}(1) = \chi(1) = 1$; i.e., $\overline{\chi}$ is an isomorphism of $A/\ker \chi$ and $\mathbb{C}$. Consequently, $A/\ker \chi$ is a field. Using 11.4.7, infer that $\ker \chi$ is a maximal ideal; i.e., $\ker \chi \in \mathrm{M}(A)$. Now, let $m \in \mathrm{M}(A)$ be some maximal ideal of A. It is clear that $m \subset \mathrm{cl}\, m$, $\mathrm{cl}\, m \lhd A$, and $1 \notin \mathrm{cl}\, m$ (because $1 \in \mathrm{Inv}(A)$, and the last set is open by the Banach Inversion Stability Theorem and 11.1.6). Therefore, the ideal m is closed. Consider the quotient algebra A/m and the coset mapping $\varphi : A \to A/m$. In view of 11.4.7 and 11.6.1, A/m is a Banach field. By the Gelfand–Mazur Theorem, there is an isometric representation $\psi : A/m \to \mathbb{C}$. Put $\chi := \psi \circ \varphi$. It is evident that $\chi \in \mathrm{X}(A)$ and $\ker \chi = \chi^{-1}(0) = \varphi^{-1}(\psi^{-1}(0)) = \varphi^{-1}(0) = m$.

To complete the proof, it suffices to show that the mapping $\chi \mapsto \ker \chi$ is one-to-one. Indeed, let $\ker \chi_1 = \ker \chi_2$ for $\chi_1, \chi_2 \in \mathrm{X}(A)$. By 2.3.11 $\chi_1 = \lambda \chi_2$ for some $\lambda \in \mathbb{C}$. Furthermore, by 11.6.3, $1 = \chi_1(1) = \lambda \chi_2(1) = \lambda$. Finally, $\chi_1 = \chi_2$. ▷

11.6.7. REMARK. Theorem 11.6.6 makes it natural to furnish $\mathrm{M}(A)$ with the inverse image topology translated from $\mathrm{X}(A)$ to $\mathrm{M}(A)$ by the mapping $\chi \mapsto \ker \chi$. In this regard, $\mathrm{M}(A)$ is referred to as the compact *maximal ideal space* of A. In other words, the character space and the maximal ideal space are often identified along the lines of 11.6.6.

11.6.8. DEFINITION. Let A be a commutative Banach algebra and let $\mathrm{X}(A)$ be the character space of A. Given $a \in A$ and $\chi \in \mathrm{X}(A)$, put $\widehat{a}(\chi) := \chi(a)$. The resulting function $\widehat{a} : \chi \mapsto \widehat{a}(\chi)$, defined on $\mathrm{X}(A)$, is the *Gelfand transform* of a.

The mapping $a \mapsto \widehat{a}$, with $a \in A$ is the *Gelfand transform* of A, denoted by $\mathscr{G}_A$ (or $\widehat{\ }$).

11.6.9. Gelfand Transform Theorem. *The Gelfand transform $\mathscr{G}_A : a \mapsto \widehat{a}$ is a representation of a commutative Banach algebra A in the algebra $C(\mathrm{X}(A),\ \mathbb{C})$. Moreover,*

$$\mathrm{Sp}(a) = \mathrm{Sp}(\widehat{a}) = \widehat{a}(\mathrm{X}(A)),$$
$$\|\widehat{a}\| = r(a),$$

with $a \in A$ and $r(a)$ standing for the spectral radius of a.

◁ The implications $a \in A \Rightarrow \widehat{a} \in C(\mathrm{X}(A),\ \mathbb{C})$, $\widehat{1} = 1$ and $a,\ b \in A \Rightarrow \widehat{ab} = \widehat{a}\widehat{b}$ follow from definitions and 11.6.3. The linearity of $\mathscr{G}_A$ raises no doubts. Consequently, the mapping $\mathscr{G}_A$ is a representation.

To begin with, take $\lambda \in \mathrm{Sp}(a)$. Then $\lambda - a$ is not invertible, and so the ideal $J_{\lambda - a} := A(\lambda - a)$ is proper in virtue of 11.4.4. By the Krull Theorem, there is a maximal ideal m of A satisfying the condition $m \supset J_{\lambda-a}$. By Theorem 11.6.6, $m = \ker \chi$ for a suitable character χ. In particular, $\chi(\lambda - a) = 0$; i.e., $\lambda = \lambda\chi(1) = \chi(\lambda) = \chi(a) = \widehat{a}(\chi)$. Consequently, $\lambda \in \mathrm{Sp}(\widehat{a})$.

If, in turn, $\lambda \in \mathrm{Sp}(\widehat{a})$ then $(\lambda - \widehat{a})$ is not invertible in $C(\mathrm{X}(A),\ \mathbb{C})$; i.e., there is a character $\chi \in \mathrm{X}(A)$ such that $\lambda = \widehat{a}(\chi)$. In other words, $\chi(\lambda - a) = 0$. Thus, the assumption $\lambda - a \in \mathrm{Inv}\,(A)$ leads to the following contradiction:

$$1 = \chi(1) = \chi((\lambda - a)^{-1}(\lambda - a)) = \chi((\lambda - a)^{-1})\chi(\lambda - a) = 0.$$

Hence, $\lambda \in \mathrm{Sp}(a)$. Finally, $\mathrm{Sp}(a) = \mathrm{Sp}(\widehat{a})$.

Using the Beurling–Gelfand formula (cf. 11.3.3 and 8.1.12), infer that

$$\begin{aligned} r(a) &= \sup\{|\lambda| :\ \lambda \in \mathrm{Sp}(a)\} = \sup\{|\lambda| :\ \lambda \in \mathrm{Sp}(\widehat{a})\} \\ &= \sup\{|\lambda| :\ \lambda \in \widehat{a}(\mathrm{X}(A))\} = \sup\{|\widehat{a}(\chi)| :\ \chi \in \mathrm{X}(A)\} = \|\widehat{a}\|, \end{aligned}$$

what was required. ▷

11.6.10. *The Gelfand transform of a commutative Banach algebra A is an isometric embedding if and only if $\|a^2\| = \|a\|^2$ for all $a \in A$.*

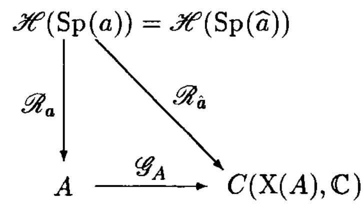
◁ $\Rightarrow$: The mapping $t \mapsto t^2$, viewed as acting on $\mathbb{R}_+$, and the inverse of the mapping on $\mathbb{R}_+$ are both increasing. Therefore, from 10.6.9 obtain

$$\begin{aligned} \|a^2\| &= \|\widehat{a}^2\|_{C(\mathrm{X}(A),\mathbb{C})} = \sup_{\chi \in \mathrm{X}(A)} |\widehat{a}^2(\chi)| = \sup_{\chi \in \mathrm{X}(A)} |\chi(a^2)| \\ &= \sup_{\chi \in \mathrm{X}(A)} |\chi(a)\chi(a)| = \sup_{\chi \in \mathrm{X}(A)} |\chi(a)|^2 \\ &= \left(\sup_{\chi \in \mathrm{X}(A)} |\chi(a)|\right)^2 = \|\widehat{a}\|^2 = \|a\|^2. \end{aligned}$$

$\Leftarrow$: By the Gelfand formula, $r(a) = \lim \|a^n\|^{1/n}$. In particular, observe that $\|a^{2^n}\| = \|a\|^{2^n}$; i.e., $r(a) = \|a\|$. By 10.6.9, $r(a) = \|\widehat{a}\|$, completing the proof. $\triangleright$

11.6.11. REMARK. It is interesting sometimes to grasp situations in which the Gelfand transform of A is faithful but possibly not isometric. The kernel of the Gelfand transform $\mathscr{G}_A$ is the intersection of all maximal ideals, called the *radical* of A. Therefore, the condition for $\mathscr{G}_A$ to be a faithful representation of A in $C(X(A), \mathbb{C})$ reads: "A is semisimple" or, which is the same, "The radical of A is trivial."

11.6.12. ***Theorem.*** *For an element a of a commutative Banach algebra A the following diagram of representations commutes:*



Moreover, $f(\widehat{a}) = f \circ \widehat{a} = f(\widehat{a})$ for $f \in H(\mathrm{Sp}(a))$.

$\triangleleft$ Take $\chi \in X(A)$. Given $z \in \mathrm{res}\,(a)$, observe that

$$\chi\left(\frac{1}{z-a}(z-a)\right) = 1 \Rightarrow \chi\left(\frac{1}{z-a}\right) = \frac{1}{\chi(z-a)} = \frac{1}{z-\chi(a)}.$$

In other words,

$$\widehat{R(a,\ z)}(\chi) = \widehat{\frac{1}{z-a}}(\chi) = \frac{1}{z-\widehat{a}(\chi)} = \frac{1}{z-\widehat{a}}(\chi) = R(\widehat{a},\ z)(\chi).$$

Therefore, appealing to the properties of the Bochner integral (cf. 5.5.9 (6)) and given $f \in H(\mathrm{Sp}(a))$, infer that

$$\begin{aligned} \widehat{f(a)} &= \mathscr{G}_A \circ \mathscr{R}_a f = \mathscr{G}_A \left(\frac{1}{2\pi i} \oint f(z) R(a,\ z)\, dz\right) \\ &= \frac{1}{2\pi i} \oint f(z) \mathscr{G}_A(R(a,\ z))\, dz = \frac{1}{2\pi i} \oint f(z) \widehat{R(a,\ z)})\, dz \\ &= \frac{1}{2\pi i} \oint f(z) R(\widehat{a},\ z)\, dz = \mathscr{R}_{\widehat{a}}(f) = f(\widehat{a}). \end{aligned}$$

Furthermore, given $\chi \in X(A)$ and using the Cauchy Integral Formula, derive the following chain of equalities

$$\begin{aligned} &f \circ \widehat{a}(\chi) = f(\widehat{a}(\chi)) = f(\chi(a)) \\ &= \frac{1}{2\pi i} \oint \frac{f(z)}{z - \chi(a)}\, dz = \frac{1}{2\pi i} \oint \chi\left(\frac{f(z)}{z-a}\right) dz \\ &= \frac{1}{2\pi i}\chi\left(\oint \frac{f(z)}{z-a}\, dz\right) = \widehat{f(a)}(\chi) = f(\widehat{a})(\chi). \ \triangleright \end{aligned}$$

11.6.13. REMARK. The theory of the Gelfand transform may be naturally generalized to the case of a commutative Banach algebra A without unity. Retain Definitions 11.6.4 and 11.6.8 verbatim. A character χ in $X(A)$ generates some character χ_e in $X(\mathscr{A}_e)$ by the rule $\chi_e(a,\ \lambda) := \chi(a) + \lambda$ $(a \in A,\ \lambda \in \mathbb{C})$. The set $\chi(\mathscr{A}_e) \setminus \{\chi_e :\ \chi \in \chi(A)\}$ is a singleton consisting of the sole element $\chi_\infty(a,\ \lambda) := \lambda$ $(a \in A,\ \lambda \in \mathbb{C})$. The space $\chi(A)$ is locally compact (cf. 9.4.19), because the mapping $\chi \in \chi(A) \mapsto \chi_e \in \chi(\mathscr{A}_e) \setminus \{\chi_\infty\}$ is a homeomorphism. Moreover, $\ker \chi_\infty = A \times 0$. Consequently, the Gelfand transform of a commutative Banach algebra without unity represents it in the algebra of continuous complex functions defined on a locally compact space and *vanishing at infinity.* Given the group algebra $(L_1(\mathbb{R}^N),\ *)$, observe that by 10.11.1 and 10.11.3 the Fourier transform coincides with the Gelfand transform, which in turn entails the Riemann–Lebesgue Lemma as well as the multiplication formula 10.11.5 (6).

11.7. The Spectrum of an Element of a C^*-Algebra

11.7.1. DEFINITION. An element a of an involutive algebra A is called *hermitian* if $a^* = a$. An element a of A is called *normal* if $a^*a = aa^*$. Finally, an element a is called *unitary* if $aa^* = a^*a = 1$ (i.e. $a,\ a^* \in \operatorname{Inv}(A)$ with $a^{-1} = a^*$ and $a^{*-1} = a$).

11.7.2. *Hermitian elements of an involutive algebra A compose a real subspace of A. Moreover, for every a in A there are unique hermitian elements $x,\ y \in A$ such that $a = x + iy$. Namely,*

$$x = \frac{1}{2}(a + a^*), \quad y = \frac{1}{2i}(a - a^*).$$

Moreover, $a^* = x - iy$.

$\triangleleft$ Only the claim of uniqueness needs examining. If $a = x_1 + iy_1$ then, using the properties of involution (cf. 6.4.13), proceed as follows: $a^* = x_1^* + (iy_1)^* = x_1^* - iy_1^* = x_1 - iy_1$. Thus, $x_1 = x$ and $y_1 = y$. $\triangleright$

11.7.3. *The unity of A is a hermitian element of A.*

◁ $1^* = 1^*1 = 1^*1^{**} = (1^*1)^* = 1^{**} = 1$ ▷

11.7.4. $a \in \mathrm{Inv}(A) \Leftrightarrow a^* \in \mathrm{Inv}(A)$. *Moreover, involution and inversion commute.*

◁ For $a \in \mathrm{Inv}(A)$ by definition $aa^{-1} = a^{-1}a = 1$. Consequently, $a^{-1*}a^* = a^*a^{-1*} = 1^*$. Using 11.7.3, infer that $a^* \in \mathrm{Inv}(A)$ and $a^{*-1} = a^{-1*}$. Repeating this argument for $a := a^*$, complete the proof. ▷

11.7.5. $\mathrm{Sp}(a^*) = \mathrm{Sp}(a)^*$. ◁▷

11.7.6. *The spectrum of a unitary element of a C^*-algebra is a subset of the unit circle.*

◁ By Definition 6.4.13, $\|a^2\| = \|a^*a\| \le \|a^*\|\,\|a\|$ for an arbitrary a. In other words, $\|a\| \le \|a^*\|$. Therefore, from $a = a^{**}$ infer that $\|a\| = \|a^*\|$. If now a is a unitary element, $a^* = a^{-1}$; then $\|a\|^2 = \|a^*a\| = \|a^{-1}a\| = 1$. Consequently, $\|a\| = \|a^*\| = \|a^{-1}\| = 1$. Whence it follows that $\mathrm{Sp}(a)$ and $\mathrm{Sp}(a^{-1})$ both lie within the unit disk. Furthermore, $\mathrm{Sp}(a^{-1}) = \mathrm{Sp}(a)^{-1}$. ▷

11.7.7. *The spectrum of a hermitian element of a C^*-algebra is real.*

◁ Take a in A arbitrarily. From the Gelfand–Dunford Theorem in an algebraic setting derive

$$\exp(a)^* = \left(\sum_{n=0}^{\infty} \frac{a^n}{n!}\right)^* = \sum_{n=0}^{\infty} \frac{(a^n)^*}{n!} = \sum_{n=0}^{\infty} \frac{(a^*)^n}{n!} = \exp(a^*).$$

If now $h = h^*$ is a hermitian element of A then, applying the holomorphic functional calculus to $a := \exp(ih)$, deduce that

$$a^* = \exp(ih)^* = \exp((ih)^*) = \exp(-ih^*) = \exp(-ih) = a^{-1}.$$

Consequently, a is a unitary element of A, and by 11.7.6 the spectrum $\mathrm{Sp}(a)$ of A is a subset of the unit circle $\mathbb{T}$. If $\lambda \in \mathrm{Sp}(h)$ then by the Spectral Mapping Theorem (also cf. 11.3.3) $\exp(i\lambda) \in \mathrm{Sp}(a) \subset \mathbb{T}$. Thus, $1 = |\exp(i\lambda)| = |\exp(i\mathrm{Re}\,\lambda - \mathrm{Im}\,\lambda)| = \exp(-\mathrm{Im}\,\lambda)$. Finally, $\mathrm{Im}\,\lambda = 0$; i.e., $\lambda \in \mathbb{R}$. ▷

11.7.8. DEFINITION. Let A be a C^*-algebra. A subalgebra B of A is called a *C^*-subalgebra* of A if $b \in B \Rightarrow b^* \in B$. In this event, B is considered with the norm induced from A.

11.7.9. Theorem. *Every closed C^*-subalgebra of a C^*-algebra is pure.*

◁ Let B be a closed (unital) C^*-subalgebra of a C^*-algebra A and $b \in B$. If $b \in \mathrm{Inv}(B)$ then it is easy that $b \in \mathrm{Inv}(A)$. Let now $b \in \mathrm{Inv}(A)$. From 11.7.4 derive $b^* \in \mathrm{Inv}(A)$. Consequently, $b^*b \in \mathrm{Inv}(A)$ and the element $(b^*b)^{-1}b^*$ is a left inverse of b. By virtue of 11.1.4, it means that $b^{-1} = (b^*b)^{-1}b^*$. Consequently, to complete the proof it suffices to show only that $(b^*b)^{-1}$ belongs to B. Since b^*b is hermitian in B, the inclusion holds: $\mathrm{Sp}_B(b^*b) \subset \mathbb{R}$ (cf. 11.7.7). Using 11.2.5, infer that $\mathrm{Sp}_A(b^*b) = \mathrm{Sp}_B(b^*b)$. Since $0 \notin \mathrm{Sp}_A(b^*b)$; therefore, $b^*b \in \mathrm{Inv}(B)$. Finally, $b \in \mathrm{Inv}(B)$. ▷

11.7.10. Corollary. *Let b be an element of a C^*-algebra A and let B be some closed C^*-subalgebra of A with $b \in B$. Then*

$$\mathrm{Sp}_B(b) = \mathrm{Sp}_A(b). \quad ◁▷$$

11.7.11. Remark. In view of 11.7.10, Theorem 11.7.9 is often referred to as the *Spectral Purity Theorem* for a C^*-algebra. It asserts that the concept of the spectrum of an element a of a C^*-algebra is absolute, i.e. independent of the choice of a C^*-subalgebra containing a.

11.8. The Commutative Gelfand–Naĭmark Theorem

11.8.1. *The Banach algebra $C(Q,\ \mathbb{C})$ with the natural involution $f \mapsto f^*$, where $f^*(q) := f(q)^*$ for $q \in Q$, is a C^*-algebra.*

$$◁\ \|f^*f\| = \sup\{|f(q)^*f(q)| :\ q \in Q\} = \sup\{|f(q)|^2 :\ q \in Q\}$$
$$= (\sup|f(Q)|)^2 = \|f\|^2\ ▷$$

11.8.2. Stone–Weierstrass Theorem. *Every unital C^*-subalgebra of the C^*-algebra $C(Q,\ \mathbb{C})$, which separates the points of Q, is dense in $C(Q,\ \mathbb{C})$.*

◁ Let A be such a subalgebra. Since $f \in A \Rightarrow f^* \in A$; therefore, $f \in A \Rightarrow \mathrm{Re}\, f \in A$ and so the set $\mathrm{Re}\, A := \{\mathrm{Re}\, f :\ f \in A\}$ is a real subalgebra of $C(Q,\ \mathbb{R})$. It is beyond a doubt that $\mathrm{Re}\, A$ contains constant functions and separates the points of Q. By the Stone–Weierstrass Theorem for $C(Q,\ \mathbb{R})$, the subalgebra $\mathrm{Re}\, A$ is dense in $C(Q,\ \mathbb{R})$. It remains to refer to 11.7.2. ▷

11.8.3. Definition. A representation of a $*$-algebra agreeing with involution $*$ is a *$*$-representation*. In other words, if $(A,\ *)$ and $(B,\ *)$ are involutive algebras and $\mathfrak{R} : A \to B$ is a multiplicative linear operator, then $\mathfrak{R}$ is called a *$*$-representation* of A in B whenever the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\mathscr{R}} & B \\ {*}\downarrow & & \downarrow{*} \\ A & \xrightarrow{\mathscr{R}} & B \end{array}$$

If $\mathfrak{R}$ is also an isomorphism then $\mathfrak{R}$ is a *$*$-isomorphism* of A and B. In the presence of norms in the algebras, the naturally understood terms "isometric $*$-representation" and "isometric $*$-isomorphism" are in common parlance.

11.8.4. Commutative Gelfand–Naĭmark Theorem. *The Gelfand transform of a commutative C^*-algebra A implements an isometric $*$-isomorphism of A and $C(\mathrm{X}(A),\ \mathbb{C})$.*

◁ Given $a \in A$, observe that

$$\|a^2\| = \|(a^2)^* a^2\|^{1/2} = \|a^* a a^* a\|^{1/2} = \|a^* a\| = \|a\|^2.$$

In virtue of 11.6.10 the Gelfand transform $\mathscr{G}_A$ is thus an isometry of A and a closed subalgebra $\widehat{A}$ of $C(\mathrm{X}(A),\ \mathbb{C})$. Undoubtedly, $\widehat{A}$ separates the points of $\mathrm{X}(A)$ and contains constant functions.

By virtue of 11.6.9 and 11.7.7, $\widehat{h}(\mathrm{X}(A)) = \mathrm{Sp}(h) \subset \mathbb{R}$ for every hermitian element $h = h^*$ of A. Now, take an arbitrary element a of A. Using 11.7.2, write $a = x + iy$, where x and y are hermitian elements. The containments $\chi(x) \in \mathbb{R}$ and $\chi(y) \in \mathbb{R}$ hold for every character χ, a member of $\mathrm{X}(A)$.

With this in mind, successively infer that

$$\begin{aligned} \mathscr{G}_A(a)^*(\chi) &= \widehat{a}^*(\chi) = \widehat{a}(\chi)^* = \chi(a)^* = \chi(x+iy)^* \\ &= (\chi(x) + i\chi(y))^* = \chi(x) - i\chi(y) = \chi(x - iy) = \chi(a^*) \\ &= \widehat{a^*}(\chi) = \mathscr{G}_A(a^*)(\chi) \quad (\chi \in \mathrm{X}(A)). \end{aligned}$$

Consequently, the Gelfand transform $\mathscr{G}_A$ is a $*$-representation and, in particular, $\widehat{A}$ is a C^*-subalgebra of $C(\mathrm{X}(A),\ \mathbb{C})$. It remains to appeal to 11.8.2 and conclude that $\widehat{A} = C(\mathrm{X}(A),\ \mathbb{C})$. ▷

11.8.5. *Assume that $\mathfrak{R} : A \to B$ is a $*$-representation of a C^*-algebra A in a C^*-algebra B. Then $\|\mathfrak{R}a\| \le \|a\|$ for $a \in A$.*

◁ Since $\mathfrak{R}(1) = 1$; therefore, $\mathfrak{R}(\mathrm{Inv}\,(A)) \subset \mathrm{Inv}\,(B)$. Hence, $\mathrm{Sp}_B(\mathfrak{R}(a)) \subset \mathrm{Sp}_A(a)$ for $a \in A$. Whence it follows from the Beurling–Gelfand formula that the inequality $r_A(a) \ge r_B(\mathfrak{R}(a))$ holds for the spectral radii. If a is a hermitian element of A then $\mathfrak{R}(a)$ is a hermitian element of B, because $\mathfrak{R}(a)^* = \mathfrak{R}(a^*) = \mathfrak{R}(a)$. If now A_0 is the least closed C^*-subalgebra containing a and B_0 is an analogous subalgebra containing $\mathfrak{R}(a)$, then A_0 and B_0 are commutative C^*-algebras. Therefore, from Theorems 11.8.4 and 11.6.9 obtain

$$\begin{aligned} \|\mathfrak{R}(a)\| &= \|\mathfrak{R}(a)\|_{B_0} = \|\mathscr{G}_{B_0}(\mathfrak{R}(a))\| = r_{B_0}(\mathfrak{R}(a)) \\ &= r_B(\mathfrak{R}(a)) \le r_A(a) = r_{A_0}(a) = \|\mathscr{G}_{A_0}(a)\| = \|a\|. \end{aligned}$$

Given $a \in A$, it is easy to observe that a^*a is a hermitian element. Thus,

$$\|\mathfrak{R}(a)\|^2 = \|\mathfrak{R}(a)^*\mathfrak{R}(a)\| = \|\mathfrak{R}(a^*a)\| \le \|a^*a\| = \|a\|^2. \ ▷$$

11.8.6. Spectral Theorem. *Let a be a normal element of a C^*-algebra A, with $\mathrm{Sp}(a)$ the spectrum of a. There is a unique isometric $*$-representation $\mathfrak{R}_a$ of $C(\mathrm{Sp}(a),\ \mathbb{C})$ in A such that $a = \mathfrak{R}_a(I_{\mathrm{Sp}(a)})$.*

◁ Let B be the least closed C^*-subalgebra of A containing a. It is clear that the algebra B is commutative by the normality of a (this algebra presents the closure of the algebra of all polynomials in a and a^*). Moreover, by 11.7.10, $\mathrm{Sp}(a) = \mathrm{Sp}_A(a) = \mathrm{Sp}_B(a)$. The Gelfand transform $\widehat{a} := \mathscr{G}_B(a)$ of a acts from $\mathrm{X}(B)$ onto $\mathrm{Sp}(a)$ by 11.6.9 and is evidently one-to-one. Since $\mathrm{X}(B)$ and $\mathrm{Sp}(a)$ are compact sets; on using 9.4.11, conclude that $\widehat{a}$ is a homeomorphism. Whence it is immediate that the mapping $\overset{\circ}{\mathfrak{R}} : f \mapsto f \circ \widehat{a}$ implements an isometric $*$-isomorphism between $C(\mathrm{Sp}(a),\ \mathbb{C})$ and $C(\mathrm{X}(B),\ \mathbb{C})$.

Using Theorem 11.3.2 and the connection between the Gelfand transform and the Riesz–Dunford integral which is revealed in 11.6.12, for the identity mapping observe that

$$\widehat{a} = \mathscr{R}_{\widehat{a}} I_{\mathbb{C}} = I_{\mathbb{C}} \circ \widehat{a} = I_{\mathbb{C}}\big|_{\widehat{a}(\mathrm{X}(B))} \circ \widehat{a} = I_{\mathbb{C}}\big|_{\mathrm{Sp}(a)} \circ \widehat{a} = I_{\mathrm{Sp}(a)} \circ \widehat{a} = \overset{\circ}{\mathfrak{R}}(I_{\mathrm{Sp}(a)}).$$

Now, put

$$\mathfrak{R}_a := \mathscr{G}_B^{-1} \circ \overset{\circ}{\mathfrak{R}}.$$

Clearly, $\mathfrak{R}_a$ is an isometric embedding and a $*$-representation. Moreover,

$$\mathfrak{R}_a(I_{\mathrm{Sp}(a)}) = \mathscr{G}_B^{-1} \circ \overset{\circ}{\mathfrak{R}}(I_{\mathrm{Sp}(a)}) = \mathscr{G}_B^{-1}(\widehat{a}) = a.$$

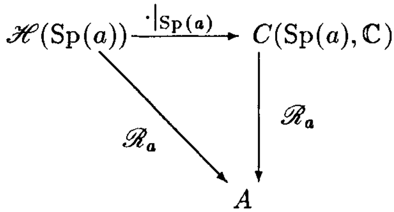
Uniqueness for such a representation $\mathfrak{R}_a$ is guaranteed by 11.8.5 and the Stone–Weierstrass Theorem implies that the C^*-algebra $C(\mathrm{Sp}(a),\ \mathbb{C})$ is its least closed (unital) C^*-subalgebra containing $I_{\mathrm{Sp}(a)}$. ▷

11.8.7. Definition. The representation $\mathfrak{R}_a : C(\mathrm{Sp}(a),\ \mathbb{C}) \to A$ of 11.8.6 is the *continuous functional calculus* (for a normal element a of A). The element $\mathfrak{R}_a(f)$ with $f \in C(\mathrm{Sp}(a),\ \mathbb{C})$ is usually denoted by $f(a)$.

11.8.8. Remark. Let f be a holomorphic function in a neighborhood about the spectrum of a normal element a of some C^*-algebra A; i.e., $f \in H(\mathrm{Sp}(a))$. Then the element $f(a)$ of A was defined by the holomorphic functional calculus. Retain the symbol f for the restriction of f to the set $\mathrm{Sp}(a)$. Then, using the continuous functional calculus, define the element $\mathfrak{R}_a(f) := \mathfrak{R}_a\big(f\big|_{\mathrm{Sp}(a)}\big)$ of A. This element, as mentioned in 11.8.7, is also denoted by $f(a)$. The use of the designation is by far not incidental (and sound in virtue of 11.6.12 and 11.8.6). Indeed, it would be weird to deliberately denote by different symbols one and the same element. This circumstance may be expressed in visual form.

Namely, let $\cdot\big|_{\mathrm{Sp}(a)}$ stand for the mapping sending a germ h, a member of $\mathscr{H}(\mathrm{Sp}(a))$, to its restriction to $\mathrm{Sp}(a)$; i.e., let $h\big|_{\mathrm{Sp}(a)}$ at a point z stand for the value of h at z (cf. 8.1.21). It is clear that $\cdot\big|_{\mathrm{Sp}(a)} : \mathscr{H}(\mathrm{Sp}(a)) \to C(\mathrm{Sp}(a),\ \mathbb{C})$.

The above connection between the continuous and holomorphic functional calculuses for a normal element a of the C^*-algebra A may be expressed as follows: The next diagram commutes



11.9. Operator $*$-Representations of a C^*-Algebra

11.9.1. DEFINITION. Let A be a (unital) Banach algebra. An element s in A' is a *state* of A, in writing $s \in S(A)$, if $\|s\| = s(1) = 1$. For $a \in A$, the set $N(a) := \{s(a) : s \in S(A)\}$ is the *numeric range* of a.

11.9.2. *The numeric range of a positive function, a member of* $C(Q, \mathbb{C})$, *lies in* $\mathbb{R}_+$.

$\triangleleft$ Let $a \geq 0$ and $\|s\| = s(1) = 1$. We have to prove that $s(a) \geq 0$. Take $z \in \mathbb{C}$ and $\varepsilon > 0$ such that the disk $B_\varepsilon(z) := z + \varepsilon\mathbb{D}$ includes $a(Q)$. Then $\|a - z\| \leq \varepsilon$ and, consequently, $|s(a-z)| \leq \varepsilon$. Hence, $|s(a) - z| = |s(a) - s(z)| \leq \varepsilon$; i.e., $s(a) \in B_\varepsilon(z)$. Observe that

$$\cap \{B_\varepsilon(z) : B_\varepsilon(z) \supset a(Q)\} = \mathrm{cl}\ \mathrm{co}(a(Q)) \subset \mathbb{R}_+.$$

Thus, $s(a) \in \mathbb{R}_+$. $\triangleright$

11.9.3. **Lemma.** *Let a be a hermitian element of a C^*-algebra. Then*

(1) $\mathrm{Sp}(a) \subset N(a)$;

(2) $\mathrm{Sp}(a) \subset \mathbb{R}_+ \Leftrightarrow N(a) \subset \mathbb{R}_+$.

$\triangleleft$ Let B be the least closed C^*-subalgebra, of the algebra A under study, which contains a. It is evident that B is a commutative algebra. By virtue of 11.6.9, the Gelfand transform $\widehat{a} := \mathscr{G}_B(a)$ provides $\widehat{a}(\mathrm{X}(B)) = \mathrm{Sp}_B(a)$. In view of 11.7.10, $\mathrm{Sp}_B(a) = \mathrm{Sp}(a)$. In other words, for $\lambda \in \mathrm{Sp}(a)$ there is a character χ of B satisfying the condition $\chi(a) = \lambda$. By 11.6.3, $\|\chi\| = \chi(1) = 1$. Using 7.5.11, find a norm-preserving extension s of χ onto A. Then s is a state of A and $s(a) = \lambda$. Finally, $\mathrm{Sp}(a) \subset N(a)$ (in particular, if $N(a) \subset \mathbb{R}_+$ then $\mathrm{Sp}(a) \subset \mathbb{R}_+$). Now, let s stand for an arbitrary state of A. It is clear that the restriction $s|_B$ is a state of B. It is an easy matter to show that $\widehat{a}$ maps $\mathrm{X}(B)$ onto $\mathrm{Sp}(a)$ in a one-to-one fashion. Consequently, B may be treated as the algebra $C(\mathrm{Sp}(a), \mathbb{C})$. From 11.9.2 derive

$s(a) = s|_B(a) \geq 0$ for $\widehat{a} \geq 0$. Thus, $\mathrm{Sp}(a) \subset \mathbb{R}_+ \Rightarrow N(a) \subset \mathbb{R}_+$, which ends the proof. ▷

11.9.4. DEFINITION. An element a of a C^*-algebra A is called *positive* if a is hermitian and $\mathrm{Sp}(a) \subset \mathbb{R}_+$. The set of all positive elements of A is denoted by A_+.

11.9.5. *In each C^*-algebra A the set A_+ is an ordering cone.*

◁ It is clear that $N(a+b) \subset N(a)+N(b)$ and $N(\alpha a) = \alpha N(a)$ for $a,\ b \in A$ and $\alpha \in \mathbb{R}_+$. Hence, 11.9.3 ensures the inclusion $\alpha_1 A_+ + \alpha_2 A_+ \subset A_+$ for $\alpha_1, \alpha_2 \in \mathbb{R}_+$. Thus, A_+ is a cone. If $a \in A_+ \cap (-A_+)$ then $\mathrm{Sp}(a) = 0$. Since a is a hermitian element, from Theorem 11.8.6 deduce that $\|a\| = 0$. ▷

11.9.6. *To every hermitian element a of a C^*-algebra A there correspond some elements a_+ and a_- of A_+ such that*

$$a = a_+ - a_-; \quad a_+ a_- = a_- a_+ = 0.$$

◁ Everything is immediate from the continuous functional calculus. ▷

11.9.7. *Kaplanski–Fukamija Lemma.* *An element a of a C^*-algebra A is positive if and only if $a = b^*b$ for some b in A.*

◁ $\Rightarrow$: Let $a \in A_+$; i.e., $a = a^*$ and $\mathrm{Sp}(a) \subset \mathbb{R}_+$. Then (cf. 11.8.6) there is a square root $b := \sqrt{a}$. Moreover, $b = b^*$ and $b^*b = a$.

$\Leftarrow$: If $a = b^*b$ then a is hermitian. Therefore, in view of 11.9.6, it may be assumed that $b^*b = u - v$, where $uv = vu = 0$ with $u \geq 0$ and $v \geq 0$ (in the ordered vector space $(A_{\mathbb{R}},\ A_+)$). Straightforward calculation yields the equalities

$$(bv)^*bv = v^*b^*bv = vb^*bv = v(u-v)v = (vu - v^2)v = -v^3.$$

Since $v \geq 0$, it follows that $v^3 \geq 0$; i.e., $(bv)^*bv \leq 0$. By Theorem 5.6.22, $\mathrm{Sp}((bv)^*bv)$ and $\mathrm{Sp}(bv(bv)^*)$ may differ only by zero. Therefore, $bv(bv)^* \leq 0$.

In virtue of 11.7.2, $bv = a_1 + ia_2$ for suitable hermitian elements a_1 and a_2. It is evident that $a_1^2,\ a_2^2 \in A_+$ and $(bv)^* = a_1 - ia_2$. Using 11.9.5 twice, arrive at the estimates

$$0 \geq (bv)^*bv + bv(bv)^* = 2\left(a_1^2 + a_2^2\right) \geq 0.$$

By 11.9.5, $a_1 = a_2 = 0$; i.e., $bv = 0$. Hence, $-v^3 = (bv)^*bv = 0$. The second appeal to 11.9.5 shows $v = 0$. Finally, $a = b^*b = u - v = u \geq 0$; i.e., $a \in A_+$. ▷

11.9.8. *Every state s of a C^*-algebra A is hermitian*; i.e.,

$$s(a^*) = s(a)^* \quad (a \in A).$$

◁ By Lemmas 11.9.7 and 11.9.3, $s(a^*a) \geq 0$ for all $a \in A$. Putting $a := a + 1$ and $a := a + i$, successively infer that

$$0 \leq s((a+1)^*(a+1)) = s(a^*a + a + a^* + 1) \Rightarrow s(a) + s(a^*) \in \mathbb{R};$$
$$0 \leq s((a+i)^*(a+i)) = s(a^*a - ia + ia^* + 1) \Rightarrow i(-s(a) + s(a^*)) \in \mathbb{R}.$$

In other words,

$$\operatorname{Im} s(a) + \operatorname{Im} s(a^*) = 0; \quad \operatorname{Re}(-s(a)) + \operatorname{Re} s(a^*) = 0.$$

Whence it follows that

$$s(a^*) = \operatorname{Re} s(a^*) + i\operatorname{Im} s(a^*) = \operatorname{Re} s(a) - i\operatorname{Im} s(a) = s(a)^*. \ ▷$$

11.9.9. *Let s be a state of a C^*-algebra A. Given $a,\ b \in A$, denote $(a,\ b)_s := s(b^*a)$. Then $(\cdot,\ \cdot)_s$ is a semi-inner product on A.*

◁ From 11.9.8 derive

$$(a,\ b)_s = s(b^*a) = s((a^*b)^*) = s(a^*b)^* = (b,\ a)_s^*.$$

Hence, $(\cdot,\ \cdot)_s$ is a hermitian form. Since $a^*a \geq 0$ for $a \in A$ in virtue of 11.9.7, $(a,\ a)_s = s(a^*a) \geq 0$ by 11.9.3. Consequently, $(\cdot,\ \cdot)_s$ is a positive-definite hermitian form. ▷

11.9.10. GNS-*Construction Theorem.* *To every state s of an arbitrary C^*-algebra A there correspond a Hilbert space $(H_s,\ (\cdot,\ \cdot)_s)$, an element x_s in H_s and a $*$-representation $\mathfrak{R}_s : A \to B(H_s)$ such that $s(a) = (\mathfrak{R}_s(a)x_s, x_s)_s$ for all $a \in A$ and the set $\{\mathfrak{R}_s(a)x_s :\ a \in A\}$ is dense in H_s.*

◁ In virtue of 11.9.9, putting $(a,\ b)_s := \underline{s(b^*a)}$ for $a,\ b \in A$, obtain a pre-Hilbert space $(A,\ (\cdot,\ \cdot)_s)$. Let $p_s(a) := \sqrt{(a,\ a)_s}$ stand for the seminorm of the space. Assume that $\varphi_s : A \to A/\ker p_s$ is the coset mapping of A onto the Hausdorff pre-Hilbert space $A/\ker p_s$ associated with A. Assume further that $\iota_s : A/\ker p_s \to H_s$ is an embedding (for instance, by the double prime mapping) of $A/\ker p_s$ onto a dense subspace of the Hilbert space H_s associated with $(A,\ (\cdot,\ \cdot)_s)$ (cf. 6.1.10 (4)). The inner product in H_s retains the previous notation $(\cdot,\ \cdot)_s$. Therefore, in particular,

$$(\iota_s\varphi_s a,\ \iota_s\varphi_s b)_s = (a,\ b)_s = s(b^*a) \quad (a,\ b \in A).$$

Given $a \in A$, consider (the image under the canonical operator representation) $L_a : b \mapsto ab$ $(b \in A)$. Demonstrate first that there are unique bounded operators $\overline{L}_a$ and $\mathfrak{R}_s(a)$ making the following diagram commutative:

$$\begin{array}{ccccc} A & \xrightarrow{\varphi_s} & A/\ker p_s & \xrightarrow{\iota_s} & H_s \\ L_a \downarrow & & \downarrow \overline{L}_a & & \downarrow \mathfrak{R}_s(a) \\ A & \xrightarrow{\varphi_s} & A/\ker p_s & \xrightarrow{\iota_s} & H_s \end{array}$$

A sought operator $\overline{L}_a$ is a solution to the equation $\mathfrak{X}\varphi_s = \varphi_s L_a$. Using 2.3.8, observe next that the necessary and sufficient condition for solvability of the equation in linear operators consists in invariance of the subspace $\ker p_s$ under L_a. Thus, examine the inclusion $L_a(\ker p_s) \subset \ker p_s$. To this end, take an element b of $\ker p_s$, i.e. $p_s(b) = 0$. By definition and the Cauchy–Bunyakovskiĭ–Schwarz inequality, deduce that

$$0 \le (L_a b,\ L_a b)_s = (ab,\ ab)_s = s((ab)^* ab)$$
$$= s(b^* a^* ab) = (a^* ab,\ b)_s \le p_s(b) p_s(a^* ab) = 0;$$

i.e., $L_a b \in \ker p_s$. Uniqueness for $\overline{L}_a$ is provided by 2.3.9, since φ_s is an epimorphism. Observe also that φ_s is an open mapping (cf. 5.1.3). Whence the continuity of $\overline{L}_a$ is immediate. Therefore, in virtue of 5.3.8 the correspondence $\iota_s \circ \overline{L}_a \circ (\iota_s)^{-1}$ may be considered as a bounded linear operator from $\iota_s(A/\ker p_s)$ to the Banach space H_s. By 4.5.10 such an operator extends uniquely to an operator $\mathfrak{R}_s(a)$ in $B(H_s)$.

Demonstrate now that $\mathfrak{R}_s : a \mapsto \mathfrak{R}_s(a)$ is a sought representation. By 11.1.6, $L_{ab} = L_a L_b$ for $a,\ b \in A$. Consequently,

$$\varphi_s L_{ab} = \varphi_s L_a L_b = \overline{L}_a \varphi_s L_b = \overline{L}_a \overline{L}_b \varphi_s.$$

Since $\overline{L}_{ab}$ is a unique solution to the equation $\mathfrak{X}\varphi_s = \varphi_s L_{ab}$, infer the equality $\overline{L}_{ab} = \overline{L}_a \overline{L}_b$, which guarantees multiplicativity for $\mathfrak{R}_s$. The linearity of $\mathfrak{R}_s$ may be verified likewise. Furthermore,

$$L_1 \varphi_s = \varphi_s L_1 = \varphi_s I_A = \varphi_s = I_{A/\ker p_s} \varphi_s = 1\varphi_s;$$

i.e., $\mathfrak{R}_s(1) = 1$.

For simplicity, put $\psi_s := \iota_s \varphi_s$. Then, on account taken of the definition of the inner product on H_s (cf. 6.1.10 (4)) and the involution in $B(H_s)$ (cf. 6.4.14 and 6.4.5), given elements $a,\ b$, and y in A, infer that

$$(\mathfrak{R}_s(a^*)\psi_s x,\ \psi_s y)_s = (\psi_s L_{a^*} x,\ \psi_s y)_s$$
$$= (L_{a^*} x,\ y)_s = (a^* x,\ y)_s = s(y^* a^* x) = s((ay)^* x) = (x,\ ay)_s$$
$$= (x,\ L_a y)_s = (\psi_s x,\ \psi_s L_a y)_s = (\psi_s x,\ \mathfrak{R}_s(a)\psi_s y)_s = (\mathfrak{R}_s(a)^* \psi_s x,\ \psi_s y)_s.$$

Now, since im ψ_s is dense in H_s it follows that $\mathfrak{R}_s(a^*) = \mathfrak{R}_s(a)^*$ for all a in A; i.e., $\mathfrak{R}_s$ is a $*$-representation.

Let $x_s := \psi_s 1$. Then

$$\mathfrak{R}_s(a)x_s = \mathfrak{R}_s(a)\psi_s 1 = \psi_s L_a 1 = \psi_s a \quad (a \in A).$$

Consequently, the set $\{\mathfrak{R}_s(a)x_s : a \in A\}$ is dense in H_s. Furthermore,

$$(\mathfrak{R}_s(a)x_s,\ x_s)_s = (\psi_s a,\ \psi_s 1)_s = (a,\ 1)_s = s(1^* a) = s(a). \ \triangleright$$

11.9.11. Remark. The construction, presented in the proof of 11.9.10, is called the GNS-*construction* or, in expanded form, the *Gelfand–Naĭmark–Segal construction*, which is reflected in the name of 11.9.10.

11.9.12. *Gelfand–Naĭmark Theorem.* *Each C^*-algebra has an isometric $*$-representation in the endomorphism algebra of a suitable Hilbert space.*

$\triangleleft$ Let A be a C^*-algebra. We have to find a Hilbert space H and an isometric $*$-representation $\mathfrak{R}$ of A in the C^*-algebra $B(H)$ of bounded endomorphisms of H. For this purpose, consider the Hilbert sum H of the family of Hilbert spaces $(H_s)_{s\in S(A)}$ which exists in virtue of the GNS-Construction Theorem; i.e.,

$$H := \bigoplus_{s\in S(A)} H_s = \left\{ h := (h_s)_{s\in S(A)} \in \prod_{s\in S(A)} H_s :\ \sum_{s\in S(A)} \|h_s\|^2_{H_s} < +\infty \right\}.$$

Observe that the inner product of $h := (h_s)_{s\in S(A)}$ and $g := (g_s)_{s\in S(A)}$ is calculated by the rule (cf. 6.1.10 (5) and 6.1.9):

$$(h,\ g) = \sum_{s\in S(A)} (h_s,\ g_s)_s.$$

Assume further that $\mathfrak{R}_s$ is a $*$-representation of A on the space H_s corresponding to s in $S(A)$. Since in view of 11.8.5 there is an estimate $\|\mathfrak{R}_s(a)\|_{B(H_s)} \le \|a\|$ for $a \in A$; therefore, given $h \in H$, infer that

$$\sum_{s\in S(A)} \|\mathfrak{R}_s(a)h_s\|^2_{H_s} \le \sum_{s\in S(A)} \|\mathfrak{R}_s(a)\|^2_{B(H_s)}\|h_s\|^2_{H_s} \le \|a\|^2 \sum_{s\in S(A)} \|h_s\|^2_{H_s}.$$

Whence it follows that the expression $\mathfrak{R}(a)h : s \mapsto \mathfrak{R}_s(a)h_s$ defines an element $\mathfrak{R}(a)h$ of H. The resulting operator $\mathfrak{R}(a) : h \mapsto \mathfrak{R}(a)h$ is a member of $B(H)$. Moreover, the mapping $\mathfrak{R} : a \mapsto \mathfrak{R}(a)$ $(a \in A)$ is a sought isometric $*$-representation of A.

Indeed, from the definition of $\mathfrak{R}$ and the properties of $\mathfrak{R}_s$ for $s \in S(A)$, it follows easily that $\mathfrak{R}$ is a $*$-representation of A in $B(H)$. Check for instance that $\mathfrak{R}$ agrees with involution. To this end, take $a \in A$ and $h,\ g \in H$. Then

$$(\mathfrak{R}(a^*)h,\ g) = \sum_{s \in S(A)} (\mathfrak{R}_s(a^*)h_s,\ g_s)_s$$

$$= \sum_{s \in S(A)} (\mathfrak{R}_s(a)^* h_s,\ g_s)_s = \sum_{s \in S(A)} (h_s,\ \mathfrak{R}_s(a) g_s)_s$$

$$= (h,\ \mathfrak{R}(a)g) = (\mathfrak{R}(a)^* h,\ g).$$

Since h and g in H are arbitrary, conclude that $\mathfrak{R}(a^*) = \mathfrak{R}(a)^*$.

It remains to establish only that the $*$-representation $\mathfrak{R}$ is an isometry, i.e. the equalities $\|\mathfrak{R}(a)\| = \|a\|$ for all $a \in A$. First, assume a positive. From the Spectral Theorem and the Weierstrass Theorem it follows that $\|a\| \in \mathrm{Sp}(a)$. In virtue of 11.9.3 (1) there is a state $s \in S(A)$ such that $s(a) = \|a\|$. Using the properties of the vector x_s corresponding to the $*$-representation $\mathfrak{R}_s$ (cf. 11.9.10) and applying the Cauchy–Bunyakovskiĭ–Schwarz inequality, infer that

$$\|a\| = s(a) = (\mathfrak{R}_s(a)x_s, x_s)_s \le \|\mathfrak{R}_s(a)x_s\|_{H_s} \|x_s\|_{H_s}$$

$$\le \|\mathfrak{R}_s(a)\|_{B(H_s)} \|x_s\|^2_{H_s} = \|\mathfrak{R}_s(a)\|_{B(H_s)} (x_s, x_s)_s$$

$$= \|\mathfrak{R}_s(a)\|_{B(H_s)} (\mathfrak{R}_s(1)x_s,\ x_s)_s = \|\mathfrak{R}_s(a)\|_{B(H_s)} s(1) = \|\mathfrak{R}_s(a)\|_{B(H_s)}.$$

From the estimates $\|\mathfrak{R}(a)\| \ge \|\mathfrak{R}_s(a)\|_{B(H_s)}$ and $\|a\| \ge \|\mathfrak{R}(a)\|$, the former obvious and the latter indicated in 11.8.5, derive

$$\|a\| \ge \|\mathfrak{R}(a)\| \ge \|\mathfrak{R}_s(a)\|_{B(H_s)} \ge \|a\|.$$

Finally, take $a \in A$. By the Kaplanski–Fukamija Lemma, a^*a is positive. So,

$$\|\mathfrak{R}(a)\|^2 = \|\mathfrak{R}(a)^* \mathfrak{R}(a)\| = \|\mathfrak{R}(a^*)\mathfrak{R}(a)\| = \|\mathfrak{R}(a^*a)\| = \|a^*a\| = \|a\|^2.$$

No further explanation is needed. ▷

Exercises

11.1. Give examples of Banach algebras and non-Banach algebras.

11.2. Let A be a Banach algebra. Take $\chi \in A^{\#}$ such that $\chi(1) = 1$ and $\chi(\mathrm{Inv}\,(A)) \subset \mathrm{Inv}\,(\mathbb{C})$. Prove that χ is multiplicative and continuous.

11.3. Let the spectrum $\mathrm{Sp}(a)$ of an element a of a Banach algebra A lie in an open set U. Prove that there is a number $\varepsilon > 0$ such that $\mathrm{Sp}(a + b) \subset U$ for all $b \in A$ satisfying $\|b\| \le \varepsilon$.

11.4. Describe the maximal ideal spaces of the algebras $C(Q, \mathbb{C})$ and $C^{(1)}([0, 1], \mathbb{C})$ with pointwise multiplication, and of the algebra of two-way infinite summable sequences $l_1(\mathbb{Z})$ with multiplication

$$(a * b)(n) := \sum_{k=-\infty}^{\infty} a_{n-k} b_k.$$

11.5. Show that a member T of the endomorphism algebra $B(X)$ of a Banach space X has a left inverse if and only if T is a monomorphism and the range of T is complemented in X.

11.6. Show that a member T of the endomorphism algebra $B(X)$ of a Banach space X has a right inverse if and only if T is an epimorphism and the kernel of T is complemented in X.

11.7. Assume that a Banach algebra A has an element with disconnected spectrum (having a proper clopen part). Prove that A has a nontrivial idempotent.

11.8. Let A be a unital commutative Banach algebra and let E be some set of maximal ideals of A. Such a set E is a ***boundary*** of A if $\|\widehat{a}\|_\infty = \sup|\widehat{a}(E)|$ for all $a \in A$. Prove that the intersection of all closed boundaries of A is also a boundary of A. This is the ***Shilov boundary*** of A.

11.9. Let A and B be unital commutative Banach algebras, with $B \subset A$ and $1_B = 1_A$. Prove that each maximal ideal of the Shilov boundary of B lies in some maximal ideal of A.

11.10. Let A and B be unital C^*-algebras and let T be a morphism from A to B. Assume further that a is a normal element of A and f is a continuous function on $\mathrm{Sp}_A(a)$. Demonstrate that $\mathrm{Sp}_B(Ta) \subset \mathrm{Sp}_A(a)$ and $Tf(a) = f(Ta)$.

11.11. Let $f \in A'$, with A a commutative C^*-algebra. Show that f is a ***positive form*** (i.e., $f(a^*a) \geq 0$ for $a \in A$) if and only if $\|f\| = f(1)$.

11.12. Describe extreme rays of the set of positive forms on a commutative C^*-algebra.

11.13. Prove that the algebras $C(Q_1, \mathbb{C})$ and $C(Q_2, \mathbb{C})$, with Q_1 and Q_2 compact, are isomorphic if and only if Q_1 and Q_2 are homeomorphic.

11.14. Let a normal element a of a C^*-algebra has real spectrum. Prove that a is hermitian.

11.15. Using the continuous functional calculus, develop a spectral theory for normal operators in a Hilbert space. Describe compact normal operators.

11.16. Let T be an algebraic morphism between C^*-algebras, and $\|T\| \leq 1$. Then $T(a^*) = (Ta)^*$ for all a.

11.17. Let T be a normal operator in a Hilbert space H. Show that there are a hermitian operator S in H and a continuous function $f : \mathrm{Sp}(S) \to \mathbb{C}$ such that $T = f(S)$. Is an analogous assertion valid in C^*-algebras?

11.18. Let A and B be C^*-algebras and let ρ be a $*$-monomorphism from A to B. Prove that ρ is an isometric embedding of A into B.

11.19. Let a and b be hermitian elements of a C^*-algebra A. Assume that $ab = ba$ and, moreover, $a \leq b$. Prove that $f(a) \leq f(b)$ for (suitable restrictions of) every increasing continuous scalar function f over $\mathbb{R}$.

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Notation Index

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